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Remarks on infinite dimensional duality

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Abstract We present an improvement of a recent duality theorem and a new result which stresses the fact that the strong duality, without assumptions on the interior of the ordering cone, is related to the normal cone.

Keywords Strong duality · Quasi-relative interior · Tangent cone · Normal cone

1 Introduction

In the papers [3] and [5] the authors present an infinite dimensional duality theory which, with the aid of a generalized constraint qualification assumption related to the notion of quasi relative interior, guarantees the existence of strong duality between a convex optimization problem and its Lagrange dual. The use of quasi relative interior, introduced by Borwein and Lewis [1], and the notions of tangent and normal cone, allows to overcome the difficulty that in many cases the interior of the set involved in the regularity condition is empty. This is the case of all optimization problems or variational inequalities connected with network equilibrium problems, the obstacle problem, the elastic-plastic torsion problem [2–4,6–14,17–20] which use positive cones of $L^{p}(\Omega)$ or Sobolev spaces. Then the usual interior conditions, as the core, the intrinsic core or the strong-quasi relative interior condition [21] are not suitable for our problem because, for example, the strong-quasi relative interior of the positive cone of $L^{p}(\Omega)$, namely the most general condition among the above mentioned ones, is empty. Also the result of Jeyakumar and Wolkowicz [16] which uses the notion of quasi relative interior, however requires that the cone defining the constraints has a non empty interior. The

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crucial assumption used by [3], which allows to obtain the strong duality results, is the so called Assumption S, namely:

$$T_{\tilde{M}}(f(x_0), \Theta_Y) \cap]-\infty, 0[\times\{\Theta_Y\} = \emptyset \tag{1}$$

where $T_{\tilde{M}}(f(x_0), \Theta_Y)$ is the tangent cone to $\tilde{M}$ at $(f(x_0), \Theta_Y)$ (see Sect. 2) and

$$\tilde{M} = \{(f(x) + \alpha, g(x) + y), x \in S\backslash K, \alpha \geq 0, y \in C\}$$

with $f : S \to \mathbb{R}$, $g : S \to Y$, S a subset of a real linear topological space X, Y a real normed space ordered by a convex cone C,

$$K = \{x \in S : g(x) \in -C\}$$

and

$$f(x_0) = \min_{x \in K} f(x). \qquad \text{(Problem 1)}$$

Assuming that $(f, g) : S \to \mathbb{R} \times Y$ is convex-like with respect to the product cone $\mathbb{R}_+ \times C$, Assumption S guarantees that the dual problem:

$$\max_{u \in C^*} \inf_{x \in S} [f(x) + \langle u, g(x)\rangle] \qquad \text{(Problem 2)}$$

where

$$C^* = \{u \in Y^* : \langle u, y\rangle \geq 0, \ \forall y \in C\}$$

is solvable, the extremal values of both Problems 1 and 2 are equal and, if $\overline{u}$ is the extremal point of Problem 2, then it results:

$$\langle \overline{u}, g(x_0)\rangle = 0.$$

Assumption S is very useful for the applications. For example, it is immediately seen that the convex obstacle problem, namely

$$f(x_0) = \min_{x \in K} f(x); \quad K = \{x \in X : -x \in -C\}$$

verifies Assumption S (see Remark 5 in [3]) and the same happens for all known equilibrium problems.

In the present paper we present a theoretical result which points out the fact that the strong duality without assumptions on the interior of the ordering cone, is connected to the normal cone (see Sect. 2). Precisely we will show the following result:

Theorem 4.1 *Let us assume that the function $(f, g) : S \to \mathbb{R} \times Y$ is convex-like, that Problem 1 is solvable and that the following assumption holds:*

There exist $\overline{x} \in K$ and $(\hat{\xi}, \hat{y}^*) \in N_M(0, \Theta_Y)$ such that

$$\hat{\xi}(f(\overline{x}) - f(x_0)) + \langle \hat{y}^*, g(\overline{x})\rangle < 0 \qquad \text{(Assumption } N\text{)}$$

where

$$M = \{(f(x) - f(x_0) + \alpha, \ g(x) + y) : x \in S, \alpha \geq 0, y \in C\}.$$

Then Problem 2 is solvable and the extremal values of both problems are equal.

From Assumption N it follows that $N_M(0, \Theta_Y)$ is not a linear subspace and hence (see Prop. 2.1), $(0, \Theta_Y) \notin \text{qri}M$. In this way we see how the strong duality is connected with the qri notion. Vice versa, if we make an assumption of the type $(0, \Theta_Y) \notin \text{qri}M$, or a similar one, immediately we get that $N_M(0, \Theta_Y)$ is not a linear subspace and hence there exists $(\hat{\xi}, \hat{y}^*) \in N_M(0, \Theta_Y) \neq (0, \Theta_Y)$.

The paper is organized in the following way. In Sect. 2 we recall some useful definitions and theorems. In Sect. 3, following a nice suggestion by C. Zalinescu, we show that in Theorem 4 by [3] only the convex-like and S assumptions are needed and in Sect. 4 we prove Theorem 4.1.

2 Preliminary concepts and results

Let X denote a real normed space and X^* the topological dual of all continuous linear functionals on X, and let C be a subset of X. Given an element $x \in X$, the set:

$$T_C(x) = \{h \in X : h = \lim_n \lambda_n(x_n - x),\ \lambda_n \in \mathbb{R},\ \lambda_n \geq 0,\ x_n \in C\ \forall n \in \mathbb{N}, \lim_n x_n = x\} \tag{2}$$

is called the tangent cone to C at x. Of course, if $T_C(x) \neq \emptyset$, then $x \in Cl\ C$. If $x \in Cl\ C$ and C is convex we have [15]:

$$T_C(x) = Cl\,\text{Cone}(C - \{x\})$$

where

$$\text{Cone}(C) = \{\lambda x : x \in C, \lambda \in \mathbb{R}, \lambda \geq 0\}.$$

Following Zarantonello [21] and Borwein and Lewis [1], we give the following defintion of quasi-relative interior for a convex set.

Definition 2.1 Let C be a convex subset of X. The quasi-relative interior of C, denoted by qri$C$, is the set of those $x \in C$ for which $T_C(x)$ is a linear subspace of X.

If we define the normal cone to C at x as the set:

$$N_C(x) = \{\xi \in X^* : \langle \xi, y - x\rangle \leq 0, \forall y \in X\},$$

the following result holds:

Proposition 2.1 *Let C be a convex subset of X and $x \in C$. Then $x \in C$ belongs to the quasi-relative interior of C, in short, $x \in qriC$, if and only if $N_C(x)$ is a linear subspace of X.*

Using the notion of qri$C$, in [3], the following separation theorem is proved.

Theorem 2.1 *Let C be a convex subset of X and $x_0 \in C \setminus qriC$. Then, there exists $\xi \neq \theta_{X^*}$ such that*

$$\langle \xi, x\rangle \leq \langle \xi, x_0\rangle, \quad \forall x \in C.$$

Vice versa, let us suppose that there exists $\xi \neq \theta_{X^*}$ and a point $x_0 \in X$ such that $\langle \xi, x\rangle \leq \langle \xi, x_0\rangle$, $\forall x \in C$, and that $Cl(T_C(x_0) - T_C(x_0)) = X$. Then $x_0 \notin qriC$.

Finally we recall the definition of convex-like function.

Definition 2.2 Let S be a nonempty subset of a linear real space X and let Y be a linear real space partially ordered by the cone C. A function $g : S \longrightarrow Y$ is called convex-like if the set $g(S) + C$ is convex.

3 An improvement of Theorem 4 of [3] (C. Zalinescu)

In the present section we show an improvement of Theorem 4 of [3]. For the reader's convenience we present the statement of Theorem 4 of [3]. Let X be a real linear topological space and S a nonempty subset of X; let $(Y, \|\cdot\|)$ be a real normed space partially ordered by a convex cone C. Let $f : S \longrightarrow \mathbb{R}$ and $g : S \longrightarrow Y$ be two functions such that the function (f, g) is convex-like with respect to the cone $\mathbb{R}_+ \times C$ of $\mathbb{R} \times Y$.

Let us consider the problem

$$\min_{x \in K} f(x) \qquad \text{(Problem 1)}$$

where

$$K = \{x \in S : g(x) \in -C\}$$

and the dual problem

$$\max_{u \in C^*} \inf_{x \in S} \{f(x) + \langle u, g(x) \rangle\}, \qquad \text{(Problem 2)}$$

where

$$C^* = \{u \in Y^* : \langle u, y \rangle \geq 0,\ \forall y \in C\}$$

is the dual cone of C.

We will say that Assumption S is fulfilled at a point $x_0 \in K$ if it results:

$$T_{\tilde{M}}(f(x_0), \Theta_Y) \cap\,]-\infty, 0[\times \{\theta_Y\} = \emptyset \qquad \text{(Assumption } S\text{)}$$

where

$$\tilde{M} = \{(f(x) + \alpha, g(x) + y) : x \in S \backslash K, \alpha \geq 0, y \in C\}.$$

Then, in [3] the following theorem is proved.

Theorem 3.1 *Under the above assumptions, let the set K be nonempty and let us assume that $qriC \neq \emptyset$, $Cl(C - C) = Y$ and there exists $\overline{x} \in S$ with $g(\overline{x}) \in -qriC$. Then, if Problem 1 is solvable and Assumption S is fulfilled at the extremal solution $x_0 \in K$, also Problem 2 is solvable, the extremal values of both problems are equal and it results:*

$$\langle \overline{\lambda}, g(x_0) \rangle = 0$$

where $\overline{\lambda}$ is the extremal point of Problem 2.

Now, following a nice suggestion of C. Zalinescu, Theorem 3.1 can be improved removing the assumptions $\text{qri}C \neq \emptyset$, $Cl(C - C) = Y$ and there exists $\overline{x} \in S$ with $g(\overline{x}) \in -\text{qri}C$. Precisely, we can prove the following

Theorem 3.2 *Let us assume that the function* $(f, g) : S \longrightarrow \mathbb{R} \times Y$ *is convex-like. Then if Problem 1 is solvable and Assumption S is fulfilled at the extremal solution* $x_0 \in S$*, also Problem 2 is solvable, the extreme values of both problems are equal and it results:*

$$\langle \overline{\lambda}, g(x_0\rangle = 0$$

where $\overline{\lambda}$ *is the extremal point of Problem 2.*

Proof Following the proof of Theorem 3.1 by [3], using only Assumption *S*, we derive that if $(\gamma, y) \in T_M(f(x_0), \Theta_Y)$ where

$$M = \{(f(x) + \alpha, g(x) + y) : x \in S, \alpha \geq 0, y \in C\} = (f, g)(S) + \mathbb{R}_+ \times C$$

then either $\gamma \geq 0$ or $y \neq \Theta_Y$. In particular

$$(-1, 0) \notin T_M(f(x_0, \Theta_Y) = \overline{\text{Cone}(M - (f(x_0, \Theta_Y)}.$$

Since this set is a closed convex cone (*M* being convex), in virtue of a well known separation theorem (see for example [15]), we obtain that there exists $(\mu, y^*) \in \mathbb{R} \times Y^*$ such that

$$-\mu < 0 \leq \mu(f(x) + \alpha - f(x_0)) + \langle y^*, g(x) + y\rangle, \ \forall x \in S, \forall \alpha \geq 0, \forall y \in C. \quad (3)$$

Setting $y^*/\mu = \overline{\lambda}$ and assuming $\alpha = 0$, $x = x_0$, we get

$$\langle \overline{\lambda}, g(x_0) + y\rangle \geq 0, \quad \forall y \in C$$

and hence, assuming $y = z - g(x_0)$ for all $z \in C$, we get

$$\langle \overline{\lambda}, y\rangle \geq 0, \quad \forall y \in C$$

namely $\overline{\lambda} \in C^*$. Moreover, from (3), assuming $\alpha = 0$ and $y = 0$, we get

$$f(x_0) \leq f(x) + \langle \overline{\lambda}, g(x)\rangle, \quad \forall x \in S.$$

Taking $x = x_0$, we obtain $\langle \overline{\lambda}, g(x_0)\rangle \geq 0$ and, since $-g(x_0 \in C$, $\langle \overline{\lambda}, g(x_0)\rangle \leq 0$, and so $\langle g(x_0), y^*\rangle = 0$.
Starting from this point the proof continues as in the next theorem. □

4 Proof of Theorem 4.1

Let us assume that Assumption *N* holds and consider the set

$$M = \{f(x) - f(x_0) + \alpha, \ g(x) + y) : x \in S, \alpha \geq 0, y \in C\} \quad (4)$$

where $x_0 \in K$ is the extremal solution to Problem 1. The normal cone $N_M(0, \Theta_Y)$ to M at $(0, \Theta_Y) \in M$ is given by the points $(\xi, y^*) \in \mathbb{R} \times Y^*$ such that:

$$\xi(f(x) - f(x_0) + \alpha) + \langle y^*, g(x) + y\rangle \leq 0, \quad \forall x \in S, \ \forall \alpha \geq 0, \ \forall y \in C. \quad (5)$$

Then, if $(\xi, y^*) \in N_M(0, \Theta_Y)$, assuming in (5) $x \in K$ and $y = -g(x) \in C$, we get

$$\xi(f(x) - f(x_0)) \leq 0, \quad \forall x \in K,$$

and hence we obtain

$$\xi \leq 0. \quad (6)$$

Now, choosing $x = x_0$, $\alpha = 0$ and $y = -g(x_0) + z$, $\forall z \in C$ we get

$$\langle y^*, z\rangle \le 0, \quad \forall z \in C,$$

namely

$$y^* \in C^-. \tag{7}$$

As a consequence, if $(\xi, y^*) \in N_M(f(x_0, \Theta_Y)$, it must be $\xi \le 0$ and $y^* \in C^-$. In virtue of Assumption *N*, i.e. there exist $\overline{x} \in K$, and $(\hat{\xi}, \hat{y}^*) \in N_M(0, \Theta_Y)$ such that

$$\hat{\xi}(f(\overline{x}) - f(x_0)) + \langle \hat{y}^*, g(\overline{x})\rangle < 0, \tag{8}$$

we get that $(\hat{\xi}, \hat{y}^*)$ is different from $(0, \Theta_Y)$.
Let us prove that $\hat{\xi} \neq 0$. In fact, if $\hat{\xi} = 0$, from (8) one could obtain $\langle \hat{y}^*, g(\overline{x})\rangle < 0$, but being $\overline{x} \in K$, it should be $\langle \hat{y}^*, g(\overline{x})\rangle \ge 0$.
Then $\hat{\xi} < 0$ and from (5), setting $\hat{y}^*/\hat{\xi} = \overline{u} \in C^*$ we get

$$f(x) + \langle \overline{u}, g(x)\rangle \ge f(x_0). \tag{9}$$

It is easy to see that $\langle \overline{u}, g(x_0)\rangle = 0$. In fact from (9) we get $\langle \overline{u}, g(x_0)\rangle \ge 0$ whereas, being $-g(x_0) \in C$, we have $\langle \overline{u}, g(x_0)\rangle \le 0$.
Hence, we obtain:

$$f(x_0) + \langle \overline{u}, g(x_0)\rangle \le f(x) + \langle \overline{u}, g(x)\rangle, \ \forall x \in S$$
$$f(x_0) + \langle \overline{u}, g(x_0)\rangle \le \inf_{x \in S}[f(x) + \langle \overline{u}, g(x)\rangle].$$

Moreover, $\forall u \in C^*$, since $g(x_0) \in -C$, $\langle u, g(x_0)\rangle \le 0, \forall u \in C^*$, then

$$\inf_{x \in S}[f(x) + \langle u, g(x)\rangle] \le f(x_0) + \langle u, g(x_0)\rangle \le f(x_0), \quad \forall u \in C^*.$$

Then

$$\sup_{u \in C^*} \inf_{x \in S}[f(x) + \langle u, g(x)\rangle] \le f(x_0) + \langle \overline{u}, g(x_0)\rangle \le \inf_{x \in S}[f(x) + \langle \overline{u}, g(x)\rangle]$$

which yields

$$\sup_{u \in C^*} \inf_{x \in S}[f(x) + \langle u, g(x)\rangle] = \inf_{x \in S}[f(x) + \langle \overline{u}, g(x)\rangle]$$

and

$$f(x_0) + \langle \overline{u}, g(x_0)\rangle \le \sup_{u \in C^*} \inf_{x \in S}[f(x) + \langle u, g(x)\rangle] \le f(x_0) + \langle \overline{u}, g(x_0)\rangle$$

and finally

$$f(x_0) = \max_{u \in C^*} \inf_{x \in S}[f(x) + \langle u, g(x)\rangle] = \inf_{x \in S}[f(x) + \langle \overline{u}, g(x)\rangle]. \tag{10}$$

The example which follows shows that Assumption *N* is not always verified.

Example Let $X = l^2 = S = Y$, $C = l^2_+$. Let $A : X \to X$ be given by $(Ax)_{n \in N} = x_n/2^n$ and let $f(x) = \langle c, x\rangle$, where $c_n = 1/n$. Let $g(x) = -Ax$ and consider the problem

$$\min_{x \in K}\langle c, x\rangle = 0 = \langle c, 0\rangle$$

where

$$K = \{x \in l^2 : -Ax \in -C\}.$$

Let us show that Assumption N is not fulfilled.
In fact it should exist a pair $(\hat{\xi}, \hat{y}^*) \in N_M(0, \theta_{l^2})$ such that

$$\hat{\xi}\langle c, \hat{x}\rangle + \langle \hat{y}^*, -A\hat{x}\rangle < 0,$$

where

$$M = \left\{(\langle c, x\rangle + \alpha, -Ax + y), x \in l^2, \alpha \geq 0, y \in l^2_+\right\}.$$

Let us check what $(\hat{\xi}, \hat{y}^*) \in N_M(0, \theta_{l^2})$ means. We would have, in particular,

$$\hat{\xi}\langle c, x\rangle + \langle \hat{y}^*, -Ax\rangle \leq 0, \quad \forall x \in l^2$$

namely:

$$\sum_{n=1}^{\infty}\left(\frac{\hat{\xi}}{n} - \frac{\hat{y}^*_n}{2^n}\right) \leq 0, \quad \forall x \in l^2.$$

By choosing $x \in l^2$ of the type $x = (0, 0, \ldots, \pm 1, \ldots, 0, 0)$ we get:

$$\frac{\hat{\xi}}{n} - \frac{\hat{y}^*_n}{2^n} = 0, \quad \text{i.e. } \hat{y}^*_n = \frac{2^n\hat{\xi}}{n}.$$

It follows that $\hat{y}^*$ belongs to l^2 if and only if $\hat{\xi} = 0$ and then also $\hat{y}^* = 0$.

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