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A new augmented Lagrangian approach to duality and exact penalization

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Abstract In this paper, we introduce a new notion of augmenting function known as indicator augmenting function to establish a minmax type duality relation, existence of a path of solution converging to optimal value and a zero duality gap relation for a nonconvex primal problem and the corresponding Lagrangian dual problem. We also obtain necessary and sufficient conditions for an exact penalty representation in the framework of indicator augmented Lagrangian.

Keywords Augmented Lagrangian · Augmenting function · Nonconvex problem · Duality

Mathematics Subject Classification (2000) 90C26 · 90C46

1 Introduction

Augmented Lagrangian with a convex quadratic augmenting function was formally introduced by Rockafellar [7,8] to eliminate the duality gap between the primal constrained optimization problem and its Lagrangian dual problem. Later Rockafellar and Wets [9] considered a general augmented Lagrangian with convex augmenting function to establish that there is no duality gap between a nonconvex primal problem and the corresponding augmented dual problem. For this purpose a dualizing parameterization function $f(x, u)$ not necessarily convex in x but convex in u and certain coercivity conditions, were imposed (see Theorem 11.59). A necessary and sufficient condition for the exact penalty representation in the framework of the augmented Lagrangian was also obtained (see [9, Theorem 11.61]).

Later Huang and Yang [2] extended the results by considering a generalized augmented Lagrangian by relaxing the convexity on the augmenting function to level boundedness

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assumption. Zero duality gap and exact penalization results were established under weaker conditions than those in [9]. Huang and Yang [2] established the existence of a path of optimal solutions generated by generalized augmented Lagrangian problems, which converged to the optimal set of the primal problem, but a minmax type relation as in Theorem 11.59 of [9] was missing in Theorem 2.1 of [2]. This result further implied the zero duality gap property between the generalized augmented Lagrangian dual problem and the primal problem. The existence and convergence of a path of optimal solutions generated by penalty type problems toward the optimal set is important for numerical methods (refer Auslender [1] and Yang and Huang [10]).

Rubinov et al. [4] studied the zero duality gap property for an augmented dual problem constructed using a family of augmenting functions. Necessary and sufficient conditions for no duality gap have been given using tools from abstract convexity (see Rubinov [5], Rubinov and Yang [6]), under the assumption that the augmenting family contains an augmenting function minorizing the primal function. Recently Nedic and Ozdaglar [3] provided a unifying geometric framework for the analysis of general classes of duality schemes and penalty methods for nonconvex constrained optimization problems.

In this paper zero duality gap relation are obtained for a nonconvex primal problem. This is achieved by considering a new form of augmenting function $\sigma(u, r)$, which is convex in u but not linear in r. It is defined as indicator function of a closed unit ball of radius r with center at 0. This function has zero as the minimum value for each fixed r but argmin of this function is not a set containing just 0. There are many advantages of considering this type of augmenting function, which we refer to as indicator augmenting function. This function will serve as augmenting function for any problem and there is no need of searching for a suitable augmenting function corresponding to a primal problem in the absence of convexity of u of the dualizing parameterization function $f(x, u)$. In Example 3.5 it is shown that the duality relation of Rockafellar and Wets [9] and Huang and Yang [2] do not hold even for a convex primal problem when f is not convex in u with a convex augmenting function but holds with indicator augmenting function. Also since $\sigma(u, r)$ is convex in u both minmax type relation and existence of optimal path converging to optimal solution can be established. Various equivalent criteria for the exact penalty criterion can be established without any extra condition as in Theorem 3.1 of Huang and Yang [2].

The indicator augmenting function differs from the augmenting function considered in [2,9] since argmin $\sigma(., r)$ is the whole of the region where it takes value 0 and not just a singleton set. The corresponding Lagrangian function is in fact quasiconcave and nonincreasing in r unlike the augmented Lagrangian of [2,9] which is concave and nondecreasing in r. Hence the dual function converges to the optimal value as the parameter r approaches 0 and not ∞.

The outline of this paper is as follows. In Sect. 2, we give a brief review of the existing concepts of augmenting function, augmented Lagrangian and augmented Lagrangian dual problem. In Sect. 3, we introduce the notion of indicator augmenting function and the corresponding notions of augmented Lagrangian, augmented Lagrangian dual function and augmented Lagrangian dual problem. We also establish both minmax type duality relation and the existence of a path of optimal solutions converging to the optimal set of the primal problem, which in turn implies a zero duality gap property for the primal problem and its indicator augmented Lagrangian dual problem. In Sect. 4, we obtain necessary and sufficient conditions for an exact penalty representation in the framework of indicator augmented Lagrangian.

2 Preliminaries

In this section we give a brief review of the concepts of Lagrangian and augmented Lagrangian theory for an unconstrained optimization problem.

Consider the following optimization problem:

$$(\mathrm{P}) \qquad \inf_{x\in R^n}\phi(x)$$

where $\phi : R^n \to \bar{R} = R \cup \{+\infty, -\infty\}$ is an extended real valued function. Suppose that $f(x,u) : R^n \times R^m \to \bar{R}$ is a dualizing parameterization function of u, that is

$$f(x,0) = \phi(x), \qquad \forall x \in R^n.$$

The Fenchel–Moreau conjugate of f, denoted by f^*, is defined as

$$f^*(\nu, y) = \sup_{(x,u)}\{\langle(\nu, y), (x,u)\rangle - f(x,u)\}.$$

The dual problem associated to (P) is

$$(\mathrm{D}) \qquad \sup_{y\in R^m}\psi(y)$$

where $\psi(y) = -f^*(0, y)$.

In fact if f is proper, lower semicontinuous (lsc in short) and convex function (ϕ is in turn convex and lsc) then under certain mild conditions there exist no duality gap, that is, $\inf_x\phi(x) = \sup_y\psi(y)$. In terms of a perturbation function $p : R^m \to \bar{R}$ defined as

$$p(u) = \inf_x f(x,u), \tag{2.1}$$

it is observed that $p(0) = \inf_x f(x,0)$ and $p^{**}(0) = \sup_y\psi(y)$. Moreover minmax results in terms of the associated Lagrangian $l(x,y) : R^n \times R^m \to \bar{R}$ given by

$$l(x,y) = \inf_u\{f(x,u) - \langle y,u\rangle\}, \tag{2.2}$$

exist in literature (for details refer to Theorem 11.50 of Rockafellar and Wets [9]).

The next question that arises is whether the no duality gap relation still holds in the absence of convexity assumption. Duality gap $\inf_x\phi(x) > \sup_y\psi(y)$ arises when p is a nonconvex function. To deal with the situation, when ϕ is nonconvex, Rockafellar and Wets in their seminal book [9] use augmenting functions to construct augmented Lagrangian functions and to show that there is no duality gap between the nonconvex primal problem and the corresponding augmented dual problem under certain coercivity assumptions. This method known as augmented Lagrangian method has been widely and successfully used in the solution of constrained optimization problems. Assuming the dualizing parameterization function f to be convex in u they introduced an *augmenting function* $\sigma : R^m \to \bar{R}$, that is proper, lsc, convex function and

$$\min\sigma(u) = 0 \quad \text{and} \quad \operatorname{argmin}\sigma(u) = \{0\}.$$

The corresponding augmented Lagrangian with penalty parameter $r > 0$ is the function $\bar{l}(x,y,r) : R^n \times R^m \times (0,\infty) \to \bar{R}$ defined by

$$\bar{l}(x,y,r) = \inf_u\{f(x,u) + r\sigma(u) - \langle y,u\rangle\}$$

and the corresponding dual function $\bar{\psi}(y,r) : R^m \times (0,\infty) \to \bar{R}$ is given by

$$\bar{\psi}(y,r) = \inf_x\bar{l}(x,y,r).$$

The corresponding augmented Lagrangian dual problem is

$$(\bar{\mathrm{D}}) \quad \sup_{(y,r)\in R^m\times(0,\infty)} \bar{\psi}(y,r).$$

It is established that the augmented Lagrangian $\bar{l}(x, y, r)$ and the corresponding dual function $\bar{\psi}(y, r)$ are concave, upper semicontinuous (usc in short) in (y, r) and nondecreasing in r. In a nonconvex setting with $\phi(u)$ as a nonconvex function and the parameterization $f(x, u)$ being nonconvex in x and convex in u the duality theorem is presented in Theorem 11.59 in [9]. Later Huang and Yang [2] extended the work in [9] by considering a generalized augmenting function that is not necessarily convex. According to them a function $\sigma : R^m \to \bar{R}$ is a *generalized augmenting function* if it is proper, lsc, level bounded on R^m (see Definition 3.2) and

$$\min \sigma(u) = 0 \quad \text{and} \quad \operatorname{argmin} \sigma(u) = \{0\}.$$

Based on this definition the notions of generalized augmented Lagrangian, generalized augmented Lagrangian dual function and generalized augmented Lagrangian dual problem are given. Also the dualizing parameterization function is not necessarily taken to be convex in u. To distinguish the two notions we will refer to the generalized augmented Lagrangian and the generalized augmented Lagrangian dual function by Huang and Yang [2] as $l^g(x, y, r)$ and $\psi^g(y, r)$, respectively and the generalized augmented Lagrangian dual problem by (D^g).

3 Duality involving indicator augmented Lagrangian

In this section we develop the new augmented Lagrangian approach in terms of indicator augmenting function.

First assume $f : R^n \times R^m \to \bar{R}$ to be any dualizing parameterization function that is not necessarily convex in u. We consider a function $\sigma(u, r) : R^{m+1} \to \bar{R}$ similar to the augmenting function considered by Rockafellar and Wets [9]. The function $\sigma(u, .)$ is not linear but convex in r and $\sigma(., r)$ is a convex function. The parameter r acts like a penalty parameter. In fact we define the function $\sigma(u, r)$ as the indicator function of closed ball of radius r in R^m with center at 0. We first recall that for a set A in R^n the indicator function $\delta_A : R^n \to \bar{R}$ is defined as

$$\delta_A(x) = \begin{cases} 0, & \text{if } x \in A \\ +\infty, & \text{if } x \notin A. \end{cases}$$

Formally we have the following definition.

Definition 3.1 A function $\sigma : R^{m+1} \to \bar{R}$ is said to be an *indicator augmenting function* if

$$\sigma(u, r) = \delta_{rC}(u)$$

where $C = \{u \in R^m | \|u\| \le 1\}$ and $\|u\|$ denotes the Euclidean norm.

Observe that even though the minimum value of $\sigma(., r) = 0$ for each fixed r, argmin σ is not a singleton set containing 0 only. Here

$$\min_u \sigma(u, r) = 0 \quad \text{and} \quad \operatorname{argmin}_u \sigma(u, r) = rC,$$

where for each $r > 0$ $rC = \{u \in R^m | \|u\| \le r\}$. Obviously $\sigma(., r)$ being an indicator function over a compact set it is proper, convex and lsc in u for each fixed r. Clearly, $\sigma(u, r) = \delta_{rC}(u)$ is a convex function of r as for $0 \le \alpha \le 1$

$$\delta_{(\alpha r_1+(1-\alpha)r_2)C}(u) \le \alpha\delta_{r_1C}(u) + (1-\alpha)\delta_{r_2C}(u).$$

In fact

$$\delta_{\max(r_1,r_2)C}(u) \le \delta_{(\alpha r_1+(1-\alpha)r_2)C}(u) \le \delta_{\min(r_1,r_2)C}(u).$$

Also

$$\alpha\delta_{rC}(u) = \delta_{rC}(u) \quad \text{for every} \quad \alpha > 0.$$

We define the perturbation function p as in (2.1) and assume that $p(0) = \inf_x f(x, 0)$ is finite throughout the sequel.

The *indicator augmented Lagrangian* is defined as

$$\begin{aligned} l^i(x, y, r) &= \inf_u\{f(x, u) + \delta_{rC}(u) - \langle y, u\rangle\} \\ &= \inf_{\|u\|\le r}\{f(x, u) - \langle y, u\rangle\} \quad x \in R^n,\ y \in R^m \text{ and } r > 0. \end{aligned}$$

The function $l^i(x, y, r)$ is similar to $l(x, y)$ given by (2.2) with the difference that the infimum is now taken over a closed ball in R^m of radius r with center at 0 instead of the whole space R^m.

The *indicator augmented Lagrangian dual function* is

$$\psi^i(y, r) = \inf_x l^i(x, y, r) \quad y \in R^m \quad \text{and} \quad r > 0.$$

The *indicator augmented Lagrangian dual problem* is

$$(\mathrm{D}^i) \quad \sup_{(y,r)\in R^m\times(0,\infty)} \psi^i(y, r).$$

Clearly the weak duality holds since for $y \in R^m$ and $r > 0$

$$\begin{aligned} \psi^i(y, r) &= \inf_x l^i(x, y, r) = \inf_x \inf_u\{f(x, u) - \langle y, u\rangle + \delta_{rC}(u)\} \\ &\le \inf_x\{f(x, 0)\} = p(0). \end{aligned}$$

Before establishing the no zero duality gap condition we first give certain properties of the generalized augmented Lagrangian and augmented dual function.

Theorem 3.2 *For any dualizing parameterization function f and the indicator augmenting function σ, the indicator augmented Lagrangian $l^i(x, y, r)$ is concave and usc in y, quasi-concave, usc and nonincreasing in r. Also it is convex in (x, r) if f is convex in (x, u). Likewise the indicator augmented Lagrangian dual function $\psi^i(y, r)$ is concave and usc in y and quasiconcave, usc and nonincreasing in r.*

Proof Obviously $l^i(x, y, r)$ is concave and usc in y. For $r_1 < r_2$ we have $\delta_{r_1C}(u) \ge \delta_{r_2C}(u)$ and hence $l^i(x, y, r_1) \ge l^i(x, y, r_2)$. Now

$$\begin{aligned} l^i(x, y, \alpha r_1 + (1-\alpha)r_2) &= \inf_u\{f(x, u) - \langle y, u\rangle + \delta_{(\alpha r_1+(1-\alpha)r_2)C}(u)\} \\ &= \inf_{\|u\|\le(\alpha r_1+(1-\alpha)r_2)}\{f(x, u) - \langle y, u\rangle\} \\ &\ge \inf_{\|u\|\le\max(r_1,r_2)}\{f(x, u) - \langle y, u\rangle\}. \end{aligned}$$

If $r_1 < r_2$ we have $l^i(x, y, r_1) \ge l^i(x, y, r_2)$ and

$$\begin{aligned} \inf_{\|u\|\le\max(r_1,r_2)}\{f(x, u) - \langle y, u\rangle\} &= l^i(x, y, r_2) \\ &= \min\{l^i(x, y, r_1), l^i(x, y, r_2)\}. \end{aligned}$$

Thus $l^i(x, y, r)$ is quasiconcave in r. The proof pertaining to the function $\psi^i(y, r)$ follows likewise. □

As observed previously the function $l^i(x, y, r)$ is similar to $l(x, y)$ given by (2.2) and in fact the following theorem gives a relation among them in terms of support function when the dualizing parameterization function f is convex.

Theorem 3.3 *For x in R^n for which $\phi(x)$ is finite and f is convex in (x, u) the indicator augmented Lagrangian is given in terms of the auxiliary Lagrangian $l(x, y)$ by*

$$l^i(x, y, r) = \sup_z\{l(x, y - z) - \rho_{rC}(z)\}$$

and the indicator augmented Lagrangian dual function is given in terms of the dual function $\psi(y)$ by

$$\psi^i(y, r) = \sup_z\{\psi(y - z) - \rho_{rC}(z)\}$$

where $\rho_{rC}(z) = \sup_{u\in rC}\langle z, u\rangle$ is the support function of the set rC.

Proof If f is convex in (x, u) it follows that $l^i(x, y, r)$ is convex in (x, r) as it is defined as the infimum over u of a convex function $f(x, u) + \delta_{rC}(u) - \langle y, u\rangle$. Observe that

$$-l^i(x, y, r) = \sup_u\{\langle y, u\rangle - f(x, u) - \delta_{rC}(u)\}$$

is the conjugate of $f(x, .)+\delta_{rC}$. By Theorem 11.23(a) of Rockafellar and Wets [9] it follows that if $\phi(x) = f(x, 0)$ is finite then

$$\begin{aligned}
-l^i(x, y, r) &= \sup_u\{\langle y, u\rangle - f(x, u)\} \oplus \sup_u\{\langle y, u\rangle - \delta_{rC}(u)\}\\
&= \inf_z\{\sup_u\{\langle y - z, u\rangle - f(x, u)\} + \sup_u\{\langle z, u\rangle - \delta_{rC}(u)\}\\
&= \inf_z\{-l(x, y - z) + \sup_{u\in rC}\langle z, u\rangle\}.
\end{aligned}$$

The proof pertaining to the function $\psi^i(y, r)$ follows likewise. □

We require the following notions from [9] to establish no zero duality gap relation.

Definition 3.2 A function $f : R^n \to \bar{R}$ is said to be *level-bounded* if, for any $\alpha \in R$, the set $\{x|f(x) \le \alpha\}$ is bounded. A function $f(x, u) : R^n \times R^m \to \bar{R}$ is said to be *level-bounded* in x *locally uniform* in u if, for each $\bar{u} \in R^m$ and $\alpha \in R$, there exists a neighborhood $U(\bar{u})$ of $\bar{u}$ along with a bounded set D in R^n, such that $\{x|f(x, u) \le \alpha\} \subset D$, for any $u \in U(\bar{u})$.

We now return to our main concern regarding the equality $\inf_x \phi(x) = \sup_{(y,r)} \psi^i(y, r)$. The nonemptyness and compactness of solution set of primal problem (P) along with the existence of a sequence converging to an optimal solution of (P) is established in the following theorem under appropriate lsc and level boundedness conditions. Another observation which forms the main crux in establishing the absence of the duality gap is that for any pair $(\bar{y}, \bar{r})$ with $\psi^i(\bar{y}, \bar{r}) > -\infty$, the indicator augmented Lagrangian dual function $\psi^i(\bar{y}, r)$ converges to the optimal value $p(0)$ of the primal problem as $r \to 0$. One of the advantage of considering the indicator augmenting function is to establish the relation $\phi(x) = \sup_{(y,r)} l^i(x, y, r)$, which in turn helps to establish minmax type relation as in (v) below. In general (as in Huang and Yang [2]) such a relation does not hold in the absence of convexity assumption on u of the dualizing parameterization function f.

Theorem 3.4 (Zero duality gap) *Consider the primal problem (P), indicator augmented Lagrangian $l^i(x, y, r)$ and indicator augmented Lagrangian dual problem (D^i). Assume that ϕ is proper, and that its dualizing parameterization function $f(x, u)$ is proper, lsc and level-bounded in x locally uniform in u. Suppose $p(u) = \inf_x f(x, u)$ and further $\psi^i(\bar{y}, \bar{r}) > -\infty$ for at least one pair $(\bar{y}, \bar{r}) \in R^m \times (0, \infty)$. Then*

(i) *the solution set of problem* (P) *is nonempty and compact;*

(ii) *there exists a sequence* $(x(r_k), u(r_k)) \in V(\bar{y}, r_k)$ *converging to* $(\bar{x}, 0)$ *as* $k \to \infty$ *corresponding to a sequence* $\{r_k\}$ *converging to 0 with* $r_k < \bar{r}$ *such that* $p(0) = \phi(\bar{x})$ *where*

$$V(\bar{y}, r) = \{(x, u) \in R^n \times R^m | f(x, u) - \langle \bar{y}, u \rangle + \delta_{rC}(u) \le \phi(x_0)\}$$

for any $x_0 \in R^n$ *with* $\phi(x_0)$ *being finite;*

(iii) $\phi(x) = \sup_{(y,r)} l^i(x, y, r)$, $\psi^i(y, r) = \inf_x l^i(x, y, r)$. *In fact if* $f(x, u)$ *is convex in* u *then* $\phi(x) = \sup_y l^i(x, y, r)$ *for every* $r > 0$*;*

(iv) $\lim_{r\to 0} \psi^i(\bar{y}, r) = p(0)$*;*

(v) $\inf_x \phi(x) = \inf_x \sup_{(y,r)} l^i(x, y, r) = \sup_{(y,r)} \inf_x l^i(x, y, r)$
$= \sup_{(y,r)} \psi^i(y, r)$;

(vi) $\bar{x} \in \arg\min_x \phi(x)$ *and* $(\bar{y}, \bar{r}) \in \arg\max_{(y,r)} \psi^i(y, r)$ *if and only if* $\inf_x l^i(x, \bar{y}, \bar{r}) = l^i(\bar{x}, \bar{y}, \bar{r}) = \sup_{(y,r)} l^i(\bar{x}, y, r)$.

Proof (i) From the assumption that the dualizing parameterization function $f(x, u)$ is lsc, and level-bounded in x, we see that ϕ is lsc and level-bounded. Hence the solution set of (P) is nonempty and compact.

(ii) Let x_0 be a point such that $\phi(x_0)$ is finite then

$$\begin{aligned} V(\bar{y}, r) &= \{(x, u) \in R^n \times R^m | f(x, u) - \langle \bar{y}, u \rangle + \delta_{rC}(u) \le \phi(x_0)\} \\ &= \{(x, u) \in R^n \times rC | f(x, u) - \langle \bar{y}, u \rangle \le \phi(x_0)\}. \end{aligned}$$

We claim $V(\bar{y}, \bar{r})$ is compact. Clearly $V(\bar{y}, \bar{r})$ is closed as f and δ_{rC} are lsc. Suppose $V(\bar{y}, \bar{r})$ is not bounded then there exists $\{(x_k, u_k)\} \in V(\bar{y}, \bar{r})$ such that $\|(x_k, u_k)\| \to \infty$ as $k \to \infty$. Since $u_k \in \bar{r}C$ which is a bounded set it follows that $\|x_k\| \to \infty$ as $k \to \infty$. Since $(x_k, u_k) \in V(\bar{y}, \bar{r})$ we have

$$f(x_k, u_k) - \langle \bar{y}, u_k \rangle \le \phi(x_0) \quad \text{for} \quad u_k \in \bar{r}C.$$

This implies that

$$f(x_k, u_k) \le \phi(x_0) + \langle \bar{y}, u_k \rangle \le \phi(x_0) + t$$

where $t = \sup_{u\in\bar{r}C} \langle \bar{y}, u \rangle$. As $f(x, u)$ is level-bounded in x locally uniform in u it follows that the sequence $\{x_k\}$ is bounded which is a contradiction. Now for $r < \bar{r}$ we have $V(\bar{y}, r) \subseteq V(\bar{y}, \bar{r})$. Since $V(\bar{y}, r)$ is closed it is compact for $r < \bar{r}$. Clearly, $(x_0, 0) \in V(\bar{y}, r)$ for all $r > 0$.

Let $\{r_k\}$ be a sequence with $r_k < \bar{r}$ such that $r_k \to 0$. Let $(x(r_k), u(r_k)) \in V(\bar{y}, r_k) \subseteq V(\bar{y}, \bar{r})$. Since $V(\bar{y}, \bar{r})$ is a compact set the sequence has a convergent subsequence. WLOG assume that the sequence $\{(x(r_k), u(r_k)\}$ converges to $(\bar{x}, \bar{u})$. As $k \to \infty$ we have $r_k C$ converges to 0, hence $u(r_k) \to 0$ as $k \to \infty$, that is $\bar{u} = 0$. Since ϕ is lsc and level bounded there exists a sequence x_k such that $\phi(x_k) \to p(0)$ as $k \to \infty$. Hence

$$\phi(\bar{x}) \le \liminf_{k\to\infty} \phi(x(r_k)) = \lim_{k\to\infty} \phi(x_k) = p(0).$$

This implies $\phi(\bar{x}) = p(0)$.

(iii) Since

$$\begin{aligned} l^i(x, y, r) &= \inf_u \{f(x, u) - \langle y, u \rangle + \delta_{rC}(u)\} \\ &\le f(x, 0) = \phi(x) \end{aligned}$$

it follows that

$$\sup_{(y,r)} l^i(x, y, r) \leq \phi(x).$$

Also we have

$$\sup_{(y,r)} l^i(x, y, r) \geq l^i(x, 0, 0) = \inf_{u=0}\{f(x, u)\} = \phi(x).$$

If $f(x, u)$ is convex in u then the function $f(x, .) + \delta_{rC}$ is lsc and convex in u. Also since $-l^i(x, y, r)$ is conjugate of $f(x, .) + \delta_{rC}$ it follows that for any $r > 0$

$$f(x, u) + \delta_{rC}(u) = \sup_y\{\langle y, u\rangle + l^i(x, y, r)\}.$$

Taking $u = 0$ we get $\phi(x) = \sup_y l^i(x, y, r)$.

(iv) Since $f(x, u)$ is level-bounded in x locally uniformly in u it follows that $p(u)$ is convex and lsc on R^m. Now for $r > 0$

$$\begin{aligned}\psi^i(\bar{y}, r) &= \inf_x \inf_{\|u\|\leq r}\{f(x, u) - \langle \bar{y}, u\rangle\}\\ &= \inf_{\|u\|\leq r}\{p(u) - \langle \bar{y}, u\rangle\}.\end{aligned}$$

From the above relation we get $\psi^i(\bar{y}, r) \to p(0)$ as $r \to 0$.

(v) Now

$$\sup_{(y,r)} \psi^i(y, r) \geq \sup_r \psi^i(\bar{y}, r) \geq \lim_{r\to 0} \psi^i(\bar{y}, r) = p(0)$$

which together with weak duality gives

$$\sup_{(y,r)} \psi^i(y, r) = p(0).$$

Hence it follows that

$$\inf_x \sup_{(y,r)} l^i(x, y, r) = \sup_{(y,r)} \inf_x l^i(x, y, r).$$

(vi) The result is obvious by (v). □

We now illustrate with an example that the relations $\phi(x) = \sup_{(y,r)} \bar{l}(x, y, r)$ and $\inf_x \phi(x) = \sup_{(y,r)} \bar{\psi}(y, r)$ do not hold in the case of the augmented Lagrangian considered by Rockafellar and Wets [9] for a convex augmenting function if f is not convex in u whereas the corresponding relations hold in the case of indicator augmenting function for a convex primal problem. However, it may be noted here that the duality relation between (P) and ($\bar{\text{D}}$) can be obtained by choosing a different augmenting function. The search for an appropriate augmenting function is another issue that can be avoided in case the indicator augmenting function is used. Even for the generalized augmented Lagrangian considered by Huang and Yang [2] the duality fails for the problem considered in this example.

Example 3.5 Consider the problem (P) with $\phi(x) = x^2$, $f(x, u) = x^2 - 2u^2$ and $\sigma(u) = |u|$ then $\bar{l}(x, y, r) = l^g(x, y, r) = -\infty$ and $\bar{\psi}(y) = \psi^g(y) = -\infty$. No duality relations can be established between the primal (P) and the augmented Lagrangian dual problem ($\bar{\text{D}}$) or generalized augmented Lagrangian dual problem (D^g). The optimal solution of the problem (P) is $\bar{x} = 0$. However, if we consider the generalized augmenting $\sigma(u, r) = \delta_{rC}(u)$ then

$$l^i(x, y, r) = \begin{cases} x^2 - 2r^2 + yr, & \text{if } y \leq 0\\ x^2 - 2r^2 - yr, & \text{if } y > 0.\end{cases}$$

and

$$\psi^i(y,r) = \begin{cases} -2r^2 + yr, & \text{if } y \le 0 \\ -2r^2 - yr, & \text{if } y > 0. \end{cases}$$

Observe that $\lim_{r \to 0} \psi^i(y,r) = p(0)$ for any $y \in R$. Here the solution set of the dual problem (D^i) is empty but the minmax type duality relation

$$\begin{aligned} \phi(\bar{x}) &= \inf{}_x \phi(x) = \inf{}_x \sup{}_{(y,r)} l^i(x,y,r) \\ &= \sup{}_{(y,r)} \inf{}_x l^i(x,y,r) = \sup{}_{(y,r)} \psi^i(y,r); \end{aligned}$$

is obviously true.

The following example justifies the strong duality theorem for primal problem (P) with nonconvex objective and the indicator augmented Lagrangian dual problem (D^i). In this example the set of optimal solutions of the dual problem is nonempty and compact.

Example 3.6 Consider the problem (P) with $\phi : R \to \bar{R}$ defined as

$$\phi(x) = \begin{cases} -x, & \text{if } x \le 0 \\ x + 1 & \text{if } x > 0. \end{cases}$$

The optimal solution of this problem is $\bar{x} = 0$. Consider a dualizing parameterization function $f(x,u) : R \times R \to \bar{R}$ given as

$$f(x,u) = \begin{cases} -x, & \text{if } x \le 0 \\ x + 1 + u - u^2 & \text{if } x > 0. \end{cases}$$

Clearly $f(x,0) = \phi(x), \forall x \in R$. For $x \in R$, $y \in R$ and $r > 0$ the function $l^i(x,y,r)$ is given by

$$l^i(x,y,r) = \begin{cases} -x - yr, & \text{if } x \le 0,\ y \ge 0 \\ -x + yr, & \text{if } x \le 0,\ y < 0 \\ x + 1 + r - r^2 - yr, & \text{if } x > 0,\ y \ge 1 \\ x + 1 - r - r^2 + yr, & \text{if } x > 0,\ y < 1. \end{cases}$$

The generalized augmented Lagrangian dual function $\psi^i(y,r)$ is given by

$$\psi^i(y,r) = \begin{cases} -yr, & \text{if } 0 \le y < 1, 2yr + 1 - r - r^2 \ge 0 \\ 1 - r - r^2 + yr, & \text{if } 0 \le y < 1, 2yr + 1 - r - r^2 < 0 \\ -yr, & \text{if } y \ge 1,\ 1 + r - r^2 \ge 0 \\ 1 + r - r^2 - yr, & \text{if } y \ge 1,\ 1 + r - r^2 < 0 \\ yr, & \text{if } y < 0,\ 1 - r - r^2 \ge 0 \\ 1 - r - r^2 + yr, & \text{if } y < 0,\ 1 - r - r^2 < 0. \end{cases}$$

Here $\phi(\bar{x}) = \inf_x \phi(x) = \sup_{(y,r)} \psi^i(y,r)$. The perturbation function $p(u)$ is given by

$$p(u) = \begin{cases} 0, & \text{if } 1 + u - u^2 \ge 0 \\ 1 + u - u^2, & \text{if } 1 + u - u^2 < 0. \end{cases}$$

It is clear that for all $y \in R^m$ and $r > 0$

$$\psi^i(y,r) \le 0 = p(0).$$

Optimal solution of (D^i) is $\{(0,r) | 1 - r - r^2 \ge 0, r > 0\} = \{(0,r) | 0 < r \le (\sqrt{5} - 1)/2\}$.

We now give an algorithm for solving the problems (P) and (D^i) in view of Theorem 3.4(vi).

1. Choose x_0, $y_0 = \bar{y}$, $r_0 = \bar{r}$, $k = 0$ and $\epsilon > 0$ where $\psi^i(\bar{y}, \bar{r}) > -\infty$ and $\phi(x_0)$ is finite.
2. Solve $\inf_x l^i(x, y_k, r_k)$ and its solution be x_{k+1} if it exists or else choose x_{k+1} such that $l^i(x, y_k, r_k) \leq \inf_x l^i(x, y_k, r_k) + \epsilon$.
3. Solve $\sup_{(y,r)} l^i(x_k, y, r)$ and its solution be (y_{k+1}, r_{k+1}) if it exists or else choose (y_{k+1}, r_{k+1}) such that $l^i(x_k, y_{k+1}, r_{k+1}) \geq \sup_{(y,r)} l^i(x_k, y, r) - \epsilon$.
4. If $\inf_x l^i(x, y_k, r_k) \approx \sup_{(y,r)} l^i(x_k, y, r)$ then x_k solves (P) and (y_{k+1}, r_{k+1}) solves (D^i) otherwise set $k = k + 1$.

We now use this algorithm to solve the problems (P) and (D^i) considered in Example 3.6. The values of x_k, y_k and r_k at each iteration are tabulated as follows.

k	x_k	y_k	r_k	$\inf_x l^i(x, y_k, r_k)$	$\sup_{(y,r)} l^i(x_k, y, r)$
0	3	-2	2	-9	4
1	ε	1	$\sqrt{\varepsilon}$	$-\sqrt{\varepsilon}$	$\varepsilon + 1$
				if $1 + \sqrt{\varepsilon} - \varepsilon \geq 0$	
2	0	1	$\sqrt{\varepsilon}$	$-\sqrt{\varepsilon}$	0
				if $1 + \sqrt{\varepsilon} - \varepsilon \geq 0$	
3	0	0	r	0	0
			$r > 0$	if $1 - r - r^2 \geq 0$	0

Hence we observe that $\bar{x} = 0$ solves the problem (P) and

$$\{(0, r) | 1 - r - r^2 \geq 0, r > 0\} = \{(0, r) | 0 < r \leq (\sqrt{5} - 1)/2\}$$

is the solution set of problem (D^i).

4 Exact penalty representation

This section is devoted to the study of exact penalty results in the framework of indicator augmented Lagrangian. We have he following notion of exact penalty representation in terms of indicator augmented Lagrangian.

Definition 4.1 (*Exact penalty representation*) A vector $\bar{y}$ is said to support an exact penalty representation for the problem of minimizing ϕ on R^n if for all $r > 0$ sufficiently small this problem is equivalent to minimizing $l^i(., \bar{y}, r)$ on R^n in the sense that

$$\inf{}_x \phi(x) = \inf{}_x l^i(x, \bar{y}, r)$$
$$\arg\min{}_x \phi(x) = \arg\min{}_x l^i(x, \bar{y}, r).$$

A value $\bar{r}$ is said to serve as an adequate penalty threshold if the property holds for all $r \in]0, \bar{r}]$.

For the problem considered in Example 3.6 it can be seen that $\bar{y} = 0$ supports an exact penalty representation. Since for $\bar{y} = 0$

$$l^i(x, \bar{y}, r) = \begin{cases} -x, & \text{if } x \leq 0, \\ x + 1 - r - r^2, & \text{if } x > 0. \end{cases}$$

it is apparent that if $1 - r - r^2 \geq 0$ then

$$\inf_x \phi(x) = \inf_x l^i(x, \bar{y}, r)$$
$$\arg\min_x \phi(x) = \arg\min_x l^i(x, \bar{y}, r).$$

Here $\bar{r} = (\sqrt{5} - 1)/2$ serves as an adequate penalty threshold.

The following theorem refines Theorem 11.61 of Rockafellar and Wets [9] and Theorem 3.1 of Huang and Yang [2]. No extra condition as in Theorem 3.1 of [2] is required to establish the equivalence.

Theorem 4.2 (Criterion for exact penalties) *In the notation and assumptions of strong duality the following assertions are equivalent:*

(i) *A vector $\bar{y}$ supports an exact penalty representation for the primal problem (P);*
(ii) *There exists an $\bar{r} > 0$ with $(\bar{y}, \bar{r}) \in \arg\max_{(y,r)} \psi^i(y, r)$;*
(iii) *There exists $\hat{r} > 0$ such that for any $r \in]0, \hat{r}]$*

$$p(u) \geq p(0) + \langle \bar{y}, u \rangle - \delta_{rC}(u) \qquad \forall u \in R^m;$$

(iv) *There exists a neighborhood U of 0 such that*

$$p(u) \geq p(0) + \langle \bar{y}, u \rangle \qquad \forall u \in U.$$

The values of $\bar{r}$ and $\hat{r}$ are the ones serving as adequate penalty threshold with respect to $\bar{y}$.

Proof

$(i) \Rightarrow (ii)$ We first assert that if $\bar{y}$ supports an exact penalty representation for the primal problem (P) with adequate penalty threshold $\bar{r}$ then $(\bar{y}, \bar{r})$ maximizes $\psi^i(y, r)$. Since $\bar{y}$ supports an exact penalty representation we have $\inf_x \phi(x) = \inf_x l^i(x, \bar{y}, r) = \psi^i(\bar{y}, r)$ for all $r \in]0, \bar{r}]$. Hence it follows that $\psi^i(\bar{y}, \bar{r}) \geq \limsup_{r\to 0} \psi^i(\bar{y}, r) \geq \inf_x \phi(x)$, as ψ^i is usc in r. By Theorem 3.4 we have $\inf_x \phi(x) = \sup_{(y,r)} \psi^i(y, r)$, hence $\psi^i(\bar{y}, \bar{r}) = \sup_{(y,r)} \psi^i(y, r)$.

$(ii) \Rightarrow (iii)$ Assume that $\psi^i(\bar{y}, \bar{r}) = \sup_{(y,r)} \psi^i(y, r)$. As $\sup_{(y,r)} \psi^i(y, r) = \inf_x \phi(x) = p(0)$ it follows that $\inf_x l^i(x, \bar{y}, \bar{r}) \geq p(0)$. Using the definition of $l^i(x, \bar{y}, \bar{r})$ we have $\inf_x \inf_u \{f(x, u) - \langle \bar{y}, u \rangle + \delta_{\bar{r}C}(u)\} \geq p(0)$ which implies

$$\inf_u \{p(u) - \langle \bar{y}, u \rangle + \delta_{\bar{r}C}(u)\} \geq p(0)$$

that is

$$p(u) \geq p(0) + \langle \bar{y}, u \rangle - \delta_{\bar{r}C}(u) \quad \forall u \in R^m.$$

For any $r \in]0, \bar{r}]$ it is obvious that

$$p(u) \geq p(0) + \langle \bar{y}, u \rangle - \delta_{rC}(u) \quad \forall u \in R^m.$$

$(iii) \Rightarrow (i)$ Clearly for any $r \in]0, \hat{r}]$, $0 \in \arg\min_u \{p(u) - \langle \bar{y}, u \rangle + \delta_{rC}(u)\}$. For a fixed $r < \hat{r}$, if we consider $g(x, u) = f(x, u) - \langle \bar{y}, u \rangle + \delta_{rC}(u)$ and $h(u) = \inf_x g(x, u)$ and $k(x) = \inf_u g(x, u)$ then $h(u) = p(u) - \langle \bar{y}, u \rangle + \delta_{rC}(u)$ and $k(x) = l^i(x, \bar{y}, r)$. According to the Rule 1.35 in [9] we have $\bar{u} \in \arg\min_u h(u)$ and $\bar{x} \in \arg\min_x g(x, \bar{u})$ if and only if $\bar{x} \in \arg\min_x k(x)$ and $\bar{u} \in \arg\min_x g(\bar{x}, u)$. Since $\bar{u} = 0 \in \arg\min_u h(u)$ the pair $(\bar{x}, \bar{u})$ with $\bar{x} \in \arg\min_x g(x, \bar{u}) = \arg\min_x \phi(x)$ also satisfy $\bar{x} \in \arg\min_x l^i(x, \bar{y}, r)$ and $\bar{u} = 0 \in \arg\min_x g(\bar{x}, u)$. Thus $\arg\min_x \phi(x) = \arg\min_x l^i(x, \bar{y}, r)$. Also we have $\inf_x \phi(x) = p(0) = \inf_u h(u) = \inf_u \inf_x g(x, u) = \inf_x \inf_u g(x, u) = \inf_x l^i(x, \bar{y}, r)$. Thus $\bar{y}$ supports an exact penalty representation for the primal problem (P) with adequate penalty threshold $\hat{r}$.

$(iii) \Rightarrow (iv)$ Choosing $U = \hat{r}C$ it follows that

$$p(u) \geq p(0) + \langle \bar{y}, u \rangle \quad \forall u \in U.$$

$(iv) \Rightarrow (iii)$ Choose any $\hat{r} > 0$ such that $\hat{r}C \subseteq U$. Then for any $r \in]\,0, \hat{r}]$

$$p(u) \geq p(0) + \langle \bar{y}, u \rangle - \delta_{rC}(u) \quad \forall u \in R^m.$$

□

Referring back to the problem considered in Example 3.6 it is observed that for $\bar{y} = 0$ which supports an exact penalty representation

$$\begin{aligned} \psi^i(\bar{y}, r) &= \begin{cases} 0, & \text{if } 1 - r - r^2 \geq 0 \\ 1 - r - r^2, & \text{if } 1 - r - r^2 < 0, \end{cases} \\ &= \begin{cases} 0, & \text{if } 0 < r \leq (\sqrt{5} - 1)/2, \\ 1 - r - r^2, & \text{if } r > (\sqrt{5} - 1)/2. \end{cases} \end{aligned}$$

Now $(\bar{y}, \bar{r}) \in \arg\max_{(y,r)} \psi^i(y, r)$ with $\bar{y} = 0$ and any $\bar{r}$ with $0 < \bar{r} \leq (\sqrt{5} - 1)/2$. Also for $r \in]\,0, \hat{r}]$ with $\hat{r} = (\sqrt{5} - 1)/2$ we have

$$p(u) \geq p(0) + \langle \bar{y}, u \rangle - \delta_{rC}(u) \quad \forall u \in R^m.$$

The neighborhood $U = [-(\sqrt{5} - 1)/2, (\sqrt{5} - 1)/2]$ of 0 with $\bar{y} = 0$ satisfies

$$p(u) \geq p(0) + \langle \bar{y}, u \rangle \quad \forall u \in U.$$

5 Conclusions

The indicator augmenting Lagrangian approach to duality and exact penalization is applicable in the case of any nonconvex optimization problem. The idea here is to define the negative of augmented Lagrangian in terms of the conjugate of the dualizing parameterization function on a restricted domain namely a closed ball with center at origin of certain radius. The radius of the closed ball acts like a penalty parameter and contracting the ball to the origin leads to the reduction of the duality gap to zero. It would be worthwhile to investigate on stability conditions that could lead to the existence of optimal solution(s) of the dual problem.

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