J Glob Optim (2009) 45:583–595
DOI 10.1007/s10898-008-9391-x

Stability of semi-infinite vector optimization problems under functional perturbations

T. D. Chuong · N. Q. Huy · J. C. Yao

Received: 11 April 2008 / Accepted: 12 December 2008 / Published online: 25 December 2008

Abstract This paper is devoted to the study of continuity properties of Pareto solution maps for parametric semi-infinite vector optimization problems (PSVO). We establish new necessary conditions for lower and upper semicontinuity of Pareto solution maps under functional perturbations of both objective functions and constraint sets. We also show that the necessary condition becomes sufficient for the lower and upper semicontinuous properties in the special case where the constraint set mapping is lower semicontinuous at the reference point. Examples are given to illustrate the obtained results.

Mathematics Subject Classification (2000) 49K40 · 46N10 · 90C29 · 90C31

Keywords Stability · Semi-infinite vector optimization · Pareto solution map · Upper semicontinuity · Lower semicontinuity

1 Introduction

Let Θ be a nonempty compact set of a Hausdorff topological space. The space $C[\Theta, \mathbb{R}^n]$ is the set of all continuous vector functions $f\colon \Theta \to \mathbb{R}^n$, where the norm of the function $\varphi \in C[\Theta, \mathbb{R}^n]$ is defined as follows:

$$\|\varphi\| := \max_{x\in\Theta} \|\varphi(x)\|_{\mathbb{R}^n},$$

T. D. Chuong
Department of Mathematics, Dong Thap University of Pedagogy, Cao Lanh, Dong Thap, Vietnam
e-mail: chuongthaidoan@yahoo.com

N. Q. Huy
Department of Mathematics, Hanoi Pedagogical University No. 2, Phuc Yen, Vinh Phuc, Vietnam
e-mail: huyngq@yahoo.com

J. C. Yao (✉)
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 804, Taiwan
e-mail: yaojc@math.nsysu.edu.tw

and where $||\cdot||_{\mathbb{R}^n}$ denotes the Euclidean norm in the finite-dimensional space $\mathbb{R}^n$. The norm on the product space $X \times Y$ is defined by

$$\|(x, y)\| := \|x\| + \|y\|.$$

Let Ω and T be nonempty compact subsets of a Hausdorff topological space. Consider *parametric semi-infinite vector optimization* problems, or *generalized parametric vector optimization* problems, under functional perturbations of both objective function and constraint set (PSVO for brevity) on the parameter space

$$P := C[\Omega, \mathbb{R}^s] \times C[\Omega \times T, \mathbb{R}^m] \times C[T, \mathbb{R}^m]$$

formulated as follows: for every triple of parameters $p := (f, g, b)$ we have the *semi-infinite vector optimization* problem

$$(\text{SVO})_p : \quad \min_{\mathbb{R}^s_+} f(x) \quad \text{subject to} \quad x \in C(p),$$

where

$$C(p) = \{x \in \Omega | g(x, t) - b(t) \in -\mathbb{R}^m_+ \ \forall t \in T\}$$

is the set of *feasible points*, $\mathbb{R}^m_+ = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m | x_j \geq 0 \quad \forall j = 1, \ldots, m\}$ the *non-negative orthant* of $\mathbb{R}^m$, and $\text{int}\mathbb{R}^m_+$ denotes the interior of $\mathbb{R}^m_+$.

Our main concern is to study lower as well as upper semicontinuity of the Pareto solution map of (PSVO) depending on the parameter p near the reference point. It is well known that semi-infinite optimization problems have attracted much attention of many researchers in the last three decades; see, e.g., [2–21] and the references therein for more comments and discussions. There are many publications devoted to the study of continuity properties of the marginal/valued function and the optimal solution mapping in parametric semi-infinite *scalar optimization* problems (see, e.g., [2–5,7–13,17,18] and the references therein), but only a few of them consider parametric semi-infinite *vector optimization* problems [6,19–21].

Chen and Craven [6] gave sufficient conditions for lower and upper semicontinuity of the *local* weak Pareto solution map of (PSVO) under functional perturbations of both the objective function and the constraint set at a given point. Under functional perturbation of only the objective function, i.e., when the constraint set mapping C is constant, Yu [21] established a necessary and sufficient condition for the lower semicontinuous property of the weak Pareto solution map. Recently, Xiang and Zhou [19] and Xiang and Yin [20] derived necessary and sufficient conditions for lower and upper semicontinuity of the Pareto solution map of (PSVO) under functional perturbation of the objective function only.

Our main goal of this paper is to establish new necessary as well as sufficient conditions for lower or upper semicontinuity of the Pareto solution map of (PSVO) under functional perturbations of both the objective function and the constraint set. Some of our results extend the corresponding results in [19,20].

The paper is organized as follows. In Sect. 2 we recall some basic definitions and preliminaries from the theory of vector optimization and set-valued analysis. In Sect. 3 we present some sufficient conditions for lower and upper semicontinuous properties of the constraint set map, which will be used in next sections. In Sect. 4 we derive necessary conditions for lower semicontinuity of the Pareto solution map. We also show that, the necessary condition obtained becomes sufficient for the lower semicontinuous property in the special case, where the constraint set mapping is lower semicontinuous at the reference point. The necessary as well as sufficient conditions for upper semicontinuity of the Pareto solution map are given in Sect. 5.

2 Preliminaries

Throughout this paper, Ω is a nonempty and compact set of a metric space and T is a nonempty compact set of a Hausdorff topological space. Let $p := (f, g, b)$ be a triple of parameters defined as in Sect. 1. Consider the following semi-infinite vector optimization problem

$$(\text{SVO})_p : \quad \min_{\mathbb{R}^s_+} f(x) \quad \text{subject to} \quad x \in C(p),$$

where $C(p) = \{x \in \Omega | g(x, t) - b(t) \in -\mathbb{R}^m_+ \ \forall t \in T\}$.

Definition 2.1

(i) We write $\bar{x} \in \mathcal{S}(p)$ (resp., $\bar{x} \in \mathcal{S}^w(p)$) to indicate that $\bar{x}$ is a *Pareto solution* (resp., a *weak Pareto solution*) of $(\text{SVO})_p$ if there is no $x \in C(p)$ satisfying $f(x) - f(\bar{x}) \in -\mathbb{R}^s_+ \backslash \{0\}$ (resp., $f(x) - f(\bar{x}) \in -\text{int}\mathbb{R}^s_+$).
(ii) We call $\mathcal{S} : P \rightrightarrows \Omega$ (resp., $\mathcal{S}^w : P \rightrightarrows \Omega$) the *Pareto solution map* of (PSVO) (resp., the *weak Pareto solution map* of (PSVO)).
(iii) The multifunction $C : P \rightrightarrows \Omega$ is said to be the *constraint set map* of (PSVO).

Let $F : X \rightrightarrows Y$ be a multifunction between Hausdorff topological spaces. We denote by $\mathcal{N}(x)$ the set of all neighborhoods of $x \in X$, and by $\text{cl}A$ the closure of A. The effective domain of F is defined by $\text{dom}F = \{x \in X | F(x) \neq \emptyset\}$.

Definition 2.2

(i) F is *upper semicontinuous* (usc for brevity) at $x_0 \in X$ if for every open set V containing $F(x_0)$, there exists $U_0 \in \mathcal{N}(x_0)$ such that $F(x) \subset V$ for all $x \in U_0$.
(ii) F is said to be *lower semicontinuous* (lsc for brevity) at $x_0 \in \text{dom}\, F$ if for any open set $V \subset Y$ satisfying $V \cap F(x_0) \neq \emptyset$, there exists $U_0 \in \mathcal{N}(x_0)$ such that $V \cap F(x) \neq \emptyset$ for all $x \in U_0$.
(iii) F is said to be *continuous* at $x_0 \in X$ if it is both upper and lower semicontinuous at x_0. F is continuous on A if it is continuous at every point belong to A.

Note that, if X, Y are metric spaces, then it is well known that (see [1, Theorem 17.20, Theorem 17.21]) F is lsc at $x_0 \in X$ if and only if for any sequence $\{x_i\} \subset X$, $x_i \to x_0$, any $y_0 \in F(x_0)$ there is a subsequence $\{x_{i_k}\}$ of $\{x_i\}$ and elements $y_k \in F(x_{i_k})$ for all k such that $y_k \to y_0$. If in addition Y is compact and F has closed values, then F is usc at $x_0 \in X$ if and only if for any sequence $\{x_i\} \subset X$ satisfying $x_i \to x_0$, $y_i \in F(x_i)$, and $y_i \to y_0$ we have $y_0 \in F(x_0)$.

Definition 2.3 (*see* [14, Def. 6.1]) Let Ω be a convex set and a function $f : \Omega \to \mathbb{R}^s$. We say that:

(i) f is $\mathbb{R}^s_+$*-convex* on Ω if for each $x_1, x_2 \in \Omega$, $t \in [0, 1]$ one has

$$f(tx_1 + (1-t)x_2) \in tf(x_1) + (1-t)f(x_2) - \mathbb{R}^s_+;$$

(ii) f is *strictly* $\mathbb{R}^s_+$*-quasiconvex* on Ω if for each $y \in \mathbb{R}^s$, $x_1, x_2 \in \Omega$, $x_1 \neq x_2$, $t \in (0, 1)$ one has

$$f(x_1), f(x_2) \in y - \mathbb{R}^s_+ \text{ implies } f(tx_1 + (1-t)x_2) \in y - \text{int}\mathbb{R}^s_+.$$

3 Continuity properties of the constraint set map

In this section, we present sufficient conditions for lower and upper semicontinuity of the constraint set mapping C of (PSVO), which will be useful in next sections. In [5,9–11,13], the reader can find some conditions implying certain stability properties of C such as the closedness, upper semicontinuity, lower semicontinuity, continuity, and metric regularity of the constraint set map of (scalar) linear and convex semi-infinite programming problems with respect to perturbations of the (scalar) linear and convex functions that define the constraints.

The following proposition ensures upper semicontinuity of the constraint set mapping C at a given point for (PSVO) with respect to perturbations of the vector functions.

Proposition 3.1 *Let $p_0 := (f_0, g_0, b_0) \in P$. The constraint set mapping C is usc at p_0.*

Proof Let $\{p_k := (f_k, g_k, b_k)\}_{k=1}^{\infty} \subset P$ be a sequence such that $p_k \to p_0$ as $k \to \infty$. For each $\{x_k\}_{k=1}^{\infty} \subset \Omega$, $x_k \in C(p_k)$, by taking a subsequence if necessary, we may assume that $x_k \to x_0$, as $k \to \infty$. It is sufficient to show that $x_0 \in C(p_0)$.

Since $p_k \to p_0$ as $k \to \infty$, it follows that for each $\epsilon > 0$, there exists k_0 such that $||p_k - p_0|| < \frac{\epsilon}{3}$ for all $k \geq k_0$. Hence, $||g_k - g_0|| < \frac{\epsilon}{3}$ and $||b_k - b_0|| < \frac{\epsilon}{3}$ for all $k \geq k_0$. This yields

$$g_0(x,t) - g_k(x,t) - \frac{1}{3}\epsilon^m \in -\mathbb{R}^m_+ \quad \forall x \in \Omega, \quad t \in T, \quad k \geq k_0, \tag{3.1}$$

$$b_k(t) - b_0(t) - \frac{1}{3}\epsilon^m \in -\mathbb{R}^m_+ \quad \forall x \in \Omega, \quad t \in T, \quad k \geq k_0, \tag{3.2}$$

where $\epsilon^m := (\epsilon, \epsilon, \ldots, \epsilon) \in \mathbb{R}^m$. By the continuity property of g_0 and the compactness property of T, there exists $\delta > 0$ such that for all $x \in \Omega$ with $d(x, x_0) < \delta$ we have

$$||g_0(x,t) - g_0(x_0,t)||_{\mathbb{R}^m} < \frac{\epsilon}{3} \quad \forall t \in T, \tag{3.3}$$

where $d(x, x_0)$ denotes the distance between the points x and x_0. This implies that

$$g_0(x_0,t) - g_0(x,t) - \frac{1}{3}\epsilon^m \in -\mathbb{R}^m_+ \quad \forall t \in T. \tag{3.4}$$

Since $x_k \to x_0$ as $k \to \infty$, there is $k_1 \geq k_0$ such that $d(x_k, x_0) < \delta$ for all $k \geq k_1$. Combining this with (3.4), (3.1) and (3.2), we get

$$g_0(x_0,t) - b_0(t) - \epsilon^m \in -\mathbb{R}^m_+ \quad \forall t \in T.$$

By the closedness of $\mathbb{R}^m_+$ and the continuity of g_0 and b_0, we have

$$g_0(x_0,t) - b_0(t) \in -\mathbb{R}^m_+ \quad \forall t \in T.$$

Thus $x_0 \in C(p_0)$. The proof is complete. □

In the special case of (PSVO) under convex perturbation functions and the compactness assumption on T, Proposition 3.1 is a direct consequence of [13, Theorem 4.2, Lemma 4.3 and Proposition 4.2].

Proposition 3.1 shows that the constraint set mapping C is always usc at every $p \in P$, but it is not true for the lower semicontinuity of C in general (see Example 4.2 below). The next result gives some sufficient conditions for lower semicontinuity of the constraint set mapping C at the reference point.

Proposition 3.2 *Let Ω be a nonempty convex compact set of a locally convex space, and let $p_0 := (f_0, g_0, b_0) \in P$. Suppose that the following conditions hold:*

(i) *for all $t \in T$, $g(\cdot, t)$ is $\mathbb{R}^m_+$-convex on Ω;*
(ii) *the Slater condition for p_0, i.e., there exist $\hat{x} \in \Omega$ such that*

$$g_0(\hat{x}, t) - b_0(t) \in -\text{int}\mathbb{R}^m_+ \quad \forall t \in T.$$

Then C is lsc at p_0.

Proof Let W be an open convex set such that $W \cap C(p_0) \neq \emptyset$. By (ii), there exists an element $\hat{x} \in C(p_0)$ satisfying

$$g_0(\hat{x}, t) - b_0(t) \in -\text{int}\mathbb{R}^m_+ \quad \forall t \in T. \tag{3.5}$$

Taking any $x_0 \in W \cap C(p_0)$ and $r \in (0, 1]$, we define

$$x_r := x_0 + r(\hat{x} - x_0) \in W.$$

By (i), $C(p_0)$ is convex. Hence,

$$x_r \in W \cap C(p_0).$$

From the convexity of $g_0(\cdot, t)$ and (3.5), it follows that

$$\begin{aligned} g_0(x_r, t) = g_0((1-r)x_0 + r\hat{x}, t) &\in (1-r)g_0(x_0, t) + rg_0(\hat{x}, t) - \mathbb{R}^m_+ \\ &\subset b_0(t) - \text{int}\mathbb{R}^m_+ \quad \forall t \in T. \end{aligned}$$

Therefore, we can choose $\epsilon > 0$ such that

$$g_0(x_r, t) - b_0(t) + \epsilon^m \in -\mathbb{R}^m_+ \quad \forall t \in T, \tag{3.6}$$

where $\epsilon^m := (\epsilon, \epsilon, \ldots, \epsilon) \in \mathbb{R}^m$. For each $p = (f, g, b) \in P$ such that $||p - p_0|| < \frac{\epsilon}{2}$, we claim that $W \cap C(p) \neq \emptyset$. Indeed, we have

$$g(x, t) - g_0(x, t) - \frac{1}{2}\epsilon^m \in -\mathbb{R}^m_+ \quad \forall x \in \Omega, t \in T, \tag{3.7}$$

$$b_0(t) - b(t) - \frac{1}{2}\epsilon^m \in -\mathbb{R}^m_+ \quad \forall x \in \Omega, t \in T. \tag{3.8}$$

From (3.6)–(3.8), we deduce

$$g(x_r, t) - b(t) \in -\mathbb{R}^m_+ \quad \forall t \in T.$$

Thus, $x_r \in C(p)$ and $W \cap C(p) \neq \emptyset$. This means that C is lower semicontinous at p_0. □

The above result implies the corresponding result of Chen and Craven [6, Lemma 3.1] in the particular case, where Ω is a nonempty convex compact subset of a finite-dimensional space.

Finally in this section, we consider a special case of (PSVO), where perturbation functions that define the constraints are real convex functions, i.e., $g(\cdot, t)$ is a convex function for every $t \in T$. Then the validity of the Slater condition for the constraint set map C at a given point p_0 in Proposition 3.2 is equivalent to the lower semicontinuity of C at p_0 by [13, Theorem 4.1(i)(v)].

4 Lower semicontinuity of the Pareto solution map

In this section we establish necessary as well as sufficient conditions for lower semicontinuity of the Pareto solution mapping $\mathcal{S}$ at the reference point.

Theorem 4.1 *Let $p_0 := (f_0, g_0, b_0) \in P$. If $\mathcal{S}$ is lsc at p_0, then for each $x_0 \in \mathcal{S}(p_0)$ and for each $V(x_0) \in \mathcal{N}(x_0)$ in Ω, there exists $\bar{x} \in V(x_0) \cap \mathcal{S}(p_0)$ such that*

$$f_0^{-1}(f_0(\bar{x})) \cap C(p_0) \subset V(x_0). \tag{4.9}$$

Moreover, if in addition the constraint set mapping C is lsc at p_0, then the converse is also true.

Proof We prove the first assertion of the theorem. Suppose to the contrary that there exist $x_0 \in \mathcal{S}(p_0)$ and $V(x_0) \in \mathcal{N}(x_0)$ in Ω such that

$$f_0^{-1}(f_0(x)) \cap C(p_0) \not\subseteq V(x_0) \quad \forall x \in V(x_0) \cap \mathcal{S}(p_0). \tag{4.10}$$

Let $V_1(x_0)$, $V_2(x_0)$ be open neighborhoods of x_0 satisfying

$$\mathrm{cl}V_1(x_0) \subset V_2(x_0) \subset \mathrm{cl}V_2(x_0) \subset V(x_0).$$

Applying Urysohn's lemma, we can construct a continuous function α on Ω such that $\alpha(x) = 0$ if $x \in \mathrm{cl}V_1(x_0)$ and $\alpha(x) = 1$ if $x \in \Omega \backslash V_2(x_0)$. For each integer number $k > 1$, define $u^k := (\frac{1}{k}, \ldots, \frac{1}{k}) \in \mathbb{R}^s$ and

$$f_k(x) := f_0(x) - \alpha(x)u^k \quad \forall x \in \Omega.$$

Then $f_k \in C[\Omega, \mathbb{R}^s] \;\; \forall k > 1$. Putting $p_k := (f_k, g_0, b_0) \in P$, we have

$$V_1(x_0) \cap \mathcal{S}(p_k) = \emptyset \quad \forall k > 1. \tag{4.11}$$

Indeed, take any $x \in V_1(x_0)$, and consider the following three cases:

(a) If $x \in V_1(x_0) \cap \mathcal{S}(p_0)$, then it follows from (4.10) that there exists $z_x \in C(p_0)\backslash V(x_0)$ such that $f_0(z_x) = f_0(x)$. Therefore,

$$\begin{aligned} f_k(z_x) - f_k(x) &= f_0(z_x) - f_0(x) - (\alpha(z_x) - \alpha(x))u^k \\ &= f_0(z_x) - f_0(x) - u^k \\ &= -u^k \in -\mathrm{int}\mathbb{R}^s_+ \subset -\mathbb{R}^s_+ \backslash \{0\}. \end{aligned}$$

This means $x \notin \mathcal{S}(p_k), \quad \forall k > 1$.

(b) If $x \in (V_1(x_0) \cap C(p_0))\backslash \mathcal{S}(p_0)$, then there exists $z_x \in C(p_0)$ such that

$$f_0(z_x) - f_0(x) \in -\mathbb{R}^s_+ \backslash \{0\}.$$

Hence,

$$\begin{aligned} f_k(z_x) - f_k(x) &= f_0(z_x) - f_0(x) - (\alpha(z_x) - \alpha(x))u^k \\ &= f_0(z_x) - f_0(x) - \alpha(z_x)u^k \in -\mathbb{R}^s_+ \backslash \{0\}, \end{aligned}$$

and so $x \notin \mathcal{S}(p_k) \quad \forall k > 1$.

(c) If $x \in V_1(x_0)\backslash C(p_0)$, then $x \notin \mathcal{S}(p_k)$ by $C(p_k) = C(p_0)\ \forall k > 1$. Combining these gives (4.11). Obviously, $p_k \to p_0$ as $k \to \infty$. This contradicts the fact that $\mathcal{S}$ is lsc at p_0, and the first assertion of the theorem is proved.

We next prove the second assertion of the theorem. Suppose that the constraint set mapping C is lsc at p_0. If $\mathcal{S}$ is not lsc at p_0, then there exist a $x_0 \in \mathcal{S}(p_0)$, an open set $V(x_0) \in \mathcal{N}(x_0)$ and a sequence $\{p_k := (f_k, g_k, b_k)\} \subset P$ such that p_k converges to p_0 and

$$\mathcal{S}(p_k) \cap V(x_0) = \emptyset \quad \forall k. \tag{4.12}$$

Choose an open set $V'(x_0) \in \mathcal{N}(x_0)$ such that $\mathrm{cl}V'(x_0) \subset V(x_0)$. By our assumption, there exists $\bar{x} \in V'(x_0) \cap \mathcal{S}(p_0)$ such that

$$f_0^{-1}(f_0(\bar{x})) \cap C(p_0) \subset V'(x_0). \tag{4.13}$$

We claim from the lower semicontinuity of C at p_0 that there exist an integer number $k_0 \geq 1,\ x_k \in C(p_k) \cap V'(x_0)$ with $d(x_k, \bar{x}) < \frac{1}{k}$, and $z_k \in C(p_k)\backslash V'(x_0)$ such that

$$f_k(z_k) - f_k(x_k) \in -\mathbb{R}^s_+\backslash\{0\} \quad \forall k \geq k_0. \tag{4.14}$$

Indeed, if our claim is false, then for each $k \geq 1$ there exists an open set $W(\bar{x}) \in \mathcal{N}(\bar{x})$, $W(\bar{x}) \subset V'(x_0)$ such that for each $x \in W(\bar{x})$ and $z \in C(p_k)\backslash V'(x_0)$ we have

$$f_k(z) - f_k(x) \notin -\mathbb{R}^s_+\backslash\{0\}. \tag{4.15}$$

Denote $\mathcal{S}(A, f_k)$ the set of Pareto solutions of f_k subject to the subset A of the set of the feasible points $C(p_k)$. By compactness of $C(p_k) \cap \mathrm{cl}V'(x_0)$ and continuity of f_k, it follows that $\mathcal{S}(C(p_k) \cap \mathrm{cl}V'(x_0), f_k) \neq \emptyset$. Consider the following two cases:

(a) If $\mathcal{S}(C(p_k)\cap\mathrm{cl}V'(x_0), f_k)\cap W(\bar{x}) \neq \emptyset$, then there exists $\bar{z} \in \mathcal{S}(C(p_k)\cap\mathrm{cl}V'(x_0), f_k)\cap W(\bar{x})$ and we have $\bar{z} \in \mathcal{S}(p_k)$. Indeed, if $\bar{z} \notin \mathcal{S}(p_k)$ then, by $\bar{z} \in \mathcal{S}(C(p_k) \cap \mathrm{cl}V'(x_0), f_k)$, there exists $z \in C(p_k)\backslash V'(x_0)$ such that

$$f_k(z) - f_k(\bar{z}) \in -\mathbb{R}^s_+\backslash\{0\},$$

contrary to (4.15) by $\bar{z} \in W(\bar{x})$. Hence, $\bar{z} \in \mathcal{S}(p_k)$ and

$$\bar{z} \in \mathcal{S}(p_k) \cap W(\bar{x}) \subset \mathcal{S}(p_k) \cap V'(x_0) \subset \mathcal{S}(p_k) \cap V(x_0).$$

This contradicts (4.12).

(b) If $\mathcal{S}(C(p_k)\cap\mathrm{cl}V'(x_0), f_k)\cap W(\bar{x}) = \emptyset$, then letting $\bar{y} \in W(\bar{x})\backslash\mathcal{S}(C(p_k)\cap\mathrm{cl}V'(x_0), f_k)$, we find an element $z_{\bar{y}} \in C(p_k) \cap \mathrm{cl}V'(x_0)$ satisfying

$$f_k(z_{\bar{y}}) - f_k(\bar{y}) \in -\mathbb{R}^s_+\backslash\{0\}. \tag{4.16}$$

Put

$$D := \{x \in C(p_k) \cap \mathrm{cl}V'(x_0) \mid f_k(x) - f_k(z_{\bar{y}}) \in -\mathbb{R}^s_+\}.$$

It is a simple matter to verify that $\mathcal{S}(D, f_k) \neq \emptyset$ and that

$$\mathcal{S}(D, f_k) \subset \mathcal{S}(C(p_k) \cap \mathrm{cl}V'(x_0), f_k).$$

Taking any $\bar{z} \in \mathcal{S}(D, f_k)$, we have $\bar{z} \in \mathcal{S}(p_k)$. Indeed, if $\bar{z} \notin \mathcal{S}(p_k)$ then, by $\bar{z} \in \mathcal{S}(C(p_k) \cap \mathrm{cl}V'(x_0), f_k)$, there exists $y \in C(p_k)\backslash V'(x_0)$ such that

$$f_k(y) - f_k(\bar{z}) \in -\mathbb{R}^s_+\backslash\{0\}. \tag{4.17}$$

By $\bar{z} \in D$, we have $f_k(\bar{z}) - f_k(z_{\bar{y}}) \in -\mathbb{R}^s_+$. Combining this with (4.16) and (4.17), gives

$$f_k(y) - f_k(\bar{y}) \in -\mathbb{R}^s_+\backslash\{0\},$$

contrary to (4.15). Hence $\bar{z} \in \mathcal{S}(p_k)$. It follows from $\bar{z} \in D$ that

$$\bar{z} \in \mathcal{S}(p_k) \cap \mathrm{cl} V'(x_0) \subset \mathcal{S}(p_k) \cap V(x_0),$$

which contradicts (4.12). This implies our claim.

Since Ω is compact, we may assume, without loss of generality, $z_k \to z_0 \in \Omega \backslash V'(x_0)$. From the upper semicontinuity of C at p_0 by Proposition 3.1, it follows that $z_0 \in C(p_0)$. Letting $k \to \infty$ in (4.14), we get

$$f_0(z_0) - f_0(\bar{x}) \in -\mathbb{R}^s_+. \tag{4.18}$$

Hence, $f_0(z_0) = f_0(\bar{x})$ by $\bar{x} \in \mathcal{S}(p_0)$. It follows from (4.13) that

$$z_0 \in f_0^{-1}(f_0(\bar{x})) \cap C(p_0) \subset V'(x_0),$$

which contradicts the fact that $z_0 \in \Omega \backslash V'(x_0)$. The proof of the second assertion of the theorem is complete. □

The following example shows that the necessary condition for lower semicontinuity of the Pareto solution map $\mathcal{S}$ in Theorem 4.1 does not become sufficient if the lower semicontinuity of C is omitted.

Example 4.2 Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 \leq x_1 \leq 0, 0 \leq x_2 \leq x_1 + 1\}$ and $T := [0, 1] \subset \mathbb{R}$. Let $f_0 : \Omega \to \mathbb{R}$, $g_0 : \Omega \times T \to \mathbb{R}$ and $b_0, b_k : T \to \mathbb{R}$ be functions, which are given as follows

$$\begin{aligned}
f_0(x) &:= x_1 \quad \forall x = (x_1, x_2) \in \Omega,\\
g_0(x, t) &:= -tx_1 + tx_2 \quad \forall (x, t) \in \Omega \times T,\\
b_0(t) &:= t \quad \forall t \in T,\\
b_k(t) &:= \begin{cases} \frac{k+1}{k} t - \frac{1}{k} & \text{if } t \in [\frac{1}{k+1}, 1] \\ 0 & \text{if } t \in [0, \frac{1}{k+1}], \quad k \geq 1. \end{cases}
\end{aligned}$$

We see that, $g_0(\cdot, t)$ is linear for all $t \in T$ and that $b_k \to b_0$. Put $p_0 := (f_0, g_0, b_0)$, $p_k := (f_k, g_k, b_k)$ with $f_k := f_0$, $g_k := g_0$ for all $k \geq 1$. It is clear that $p_k \to p_0$. We obtain

$$C(p_0) = \Omega, \mathcal{S}(p_0) = \mathcal{S}^w(p_0) = \{(-1, 0)\},$$

$$C(p_k) = \{(0, 0)\}, \mathcal{S}(p_k) = \{(0, 0)\} \quad \forall k \geq 1.$$

It is easy to check that inclusion (4.9) is fulfilled and that C is not lsc at p_0. Actually $\mathcal{S}$ is not lsc at p_0 as well.

In next example we show that the Slater condition is not sufficient for the lower semicontinuity of $\mathcal{S}$.

Example 4.3 Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1 - x_1\}$ and $T := [0, 1] \subset \mathbb{R}$. Let $f_0 : \Omega \to \mathbb{R}$, $g_0 : \Omega \times T \to \mathbb{R}$, and $b_0, b_k : T \to \mathbb{R}$, $k \geq 1$ be defined as follows

$$\begin{aligned}
f_0(x) &:= x_1 \quad \forall x = (x_1, x_2) \in \Omega,\\
g_0(x, t) &:= (t-1)x_1 + tx_2 \quad \forall (x, t) \in \Omega \times T,\\
b_0(t) &:= t \quad \forall t \in T,\\
b_k(t) &:= \begin{cases} \frac{k+1}{k} t - \frac{1}{k} & \text{if } t \in [\frac{1}{k+1}, 1] \\ 0 & \text{if } t \in [0, \frac{1}{k+1}]. \end{cases}
\end{aligned}$$

We see that, $g_0(\cdot, t)$ is linear for all $t \in T$ and $b_k \to b_0$. Put $p_0 := (f_0, g_0, b_0)$, $p_k := (f_k, g_k, b_k)$ with $f_k := f_0, g_k := g_0$ for all $k \geq 1$. It is clear that $p_k \to p_0$. Choosing $\hat{x} = (\frac{1}{4}, \frac{1}{4}) \in \Omega$, we have

$$g_0(\hat{x}, t) = \frac{1}{2}t - \frac{1}{4} < t = b_0(t) \quad \forall t \in T.$$

Therefore, the Slater condition holds for p_0. It follows from Proposition 3.2 that C is lsc at p_0. We have

$$\begin{aligned} C(p_0) &= \Omega, \\ C(p_k) &= \left\{(x_1, x_2) \,\middle|\, 0 \leq x_1 \leq \frac{1}{k+1}, 0 \leq x_2 \leq kx_1\right\} \\ &\quad \cup \left\{(x_1, x_2) \,\middle|\, \frac{1}{k+1} \leq x_1 \leq 1, 0 \leq x_2 \leq 1 - x_1\right\} \quad \forall k \geq 1, \\ \mathcal{S}(p_0) &= \{(0, x_2) \mid 0 \leq x_2 \leq 1\}, \mathcal{S}(p_k) = \{(0, 0)\} \quad \forall k \geq 1. \end{aligned}$$

Take $x_0 = (0, \frac{1}{2}) \in \mathcal{S}(p_0)$ and $V(x_0) = B(x_0, \frac{1}{4}) \cap \Omega$. We see that inclusion (4.9) is not true. It is easy to see that $\mathcal{S}$ is not lsc at p_0.

The following result is immediate from Theorem 4.1 by taking $g(x, t) := (0, \ldots, 0) \in \mathbb{R}^m$ and $b(t) := (1, \ldots, 1) \in \mathbb{R}^m$ for all $x \in \Omega$ and for all $t \in T$.

Corollary 4.4 [19, Theorem 4.2], [20, Theorem 3.3] *Let $p_0 \in P$. If $C(p) = \Omega$ for all $p \in P$, then $\mathcal{S}$ is lsc at p_0 if and only if for each $x_0 \in \mathcal{S}(p_0)$ and for each $V(x_0) \in \mathcal{N}(x_0)$ in Ω there exists $\bar{x} \in V(x_0) \cap \mathcal{S}(p_0)$ such that*

$$f_0^{-1}(f_0(\bar{x})) \cap [\Omega \setminus V(x_0)] = \emptyset.$$

Corollary 4.5 *Let Ω be a nonempty convex compact set of a locally convex space, and let $p_0 = (f_0, g_0, b_0) \in P$. Suppose that the following conditions hold:*

(i) *for all $t \in T$, $g(\cdot, t)$ is $\mathbb{R}^m_+$-convex on Ω;*
(ii) *the Slater condition for p_0;*
(iii) *for each $x_0 \in \mathcal{S}(p_0)$, there exists $\sigma \in \text{int}\mathbb{R}^s_+$ such that*

$$\text{argmin}\{\langle \sigma, f_0 \rangle(x) \mid x \in C(p_0)\} = \{x_0\}.$$

Then $\mathcal{S}$ is lsc at p_0.

Proof Since $g(\cdot, t)$ is $\mathbb{R}^m_+$-convex on Ω for all $t \in T$ and the Slater condition holds for p_0, it follows from Proposition 3.2 that C is lsc at p_0. It remains to show that (4.9) holds. If (4.9) does not hold, then there exists $V(x_0) \in \mathcal{N}(x_0)$ such that for each $\bar{x} \in V(x_0) \cap \mathcal{S}(p_0)$ we have

$$f_0^{-1}(f_0(\bar{x})) \cap C(p_0) \not\subset V(x_0). \tag{4.19}$$

By taking $\bar{x} = x_0$, there exists $x_1 \in C(p_0)$ such that $f_0(x_1) = f_0(x_0)$ and $x_1 \notin V(x_0)$, which contradicts the assumption (iii). □

From the condition (iii) in Corollary 4.5 we see that (4.9) relaxes the requirement for the unique solution of $\mathcal{S}$ at the reference point. Let us examine a special case of (PSVO) with convex perturbation functions defined the objective and constraints. In this case, we recall that the validity of the Slater condition for the constraint set map C at a given point p_0 is equivalent to the lower semicontinuity of C at p_0. Then Corollary 4.5 slightly generalizes [5, Proposition 4(iv)].

Note that, the sufficient conditions in Corollary 4.5 are similar to the sufficient conditions for the lower semicontinuity of weak Pareto solution map $\mathcal{S}^w$, which are given in [6, Theorem 3.2] with (iii) replaced by the following *coercivity condition*: there exist $\sigma \in \text{int}\mathbb{R}^s_+$ depending on x_0 and a positive increasing function τ depending on σ and x_0 such that $\tau(0) = 0$ and

$$\sigma(f_0(x) - f_0(x_0)) \geq \tau(||x - x_0||_{\mathbb{R}^n}) \quad \forall x \in \Omega.$$

It is not difficult to verify that the coercivity condition implies (iii).

Corollary 4.6 *Let $p_0 := (f_0, g_0, b_0) \in P$. Suppose that, the constraint set mapping C is lsc at p_0. Assume that the function $f_0 \in C[\Omega, \mathbb{R}^s]$ has one of the following two properties:*

(i) *Ω is convex and f_0 is strictly $\mathbb{R}^s_+$-quasiconvex on Ω.*
(ii) *f_0 is injective, i.e., $f_0(x_1) \neq f_0(x_2)$ whenever $x_1 \neq x_2$.*

Then $\mathcal{S}$ is lsc at p_0.

Proof Take any $x_0 \in \mathcal{S}(p_0)$ and $V(x_0) \in \mathcal{N}(x_0)$. In view of Theorem 4.1, to obtain the lower semicontinuity of $\mathcal{S}$ at p_0, it is sufficient to verify that

$$f_0^{-1}(f_0(x_0)) \cap C(p_0) = \{x_0\}. \tag{4.20}$$

Obviously, (ii) implies (4.20). Suppose that (i) holds and that $f_0^{-1}(f_0(x_0)) \cap C(p_0) \neq \{x_0\}$. Then there exists $x_1 \in C(p_0) \backslash \{x_0\}$ such that $f_0(x_1) = f_0(x_0)$. Clearly,

$$f_0(x_0) \in f_0(x_0) - \mathbb{R}^s_+,$$
$$f_0(x_1) \in f_0(x_1) - \mathbb{R}^s_+ = f_0(x_0) - \mathbb{R}^s_+.$$

Since $C(p_0)$ is convex, it follows that $z_0 = \frac{x_0+x_1}{2} \in C(p_0)$. By the strict quasiconvexity of f_0 we have

$$\begin{aligned} f_0(z_0) = f_0\left(\frac{x_0 + x_1}{2}\right) &\in f_0(x_0) - \text{int}\mathbb{R}^s_+ \\ &\subset f_0(x_0) - \mathbb{R}^s_+ \backslash \{0\}. \end{aligned}$$

This contradicts the fact that, $x_0 \in \mathcal{S}(p_0)$, and hence (4.20) follows. The proof is complete. □

5 Upper semicontinuity of the Pareto solution map

In this section we derive necessary and sufficient conditions for the upper semicontinuity of the Pareto solution mapping $\mathcal{S}$ at the reference point.

Theorem 5.1 *Let $p_0 := (f_0, g_0, b_0) \in P$. If $\mathcal{S}$ is usc at p_0, then $\mathcal{S}(p_0) = \mathcal{S}^w(p_0)$. Moreover, if in addition the constraint set mapping C is lsc at p_0, then the converse is also true.*

Proof We prove the first assertion of the theorem. Suppose to the contrary that $\mathcal{S}(p_0) \neq \mathcal{S}^w(p_0)$. Then by $\mathcal{S}(p_0) \subset \mathcal{S}^w(p_0)$, there exists some $\bar{x} \in \mathcal{S}^w(p_0) \backslash \mathcal{S}(p_0)$. Let

$$\alpha(x) := \frac{1}{1 + d(x, \bar{x})} \quad \forall x \in \Omega.$$

Obviously, the function α is continuous on Ω. For each real number $k > 1$, let $u^k := (\frac{1}{k}, \ldots, \frac{1}{k}) \in \mathbb{R}^s$ and

$$f_k(x) := f_0(x) - \alpha(x)u^k \quad \forall x \in \Omega,$$

then we have $f_k \in C[\Omega, \mathbb{R}^s]$ $\forall k > 1$. Put $p_k := (f_k, g_0, b_0) \in P$. We claim that

$$\bar{x} \in \mathcal{S}(p_k) \ \ \forall k > 1. \tag{5.21}$$

Indeed, if there exists $k_0 > 1$ such that $\bar{x} \notin \mathcal{S}(p_{k_0})$, then there is $z \in C(p_0)$ satisfying

$$f_{k_0}(z) - f_{k_0}(\bar{x}) \in -\mathbb{R}^s_+ \setminus \{0\}.$$

Thus, we have

$$f_0(z) - f_0(\bar{x}) + (1 - \alpha(z))u^{k_0} \in -\mathbb{R}^s_+ \setminus \{0\},$$

and hence

$$f_0(z) - f_0(\bar{x}) \in -\text{int}\mathbb{R}^s_+.$$

This contradicts the fact that, $\bar{x} \in \mathcal{S}^w(p_0)$, which proves (5.21). Take an open set W such that $\mathcal{S}(p_0) \subset W$ and $\bar{x} \notin W$. By the upper semicontinuity of $\mathcal{S}$, we have $\bar{x} \in W$, which is impossible, and the first assertion of the theorem follows.

We next prove the second assertion of the theorem. Suppose that, the constraint set mapping C is lsc at p_0. If $\mathcal{S}$ is not usc at p_0, then there exist an open set W containing $\mathcal{S}(p_0)$, a sequence $\{p_k := (f_k, g_k, b_k)\} \subset P$ converging to p_0, and $x_k \in \mathcal{S}(p_k)$ such that $x_k \notin W$ for all $k \geq 1$. Since Ω is compact, by taking a convergent subsequence if necessary, we can assume that, $x_k \to x_0$. From the upper semicontinuity of C at p_0 by Proposition 3.1, it follows that $x_0 \in C(p_0)$. Hence, $x_0 \notin \mathcal{S}^w(p_0)$ by $\mathcal{S}(p_0) = \mathcal{S}^w(p_0)$. This implies that there exists $z_0 \in C(p_0)$ such that

$$f_0(z_0) - f_0(x_0) \in -\text{int}\mathbb{R}^s_+.$$

By the lower semicontinuity of C at p_0, there exists $z_k \in C(p_k)$ such that $z_k \to z_0$ as $k \to \infty$. Hence

$$f_k(z_k) - f_k(x_k) \in -\text{int}\mathbb{R}^s_+ \subset -\mathbb{R}^s_+ \setminus \{0\}$$

for all sufficiently large k, which is contrary to $x_k \in \mathcal{S}(p_k)$. The proof of the second assertion of the theorem is complete. □

Note that, the necessary condition for the upper semicontinuity of the Pareto solution map $\mathcal{S}$ in Theorem 5.1 does not become sufficient if the lower semicontinuity of C is omitted. Indeed, we showed in Example 4.2 that C is not lsc at p_0. It is easily seen that $\mathcal{S}$ is not upper semicontinuous at p_0.

Corollary 5.2 *Let Ω be a nonempty convex compact set of a locally convex space and let $p_0 = (f_0, g_0, b_0) \in P$. If $\mathcal{S}$ is usc at p_0, then $\mathcal{S}(p_0) = \mathcal{S}^w(p_0)$. Moreover, if in addition $g(\cdot, t)$ is $\mathbb{R}^m_+$-convex on Ω for all $t \in T$, and the Slater condition holds for p_0, then the converse is true as well.*

Proof Applying Proposition 3.2, we have that, C is lsc at p_0. Then our assertions are immediate from Theorem 5.1. The proof is complete. □

The following result is immediate from Theorem 5.1 by taking $g(x,t) := (0,\ldots,0) \in \mathbb{R}^m$ and $b(t) := (1,\ldots,1) \in \mathbb{R}^m$, for all $x \in \Omega$ and for all $t \in T$.

Corollary 5.3 [19, Theorem 3.1] *Let $p_0 \in P$. If $C(p) = \Omega$ for all $p \in P$, then $\mathcal{S}$ is usc at p_0 if and only if $\mathcal{S}(p_0) = \mathcal{S}^w(p_0)$.*

Corollary 5.4 *Let $p_0 := (f_0, g_0, b_0) \in P$. Suppose that the constraint set mapping C is lsc at p_0. If Ω is convex and $f_0 \in C[\Omega, \mathbb{R}^s]$ is strictly $\mathbb{R}^s_+$-quasiconvex on Ω, then $\mathcal{S}$ is usc at p_0.*

Proof The equality $\mathcal{S}(p_0) = \mathcal{S}^w(p_0)$ follows from [14, Proposition 5.13]. Applying Theorem 5.1, we obtain the upper semicontinuity of $\mathcal{S}$ at p_0.

Corollary 5.5 *Let $p_0 := (f_0, g_0, b_0) \in P$. Suppose that, the following conditions hold*

(i) *the constraint set mapping C is lsc at p_0;*
(ii) *$\mathcal{S}(p_0) = \mathcal{S}^w(p_0)$;*
(iii) *for each $x_0 \in \mathcal{S}(p_0)$ and for each $V(x_0) \in \mathcal{N}(x_0)$, there exists $\bar{x} \in V(x_0) \cap \mathcal{S}(p_0)$ such that $f_0^{-1}(f_0(\bar{x})) \cap C(p_0) \subset V(x_0)$. Then, $\mathcal{S}$ is continuous at p_0.*

Proof The proof is immediate from Theorem 4.1 and Theorem 5.1, so can be omitted. □

Acknowledgements We would like to thank Professors Boris S. Mordukhovich and Nguyen Dong Yen for several useful discussions on the topic of this paper. We are thankful to two anonymous referees for their valuable comments and suggestions. This work was supported by research grant from the National Sun Yat-sen University and the grant NSC 97-2115-M-110-001.

References

1. Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis. A Hitchhiker's Guide, 3rd edn. Springer, Berlin (2006)
2. Brosowski, B.: Parametric semi-infinite linear programming. I. Continuity of the feasible set and of the optimal value. Sensitivity, stability and parametric analysis. Math. Program. Stud. **21**, 18–42 (1984)
3. Cánovas , M.J., López, M.A., Parra, J., Todorov, M.I.: Stability and well-posedness in linear semi-infinite programming. SIAM J. Optim. **10**(1), 82–98 (1999)
4. Cánovas, M.J., López, M.A., Parra, J., Toledo, F.J.: Lipschitz continuity of the optimal value via bounds on the optimal set in linear semi-infinite optimization. Math. Oper. Res. **31**(3), 478–489 (2006)
5. Cánovas, M.J., Klatte, D., López, M.A., Parra, J.: Metric regularity in convex semi-infinite optimization under canonical perturbations. SIAM. J Optim. **18**(3), 717–732 (2007)
6. Chen, G.Y., Craven, B.D.: Existence and continuity of solutions for vector optimization. J. Optim. Theory Appl. **81**(3), 459–468 (1994)
7. Colgen, R., Schnatz, K.: Continuity properties in semi-infinite parametric linear optimization. Numer. Funct. Anal. Optim. **3**(4), 451–460 (1981)
8. Gayá, V.E., López, M.A., Vera de Serio, V.N.: Stability in convex semi-infinite programming and rates of convergence of optimal solutions of discretized finite subproblems. Optimization **52**(6), 693–713 (2003)
9. Goberna, M.A.: Linear semi-infinite optimization: recent advances. Continuous optimization, 3–22, Appl. Optim., 99. Springer, New York (2005)
10. Goberna, M.A., López, M.A., Todorov, M.: Stability theory for linear inequality systems. SIAM J. Matrix Anal. Appl. **17**(4), 730–743 (1996)
11. Goberna, M.A., López, M.A., Todorov, M.: Stability theory for linear inequality systems. II. Upper semicontinuity of the solution set mapping. SIAM J. Optim. **7**(4), 1138–1151 (1997)
12. Goberna, M.A., Gómez, S., Guerra, F., Todorov, M.I.: Sensitivity analysis in linear semi-infinite programming: perturbing cost and right-hand-side coefficients. Eur. J. Oper. Res. **181**(3), 1069–1085 (2007)
13. López, M.A., Vera de Serio, V.N.: Stability of the feasible set mapping in convex semi-infinite programming semi-infinite programming (Alicante, 1999), 101–120. Nonconvex Optim. Appl., 57. Kluwer Acad. Publ., Dordrecht (2001)

14. Luc, D.T.: Theory of Vector Optimization Lecture Notes in Economics and Mathematical Systems, 319. Springer, Berlin (1989)
15. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation. I. Basic theory, Grundlehren der Mathematischen Wissenschaften, 330. Springer, Berlin (2006)
16. Mordukhovich, B.S.: Variational Anlysis and Generalized Differentiation. II. Applications, Grundlehren der Mathematischen Wissenschaften, 331. Springer, Berlin (2006)
17. Reemtsen, R., Rückmann, J.-J. (eds.): Semi-Infinite Programming, Nonconvex Optimization and its Applications, 25. Kluwer Academic Publishers, Boston, MA (1998)
18. Weber, G.-W.: Generalized Semi-Infinite Optimization and Related Topics, Research and Exposition in Mathematics, 29. Heldermann, Lemgo (2003)
19. Xiang, S.W., Zhou, Y.H.: Continuity properties of solutions of vector optimization. Nonlinear Anal. **64**, 2496–2506 (2006)
20. Xiang, S.W., Yin, W.S.: Stability results for efficient solutions of vector optimization problems. J. Optim. Theory Appl. **134**, 385–398 (2007)
21. Yu, J.: Essential weak efficient solution in multiobjective optimization problems. J. Math. Anal. Appl. **166**, 230–235 (1992)