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On the existence of solutions to quasivariational inclusion problems

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Abstract We establish sufficient existence conditions for general quasivariational inclusion problems, which contain most of variational inclusion problems and quasiequilibrium problems considered in the literature. These conditions are shown to extend recent existing results and sharpen some of them even for particular cases.

Keywords Quasivariational inclusion problems · Quasiequilibrium problems · Implicit variational inequalities · The solution existence · Fixed points · Nash equilibria · Traffic networks

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1 Introduction

Equilibrium problems were introduced in [7] as generalizations of variational inequalities and optimization problems. This problem setting proved to be rather general, including also many other optimization-related problems such as fixed-point and coincidence-point problems, complementarity problems, Nash equilibria, minimax problems, traffic networks, etc. On the

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other hand, this setting proved to be suitable for applying analytic tools in consideration. For the last decade there have been a number of generalizations of the equilibrium-problem formulation. A turning point was the quasiequilibrium problem with constraint sets depending also on state variables. The starting point for this kind of constraint sets was [6], where the authors investigated random impulse control problems. Further generalizations were variational inclusion and quasivariational inclusion problems, see e.g. [2,3,17,19,24,26,27,38,41–44]. It should be noted that the term "inclusion" appeared in recent papers also in other meanings. In [13,31] "variational inclusion" means a multivalued variational inequality. Variational inclusions studied in [1,8,9] are problems of finding the zeroes of maximal monotone mappings.

For the above-mentioned problems we can observe that the solution existence was always the first topic and attracted the attention of many mathematicians. Existence results for various types of equilibrium problems were the contributions of [4,5,10,11,15,16,18,20,21,34–37] among others. For quasivariational inclusions, existence conditions were developed in [17,19,24,26,27,38,43,44].

The aim of the present paper is to establish new existence results for quasivariational inclusion problems discussed in [2,3,17,24,26,27]. This problem setting proved to include, as particular cases, most of quasivariational inclusion and quasiequilibrium problems in the literature. We try to get sufficient conditions for the solution existence so that when applied to particular cases they are stronger than several recent results, e.g. in [17,22,32,33,35–37]. About the main tools for proving existence results in quasivariational inclusions and their special cases, we observe the KKM-Fan theorem in the first place, see e.g. [11,17,22,23,26–28]. Several fixed-point theorems such as those of Kakutani, Tarafdar, Park, Kim-Tan are also important and convenient tools, see e.g [18,20,21,23,32,33,44]. Maximal-element theorems are used in e.g. [4,16,19,25]. Minimax theorems may also be employed, see e.g. [32] for applying Kneser's theorem ([30]). Existence results for problems of other kinds may be applied also to get corresponding results, e.g. in [29] an existence theorem in game theory is applied to prove existence conditions for quasivariational inequalities. Each tool has advantages in some appropriate situations. In this paper we make use of a fixed-point theorem in [40]. It turns out that this theorem is suitable to our problem setting and helps to get new results or to sharpen some recent existing ones.

The layout of the paper is as follows. In the rest of this section we state our problems under consideration and supply some preliminaries. Section 2 is devoted to the main result. In Sect. 3 we discuss its consequences in particular cases to explain advantages and possibilities of applications.

Our problem setting is as follows. In the sequel, if not stated otherwise, let X, Y and Z be real topological vector spaces; let X be Hausdorff and $A, B \subseteq X$ be nonempty closed convex subsets. Let $C : A \to 2^Y$, $S_1 : A \to 2^B$, $S_2 : A \to 2^B$ and $T : A \times B \to 2^Z$ be multifunctions such that $C(x)$ is a closed convex cone with nonempty interior and $C(x) \neq Y$, for each $x \in A$. Let $F : T(A \times B) \times B \times A \to 2^Y$ and $G : T(A \times B) \times A \to 2^Y$ be multifunctions. We always assume that the images of the above multifunctions are nonempty. For subsets U and V and points x, y under consideration we adopt the notations

$$\begin{aligned}
r_1(U, V) \quad \text{means} \quad U \cap V \neq \emptyset;\ \ \bar{r}_1(U, V) \quad \text{means} \quad U \cap V = \emptyset;\\
r_2(U, V) \quad \text{means} \quad U \subseteq V;\ \ \bar{r}_2(U, V) \quad \text{means} \quad U \not\subseteq V;\\
\alpha_1(x, U) \quad \text{means} \quad \forall x \in U;\ \ \alpha_2(x, U) \quad \text{means} \quad \exists x \in U.
\end{aligned}$$

For each $r \in \{r_1, r_2\}$, $\alpha \in \{\alpha_1, \alpha_2\}$, we consider the following quasivariational inclusion problem

($\mathrm{IP}_{r\alpha}$) Find $\bar{x} \in S_1(\bar{x})$ such that, for all $y \in S_2(\bar{x})$,

$$\alpha(\bar{t}, T(\bar{x}, y)), r(F(\bar{t}, y, \bar{x}), G(\bar{t}, \bar{x})).$$

This setting is not explicit but helps to shorten remarkably the presentation , since ($\mathrm{IP}_{r\alpha}$) includes four different problems depending on combinations of values of r and α. We discuss several particular cases.

(a) Let $A = B$, $C(x) \equiv C$, $G(t, x) = F(t, x, x) + C$, where $C \subseteq Y$ is a closed convex cone. Then ($\mathrm{IP}_{r_2\alpha_1}$) collapses to the quasivariational inclusion problem studied in [38]:

(IP) Find $\bar{x} \in S_1(\bar{x})$ such that, for all $y \in S_2(\bar{x})$ and all $\bar{t} \in T(\bar{x}, y)$,

$$F(\bar{t}, y, \bar{x}) \subseteq F(\bar{t}, \bar{x}, \bar{x}) + C.$$

If T has the special form $(x, y) \mapsto T(x, x)$, (IP) is a quasivariational inclusion of the Minty type (MP). While if T is of the form $(x, y) \mapsto T(y, y)$, (IP) is a quasivariational inclusion of the Stampacchia type. For instance, if $Y = R$, $C = R_+$ and $G(t, x) \equiv R_+$, then (IP) becomes the quasiequilibrium problem of the Minty type dealt with in [37].

(b) With $A = B$, $S_1(x) = S_2(x) := S(x)$ and $C(x) \equiv C$, $G(t, x) = F(t, x, x) + C$ and T given by $(x, y) \mapsto T(x, x) := T(x)$, ($\mathrm{IP}_{r_2\alpha_1}$) becomes the upper variational inclusion problem investigated in [43]:

(UIP) Find $\bar{x} \in S(\bar{x})$ such that, for all $y \in S(\bar{x})$ and all $\bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq F(\bar{t}, \bar{x}, \bar{x}) + C.$$

(c) Consider the following quasiequilibrium problem studied by many authors:

(QEP) Find $\bar{x} \in S(\bar{x})$ such that, for all $y \in S(\bar{x})$ and all $\bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq C.$$

It is clear that (IP) and (UIP) do not include (QEP), without severe assumptions on F. However, our ($\mathrm{IP}_{r_2\alpha_1}$) does (by choosing special forms of involved multifunctions as in (b), except G which now is $G(t, x) \equiv C$).

(d) If $A = B$, $S_1(x) = S_2(x) := S(x)$, T is of the form $(x, y) \mapsto T(x, x) := T(x)$ and $G(t, x) = C(x)$, then ($\mathrm{IP}_{r_2\alpha_1}$) and ($\mathrm{IP}_{r_1\alpha_2}$) coincide with the following quasiequilibrium problems, respectively, considered in [36]:

(EP1) Find $\bar{x} \in A$ such that $\bar{x} \in S(\bar{x})$ and, for all $y \in S(\bar{x})$ and all $\bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq C(\bar{x});$$

(EP2) Find $\bar{x} \in A$ such that $\bar{x} \in S(\bar{x})$ and, for all $y \in S(\bar{x})$ and all $\bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \cap C(\bar{x}) \neq \emptyset.$$

If we replace the definition of G by $G(t, x) = Y \backslash -\mathrm{int} C(x)$, then ($\mathrm{IP}_{r_2\alpha_1}$) and ($\mathrm{IP}_{r_1\alpha_2}$) collapse to other equilibrium problems in [36]:

(EP3) Find $\bar{x} \in A$ such that $\bar{x} \in S(\bar{x})$ and, for all $y \in S(\bar{x})$ and all $\bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \subseteq Y \backslash -\mathrm{int} C(\bar{x});$$

(EP4) Find $\bar{x} \in A$ such that $\bar{x} \in S(\bar{x})$ and, for all $y \in S(\bar{x})$ and all $\bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \cap Y \backslash -\mathrm{int} C(\bar{x}) \neq \emptyset.$$

For the more special case, where $A = B \equiv S_1(x) = S_2(x) := S(x)$, we have the corresponding four problems (EP1–EP4) investigated in [35].

(e) If $A = B = S_1(x) = S_2(x)$, T has the form $(x, y) \mapsto T(x, x) := T(x)$, $Z = L(X, Y)$ (the space of the continuous linear mappings from X into Y equipped with either the topology of pointwise convergence or that of bounded convergence), F is single-valued and $G(t, x) = Y \backslash -\text{int}C(x)$, then $(\text{IP}_{r_2\alpha_2})$ and $(\text{IP}_{r_1\alpha_2})$ become the implicit vector variational inequality in [32,33]:

(IVI) Find $\bar{x} \in A$ such that, for all $y \in A$, there exists $\bar{t} \in T(\bar{x})$,

$$F(\bar{t}, y, \bar{x}) \notin -\text{int}C(\bar{x}).$$

(f) The following quasivariational inequality commonly interested in the literature, see e.g. [15]:

(QVI) Find $\bar{x} \in S(\bar{x})$ such that, for all $y \in S(\bar{x})$, there exists $\bar{t} \in T(\bar{x})$,

$$(\bar{t}, y - \bar{x}) \notin -\text{int}C(\bar{x}),$$

where (t, x) denotes the image of $t \in L(X, Y)$ at x (this notation is used also for other routine meanings, but the context will eliminate any threat of confusion), is clear a particular case of $(\text{IP}_{r_2\alpha_2})$ and $(\text{IP}_{r_1\alpha_2})$.

We recall the definitions of semicontinuity properties of a multifunction $I : X \to 2^Y$, where X and Y are topological spaces. I is said to be upper semicontinuous (usc) at $x_0 \in X$ if, for each open subset U containing $I(x_0)$, there is a neighborhood N of x_0 such that $I(N) \subseteq U$. $I : X \to 2^Y$ is called lower semicontinuous (lsc) at $x_0 \in X$ if, for each open subset U with $I(x_0) \cap U \neq \emptyset$, there is a neighborhood N of x_0 such that, $I(x) \cap U \neq \emptyset$, for all $x \in N$. $I : X \to 2^Y$ is termed closed at x_0 if, for any $x_\gamma \to x_0$ and $y_\gamma \in I(x_\gamma)$ with $y_\gamma \to y_0$, one has $y_0 \in I(x_0)$. If I is closed at each $x \in A$ we say that I is closed in A. In particular, if $A = \text{dom}I := \{x \in X : I(x) \neq \emptyset\}$ we say simply that I is closed. A similar convention is adopted for other properties of I.

Our main tool for proving the existence conditions is the following theorem.

Theorem 1.1 ([40]) *Let X be a Hausdorff topological vector space, $A \subseteq X$ be nonempty convex and $D \subseteq A$ be a nonempty compact subset. Let $S : A \to 2^A$ and $L : A \to 2^A$ be multifunctions. Assume that*

(a) for all $x \in A$, $L(x)$ is convex and $S(x) \subseteq L(x)$;
(b) for all $x \in D$, $S(x) \neq \emptyset$;
(c) for all $y \in A$: $S^{-1}(y)$ is open in A;
(d) for each finite subset N of A there is a compact convex subset L_N such that, $N \subseteq L_N \subseteq A$ and $S(x) \cap L_N \neq \emptyset$ for all $x \in L_N \backslash D$.

Then L has fixed points.

Remark 1.1 The coercivity condition (d) in Theorem 1.1 can be replaced by another coercivity as follows.

(d′) There is a nonempty compact convex subset $K \subseteq A$ such that, for all $x \in A \backslash D$, $y \in K$ exists with $x \in S^{-1}(y)$.

Indeed, assume (d′) and let $N \subseteq A$ be finite. Take $L_N = \text{co}(K \cup N)$, then for any $x \in L_N \backslash D \subseteq A \backslash D$, $y \in K \subseteq L_N$ exists such that $x \in S^{-1}(y)$. Hence $y \in S(x) \cap K \subseteq S(x) \cap L_N$, i.e. (d) is satisfied.

2 The main result

Theorem 2.1 *For problem (IP$_{r_i\alpha_j}$), $i, j = 1, 2$, assume the existence of $H : T(A \times B) \times B \times A \to 2^Y$ satisfying the following conditions*

(i) if, $\alpha_j(t, T(x, y)), r_i(H(t, y, x), G(t, x))$, then $\alpha_j(t, T(x, y)), r_i(F(t, y, x), G(t, x))$;
(ii) for all $x \in A$, the set $\{y \in A \mid \alpha_{3-j}(t, T(x, y)), \bar{r}_i(H(t, y, x), G(t, x))\}$ is convex and $\alpha_j(t, T(x, x)), r_i(H(t, x, x), G(t, x))$;
(iii) for all $y \in A$, the set $\{x \in A \mid \alpha_j(t, T(x, y)), r_i(H(t, y, x), G(t, x))\}$ is closed;
(iv) $S_1(.)$ is closed and, for all $x, y \in A$, $co(S_2(x)) \subseteq S_1(x)$, $S_2(x) \cap A \neq \emptyset$ and $S_2^{-1}(y)$ is open in A;
(v) there is a nonempty, compact subset $D \subseteq A$ such that, for each finite subset N of A, a compact convex subset L_N with $N \subseteq L_N \subseteq A$ exists satisfying, for all $x \in L_N \backslash D$, $S_2(x) \cap L_N \neq \emptyset$ and, for $x \in S_1(x) \cap (L_N \backslash D)$, there is $y \in S_2(x) \cap L_N$ with $\alpha_{3-j}(t, T(x, y)), \bar{r}_i(H(t, y, x), G(t, x))$.

Then problem (IP$_{r_i\alpha_j}$)has a solution.

Proof For $x \in A$, $i, j = 1, 2$ and $k = 1, 2$, set

$$
\begin{aligned}
E &= \{x \in A \mid x \in S_1(x)\},\\
P_1(x) &= \{y \in A \mid \alpha_{3-j}(t, T(x, y)), \bar{r}_i(F(t, y, x), G(t, x))\},\\
P_2(x) &= \{y \in A \mid \alpha_{3-j}(t, T(x, y)), \bar{r}_i(H(t, y, x), G(t, x))\},\\
\Phi_k(x) &= \begin{cases} S_2(x) \cap P_k(x) & \text{if } x \in E,\\ A \cap S_2(x) & \text{if } x \in A \backslash E,\end{cases}\\
Q(x) &= \begin{cases} (\mathrm{co}S_2(x)) \cap P_2(x) & \text{if } x \in E,\\ A \cap \mathrm{co}S_2(x) & \text{if } x \in A \backslash E.\end{cases}
\end{aligned}
$$

We will apply Theorem 1.1 with $L = Q$ and $S = \Phi_2$, showing that Q has no fixed point and assumptions (a), (c) and (d) of this theorem are satisfied and hence assumption (b) must be violated. For (a) we see from (i) of Theorem 2.1 that, $\forall x \in A$, $P_1(x) \subseteq P_2(x)$, whence $\Phi_1(x) \subseteq \Phi_2(x) \subseteq Q(x)$ by the definition of Q. Moreover, $Q(x)$ is convex by (ii).

For (c) we have, for all $y \in A$,

$$
\begin{aligned}
\Phi_2^{-1}(y) &= [E \cap S_2^{-1}(y) \cap P_2^{-1}(y)] \cup [(A \backslash E) \cap S_2^{-1}(y)]\\
&= [(A \backslash E) \cup P_2^{-1}(y)] \cap S_2^{-1}(y).
\end{aligned}
$$

Consequently,

$$A \backslash \Phi_2^{-1}(y) = [E \cap (A \backslash P_2^{-1}(y))] \cup (A \backslash S_2^{-1}(y)). \tag{1}$$

It suffices to verify the closedness of this set. By the closedness of $S_1(.)$ assumed in (iv) it is not hard to see that E is closed. $A \backslash S_2^{-1}(y)$ is also closed by (iv). The remaining term in (1) is

$$A \backslash P_2^{-1}(y) = \{x \in A \mid \alpha_j(t, T(x, y)), r_i(H(t, y, x), G(t, x))\},$$

which is closed by (iii). Thus, $A \backslash \Phi_2^{-1}(y)$ is closed.

To see (d) we have D and L_N for each N by assumption (v). Let $x \in L_N \backslash D$ be arbitrary. If $x \in A \backslash E$, one has $\Phi_2(x) \cap L_N = A \cap S_2(x) \cap L_N = S_2(x) \cap L_N \neq \emptyset$ by (v). If $x \in E$, then $x \in S_1(x) \cap (L_N \backslash D)$ and, by (v), there is $y \in S_2(x) \cap L_N$ such that $y \in P_2(x)$. Hence $y \in \Phi_2(x)$ and $\Phi_2(x) \cap L_N \neq \emptyset$.

Finally, suppose that Q has a fixed point $x_0 \in A$. If $x_0 \in E$, then $x_0 \in P_2(x_0)$, i.e. $\alpha_{3-j}(t, T(x_0, x_0)), \bar{r}_i(H(t, x_0, x_0), G(t, x_0))$, contradicting (ii). If $x_0 \in A\backslash E$, then $x_0 \in \mathrm{co}(S_2(x_0)) \subseteq S_1(x_0)$, i.e. $x_0 \in E$, again a contradiction.

The above argument implies that (b) of Theorem 1.1 must be violated, i.e. there is $x_0 \in D \subseteq A$ such that $\Phi_2(x_0) = \emptyset$ and hence $\Phi_1(x_0) = \emptyset$. If $x_0 \in A\backslash E$ then $A \cap S_2(x_0) = \Phi_1(x_0) = \emptyset$, contradicting (iv). So $x_0 \in E$ and $\emptyset = \Phi_1(x_0) = S_2(x_0) \cap P_1(x_0)$. Consequently, for all $y \in S_2(x_0)$, $y \notin P_1(x_0)$, i.e., $\alpha_j(t, T(x_0, y)), r_i(F(t, y, x_0), G(t, x_0))$, which means that x_0 is a solution of $(\mathrm{IP}_{r_i \alpha_j})$. □

Remark 2.1

(i) Assumption (v) is a coercivity condition. If A is compact, (v) is satisfied with $D = A$. So we can omit (v). Moreover, due to Remark 1.1, assumption (v) can be replaced by

  (v′) there are nonempty compact convex subset $K \subseteq A$ and nonempty compact subset $D \subseteq A$ such that, $\forall x \in A\backslash D$, $S_2(x) \cap K \neq \emptyset$ and, if $x \in S_1(x) \cap (A\backslash D)$, there exists $y \in S_2(x) \cap K$ with $\alpha_{3-j}(t, (x, y)), \bar{r}_i(H(t, y, x), G(t, x))$.

(ii) In the case where $r_i = r_2, \alpha_j = \alpha_1$, (iii) of Theorem 2.1 will be satisfied if, for all $y \in A, T(., y)$ and $H(., y, .)$ are lsc and G is closed. Indeed, we will verify that the following set is closed:

$$M_1 = \{x \in A \mid \forall t \in T(x, y), H(t, y, x) \subseteq G(t, x)\}.$$

Let $x_\gamma \in M_1$ and $x_\gamma \to x_0$. By the assumed lower semicontinuity, $\forall t_0 \in T(x_0, y)$, $\forall w_0 \in H(t_0, y, x_0), \exists t_\gamma \in T(x_\gamma, y) : t_\gamma \to t_0, \exists w_\gamma \in H(t_\gamma, y, x_\gamma) \subseteq G(t_\gamma, x_\gamma)$ such that $w_\gamma \to w_0$. Since G is closed, $w_0 \in G(t_0, x_0)$. Hence, for all $t_0 \in T(x_0, y)$, $H(t_0, y, x_0) \subseteq G(t_0, x_0)$, i.e. $x_0 \in M_1$.
Similarly, in the remaining cases (iii) of Theorem 2.1 will be satisfied:

- for $r_i = r_1$ and $\alpha_j = \alpha_1$: if, for all $y \in A, T(., y)$ is lsc; $H(., y, .)$ is usc and compact-valued and G is closed;
- for $r_i = r_1$ and $\alpha_j = \alpha_2$: if, for all $y \in A, T(., y)$ and $H(., y, .)$ are usc and compact-valued and G is closed;
- for $r_i = r_2$ and $\alpha_j = \alpha_2$: if, for all $y \in A, T(., y)$ is usc and compact-valued; $H(., y, .)$ is lsc and G is closed.

(iii) In the case where $r_i = r_2$ and $\alpha_j = \alpha_1$, if we replace, in assumptions (ii), (iii) and (v) of Theorem 2.1, multifunction H by F, then we can omit assumption (i) to get a consequence, called Theorem 2.1_F for convenience. This Theorem 2.1_F is different from Theorem 3.1 of [17] for the same problem $(\mathrm{IP}_{r_2 \alpha_1})$ and may be more applicable in some cases as shown by the following example.

Example 2.1 Let $X = Y = Z = R$, $A = B = (-\infty, 1]$, $S_1(x) = S_2(x) \equiv (-\infty, 1]$, $T(x, y) = \{x\}$, $G(t, x) \equiv R_+$ and

$$F(t, y, x) = \begin{cases} \{y^2\} & \text{if } y < 0, \\ \{xy\} & \text{if } 0 \leq y \leq 1. \end{cases}$$

Since A is not compact, Theorem 3.1 of [17] cannot be applied. For the assumptions of Theorem 2.1_F, only the coercivity condition is not clear and needs to be checked. Take $D = [0, 1]$. For any finite subset $N \subseteq A$, choose $L_N = \{x \in A \mid 1 \geq x \geq \min N\}$. Then for each $x \in L_N\backslash D$, $S_2(x) \cap L_N = L_N \neq \emptyset$ and, for $y = 1 \in S_2(x) \cap L_N$, $F(t, y, x) =$

$\{xy\} \not\subseteq G(t,x) = R_+$, as $x < 0$. Now that all the assumptions of Theorem 2.1_F are satisfied, $(\mathrm{IP}_{r_2\alpha_1})$ has solutions. (Direct computations give the solution set being [0,1].)

Moreover, with assumption (v′) replacing (as mentioned in Remark 2.1 (i)) the coercivity condition of Theorem 2.1_F, by Remark 2.1 (ii) we see that Theorem 2.1_F sharpens Theorem 4.2 of [36], since the semicontinuity assumptions of Theorem 4.2 are more restrictive than the counterpart of Theorem 2.1_F. The following example yields a special case of quasiequilibrium problems where Theorem 2.1_F can be applied but Theorem 4.2 of [36] cannot.

Example 2.2 Let $X = Y = Z = R$, $A = B = [0, 1]$, $S_1(x) = S_2(x) \equiv [0, 1]$, $G(t, x) \equiv R_+$,

$$T(x,y) = \begin{cases} [0, 0.5] & \text{if } 0 \le x < 0.5, \\ [0.5, 1] & \text{if } 0.5 \le x \le 1, \end{cases}$$

$$F(t,y,x) = \begin{cases} [0, 0.5] & \text{if } 0 \le y < 0.5, \\ [1, 1.5] & \text{if } 0.5 \le y \le 1. \end{cases}$$

It is clear that, for any $y \in [0, 1]$, $T(., y)$ and $F(., y, .)$ is not lsc in [0,1]. Consequently, Theorem 4.2 of [36] is out of work. On the other hand, it is equally evident that the assumptions of Theorem 2.1_F are fulfilled. By direct calculations, [0,1] is seen to be the solution set of $(\mathrm{IP}_{r_2\alpha_1})$.

3 Applications

The main results in Sect. 2 imply clearly existence conditions for various particular cases, e.g. those mentioned in Sect. 1. Here we derive consequences only for several important problems as examples and compare them with recent papers to see advantages of our results.

3.1 Equilibrium problems

We discuss equilibrium problems (EP1) and (EP2) encountered in Sect. 1(d) and studied in [36].

Corollary 3.1 *For problem (EP1) assume the existence of* $H : T(A) \times A \times A \to 2^Y$ *satisfying the following conditions*

(i) if, $\forall t \in T(x)$, $H(t, y, x) \subseteq C(x)$, *then* $\forall t \in T(x)$, $F(t, y, x) \subseteq C(x)$;
(ii) for all $x \in A$, *the set* $\{y \in A \mid \exists t \in T(x), H(t, y, x) \not\subseteq C(x)\}$ *is convex and, for all* $t \in T(x)$, $H(t, x, x) \subseteq C(x)$;
(iii) for all $y \in A$, *the set* $\{x \in A \mid \forall t \in T(x), H(t, y, x) \subseteq C(x)\}$ *is closed;*
(iv) $S(.)$ *is closed and* $S^{-1}(y)$ *is open in* A *for all* $y \in A$;
(v) there is a nonempty compact subset $D \subseteq A$ *such that, for each finite subset* N *of* A, *a compact convex subset* L_N *of* A *exists containing* N *and satisfying,* $\forall x \in L_N \backslash D$, $\exists y \in L_N$, $H(t, y, x) \not\subseteq C(x)$ *for some* $t \in T(x)$.

Then problem (EP1) is solvable.

Proof We simply apply Theorem 2.1 with $r_i = r_2, \alpha_j = \alpha_1, A = B, S_1(x) = S_2(x)$, $T(x, y) = T(x)$ and $G(t, x) = C(x)$. □

Corollary 3.2 *For problem (EP2) assume (iv) of Corollary 3.1 and that* $H : T(A) \times A \times A \to 2^Y$ *exists satisfying*

(i) *if,* $\exists t \in T(x), H(t, y, x) \cap C(x) \neq \emptyset$*, then* $\exists t \in T(x), F(t, y, x) \cap C(x) \neq \emptyset$*;*
(ii) *for all* $x \in A$*, the set* $\{y \in A \mid \forall t \in T(x), H(t, y, x) \cap C(x) \neq \emptyset\}$ *is convex and there exists* $t \in T(x)$ *such that* $H(t, x, x) \subseteq C(x)$*;*
(iii) *for all* $y \in A$*, the set* $\{x \in A \mid \exists t \in T(x), H(t, y, x) \cap C(x) \neq \emptyset\}$ *is closed;*
(iv) *it is (v) of Corollary 3.1 with "*$H(t, y, x) \not\subseteq C(x)$ *for some* $t \in T(x)$*" replaced by "*$H(t, y, x) \cap C(x) = \emptyset$ *for all* $t \in T(x)$*".*

Then problem (EP2) has solutions.

Remark 3.1 Corollaries 3.1 and 3.2 sharpen Theorems 4.11 and 4.12, respectively, of [35]. Corollary 3.1 improves Theorem 4.2 of [36]. The convexity and semicontinuity assumptions in these theorems are stronger than the corresponding assumptions in our corollaries. That is why these theorems are not applicable in the following example while our corollaries are.

Example 3.1 Let $X = Y = Z = R$, $A = [0, 1]$, $C(x) \equiv R_+$, $T(x) = \{x\}$ and $F(t, y, x) = \{1 - (y - 0.5)^2\}$. Then F is not $C(x)$-quasiconvex for $x \in A$ as assumed in [36]. (Recall that a multifunction $y \mapsto Q(y, x)$ is called $C(x)$-quasiconvex if, for all $y_1, y_2 \in A$ and all $\lambda \in [0, 1]$,

$$Q(y_1, x) \subseteq Q((1 - \lambda)y_1 + \lambda y_2, x) + C(x)$$

or

$$Q(y_2, x) \subseteq Q((1 - \lambda)y_1 + \lambda y_2, x) + C(x).$$

Indeed, choose $y_1 = 0$, $y_2 = 1$ and $\lambda = 0.5$. Then

$$F(t, y_1, x) = \{0.75\} \not\subseteq F(t, 0.5y_1 + 0.5y_2, x) + C(x) \equiv [1, +\infty],$$
$$F(t, y_2, x) = \{0.75\} \not\subseteq [1, +\infty],$$

i.e. F is not $C(x)$-quasiconvex. Hence Theorem 4.2 of [36] is not applicable. All the assumptions of Corollary 3.1 are easily seen to be satisfied with $H(t, y, x) = F(t, y, x)$. (For assumption (ii) note that $\{y \in A \mid \exists t \in T(x), H(t, y, x) \not\subseteq C(x)\}$ is empty and then convex.) So by Corollary 3.1, (EP1) has solutions. Direct computations show that the solution set is [0,1].

Pass now to the equilibrium problem of the Minty type (MP) mentioned in Sect. 1 (a) and investigated in [37].

Corollary 3.3 *For problem (MP) assume (iv) of Theorem 2.1 and that* $H : T(A \times A) \times A \times A \to 2^R$ *exists satisfying*

(i) *if,* $\forall s \in T(x, y), H(s, y, x) \subseteq R_+$*, then* $\forall t \in T(x, y), F(t, y, x) \subseteq R_+$*;*
(ii) *for all* $x \in A$*, the set* $\{y \in A \mid \exists t \in T(x, y), H(t, y, x) \not\subseteq R_+\}$ *is convex and* $H(t, x, x) \subseteq R_+$ *for all* $t \in T(x, x)$*;*
(iii) *for all* $y \in A$*, the set* $\{x \in A \mid \forall t \in T(x, y), H(t, y, x) \subseteq R_+\}$ *is closed;*
(iv) *there is a nonempty compact subset* $D \subseteq A$ *such that, for each finite subset* N *of* A*, a compact convex subset* L_N *with* $N \subseteq L_N \subseteq A$ *and, for* $x \in L_N \backslash D$*,* $S_2(x) \cap L_N \neq \emptyset$ *and furthermore, for* $x \in S_1(x) \cap (L_N \backslash D)$*,* $y \in S_2(x) \cap L_N$ *exists so that* $H(t, y, x) \not\subseteq R_+$ *for some* $t \in T(x, y)$*.*

Then (MP) has a solution.

Proof Employ Theorem 2.1 with $r_i = r_2, \alpha_j = \alpha_1$ and $G(t, x) = R_+$. □

Remark 3.2 When applied to the case where $H(t, y, x) = \{\sup F(T(x, x), y, x)\}$, Corollary 3.3 is stronger than Theorem 4.1 of [37], since its assumptions are more relaxed. The example below gives a case where this Theorem 4.1 cannot be employed but our Corollary 3.3 can be easily.

Example 3.2 Let $X = R,\ A = (-\infty, 3],\ S_1(x) = S_2(x) \equiv A,\ T(x, y) \equiv R$ and

$$F(t, y, x) = \begin{cases} [x - y + 1, 6] & \text{if } y \geq 0, \\ \{0\} & \text{if } y < 0. \end{cases}$$

Then, we have

$$\inf F(T(x, y), y, x) = \min\{0, x - y + 1\},$$
$$\sup F(T(x, x), y, x) = \begin{cases} 6 & \text{if } y \geq 0, \\ 0 & \text{if } y < 0. \end{cases}$$

Therefore, $\inf F(T(x, y), y, x) < 0$ does not imply $\sup F(T(x, x), y, x) < 0$. Moreover, $0 \notin F(t, 1, 1)$. Hence, the assumptions of Theorem 4.1 of [37] are not satisfied. As opposed to this, the assumptions of Corollary 3.3 are easy to be checked with $D = [0, 3]$ and

$$H(t, y, x) = \begin{cases} [x - y, 6] & \text{if } y \geq 0, \\ \{0\} & \text{if } y < 0. \end{cases}$$

3.2 Implicit variational inequalities

Passing to a particular case where F is single-valued, we apply a result in Sect. 2 to the implicit variational inequality (IVI) stated in Sect. 1 (e) and studied in [32,33].

Corollary 3.4 *For problem (IVI) assume that the dual topological spaces X^* and Y^* of X and Y, respectively, separate points and that $H : L(X, Y) \times A \times A \to 2^Y$ exists such that*

(i) if, $\exists t \in T(x),\ H(t, y, x) \subseteq Y \backslash -intC(x)$, then $\exists t \in T(x),\ F(t, y, x) \in Y \backslash -\mathrm{int}C(x)$;
(ii) for all $x \in A$, the set $\{y \in A \mid \forall t \in T(x),\ H(t, y, x) \not\subseteq Y \backslash -\mathrm{int}C(x)\}$ is convex and, $t \in T(x)$ exists such that $H(t, x, x) \subseteq Y \backslash \mathrm{int}C(x)$;
(iii) for all $y \in A$, the set $\{x \in A \mid \exists t \in T(x),\ H(t, y, x) \subseteq Y \backslash -\mathrm{int}C(x)\}$ is closed;
(iv) there is a nonempty compact subset $D \subseteq A$ such that, for each finite subset $N \subseteq A$, there is a compact convex subset L_N with $N \subseteq L_N \subseteq A$ and, $\forall x \in L_N \backslash D$, $\exists y \in L_N, \forall t \in T(x),\ H(t, y, x) \not\subseteq Y \backslash -intC(x)$.

Then problem (IVI) is solvable.

Proof One simply employs Theorem 2.1 with $r_i = r_2, \alpha_j = \alpha_2, A = B, S_1(x) = S_2(x) = A, T(x, y) = T(x)$ and $G(t, x) = Y \backslash -\mathrm{int}C(x)$. □

Remark 3.3 Theorem 3.2 of [32] and Theorem 3.1 of [33] are weaker than Corollary 3.4, since the convexity and semicontinuity assumptions there are stronger than our counterparts as explained now. Recall first some notions used in [32,33]. Let A, T, C and F be as in the formulation of (IVI). T is said to be generalized upper hemicontinuous (guhc) with respect to (wrt) F if, for all $x, y \in A$ and all $\lambda \in [0, 1]$, the multifunction $\lambda \mapsto F(T((1-\lambda)x+\lambda y), x, y)$ is usc at 0^+. For $t \in L(X, Y)$ and $x \in A$, $F(t, ., x)$ is called $C(x)$-convex if, for all $y, z \in A$ and all $\lambda \in [0, 1]$,

$$F(t, (1 - \lambda)y + \lambda z, x) \in (1 - \lambda)F(t, y, x) + \lambda F(t, z, x) - C(x).$$

T is termed generalized C-pseudomonotone wrt F if, for all $x, y \in A$,

$$[\exists t \in T(x), F(t, y, x) \notin -\text{int}C(x)] \Longrightarrow [\forall t \in T(y), -F(t, x, y) \notin -\text{int}C(x)].$$

Proposition 3.1 *Let A, T, C and F be as in the formulation of (IVI). Let $H(t, y, x) = \{-F(s, y_\lambda, x) \mid s \in T(y_\lambda), \lambda \in [0, 1]\}$, where $y_\lambda = (1-\lambda)x + \lambda y$. (Then H does not depend on t.)*

(a) Assume that

(i) T is guhc with respect to F;
(ii) for each $t \in L(X, Y)$ and $x \in A$, $F(t, ., x)$ is $C(x)$-convex;
(iii) for all $x, y \in A$ and all $t \in T(x)$, $F(t, y, y) \in C(x)$;
(iv) for all $t \in L(X, Y)$, all $x, y \in A$ and all $\lambda \in [0, 1]$,

$$F(t, y, (1-\lambda)x + \lambda y) = (1-\lambda)F(t, y, x).$$

Then assumption (i) of Corollary 3.4 is satisfied.

(b) In addition to the assumptions in (a), impose that T is generalized C-pseudomonotone wrt F. Then, (ii) of Corollary 3.4 is fulfilled.

(c) If $Y \setminus -\text{int}C(.)$ is closed and, for each $t \in L(X, Y)$ and each $y \in A$, $F(t, y, .)$ is continuous then (iii) of Corollary 3.4 is satisfied.

Proof Ad absurdum suppose the existence of $t \in T(x)$ such that

$$H(t, y, x) \subseteq Y \setminus -\text{int}C(x) \tag{2}$$

but, $F(t, y, x) \in -\text{int}C(x)$ for all $t \in T(x)$. By (2) and the definition of H one has, for all $\lambda \in [0, 1]$ and all $s \in T(y_\lambda)$,

$$-F(s, y_\lambda, x) \notin -\text{int}C(x). \tag{3}$$

Define $I : [0, 1] \to 2^Y$ by

$$I(\lambda) = \{F(t, y, x) \mid t \in T(y_\lambda)\}.$$

Due to (2), $I(0) \subseteq -\text{int}C(x)$. Assumption (i) implies the existence of $\lambda_0 \in (0, 1]$ such that $I(\lambda) \subseteq -\text{int}C(x)$, for all $\lambda \in [0, \lambda_0)$. Hence, for each $\lambda \in (0, \lambda_0)$ and $s \in T(y_\lambda)$,

$$F(s, y, x) \in -\text{int}C(x). \tag{4}$$

For any fixed $\lambda \in (0, \lambda_0)$, from (ii) one has, for all $s \in T(y_\lambda)$,

$$F(s, y_\lambda, y_\lambda) \in (1-\lambda)F(s, x, y_\lambda) + \lambda F(s, y, y_\lambda) - C(x). \tag{5}$$

(iii), (iv) together with (4), (5) imply that

$$\begin{aligned} -(1-\lambda)F(s, x, y_\lambda) &\in \lambda F(s, y, y_\lambda) - F(s, y_\lambda, y_\lambda) - C(x) \\ &\subseteq \lambda(1-\lambda)F(s, y, x) - C(x) - C(x) \\ &\subseteq -\text{int}C(x), \end{aligned}$$

for all $s \in T(y_\lambda)$, which contradicts (3).

(b) First we prove that, for each $x \in A$, the set

$$\begin{aligned} M(x) : &= \{y \in A \mid \forall t \in T(x), H(t, y, x) \not\subseteq Y \backslash -\operatorname{int}C(x)\} \\ &= \{y \in A \mid \exists \lambda \in [0,1], \exists s \in T(y_\lambda), -F(t, y_\lambda, x) \in -\operatorname{int}C(x)\} \end{aligned}$$

is convex. For arbitrarily fixed $y, z \in M(x)$ and $\lambda \in [0,1]$, we have to show that $y^* = (1-\lambda)z + \lambda y \in M(x)$. By the definition of $M(x)$, there are $\lambda_1, \lambda_2 \in [0,1]$, $s_1 \in T(y_{\lambda_1})$ and $s_2 \in T(z_{\lambda_2})$ such that

$$\begin{aligned} -F(s_1, y_{\lambda_1}, x) &\in -\operatorname{int}C(x), \\ -F(s_2, z_{\lambda_2}, x) &\in -\operatorname{int}C(x). \end{aligned}$$

Due to the assumed C-pseudomonotonicity of T, one has, for all $t \in T(x)$,

$$\begin{aligned} F(t, x, y_{\lambda_1}) &\in -\operatorname{int}C(x), \\ F(t, x, z_{\lambda_2}) &\in -\operatorname{int}C(x). \end{aligned}$$

This and assumption (ii) together imply that, for each $\gamma \in [0,1]$,

$$\begin{aligned} F(t, x, (1-\gamma)z_{\lambda_2} + \gamma y_{\lambda_1}) &\in (1-\gamma)F(t, x, z_{\lambda_2}) + \gamma F(t, x, y_{\lambda_1}) - C(x) \\ &\subseteq -\operatorname{int}C(x). \end{aligned} \tag{6}$$

Without loss of generality assume that $\lambda_1 \geq \lambda_2$. Setting

$$\begin{aligned} \gamma_0 &= \frac{\lambda\lambda_2}{\lambda_1 + \lambda(\lambda_2 - \lambda_1)}, \\ \lambda_0 &= \frac{\lambda_1(1-\lambda_2) + \lambda(\lambda_2 - \lambda_1)}{\lambda_1 + \lambda(\lambda_2 - \lambda_1)}, \end{aligned}$$

we see that $\gamma_0, \lambda_0 \in [0,1]$. Set $y_0 = (1-\lambda_0)y^* + \lambda_0 x$ and substitute γ_0 into (6) we obtain, for $t \in T(x)$,

$$F(t, x, y_0) \in -\operatorname{int}C(x).$$

By a similar argument as that of part (a), using assumptions (i–iv) we see the existence of $\hat{\lambda} \in [0,1]$ such that, for all $s \in T(y_{0\hat{\lambda}})$,

$$-F(s, y_{0\hat{\lambda}}, x) \in -\operatorname{int}C(x). \tag{7}$$

Choosing $\bar{\lambda} = \hat{\lambda}(1-\lambda_0) \in [0,1]$ one gets from (7) that, $\forall s \in T(y^*_{\bar{\lambda}})$,

$$-F(s, y^*_{\bar{\lambda}}, x) \in -\operatorname{int}C(x)$$

(by our convention, $y^*_{\bar{\lambda}} = (1-\bar{\lambda})x + \bar{\lambda}y^*$). This means that $y^* \in M(x)$.

Next we have to check that

$$H(t, x, x) \subseteq Y \backslash -\operatorname{int}C(x).$$

This is derived from the fact that, for all $s \in T(x)$,

$$-F(t, x, x) \notin -\operatorname{int}C(x), \tag{8}$$

which in turn follows from assumption (iii) and the C-pseudomonotonicity of T.

(c) Consider arbitrarily fixed $y \in A$ and $x_\lambda \to x_0$, where x_λ is in the set

$$N(y) := \{x \in A \mid \forall t \in T(y), -F(t, y, x) \notin -\operatorname{int}C(x)\}.$$

Then, for all λ and for $t \in T(y)$,

$$-F(t, y, x_\lambda) \in Y \setminus -\text{int}C(x_\lambda).$$

Since $F(t, y, .)$ is continuous and $Y \setminus -\text{int}C(.)$ is closed, we have, for all $t \in T(y)$,

$$-F(t, y, x_0) \in Y \setminus -\text{int}C(x_0),$$

i.e. $x_0 \in N(y)$ and hence $N(y)$ is closed. Now we consider the set in assumption (iii)

$$\begin{aligned}\widetilde{M}(y) &= \{x \in A \mid \exists t \in T(x), H(t, y, x) \subseteq Y \setminus -\text{int}C(x)\}\\ &= \{x \in A \mid \forall \theta \in [0, 1], \forall s \in T(y_\theta), -F(s, y_\theta, x) \notin -\text{int}C(x)\}\\ &= \bigcap_{\theta \in [0,1]} \{x \in A \mid \forall s \in T(y_\theta), -F(s, y_\theta, x) \notin -\text{int}C(x)\}\\ &:= \bigcap_{\theta \in [0,1]} N(y_\theta).\end{aligned}$$

$\widetilde{M}(y)$ is closed since so is $N(y_\theta)$, for all $\theta \in [0, 1]$. □

Remark 3.4 The assumptions of Proposition 3.1 are (or are slightly) weaker than those of Theorem 3.1 of [33]. Indeed, note first that, since each continuous linear mapping from X into Y with the original topologies is still continuous when X and Y are equipped with the weak topologies, the space $L(X, Y)$ is the same for these two cases. Observe next that if T is guhc with respect to F when Z is endowed with the original topology, then so is T when Z is equipped with the weak topology. Then by Proposition 3.1, Corollary 3.4 with all the topologies in X, Y and Z are the weak topologies, contains directly Theorem 3.1 of [33], since its assumptions are more relaxed.

The following example shows that this containment is proper, since it gives a case where the assumptions of Corollary 3.4 are satisfied but those of Theorem 3.1 of [33] are not.

Let $X = Y = Z = R$, $A = [0, 1]$, $C(x) \equiv R_+$, $F(t, y, x) = t$ and

$$T(x) = \begin{cases} [0, 0.5] & \text{if } x = 1, \\ [0.5, 1] & \text{if } 0 \le x < 1. \end{cases}$$

Then the multifunction $\lambda \mapsto \{F(t, y, x) \mid t \in T(y_\lambda)\}$ is not usc at 0^+, since

$$\{F(t, y, x) \mid t \in T(y_\lambda)\} = T(y_\lambda) = T(1-\lambda) = \begin{cases} [0, 0.5] & \text{if } \lambda = 1, \\ [0.5, 1] & \text{if } 0 \le \lambda < 1. \end{cases}$$

This means that an assumption of Theorem 3.1 of [33] is not satisfied. However, choosing $H(t, y, x) = F(t, y, x)$ it is easy to see that all the assumptions of Corollary 3.4 are satisfied. Direct computations yield the solution set being [0,1].

3.3 Nash equilibria

Let $I = \{1, 2, \ldots, n\}$ be a set of players. An n-person non-cooperative game Γ is $2n$-tuple $(X_1, X_2, ..., X_n; f_1, f_2, \ldots, f_n)$, where the ith player has a nonempty strategy set X_i and $f_i : X := \prod_{i \in I} X_i \to R$ is his payoff function. For a point $x \in X$, we denote by $x_{\hat{i}}$ its projection onto $X_{\hat{i}} = \prod_{j \neq i} X_j$. A point $\bar{x} = (\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}) \in X$ is called a Nash equilibrium point of game Γ if, for all $y_i \in X_i$ and all $i \in I$,

$$f_i(\bar{x}) \ge f_i(\bar{x}_{\hat{i}}, y_i).$$

We define $\Phi_\Gamma : X \times X \to R$ by $\Phi_\Gamma(x, y) = \sum_{i=1}^n (f_i(x) - f_i(x_{\hat{i}}, y_i))$. Then, $\bar{x}$ is a Nash equilibrium point of Γ if and only if $\bar{x}$ is a solution of the vector equilibrium problem of

(VEP) finding $\bar{x} \in X$ such that, for all $y \in X$, $\Phi_\Gamma(\bar{x}, y) \geq 0$.

Clearly (VEP) is a particular case of ($\text{IP}_{r_2\alpha_1}$) with $A = B = X$, $S_1(x) = S_2(x) = X$, $T(x, y) = \{x\}$, $f(t, y, x) = \Phi_\Gamma(x, y)$ and $g(t, x) = [0, +\infty)$. By Theorem 2.1 we easily obtain

Corollary 3.5 *Assume that each X_i, $i = 1, \ldots, n$, is a compact convex subset of a Hausdorff topological vector space and the following conditions hold*

(i) the set $\{y \in X \mid \Phi_\Gamma(x, y) < 0\}$ is convex for each $x \in X$;
(ii) the set $\{x \in X \mid \Phi_\Gamma(x, y) \geq 0\}$ is closed for each $y \in X$.

Then the game Γ has a Nash equilibrium point.

3.4 Traffic networks

We describe first our traffic problem. Let the network consist of nodes and links (or arcs). Let $W = (W_1, \ldots, W_l)$ be the set of pairs. Each of them consists of an origin node and a destination one. So the pairs are called O/D pairs. Assume that P_j, $j = 1, ..., l$, is the set of paths connecting the pair W_j and that P_j contains $r_j \geq 1$ paths. Let $m = r_1 + \cdots + r_l$ and $f = (f_1, \ldots, f_m)$ denote the path (vector) flow. In [14] it was proposed that restrictions of the capacity of the paths should be taken into account. Hence we assume that the capacity constraint of paths is of the form

$$A = \{f \in R^m | 0 \leq f_s \leq \Gamma_s, \quad s = 1, \ldots, m\}.$$

Let the vector cost $T(f) = (T_1(f), \ldots, T_m(f))$ be a multifunction of the flow f, as considered in [2,3,12,22,23,39].

Assume further that travel demand g_j of the O/D pair W_j depends on the equilibrium (vector) flow $\bar{f}$ as explained in [12,39]: $g_j(\bar{f})$, where $g_j(.)$ is continuous. Denote the travel vector demand by $g = (g_1, \ldots, g_l)$ and the Kronecker numbers by

$$\phi_{js} = \begin{cases} 1 & \text{if } s \in P_j, \\ 0 & \text{if } s \notin P_j, \end{cases}$$
$$\phi = \left(\phi_{js}\right), \quad j = 1, \ldots, l; \quad s = 1, \ldots, m.$$

Then the set of all feasible path flows is

$$S(\bar{f}) = \{f \in A \mid \phi f = g(\bar{f})\}.$$

However, we are interested in flows satisfying the demands with tolerances as follows. Let $\varepsilon : R^m \to R_+$ be a continuous function. We define the set of the feasible path flows as

$$S(\bar{f}) := \{f \in A \mid \phi f \in B(g(\bar{f}), \varepsilon(\bar{f}))\},$$

where $B(g, \varepsilon)$ stands for the closed ball centered at g and of radius ε. It is not hard to check directly that $S(.)$ satisfies the assumption (iv) of Corollary 3.1.

For the case of multivalued cost, the following generalized Wardrop equilibrium was proposed in [22,23].

Definition 3.1

(i) A feasible path flow $\bar{f}$ is said to be a weak equilibrium flow if, $\forall W_j, \forall q$, $s \in P_j, \exists t \in T(\bar{f})$,

$$[t_q < t_s] \Longrightarrow [\bar{f}_q = \Gamma_q \quad \text{or} \quad \bar{f}_s = 0],$$

where $j = 1, \ldots, l$ and $q, s = 1, \ldots, r_j$.

(ii) A feasible path flow $\bar{f}$ is called a strong equilibrium flow if (i) is satisfied with $\exists t \in T(\bar{f})$ replaced by $\forall t \in T(\bar{f})$.

In [22], it is proved that a feasible path flow $\bar{f}$ is a weak equilibrium flow if and only if $\bar{f}$ is a solution of the quasivariational inequality

$$(\text{QVI}') \quad \text{Find } \bar{f} \in S(\bar{f}) \text{ such that,} \quad \forall f \in S(\bar{f}), \exists \bar{t} \in T(\bar{f}),$$

$$\langle \bar{t}, f - \bar{f} \rangle \geq 0,$$

which is a special case of problem (f) in Sect. 1. Similarly, $\bar{f}$ is a strong equilibrium flow if and only if $\bar{f}$ is a solution of (QVI$'$) with $\exists \bar{t}$ replaced by $\forall \bar{t}$, which is denoted by (QVI$''$), also a special case of our (IP$_{r_2\alpha_1}$) and (IP$_{r_1\alpha_1}$).

Corollary 3.6 *For the traffic network problem assume that*

(i) for all $f \in A$, the set $\{h \in A \mid \exists t \in T(f), \langle t, h - f \rangle < 0\}$ is convex;
(ii) for all $h \in A$, the set $\{f \in A \mid \forall t \in T(f), \langle t, h - f \rangle \geq 0\}$ is closed.

Then, there is a strong equilibrium flow.

Proof It suffices to verify that problem (QVI$''$) has a solution. We apply Corollary 3.1 with $F(t, f, \bar{f}) = H(t, f, \bar{f}) = \langle t, f - \bar{f} \rangle$ and $C(f) = R_+$. From the compactness of A and the remark on function $S(.)$, it is easy to check that all the assumptions of Corollary 3.1 are satisfied. So (QVI”) has a solution, which is a strong equilibrium flow. □

Similarly, we have the following result.

Corollary 3.7 *For the traffic network assume that, for all $h \in A$, the set $\{f \in A \mid \exists t \in T(f), \langle t, h - f \rangle \geq 0\}$ is closed. Then, there is a weak equilibrium flow.*

By Remark 2.1(ii) the closedness assumptions in these corollaries are more relaxed than the corresponding semicontinuity ones. Therefore, Corollaries 3.6 and 3.7 improve their counterparts in [22] as illustrated by the next example.

Example 3.3 Let $W = \{(1, 1'), \ldots, (n, n')\}$ and $P_j = \{r_{j1}, r_{j2}\}$ for $j = 1, \ldots, n$, where r_{j1} and r_{j2} connect the O/D pair (j, j'). Assume that the capacity constraint is

$$A = \{f \in R^{2n} \,|\, 0 \leq f_{jk} \leq 2, \quad j = 1, \ldots n, k = 1, 2\}.$$

and the $2n$-dimensional vector cost $T(f)$ is defined by

$$T_{jk}(f) = \begin{cases} \{0\} & \text{if } 0 \leq f_{jk} \leq 1, \\ \{k\} & \text{if } 1 < f_{jk} \leq 2. \end{cases}$$

Then, the assumptions of Corollary 3.6 are satisfied. Indeed, to checked them we denote $\mathcal{D} = \{jk \mid 1 < f_{jk} \leq 2\}$. For any $f \in A$ we show that

$$M := \{h \in A \mid \exists t \in T(f), \langle t, h - f\rangle < 0\} = \left\{h \in A \mid \sum_{jk\in\mathcal{D}} kh_{jk} < \sum_{jk\in\mathcal{D}} kf_{jk}\right\}$$

is a convex subset of R^{2n}. Let $\lambda \in [0, 1]$, $h, h' \in M$. Then

$$\begin{aligned}\sum_{jk\in\mathcal{D}} k(\lambda h_{jk} + (1-\lambda)h'_{jk}) &= \lambda \sum_{jk\in\mathcal{D}} kh_{jk} + (1-\lambda) \sum_{jk\in\mathcal{D}} kh'_{jk}\\ &< \lambda \sum_{jk\in\mathcal{D}} kf_{jk} + (1-\lambda) \sum_{jk\in\mathcal{D}} kf_{jk}\\ &= \sum_{jk\in\mathcal{D}} kf_{jk},\end{aligned}$$

i.e. $\lambda h + (1-\lambda)h' \in M$. Next, we show that the set

$$N := \{f \in A \mid \forall t \in T(f), \langle t, h - f\rangle \geq 0\} = \left\{f \in A \mid \sum_{jk\in\mathcal{D}} kf_{jk} \leq \sum_{jk\in\mathcal{D}} kh_{jk}\right\}$$

is closed in R^{2n} for all $h \in A$. Let $f^{(m)} \in N$, $f^{(m)} \to f$. Then $f^{(m)}_{jk} \to f_{jk}$ for all jk satisfying $f_{jk} \in (1, 2]$. Let $\epsilon > 0$ be arbitrary. There exists m_ϵ such that

$$f_{jk} < f^{(m_\epsilon)}_{jk} + \epsilon / \sum_{jk\in\mathcal{D}} k.$$

Hence

$$\sum_{jk\in\mathcal{D}} kf_{jk} < \sum_{jk\in\mathcal{D}} kf^{(m_\epsilon)}_{jk} + \epsilon \leq \sum_{jk\in\mathcal{D}} kh_{jk} + \epsilon.$$

Since ϵ is arbitrary, one has

$$\sum_{jk\in\mathcal{D}} kf_{jk} \leq \sum_{jk\in\mathcal{D}} kh_{jk},$$

i.e. $f \in N$. Thus, N is closed.

By Corollary 3.6, a strong equilibrium flow exists, although T is neither usc nor lsc, nor generalized lower hemicontinuous (whence Corollary 4.1 of [22] is not applicable). Note that, in this case weak and strong equilibrium flows coincide as $T(.)$ is single-valued.

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