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Browder-Tikhonov regularization for a class of evolution second order hemivariational inequalities

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Abstract In this paper, we consider a class of evolution second order hemivariational inequalities with non-coercive operators which are assumed to be known approximately. Using the so-called Browder-Tikhonov regularization method, we prove that the regularized evolution hemivariational inequality problem is solvable. We construct a sequence based on the solvability of the regularized evolution hemivariational inequality problem and show that every weak cluster of this sequence is a solution for the evolution second order hemivariational inequality.

Keywords Evolution hemivariational inequalities · Evolution inclusion · Pseudomonotone with respect to $D(L)$ · Regularization · Convergence · Duality mapping

Mathematics Subject Classification (2000) 49J40 · 49J52 · 40A30

1 Introduction

Let H be a separable Hilbert space and V be a reflexive, separable Banach space which is a dense subspace of H. We suppose that V compactly embeds into H. Identifying H with its dual space H^*, we obtain $V \subset H \subset V^*$ forms an evolution triple with all embeddings being continuous and dense, where V^* is the dual space of V (see [23]). A concrete example is

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$V = W_0^{1,p}(\Omega)$ with $2 \leq p < \infty$ and $H = L^2(\Omega)$, where $\Omega \subset R^n$ is a bounded domain with Lipschitz boundary. We denote by $\langle \cdot, \cdot \rangle$ the duality between V and V^*, and by $\| \cdot \|_E$ the norm in the space E being V and V^*, respectively. Given a fixed number $0 < T < +\infty$, we denote by $L^p(0, T; V)$ the space of strongly measurable Banach-valued functions $f : [0, T] \to V$ such that $\int_0^T \|f(t)\|_V^p dt < \infty$. In this paper, let $\mathscr{V} = L^2(0, T; V)$, $\mathscr{H} = L^2(0, T; H)$ and $\mathscr{V}^* = L^2(0, T; V^*)$ be the dual space of $\mathscr{V}$. Clearly we have $\mathscr{V} \subset \mathscr{H} \subset \mathscr{V}^*$ with dense and continuous embeddings due to $V \subset H \subset V^*$ being an evolution triple (see [23]). The pairing between $\mathscr{V}$ and $\mathscr{V}^*$ is denoted by

$$\langle\langle f, g \rangle\rangle = \int_0^T \langle f(t), g(t) \rangle dt \quad \text{for all} \quad f \in \mathscr{V}^* \text{ and } g \in \mathscr{V}.$$

The problem under consideration is the following evolution second order hemivariational inequality:

$$\begin{cases} \text{Find } u \in \mathscr{V} \text{ such that } u' \in \mathscr{V},\, u'' \in \mathscr{V}^* \text{ and} \\ \langle u''(t) + Au'(t) + Bu(t) - f(t), v \rangle + G_2^\circ(t, u', v) \geq 0 \quad \text{for all } v \in V \text{ and a.e. } t \in [0, T] \\ u(0) = \theta \\ u'(0) = \theta, \end{cases} \tag{1.1}$$

where A and B are nonlinear operators from Banach space V into its dual V^*, $f \in \mathscr{V}^*$, θ is the zero element of Banach space V, and $G_2^\circ(t, u, v)$ denotes the generalized directional derivative of G at u in the direction v with respect to the second variable u.

It can be seen that the evolution second order hemivariational inequality (1.1) is equivalent to the following evolution second order inclusion:

$$\begin{cases} \text{Find } u \in \mathscr{V} \text{ such that } u' \in \mathscr{V},\, u'' \in \mathscr{V}^* \text{ and} \\ f(t) \in u''(t) + Au'(t) + Bu(t) + \partial_2 G(t, u'(t)) \quad \text{for a.e. } t \in [0, T] \\ u(0) = \theta \\ u'(0) = \theta, \end{cases} \tag{1.2}$$

where $\partial_2 G(t, \cdot)$ denotes the Clarke's generalized gradient of the locally Lipschitz functional $G(t, \cdot)$ with respect to the second variable.

The notion of the hemivariational inequality was introduced by Panagiotopoulos in the early 1980s as variational expressions for several classes of mechanical problems with non-smooth and nonconvex energy superpotentials (see [16,18,19]). The derivation of hemivariational inequality is based on the mathematical notion of the generalized gradient of Clarke (see [8]). The hemivariational inequalities appear in a variety of mechanical problems, for example, the unilateral contact problems in nonlinear elasticity, the problems describing the adhesive and frictional effects, the nonconvex semipermeability problems, the masonry structures, and the delamination problems in multilayered composites, see [15,16,19,20] for detailed descriptions. Recently, many kinds of the hemivariational inequalities have been studied under some suitable hypotheses. Using the method of subsolution and supersolution, Carl et al. [3,5,6] got the existence results of solutions for some quasi-linear hemivariational inequalities and quasi-linear evolution hemivariational inequalities. In [22], Xiao and Huang studied a new class of generalized quasi-variational-like hemivariational inequalities with multi-valued η-pseudomonotone operators and obtained some new existence theorems of solutions for the generalized quasi-variational-like hemivariational inequalities. Ochal [17] considered a class of evolution hemivariational inequalities of second order with a time-dependent pseudomonotone operator and nonmonotone multivalued perturbations, presented the existence of solutions and discussed some useful examples which indicated the practical importance of their theoretical findings. We can refer to [4,9,14,21] for more related works.

On the other hand, many authors have increasingly paid their attention to obtaining reasonable approximations to solutions of hemivariational inequalities. In [12] and [13], Liu devoted to the regularization of a class of evolution hemivariational inequalities and obtained a strongly convergent approximation procedure by means of the so-called Browder-Tikhonov regularization method.

In this paper, instead of exact data (A, B, G, f), we assume that only the noisy data $(A_{\alpha_n}, B_{\beta_n}, G_{\gamma_n}, f_{\theta_n})$ are available, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\theta_n\}$ are sequences of positive reals. By using the so-called Browder-Tikhonov regularization method, we prove that the regularized evolution hemivariational inequality problem is solvable under the relationship between exact data and noisy data. We construct a sequence based on the solvability of the regularized evolution hemivariational inequality problem and show that every weak cluster of this sequence is a solution for the second order evolution hemivariational inequality (1.1).

2 Preliminaries

Suppose that V is a Banach space and $g : V \to R$ is a locally Lipschitz functional on V. For a given point $u \in V$ and any other vector $v \in V$, the generalized directional derivative of g at u in the direction v, denoted by $g^\circ(u, v)$ is given by

$$g^\circ(u, v) = \limsup_{w\to u\, t\downarrow 0} \frac{g(w + tv) - g(w)}{t},$$

where $w \in V$ and λ is a positive scalar (see [8]). The Clarke's generalized gradient of g at u (see [8]), denoted by $\partial g(u)$ is defined by

$$\partial g(u) = \{w \in V^* : g^\circ(u, v) \geq \langle w, v\rangle, \forall v \in V\}.$$

A locally Lipschitz functional $g : V \to R$ is said to be regular at point $u \in V$ if the directional derivative $g'(u, v)$ exists and $g^\circ(u, v) = g'(u, v)$ for all $v \in V$. We say that g is regular on V if g is regular at any point $u \in V$.

Let $\mathscr{W} = \{w \in \mathscr{V} : w' \in \mathscr{V}^*\}$, where the derivative $w' = \partial w/\partial t$ is understood in the sense of vector-valued distributions (see [23]), which is characterized by

$$\int_0^T w'(t)\phi(t)dt = -\int_0^T w(t)\phi'(t)dt,\ \forall\phi \in C_0^\infty[0, T].$$

The space $\mathscr{W}$ endowed with the graph norm

$$\|w\|_{\mathscr{W}} = \|w\|_{\mathscr{V}} + \|w'\|_{\mathscr{V}^*}, \quad \forall w \in \mathscr{W}$$

is a Banach space which is separable and reflexive due to the separability and reflexivity of $\mathscr{V}$ and $\mathscr{V}^*$, respectively (see Propositions 23.7(c) and 23.23(i) in [23]).

Let $L : D(L) \subset \mathscr{V} \to \mathscr{V}^*$ be the operator defined by $Lu = u'(\partial u/\partial t)$ with

$$D(L) = \{u \in \mathscr{W} : u(0) = \theta\}.$$

It can be shown that $L: D(L) \subset \mathscr{V} \to \mathscr{V}^*$ is a closed, densely defined and linear maximal monotone operator (see [23]) with

$$\langle\langle Lu, v\rangle\rangle = \int_0^T \langle u'(t), v(t)\rangle dt, \quad \forall u \in D(L), v \in \mathscr{V}.$$

Thus, the Problem (1.1) we consider becomes the following initial value problem:

$$\begin{cases} \text{Find } u \in \mathscr{V} \text{ such that } u' \in \mathscr{W} \text{ and} \\ \langle u''(t) + Au'(t) + Bu(t) - f(t), v\rangle + G_2^\circ(t, u', v) \geq 0 \quad \text{for all } v \in V \text{ and a.e. } t \in [0, T] \\ u(0) = \theta \\ u'(0) = \theta, \end{cases} \tag{2.1}$$

which is equivalent to the following evolution inclusion

$$\begin{cases} \text{Find } u \in \mathscr{V} \text{ such that } u' \in \mathscr{W} \text{ and} \\ f(t) \in u''(t) + Au'(t) + Bu(t) + \partial_2 G(t, u'(t)) \quad \text{for a.e. } t \in [0, T] \\ u(0) = \theta \\ u'(0) = \theta. \end{cases} \tag{2.2}$$

In the sequel, to formulate our main results, we present some important definitions, lemmas, theorems and hypotheses.

Definition 2.1 (*see* [10]) Let V be a real reflexive Banach space with its dual V^*. A mapping $T: D(T) \subset V \to 2^{V^*}$ is said to be pseudomonotone if the following conditions hold:

(a) The set Tu is nonempty, bounded, closed and convex for all $u \in D(T)$;
(b) T is upper semi-continuous from each finite dimensional subspace F of V into V^* endowed with the weak topology;
(c) if $\{u_i\}$ is a sequence in $D(T)$ converging weakly to u of $D(T)$, and if $u_i^* \in Tu_i$ is such that $\limsup\langle u_i^*, u_i - u\rangle \leq 0$, then for each element $v \in D(T)$ there exists $u^*(v) \in Tu$ with the property that

$$\liminf\langle u_i^*, u_i - u\rangle \geq \langle u^*(v), u - v\rangle.$$

Definition 2.2 (*see* [10]) Let V be a real reflexive Banach space with its dual V^*, $L: D(L) \subset V \to V^*$ be a linear densely defined maximal monotone mapping. A mapping $T: D(T) \subset V \to 2^{V^*}$ is said to be pseudomonotone with respect to $D(L)$ if and only if (a), (b) and the following condition holds:

(d) if $\{u_i\} \in D(L) \cap D(T)$ is such that $u_i \to u$ weakly in V, $Lu_i \to Lu$ weakly in V^*, $u_i^* \in T(u_i)$, $u_i^* \to u^*$ weakly in V^* and $\limsup\langle u_i^*, u_i - u\rangle \leq 0$, then $u^* \in T(u)$ and $\langle u_i^*, u_i\rangle \to \langle u^*, u\rangle$.

Definition 2.3 (*see* [2,16,23]) A mapping $T : D(T) \subset V \to 2^{V^*}$ is said to have (S_+) property (be of class (S_+)) if and only if (a), (b) and the following condition holds:

(e) for any sequence $\{u_i\}$ in $D(T)$ converging weakly to $u \in D(T)$, $w_i \in T(u_i)$, the condition

$$\limsup\langle w_i, u_i - u\rangle \leq 0$$

implies the strong convergence of $\{u_i\}$ to u in V and there exists a subsequence $\{w_{n_i}\}$ of $\{w_i\}$ such that $\{w_{n_i}\}$ converges weakly to $w \in Tu$ in V^*.

It is well known that the conditions

$$\|J(u)\|_{V^*} = \|u\|_V \text{ and } \langle J(u), u\rangle = \|u\|_V^2, \quad \forall u \in V \tag{2.3}$$

determine a unique mapping J from V to V^*, which is called the duality mapping. In our case it is bijective bicontinuous, strictly monotone and of class (S_+). For more details we can refer to [23].

Definition 2.4 (*see* [1]) Let $L : D(L) \subset V \to V^*$ be a linear densely defined maximal monotone mapping. A multivalued mapping $T : D(T) \subset V \to 2^{V^*}$ is said to have (S_+) property with respect to $D(L)$ (be of class (S_+) with respect to $D(L)$) if and only if (a), (b) and the following condition holds:

(f) for any sequence $\{u_i\}$ in $D(T) \cap D(L)$ converging weakly to $u \in D(T)$, Lu_n converging weakly to Lu in V^* and for any sequence $\{w_i\}$ in V^* with $w_i \in T(u_i)$ for each $i \geq 1$, the condition

$$\limsup \langle w_i, u_i - u\rangle \leq 0$$

implies the strong convergence of $\{u_i\}$ to u in V and there exists a subsequence $\{w_{n_i}\}$ of $\{w_i\}$ such that $\{w_{n_i}\}$ converges weakly to $w \in Tu$ in V^*.

Lemma 2.1 (see [1]) *Let V be a reflexive Banach space and $T : D(T) \subset V \to 2^{V^*}$ be a multivalued mapping. If T is demicontinuous and has (S_+) property, then T is pseudomonotone.*

Theorem 2.1 (see [10]) *Let V be a real reflexive, strictly convex Banach space with dual space V^* and let $L : D(L) \subset V \to V^*$ be a closed, densely and maximal monotone operator. If the multivalued mapping $T : V \to 2^{V^*}$ is bounded, coercive and pseudomonotone with respect to $D(L)$, then $L + T$ is surjective, i.e., $R(L + T) = V^*$.*

Definition 2.5 Let m be a positive constant. By $\mathbf{A(m)}$ we denote the class of mappings $A : V \to V^*$, which is bounded, demicontinuous and satisfies the following strong monotonicity condition:

$$\langle Au - Av, u - v\rangle \geq m\|u - v\|_V^2, \quad \forall u, v \in V. \tag{2.4}$$

By $\mathbf{G(m)}$, we denote the class of locally Lipschitz functional $G(t, \cdot) : V \to R$ for every $t \in [0, T]$ which satisfies the following two conditions:

(i) There exists a constant $h > 0$, such that for all $u \in V$, $t \in [0, T]$ and $\xi \in \partial_2 G(t, u)$,

$$\|\xi\|_{V^*} \leq h(1 + \|u\|_V); \tag{2.5}$$

(ii) For any $t \in (0, T)$, $u, v \in V$,

$$\langle u^* - v^*, u - v\rangle \geq -m\|u - v\|_V^2, \quad \forall u^* \in \partial_2 G(t, u) \text{ and } v^* \in \partial_2 G(t, v). \tag{2.6}$$

The conditions (i) and (ii) are called growth condition and relaxed monotonicity condition, respectively.

Remark 2.1 If $A \in \mathbf{A(m)}$ and $G(t, \cdot) \in \mathbf{G(m)}$ for any $t \in [0, T]$, then we obviously have that $A + \partial_2 G(t, \cdot)$ is monotone but may not be coercive in general.

Let $\{\epsilon_n\}$ with $\epsilon_n > 0$ be a sequence of positive reals which is (strictly) decreasing and converges to zero. Let $\alpha_n, \beta_n, \gamma_n$ and θ_n be four sequences of positive reals. The hypotheses of evolution hemivariational inequality (1.1) between the exact data and the noisy data are as follows:

H(1) There exists a constant c_1 such that

$$\|A_{\alpha_n} z - Az\|_{V^*} \le \alpha_n c_1 \|z\|_V, \forall z \in V;$$

H(2) There exists a constant c_2 such that

$$\|B_{\beta_n} z - Bz\|_{V^*} \le \beta_n c_2 \|z\|_V, \forall z \in V;$$

H(3) There exists a constant c_3 such that for all $z \in V, t \in [0, T]$

$$H(\partial_2 G_{\gamma_n}(t, z), \partial_2 G(t, z)) \le \gamma_n c_3 \|z\|_V,$$

where $H(Q, S) = \max\{\sup_{x \in Q} d(x, S), \sup_{y \in S} d(y, Q)\}$ is the Hausdorff distance between the sets Q and S;

H(4) For $f_{\theta_n} \in \mathscr{V}^*$, we have

$$\|f_{\theta_n} - f\|_{\mathscr{V}^*} \le \theta_n;$$

H(5) For $n \to \infty$,

$$\alpha_n, \beta_n, \gamma_n, \theta_n, \frac{\alpha_n}{\epsilon_n}, \frac{\beta_n}{\epsilon_n}, \frac{\gamma_n}{\epsilon_n}, \frac{\theta_n}{\epsilon_n} \to 0.$$

Example 2.1 Let $V = H$ be a Hilbert space and $G : [0, T] \times H \to R$ be a mapping such that $G(t, \cdot) \in \mathbf{G(m)}$ for any $t \in [0, T]$ and G is regular with respect to the second variable. Let $G_{\gamma_n} : [0, T] \times H \to R$ be a mapping defined by

$$G_{\gamma_n}(t, u) = G(t, u) + \gamma_n \|u\|^2,$$

where $\gamma_n > 0$ is a real sequence. By the regularity of mapping G and $\|u\|^2$, Proposition 2.174 of [7] (cf. p. 68) implies that

$$\partial_2 G_{\gamma_n}(t, u) = \partial_2 G(t, u) + \gamma_n \partial(\|u\|^2), \tag{2.7}$$

where $\partial(\|u\|^2)$ is the Clarke's generalized gradient of $\|u\|^2$. It is well known that, for any $u \in H$,

$$\partial(\|u\|^2) = 2u,$$

which together with (2.7) implies

$$\partial_2 G_{\gamma_n}(t, u) = \partial_2 G(t, u) + 2\gamma_n u. \tag{2.8}$$

Now we can conclude that

$$H(\partial_2 G_{\gamma_n}(t, u), \partial_2 G(t, u)) \le 2\gamma_n \|u\| \tag{2.9}$$

and thus, $H(3)$ is satisfied with $c_3 = 2$. In fact, for any $w \in \partial_2 G_{\gamma_n}(t, u)$, $v \in \partial_2 G(t, u)$, it follows from (2.8) that there exists a $v' \in \partial_2 G(t, u)$ such that $w = v' + 2\gamma_n u$ and so

$$\|w - v\| = \|v' + 2\gamma_n u - v\| \le \|v' - v\| + 2\gamma_n \|u\|,$$

which implies that

$$\sup_{w\in\partial_2 G_{\gamma_n}(t,u)} d(w, \partial_2 G(t,u)) \le \sup_{v'\in\partial_2 G(t,u)} d(v', \partial_2 G(t,u)) + 2\gamma_n\|u\| \\ = 2\gamma_n\|u\| \tag{2.10}$$

and

$$\sup_{v\in\partial_2 G(t,u)} d(v, \partial_2 G_{\gamma_n}(t,u)) \le \sup_{v\in\partial_2 G(t,u)} d(v, \partial_2 G(t,u)) + 2\gamma_n\|u\| \\ = 2\gamma_n\|u\|. \tag{2.11}$$

It follows from (2.10), (2.11) and the definition of Hausdorff distance that

$$H(\partial_2 G_{\gamma_n}(t,u), \partial_2 G(t,u)) \le 2\gamma_n\|u\|.$$

This shows that (2.9) holds.

Lemma 2.2 (see [11]) *Let $C \subset V$ be non-empty, closed and convex, $C^* \subset V^*$ be non-empty, closed, convex and bounded, $\varphi : V \to R$ be proper, convex and lower semi-continuous and $y \in C$ be arbitrary. Assume that for each $x \in C$ there exists $x^*(x) \in C^*$ such that*

$$\langle x^*(x), x - y\rangle \ge \varphi(y) - \varphi(x).$$

Then there exists $y^* \in C^*$ such that

$$\langle y^*, x - y\rangle \ge \varphi(y) - \varphi(x), \quad \forall x \in C.$$

3 Main results

Denote an operator $K : \mathscr{V} \to C(0, T; V)$ by

$$Kv(t) = \int_0^t v(s)ds \quad \text{for all } v \in \mathscr{V}.$$

It is easy to see that K is a bounded and continuous operator from $\mathscr{V}$ to $C(0, T; V)$ (see [23]). Consider the following evolution first order hemivariational inequality:

$$\begin{cases} \text{Find } z \in \mathscr{W} \text{ such that} \\ \langle z'(t) + Az(t) + BKz(t) - f(t), v\rangle + G_2^\circ(t, z(t), v) \ge 0 \quad \text{for all } v \in \text{ and a.e. } t \in [0, T] \\ z(0) = \theta. \end{cases} \tag{3.1}$$

We observe that z is a solution to (3.1) if and only if $u \coloneqq Kz$ satisfies (2.1). Therefore, in what follows, we will consider the evolution hemivariational inequality (3.1).

Lemma 3.1 *$z \in \mathscr{W}$ is a solution of evolution hemivariational inequality (3.1) if and only if z is a solution of the following evolution inclusion: Find $z \in D(L)$ such that*

$$f(t) \in Lz(t) + Az(t) + BKz(t) + \partial_2 G(t, z(t)) \quad \textit{for a.e. } t \in [0, T]. \tag{3.2}$$

Proof **Sufficiency** Let $z \in D(L)$ be a solution of evolution inclusion (3.2). Then $z \in \mathscr{W}$, $z(0) = \theta$ and

$$f(t) \in Lz(t) + Az(t) + BKz(t) + \partial_2 G(t, z(t)) \quad \text{for a.e. } t \in [0, T].$$

So there exists a $w(t) \in \partial_2 G(t, z(t))$ such that

$$Lz(t) + Az(t) + BKz(t) + w(t) = f(t) \quad \text{for a.e. } t \in [0, T].$$

Scalar multiplying the above equality by $v \in V$, we have

$$\langle z'(t) + Az(t) + BKz(t) + w(t), v\rangle = \langle f(t), v\rangle \quad \text{for a.e. } t \in [0, T].$$

Since the Clarke generalized gradient $\partial_2 G(t, z(t))$ is given by

$$\partial_2 G(t, z(t)) = \{w(t) \in V^* : G_2^\circ(t, z(t), v) \geq \langle w(t), v\rangle, \forall v \in V\},$$

we obtain

$$\langle z'(t) + Az(t) + BKz(t) - f(t), v\rangle + G_2^\circ(t, z(t), v) \geq 0 \quad \text{for all } v \in V \text{ and a.e. } t \in [0, T],$$

which implies that z is a solution of the evolution hemivariational inequality (3.1).

Necessity Let $z \in \mathscr{W}$ be a solution of evolution hemivariational inequality (3.1). Then $z(0) = \theta$ and so $z \in D(L)$. It follows that

$$\langle z'(t) + Az(t) + BKz(t) - f(t), v\rangle + G_2^\circ(t, z(t), v) \geq 0 \quad \text{for all } v \in V \\ \text{and a.e. } t \in [0, T].$$

From the fact that

$$G_2^\circ(t, z(t), v) = \max\{\langle w(t), v\rangle : w(t) \in \partial_2 G(t, z(t))\},$$

we have that, for each $v \in V$, there exists a $w(t, v) \in \partial_2 G(t, z(t))$ such that

$$\langle Lz(t) + Az(t) + BKz(t) + w(t, v) - f(t), v\rangle \geq 0, \quad \text{for a.e. } t \in [0, T]. \tag{3.3}$$

By virtue of Proposition 1.5 of [8] (cf. p. 73), we get that for any $t \in (0, T)$, $\partial_2 G(t, z(t))$ is a non-empty, closed, convex and bounded subset in V^*, which implies that for any $t \in (0, T)$,

$$\{Lz(t) + Az(t) + BKz(t) + w(t) - f(t) : w(t) \in \partial_2 G(t, z(t))\}$$

is nonempty, closed, convex and bounded in V^*. So, it follows from Lemma 2.2 with $\varphi = 0$ and (3.3) that there exists $w(t) \in \partial_2 G(t, z(t))$ such that

$$\langle Lz(t) + Az(t) + BKz(t) + w(t) - f(t), v\rangle \geq 0, \quad \text{for all } v \in V \text{ and a.e. } t \in [0, T].$$

which implies that

$$f(t) = Lz(t) + Az(t) + BKz(t) + w(t) \in Lz(t) + Az(t) + BKz(t) + \partial_2 G(t, z(t)) \\ \text{for a.e. } t \in (0, T),$$

i.e. z is a solution of evolution inclusion (3.2). This completes the Proof of Lemma 3.1. □

Now, we consider the following regularized evolution hemivariational inequality of problem (3.1):

$$\begin{cases} \text{Find } z_n \in \mathscr{W} \text{ such that} \\ \langle z_n'(t), v\rangle + \langle A_{\alpha_n} z_n(t) + B_{\beta_n} K z_n(t) + \epsilon_n J(z_n(t)) - f_{\theta_n}(t), v\rangle \\ \qquad + G_{\gamma_n}{}_2^\circ(t, z_n(t), v) \geq 0 \quad \text{for all } v \in V \text{ and a.e. } t \in [0, T] \\ z_n(0) = \theta, \end{cases} \tag{3.4}$$

where the operator $J : V \to V^*$ is the regularizing operator, ϵ_n is the regularization parameter and z_n is the regularized solution to the problem (3.1). Here the symbol $(\alpha_n, \beta_n, \gamma_n, \theta_n, \epsilon_n)$ shows the influence of the error parameters $\alpha_n, \beta_n, \gamma_n, \theta_n$ and the regularization parameters ϵ_n.

Corollary 3.1 *$z_n \in \mathscr{W}$ is a solution of regularized evolution hemivariational inequality (3.4) if and only if z_n is a solution of the following regularized evolution inclusion: Find $z_n \in D(L)$ such that*

$$f_{\theta_n}(t) \in Lz_n(t) + A_{\alpha_n} z_n(t) + B_{\beta_n} K z_n(t) + \epsilon_n J(z_n(t)) + \partial_2 G_{\gamma_n}(t, z_n(t)) \quad \text{for a.e. } t \in [0, T]. \tag{3.5}$$

Proof The proof is similar to Lemma 3.1. We omit it here. □

Theorem 3.1 *Assume that the hypotheses $H(1) - H(5)$ hold, the operators $A_{\alpha_n} \in A(m)$, $G_{\gamma_n} \in G(m)$ and B_{β_n} is a bounded, linear, monotone and symmetric operator. Then, for each $n \in N$ and given $f_{\theta_n} \in \mathscr{V}$ there exists a unique solution of the regularized evolution inclusion (3.5).*

In order to prove Theorem 3.1, we first define the operators $\mathscr{A}, \mathscr{A}_{\alpha_n}, \mathscr{B}, \mathscr{B}_{\beta_n}, \mathscr{J} : \mathscr{V} \to \mathscr{V}^*$ and $\mathscr{N}, \mathscr{N}_{\gamma_n} : \mathscr{V} \to 2^{\mathscr{V}^*}$, respectively, as follows:

$$(\mathscr{A}v)(\cdot) = A(v(\cdot)); \quad (\mathscr{A}_{\alpha_n}v)(\cdot) = A_{\alpha_n}(v(\cdot));$$
$$(\mathscr{B}v)(\cdot) = B(Kv(\cdot)); \quad (\mathscr{B}_{\beta_n}v)(\cdot) = B_{\beta_n}(Kv(\cdot));$$
$$(\mathscr{J}v)(\cdot) = J(v(\cdot))$$

and

$$\mathscr{N}v = \{w \in \mathscr{V}^* : w(t) \in \partial_2 G(t, v(t)) \quad \text{for a.e. } t \in [0, T]\}$$

with

$$\mathscr{N}_{\gamma_n}v = \{w \in \mathscr{V}^* : w(t) \in \partial_2 G_{\gamma_n}(t, v(t)) \quad \text{for a.e. } t \in [0, T]\}$$

for all $v \in \mathscr{V}$.

Lemma 3.2 *Assume that $A, A_{\alpha_n} \in A(m)$ and $G, G_{\gamma_n} \in G(m)$. Suppose that B and B_{β_n} are bounded, linear, monotone and symmetric operators. Assume that the operators $\mathscr{A}, \mathscr{A}_{\alpha_n}, \mathscr{B}, \mathscr{B}_{\beta_n}, \mathscr{N}$ and $\mathscr{N}_{\gamma_n}$ are defined as above. Then*

(1) *$\mathscr{A}$ and $\mathscr{A}_{\alpha_n}$ are bounded and demicontinuous operators from $\mathscr{V}$ to $\mathscr{V}^*$;*
(2) *$\mathscr{B}$ and $\mathscr{B}_{\beta_n}$ are linear, bounded and monotone operators from $\mathscr{V}$ to $\mathscr{V}^*$;*
(3) *$\mathscr{N}$ and $\mathscr{N}_{\gamma_n}$ are bounded and weakly closed multivalued operators from $\mathscr{V}$ into $\mathscr{V}^*$;*
(4) *$\mathscr{A} + \mathscr{N}$ and $\mathscr{A}_{\alpha_n} + \mathscr{N}_{\gamma_n}$ are monotone from $\mathscr{V}$ into $\mathscr{V}^*$;*
(5) *$\mathscr{J}$ is the duality mapping from $\mathscr{V}$ to $\mathscr{V}^*$.*

Proof Since $A, A_{\alpha_n} \in A(m)$, we can easily deduce that $\mathscr{A}$ and $\mathscr{A}_{\alpha_n}$ are bounded by their definition and demicontinuous by Lemma 1 of Berkovits and Mustonen [1], which proves that (1) holds. Due to the operators B and K are all linear we get $\mathscr{B}$ is linear. By the definition of $\mathscr{B}$, we have

$$\begin{aligned}\|\mathscr{B}z\|^2_{\mathscr{V}^*} &= \int_0^T \|BKz(t)\|^2_{V^*}dt \\ &\le \|B\|^2_{\mathscr{L}(V,V^*)}\int_0^T \|\int_0^t z(s)ds\|^2_V dt \\ &\le \|B\|^2_{\mathscr{L}(V,V^*)}\int_0^T T^2\|z(t)\|^2_V dt \\ &= \|B\|^2_{\mathscr{L}(V,V^*)}T^2\|z\|^2_{\mathscr{V}},\end{aligned}$$

which implies that $\mathscr{B}$ is a bounded operator in terms of boundedness of the operator B. Furthermore, lets us note that B is monotone and symmetric. Thus, for any $y, z \in \mathscr{V}$,

$$\begin{aligned}\langle\langle \mathscr{B}(y) - \mathscr{B}(z), y - z\rangle\rangle &= \int_0^T \langle BKy(t) - BKz(t), y(t) - z(t)\rangle dt \\ &= \int_0^T \langle BKy(t) - BKz(t), (Ky)'(t) - (Kz)'(t)\rangle dt \\ &= \frac{1}{2}\int_0^T \frac{d}{dt}\langle BKy(t) - BKz(t), Ky(t) - Kz(t)\rangle dt \\ &\ge 0,\end{aligned}$$

which implies that $\mathscr{B}$ is monotone. Similarly, we can prove that $\mathscr{B}_{\beta_n}$ is a linear, bounded and monotone operator which proves that (2) holds. Since $G \in G(m)$, under the assumption (2.5) we can get that $\mathscr{N}$ is weakly closed in $\mathscr{V}^*$ by Ochal [17]. Again by the assumption (2.5), for any $z \in \mathscr{V}$ and $\zeta \in \mathscr{N}z$, we have

$$\|\zeta(t)\|_{V^*} \le h(1 + \|z(t)\|_V) \quad \text{for a.e.} \quad t \in [0, T]$$

and so

$$\begin{aligned}\|\zeta\|^2_{\mathscr{V}^*} &= \int_0^T \|\zeta(t)\|^2_{V^*}dt \le \int_0^T h^2(1 + \|z(t)\|_V)^2 dt \le k_1 + k_2\int_0^T \|z(t)\|^2_V dt \\ &= k_1 + k_2\|z\|^2_{\mathscr{V}},\end{aligned}$$

where k_1, k_2 are two constants, which implies that $\mathscr{N}$ is a bounded operator from $\mathscr{V}$ into $\mathscr{V}^*$. Similarly, we also can prove that $\mathscr{N}_{\gamma_n}$ is a bounded and weakly closed multivalued operator which proves that (3) holds. For (4), we can be easily prove it due to $A, A_{\alpha_n} \in A(m)$ and $G, G_{\gamma_n} \in G(m)$. In fact, since $A \in A(m)$ and $G \in G(m)$, by the definitions of $\mathscr{A}$ and $\mathscr{N}$, we get

$$\begin{aligned}\langle\langle \mathscr{A}y - \mathscr{A}z, y - z\rangle\rangle &= \int_0^T \langle Ay(t) - Az(t), y(t) - z(t)\rangle dt \ge \int_0^T m\|y(t) - z(t)\|^2_V dt \\ &= m\|y - z\|^2_{\mathscr{V}} \qquad (3.6)\end{aligned}$$

for any $y, z \in \mathscr{V}$ and

$$\langle\langle w_y - w_z, y - z\rangle\rangle = \int_0^T \langle w_y(t) - w_z(t), y(t) - z(t)\rangle dt \geq \int_0^T -m\|y(t) - z(t)\|_V^2 dt$$
$$= -m\|y - z\|_{\mathscr{V}}^2 \tag{3.7}$$

for any $y, z \in \mathscr{V}$, $w_y \in \mathscr{N}(y)$ and $w_z \in \mathscr{N}(z)$. Adding the above two inequalities (3.6) and (3.7), we obtain

$$\langle\langle \mathscr{A}y + w_y - (\mathscr{A}z + w_z), y - z\rangle\rangle \geq 0$$

for any $y, z \in \mathscr{V}$, $w_y \in \mathscr{N}(y)$ and $w_z \in \mathscr{N}(z)$, which implies that $\mathscr{A} + \mathscr{N}$ is monotone. The proof for the monotonicity of the operator $\mathscr{A}_{\alpha_n} + \mathscr{N}_{\gamma_n}$ is similar. By the definition of operator $\mathscr{J}$, it follows from the condition (2.3) that

$$\langle\langle \mathscr{J}z, z\rangle\rangle = \int_0^T \langle J(z(t)), z(t)\rangle dt = \int_0^T \|z(t)\|_V^2 dt = \|z\|_{\mathscr{V}}^2 \tag{3.8}$$

and

$$\|\mathscr{J}z\|_{\mathscr{V}^*}^2 = \int_0^T \|J(z(t))\|_{V^*}^2 dt = \int_0^T \|z(t)\|_V^2 dt = \|z\|_{\mathscr{V}}^2, \tag{3.9}$$

which imply that $\mathscr{J}$ is the duality mapping from $\mathscr{V}$ to $\mathscr{V}^*$. This completes the Proof of Lemma 3.2. □

Lemma 3.3 *Under the hypotheses $H(1)$, $H(2)$ and $H(3)$ imposed on the operators A, B, G and their noisy data A_{α_n}, B_{β_n}, G_{γ_n}, we can obtain the following results: for any $z \in \mathscr{V}$,*

$$\|\mathscr{A}_{\alpha_n}z - \mathscr{A}z\|_{\mathscr{V}^*} \leq \alpha_n c_1\|z\|_{\mathscr{V}} \quad \textit{and} \quad \|\mathscr{B}_{\beta_n}z - \mathscr{B}z\|_{\mathscr{V}^*} \leq \beta_n c_2 T\|z\|_{\mathscr{V}} \tag{3.10}$$

and for $\epsilon_n > 0$, there exist $\overline{w}_n \in \mathscr{N}_{\gamma_n}z$ and $\hat{w}_n \in \mathscr{N}z_n$ such that

$$\|\overline{w}_n - w\|_{\mathscr{V}^*} \leq \gamma_n c_3\|z\|_{\mathscr{V}} + T^{1/2}\epsilon_n, \quad \forall w \in \mathscr{N}z \tag{3.11}$$

and

$$\|w_n - \hat{w}_n\|_{\mathscr{V}^*} \leq \gamma_n c_3\|z_n\|_{\mathscr{V}} + T^{1/2}\epsilon_n, \quad \forall w_n \in \mathscr{N}_{\gamma_n}z_n. \tag{3.12}$$

Proof For any $z \in \mathscr{V}$, by the hypotheses $H(1)$ and $H(2)$,

$$\|\mathscr{A}_{\alpha_n}z - \mathscr{A}z\|_{\mathscr{V}^*}^2 = \int_0^T \|A_{\alpha_n}z(t) - Az(t)\|_{V^*}^2 dt$$
$$\leq \int_0^T \alpha_n^2 c_1^2\|z(t)\|_V^2 dt$$
$$= \alpha_n^2 c_1^2\|z\|_{\mathscr{V}}^2$$

and

$$\begin{aligned}\|\mathscr{B}_{\beta_n} z - \mathscr{B} z\|_{\mathscr{V}^*}^2 &= \int_0^T \|B_{\beta_n} Kz(t) - BKz(t)\|_{V^*}^2 dt \\ &\le \int_0^T \beta_n^2 c_2^2 \|\int_0^t z(s) ds\|_V^2 dt \\ &\le \beta_n^2 c_2^2 T^2 \|z\|_{\mathscr{V}}^2,\end{aligned}$$

which prove that (3.10) holds. By definitions of $\mathscr{N}_{\gamma_n}$ and $\mathscr{N}$, one has $w_n(t) \in \partial_2 G_{\gamma_n}(t, z(t))$ and $w(t) \in \partial_2 G(t, z(t))$ for all $t \in [0, T]$ when $w_n \in \mathscr{N}_{\gamma_n} z$ and $w \in \mathscr{N} z$. So from hypothesis $H(3)$, for $\epsilon_n > 0$ and $w \in \mathscr{N} z$, we can choose a $\overline{w}_n \in \mathscr{N}_{\gamma_n} z$ such that

$$\begin{aligned}\|\overline{w}_n - w\|_{\mathscr{V}^*}^2 &= \int_0^T \|\overline{w}_n(t) - w(t)\|_{V^*}^2 dt \\ &\le \int_0^T (d(\partial_2 G_{\gamma_n}(t, z(t)), w(t)) + \epsilon_n)^2 dt \\ &\le \int_0^T (H(\partial_2 G_{\gamma_n}(t, z(t)), \partial_2 G(t, z(t))) + \epsilon_n)^2 dt \\ &\le \int_0^T (\gamma_n c_3 \|z(t)\|_V + \epsilon_n)^2 dt \\ &= \int_0^T (\gamma_n^2 c_3^2 \|z(t)\|_V^2 + 2\epsilon_n \gamma_n c_3 \|z(t)\|_V + \epsilon_n^2) dt \\ &\le \gamma_n^2 c_3^2 \|z\|_{\mathscr{V}}^2 + 2\epsilon_n \gamma_n c_3 T^{1/2} \|z\|_{\mathscr{V}} + T \epsilon_n^2 \\ &= (\gamma_n c_3 \|z\|_{\mathscr{V}} + T^{1/2} \epsilon_n)^2,\end{aligned}$$

which proves that (3.11) holds. Similarly, we can prove that there exists a $\hat{w}_n \in N z_n$ such that (3.12) holds. This completes the proof of Lemma 3.3. □

Lemma 3.4 *$z_n \in D(L)$ is a solution of regularized evolution inclusion (3.5) if and only if z_n solves the following problem: Find $z_n \in D(L)$ such that*

$$f_{\theta_n} \in L z_n + S_n z_n \tag{3.13}$$

with the operator $S_n : \mathscr{V} \to 2^{\mathscr{V}^*}$ given by

$$S_n z_n = \mathscr{A}_{\alpha_n} z_n + \mathscr{B}_{\beta_n} z_n + \epsilon_n \mathscr{J} z_n + \mathscr{N}_{\gamma_n} z_n.$$

Similarly, $z \in D(L)$ is a solution of the evolution inclusion (3.2) if and only if z solves the following problem: Find $z \in D(L)$ such that

$$f \in Lz + Sz \tag{3.14}$$

with the operator $S : \mathscr{V} \to 2^{\mathscr{V}^*}$ *given by*

$$Sz = \mathscr{A}z + \mathscr{B}z + \mathscr{N}z.$$

Proof The proof is obvious and so we omit it here. □

Proof of Theorem 3.1 By Lemma 3.4, we need only to prove the existence of solutions for problem (3.13). It follows from Theorem 2.1 that we need only to prove that the operator S_n is a bounded, coercive and pseudomonotone mapping with respect to $D(L)$.

Claim 1 *S_n is bounded.*

From Lemma 3.2, we have $A_{\alpha_n}, \mathscr{B}_{\beta_n}, \mathscr{N}_{\gamma_n}$ are bounded operators and $\mathscr{J}$ is duality mapping from $\mathscr{V}$ to $\mathscr{V}^*$. So $\mathscr{J}$ is bounded mapping from $\mathscr{V}$ to $\mathscr{V}^*$ which implies that the operator $S_n = \mathscr{A}_{\alpha_n} + \mathscr{B}_{\beta_n} + \epsilon_n \mathscr{J} + \mathscr{N}_{\gamma_n}$ is bounded from $\mathscr{V}$ into $\mathscr{V}^*$.

Claim 2 *S_n is coercive.*

By the assumption B_{β_n} is a linear, monotone and symmetric operator, we get

$$\begin{aligned} \langle\langle \mathscr{B}_{\beta_n} z, z\rangle\rangle &= \int_0^T \langle B_{\beta_n} Kz(t), z(t)\rangle dt \\ &= \int_0^T \langle B_{\beta_n} Kz(t), (Kz)'(t)\rangle dt \\ &= \int_0^T \frac{1}{2}\langle B_{\beta_n} Kz(t), Kz(t)\rangle dt \\ &\geq 0. \end{aligned} \tag{3.15}$$

Since $A_{\alpha_n} \in A(m)$, $G_{\gamma_n} \in G(m)$, for any $z \in \mathscr{V}$ and $w \in \mathscr{N}_{\gamma_n}(z)$, it follows from (2.4) and (2.6) that

$$\begin{aligned} \langle\langle \mathscr{A}_{\alpha_n} z, z\rangle\rangle &= \int_0^T \langle A_{\alpha_n} z(t), z(t)\rangle dt \\ &\geq \int_0^T \left(m\|z(t)\|_V^2 - \|A_{\alpha_n}(\theta)\|_{V^*}\|z(t)\|_V\right) dt \\ &\geq m\|z\|_{\mathscr{V}}^2 - k_3\|z\|_{\mathscr{V}} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \langle\langle w, z\rangle\rangle &= \int_0^T \langle w(t), z(t)\rangle dt \\ &\geq \int_0^T \left(-m\|z(t)\|_V^2 - \|w'\|_{V^*}\|z(t)\|_V\right) dt \\ &\geq -m\|z\|_{\mathscr{V}}^2 - k_4\|z\|_{\mathscr{V}}, \end{aligned} \tag{3.17}$$

where $w' \in \mathscr{N}_{\gamma_n}(\theta)$, k_3 and k_4 are two constants. Combining (3.8) with (3.15–3.17), for any $z \in \mathscr{V}$ and $w \in \mathscr{N}_{\gamma_n}(z)$, we obtain

$$\langle\langle \mathscr{A}_{\alpha_n} z + \mathscr{B}_{\beta_n} z + \epsilon_n \mathscr{J} z + w, z \rangle\rangle \geq \epsilon_n \|z\|^2_{\mathscr{V}} - (k_3 + k_4)\|z\|_{\mathscr{V}}$$

and so

$$\inf \left\{ \frac{\langle\langle z^*, z\rangle\rangle : z^* \in S_n z}{\|z\|_{\mathscr{V}}} \right\} \to \infty \quad \text{when} \quad \|z\|_{\mathscr{V}} \to \infty,$$

which proves the coercivity of S_n for all $\epsilon_n > 0$.

Claim 3 *S_n is pseudomonotone with respect to $D(L)$.*

To show S_n is pseudomonotone with respect to $D(L)$, we need only to prove that S_n is demicontinuous and has (S_+) property with respect to $D(L)$. Similar to the Proof of Theorem 4 in [17], we easily show that conditions (a) and (b) in the definition of (S_+) property with respect to $D(L)$ hold. Now let $\{z_k\} \in D(L)$ with $z_k \to z$ weakly in $\mathscr{V}$, $Lz_k \to Lz$ weakly in $\mathscr{V}^*$ and $w_k \in \mathscr{N}_{\gamma_n} z_k$ such that

$$\limsup_{k\to\infty} \langle\langle \mathscr{A}_{\alpha_n} z_k + \mathscr{B}_{\beta_n} z_k + \epsilon_n \mathscr{J} z_k + w_k, z_k - z\rangle\rangle \leq 0. \tag{3.18}$$

By (2) in Lemma 3.2, the operator $\mathscr{B}_{\beta_n}$ is monotone. Thus, it follows from the weak convergence of z_k that

$$\begin{aligned}\lim_{k\to\infty} \langle\langle \mathscr{B}_{\beta_n}(z_k), z_k - z\rangle\rangle &= \lim_{k\to\infty} \langle\langle \mathscr{B}_{\beta_n}(z_k) - \mathscr{B}_{\beta_n}(z), z_k - z\rangle\rangle \\ &\quad + \lim_{k\to\infty} \langle\langle \mathscr{B}_{\beta_n}(z), z_k - z\rangle\rangle \geq 0. \end{aligned} \tag{3.19}$$

Since $A_{\alpha_n} \in A(m)$, $G_{\gamma_n} \in G(m)$, by (4) in Lemma 3.2, we know that $\mathscr{A}_{\alpha_n} + \mathscr{N}_{\gamma_n}$ is also monotone and so

$$\begin{aligned}\lim_{k\to\infty} \langle\langle \mathscr{A}_{\alpha_n} z_k + w_k, z_k - z\rangle\rangle &= \lim_{k\to\infty} \langle\langle \mathscr{A}_{\alpha_n} z_k + w_k - (\mathscr{A}_{\alpha_n} z + w), z_k - z\rangle\rangle \\ &\quad + \lim_{k\to\infty} \langle\langle \mathscr{A}_{\alpha_n} z + w, z_k - z\rangle\rangle \geq 0. \end{aligned} \tag{3.20}$$

It follows from (3.18–3.20) that

$$\limsup_{k\to\infty} \langle\langle \epsilon_n \mathscr{J} z_k, z_k - z\rangle\rangle \leq 0. \tag{3.21}$$

Again by Lemma 3.2, $\mathscr{J}$ is the duality mapping from $\mathscr{V}$ to $\mathscr{V}^*$ and so it is bijective bicontinuous strictly monotone and of class (S_+). Therefore, it follows from the above inequality (3.21) that

$$z_k \to z \quad \text{and} \quad \mathscr{J} z_k \to \mathscr{J} z \quad \text{as} \quad k \to \infty. \tag{3.22}$$

Again due to $A_{\alpha_n} \in A(m)$, $G_{\gamma_n} \in G(m)$, by (1–3) in Lemma 3.2, we know that $\mathscr{A}_{\alpha_n}$ is demicontinuous, $\mathscr{B}_{\beta_n}$ is linear and bounded (thus it is continuous), and $\mathscr{N}_{\gamma_n}$ is weakly closed from $\mathscr{V}$ into $\mathscr{V}^*$. It follows that

$$\mathscr{A}_{\alpha_n}(z_k) \rightharpoonup \mathscr{A}_{\alpha_n}(z) \quad \text{as} \quad k \to \infty, \tag{3.23}$$

$$\mathscr{B}_{\beta_n}(z_k) \to \mathscr{B}_{\beta_n}(z) \quad \text{as} \quad k \to \infty \tag{3.24}$$

and there exist a $w \in \mathscr{N}_{\gamma_n}(z)$ such that

$$w_k \rightharpoonup w \in \mathscr{N}_{\gamma_n}(z) \quad \text{as} \quad k \to \infty. \tag{3.25}$$

Combining (3.22–3.25), there exists $w \in \mathcal{N}_{\gamma_n}(z)$ such that

$$\mathcal{A}_{\alpha_n} z_k + \mathcal{B}_{\beta_n} z_k + \epsilon_n \mathcal{J} z_k + w_k \rightharpoonup \mathcal{A}_{\alpha_n} z + \mathcal{B}_{\beta_n} z + \epsilon_n \mathcal{J} z + w \in S_n z,$$

which implies that S_n has (S_+) property with respect to $D(L)$. Therefore, the operator S_n is a demicontinuous mapping of class (S_+) with respect to $D(L)$ and so it is pseudomonotone with respect to $D(L)$.

From the Claims 1–3, S_n is bounded, coercive and pseudomonotone with respect to $D(L)$. It follows from Theorem 2.1 that $L + S_n$ is surjective, i.e., for each $f_{\theta_n} \in \mathcal{V}^*$, there exists $z_n \in D(L)$ such that

$$f_{\theta_n} \in L z_n + S_n z_n = L z_n + \mathcal{A}_{\alpha_n} z_n + \mathcal{B}_{\beta_n} z_n + \epsilon_n \mathcal{J} z_n + \mathcal{N}_{\gamma_n} z_n,$$

which implies that z_n is a solution of (3.5). By Lemma 3.2, the operators $\mathcal{A}_{\alpha_n} + \mathcal{N}_{\gamma_n}, \mathcal{B}_{\beta_n}$ are monotone and $\mathcal{J}$ is duality mapping from $\mathcal{V}$ to $\mathcal{V}^*$ so it is strictly monotone. Therefore, the operator $L + S_n = L + \mathcal{A}_{\alpha_n} + \mathcal{B}_{\beta_n} + \epsilon_n \mathcal{J} + \mathcal{N}_{\gamma_n}$ is strictly monotone from $\mathcal{V}$ to $\mathcal{V}^*$ which implies that the uniqueness of the solution for regularized evolution inclusion (3.5). This completes the Proof of Theorem 3.1. □

Theorem 3.2 *Assume that the mappings* $A, A_{\alpha_n} \in A(m)$ *and* $G, G_{\gamma_n} \in G(m)$. *Suppose that the mappings* B, B_{β_n} *are bounded, linear, monotone and symmetric operators and the hypotheses* $H(1) - H(5)$ *hold. If the evolution inclusion* (3.2) *is solvable, then the solution* $\{z_n\}$ *for the regularized evolution inclusion* (3.5) *is uniformly bounded in* $\mathcal{W}$.

Proof Since z_n solves the regularized evolution inclusion (3.5), we obtain $z_n \in D(L)$ and

$$f_{\theta_n} \in L z_n + \mathcal{A}_{\alpha_n} z_n + \mathcal{B}_{\beta_n} z_n + \epsilon_n \mathcal{J} z_n + \mathcal{N}_{\gamma_n} z_n. \tag{3.26}$$

By the assumption that the evolution inclusion (3.2) is solvable, from Lemma 3.4 there exists at least a $z \in D(L)$ such that

$$-f \in -Lz - \mathcal{A} z - \mathcal{B} z - \mathcal{N} z. \tag{3.27}$$

Summing-up (3.26) and (3.27) side by side, we get

$$f_{\theta_n} - f \in L z_n - L z + \mathcal{A}_{\alpha_n} z_n - \mathcal{A} z + \mathcal{B}_{\beta_n} z_n - \mathcal{B} z + \epsilon_n \mathcal{J} z_n + \mathcal{N}_{\gamma_n} z_n - \mathcal{N} z,$$

which implies that there exist $w_n \in \mathcal{N}_{\gamma_n} z_n$ and $w \in \mathcal{N} z$ such that

$$\begin{aligned} \langle\langle f_{\theta_n} - f, z_n - z \rangle\rangle = {} & \langle\langle L z_n - L z, z_n - z \rangle\rangle + \langle\langle \mathcal{A}_{\alpha_n} z_n - \mathcal{A} z, z_n - z \rangle\rangle \\ & + \langle\langle \mathcal{B}_{\beta_n} z_n - \mathcal{B} z, z_n - z \rangle\rangle + \langle\langle \epsilon_n \mathcal{J} z_n, z_n - z \rangle\rangle \\ & + \langle\langle w_n - w, z_n - z \rangle\rangle. \end{aligned}$$

Therefore, for any $\overline{w}_n \in \mathcal{N}_{\gamma_n} z$,

$$\begin{aligned} \langle\langle \epsilon_n \mathcal{J} z_n, z_n - z \rangle\rangle = {} & \langle\langle f_{\theta_n} - f, z_n - z \rangle\rangle - \langle\langle L z_n - L z, z_n - z \rangle\rangle - \langle\langle \mathcal{A}_{\alpha_n} z - \mathcal{A} z, z_n - z \rangle\rangle \\ & - \langle\langle (\mathcal{A}_{\alpha_n} z_n + w_n) - (\mathcal{A}_{\alpha_n} z + \overline{w}_n), z_n - z \rangle\rangle - \langle\langle \overline{w}_n - w, z_n - z \rangle\rangle \\ & - \langle\langle \mathcal{B}_{\beta_n} z_n - \mathcal{B}_{\beta_n} z, z_n - z \rangle\rangle - \langle\langle \mathcal{B}_{\beta_n} z - \mathcal{B} z, z_n - z \rangle\rangle. \end{aligned}$$

By the monotonicity of L, $\mathcal{A}_{\alpha_n} + \mathcal{N}_{\gamma_n}$ and $\mathcal{B}_{\beta_n}$, from hypotheses $H(4)$, we have

$$\begin{aligned} \epsilon_n \|z_n\|_{\mathcal{V}}^2 \le {} & \epsilon_n \|z_n\|_{\mathcal{V}} \|z\|_{\mathcal{V}} + \theta_n \|z_n - z\|_{\mathcal{V}} + \|\mathcal{A}_{\alpha_n} z - \mathcal{A} z\|_{\mathcal{V}^*} \|z_n - z\|_{\mathcal{V}} \\ & + \|\overline{w}_n - w\|_{\mathcal{V}^*} \|z_n - z\|_{\mathcal{V}} + \|\mathcal{B}_{\beta_n} z - \mathcal{B} z\|_{\mathcal{V}^*} \|z_n - z\|_{\mathcal{V}}. \end{aligned}$$

Since for any $\overline{w}_n \in \mathscr{N}_{\gamma_n} z$, the above inequality holds, by Lemma 3.3, we can choose $\overline{w}_n \in \mathscr{N}_{\gamma_n} z$ such that

$$\epsilon_n \|z_n\|^2_{\mathscr{V}} \le \epsilon_n \|z_n\|_{\mathscr{V}} \|z\|_{\mathscr{V}} + \theta_n \|z_n - z\|_{\mathscr{V}} + \alpha_n c_1 \|z\|_{\mathscr{V}} \|z_n - z\|_{\mathscr{V}} + (\gamma_n c_3 \|z\|_{\mathscr{V}} + T^{1/2}\epsilon_n)\|z_n - z\|_{\mathscr{V}} + \beta_n c_2 T \|z\|_{\mathscr{V}} \|z_n - z\|_{\mathscr{V}},$$

which implies that

$$\|z_n\|^2_{\mathscr{V}} \le \frac{\theta_n}{\epsilon_n}\|z_n - z\|_{\mathscr{V}} + \frac{\alpha_n}{\epsilon_n} c_1 \|z\|_{\mathscr{V}} \|z_n - z\|_{\mathscr{V}} + \frac{\gamma_n}{\epsilon_n} c_3 \|z\|_{\mathscr{V}} \|z_n - z\|_{\mathscr{V}} + \frac{\beta_n}{\epsilon_n} c_2 T \|z\|_{\mathscr{V}} \|z_n - z\|_{\mathscr{V}} + \|z_n\|_{\mathscr{V}} \|z\|_{\mathscr{V}} + T^{1/2}\|z_n - z\|_{\mathscr{V}}.$$

In view of hypothesis $H(5)$, the above inequality confirms a constant C independent of n such that

$$\|z_n\|_{\mathscr{V}} \le C. \tag{3.28}$$

In order to show that $\{z_n\}$ is uniformly bounded in $\mathscr{W}$, we still have to prove that there exists a constant C' such that $\|Lz_n\|_{\mathscr{V}^*} \le C'$. In terms of (3.26), there exists $w_n \in \mathscr{N}_{\gamma_n} z_n$ such that

$$f_{\theta_n} = Lz_n + \mathscr{A}_{\alpha_n} z_n + \mathscr{B}_{\beta_n} z_n + \epsilon_n \mathscr{J} z_n + w_n,$$

which implies that

$$\|Lz_n\|_{\mathscr{V}^*} \le \|\mathscr{A}_{\alpha_n} z_n - \mathscr{A} z_n\|_{\mathscr{V}^*} + \|\mathscr{A} z_n\|_{\mathscr{V}^*} + \|\mathscr{B}_{\beta_n} z_n - \mathscr{B} z_n\|_{\mathscr{V}^*} + \|\mathscr{B} z_n\|_{\mathscr{V}^*} + \|f_{\theta_n} - f\|_{\mathscr{V}^*} + \|f\|_{\mathscr{V}^*} + \epsilon_n \|z_n\|_{\mathscr{V}} + \|w_n - \hat{w}_n\|_{\mathscr{V}^*} + \|\hat{w}_n\|_{\mathscr{V}^*}$$

for any $\hat{w}_n \in \mathscr{N} z_n$. Since $\hat{w}_n \in \mathscr{N} z_n$ is arbitrary, by (3.10) and (3.12) in Lemma 3.3, we can choose $\hat{w}_n \in \mathscr{N} z_n$ such that

$$\|Lz_n\|_{\mathscr{V}^*} \le \alpha_n c_1 \|z_n\|_{\mathscr{V}} + \|\mathscr{A} z_n\|_{\mathscr{V}^*} + \beta_n c_2 \|z_n\|_{\mathscr{V}} + \|\mathscr{B} z_n\|_{\mathscr{V}^*} + \theta_n + \|f\|_{\mathscr{V}^*} + \epsilon_n \|z_n\|_{\mathscr{V}} + \gamma_n c_3 \|z_n\|_{\mathscr{V}} + T^{1/2}\epsilon_n + \|\hat{w}_n\|_{\mathscr{V}^*}. \tag{3.29}$$

From (1–3) in Lemma 3.2 the operators $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{N}$ are bounded. So by means of hypotheses $H(5)$ and (3.28), the inequality (3.29) implies that there exists a constant C' also independent of n such that

$$\|Lu_n\|_{\mathscr{V}^*} \le C'.$$

This together with (3.28) completes the Proof of Theorem 3.2. □

Theorem 3.3 *Assume that all hypotheses of Theorem 3.2 hold. Then there exists a convergent subsequence of $\{z_n\}$ such that every weak cluster of sequence $\{z_n\}$ in $\mathscr{W}$ is a solution of evolution inclusion (3.2).*

Proof By Theorem 3.2, $\{z_n\}$ is uniformly bounded in $\mathscr{W}$ and so it is weakly compact due to the reflexivity of the space $\mathscr{W}$. It follows that there exists a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that $z_n \rightharpoonup z$ in $\mathscr{W}$, which implies that

$$z_n \rightharpoonup z \quad \text{and} \quad Lz_n \rightharpoonup Lz \quad \text{as} \quad n \to \infty. \tag{3.30}$$

Since z_n is solution to the regularized inclusion (3.5), there exists a $w_n \in \mathscr{N}_{\gamma_n} z_n$ such that

$$f_{\theta_n} = Lz_n + \mathscr{A}_{\alpha_n} z_n + \mathscr{B}_{\beta_n} z_n + \epsilon_n \mathscr{J} z_n + w_n. \tag{3.31}$$

Thus, for any $\hat{w}_n \in \mathscr{N} z_n$, $w \in \mathscr{N} z$, we get

$$\begin{aligned}\langle\langle \epsilon_n \mathscr{J} z_n, z_n - z\rangle\rangle &= \langle\langle f_{\theta_n} - f, z_n - z\rangle\rangle + \langle\langle f, z_n - z\rangle\rangle - \langle\langle Lz_n - Lz, z_n - z\rangle\rangle \\ &\quad - \langle\langle Lz, z_n - z\rangle\rangle - \langle\langle \mathscr{A}_{\alpha_n} z_n - \mathscr{A} z_n, z_n - z\rangle\rangle - \langle\langle w_n - \hat{w}_n, z_n - z\rangle\rangle \\ &\quad - \langle\langle \mathscr{A} z_n + \hat{w}_n - (\mathscr{A} z + w), z_n - z\rangle\rangle - \langle\langle \mathscr{A} z + w, z_n - z\rangle\rangle \\ &\quad - \langle\langle \mathscr{B}_{\beta_n} z_n - \mathscr{B} z_n, z_n - z\rangle\rangle - \langle\langle \mathscr{B} z_n - \mathscr{B} z, z_n - z\rangle\rangle \\ &\quad - \langle\langle \mathscr{B} z, z_n - z\rangle\rangle. \end{aligned} \tag{3.32}$$

Due to $\hat{w}_n \in \mathscr{N} z_n$ is arbitrary, we can choose $\hat{w}_n \in \mathscr{N} z_n$ by (3.12) in Lemma 3.3 such that

$$\|\hat{w}_n - w\|_{\mathscr{V}^*} \leq \gamma_n c_3 \|z\|_{\mathscr{V}} + T^{1/2} \epsilon_n. \tag{3.33}$$

By using the weak convergence of $\{z_n\}$, the monotonicity of L, $\mathscr{A} + \mathscr{N}$ and $\mathscr{B}$, the hypotheses $H(4)$ and $H(5)$, and Lemma 3.3, we deduce from (3.32) that

$$\limsup_{n\to\infty} \langle\langle \epsilon_n \mathscr{J} z_n, z_n - z\rangle\rangle \leq 0,$$

which by the (S_+) property of $\mathscr{J}$ implies that

$$z_n \to z. \tag{3.34}$$

Furthermore, for any weak cluster z of sequence $\{z_n\}$, there exists a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that $z_n \rightharpoonup z$ in $\mathscr{W}$. By the same arguments as above we can always obtain $z_n \to z$. From Lemma 3.2, the operator $\mathscr{A}$ is demicontinuous, the operator $\mathscr{B}$ is linear and bounded which implies that it is continuous, and the operator $\mathscr{N}$ is weakly closed. By the convergence of $\{z_n\}$, we obtain that there exists a $w \in \mathscr{N} z$ such that

$$\mathscr{A} z_n \rightharpoonup \mathscr{A} z, \quad \mathscr{B} z_n \to \mathscr{B} z, \quad \hat{w}_n \rightharpoonup w \quad \text{as} \quad n \to \infty. \tag{3.35}$$

Under the hypotheses $H(4)$ and $H(5)$, it follows from Lemma 3.3 that

$$\begin{aligned} &f_{\theta_n} \to f,\ \epsilon_n \mathscr{J} z_n \to \Theta,\ \mathscr{A}_{\alpha_n} z_n - \mathscr{A} z_n \to \Theta,\ \mathscr{B}_{\beta_n} z_n - \mathscr{B} z_n \to \Theta, \\ &w_n - \hat{w}_n \to \Theta\ (n \to \infty), \end{aligned} \tag{3.36}$$

where Θ is the zero element of $\mathscr{V}^*$. Therefore, from (3.35) and (3.36),

$$\mathscr{A}_{\alpha_n} z_n \rightharpoonup \mathscr{A} z, \quad \mathscr{B}_{\beta_n} z_n \to \mathscr{B} z, \quad w_n \rightharpoonup w \in \mathscr{N} z \quad \text{as} \quad n \to \infty. \tag{3.37}$$

Letting $n \to \infty$ in (3.31), by (3.30), (3.36) and (3.37), we conclude that

$$f = Lz + \mathscr{A} z + \mathscr{B} z + w \in Lz + \mathscr{A} z + \mathscr{B} z + \mathscr{N} z,$$

which together with Lemma 3.4 implies that z is a solution of evolution inclusion (3.2). This completes the Proof of Theorem 3.3. □

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References

1. Berkovits, J., Mustonen, V.: Monotonicity methods for nonlinear evolution equations. Nonlinear Anal. TMA **27**, 1397–1405 (1996)
2. Carl, S., Heikkilä, S.: Nonlinear Differential Equations in Ordered Spaces. Chapman & Hall/CRC, Boca Raton, FL (2000)

3. Carl, S., Motreanu, D.: Extremal solutions of quasilinear parabolic inclusions with generalized Clarke's gradient. J. Differ. Equ. **191**, 206–233 (2003)
4. Carl, S., Naniewicz, Z.: Vector quasi-hemivariational inequalities and discontinuous elliptic systems. J. Global Optim. **34**, 609–634 (2006)
5. Carl, S., Le, V.K., Motreanu, D.: Existence and comparison results for quasilinear evolution hemivariational inequalities. Electron. J. Differ. Equ. **57**, 1–17 (2004)
6. Carl, S., Le, V.K., Motreanu, D.: The sub-supersolutio method and extremal solutions for quasilinear hemivariational inequalities. Differ. Integral Equ. **17**, 165–178 (2004)
7. Carl, S., Le, V.K., Motreanu, D.: Nonsmooth Variational Problems and their Inequalities, Comparison Principles and Applications. Springer-Verlag, Berlin (2005)
8. Clarke, F.H.: Optimization and Nonsmooth Analysis. SIAM, Philadelphia (1990)
9. Denkowski, Z., Migorski, S.: Existence of solutions to evolution second order hemivariational inequalities with multivalued damping. Syst. Model. Optim. **166**, 203–215 (2005)
10. Denkowski, Z., Migorski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Applications. Kluwer Academic Publishers, Boston, Dordrecht, London (2003)
11. Giannessi, F., Khan, A.A.: Regularization of non-coercive quasi variational inequalities. Control Cybern. **29**, 91–110 (2000)
12. Liu, Z.H.: Some convergence results for evolution hemivariational inequalities. J. Global Optim. **29**, 85–95 (2004)
13. Liu, Z.H.: Browder-Tikhonov regularization on non-coercive evolution hemivariational inequalities. Inverse Probl. **21**, 13–20 (2005)
14. Migorski, S.: Boundary hemivariational inequalities of hyperbolic type and applications. J. Global Optim. **31**, 505–533 (2005)
15. Motreanu, D., Panagiotopoulos, P.D.: Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities and Applications, *Nonconvex Optimization and Its Applications*, vol. 29. Kluwer Academic, Dordrecht (1999)
16. Naniewicz, Z., Panagiotopoulos, P.D.: Mathematical Theory of Hemivariational Inequalities and Applications. Marcel Dekker, New York (1995)
17. Ochal, A.: Existence results for evolution hemivariational inequalities of second order. Nonlinear Anal. TMA **60**, 1369–1391 (2005)
18. Panagiotopoulos, P.D.: Coercive and semicoercive hemivariational inequalities. Nonlinear Anal. TMA **16**, 209–231 (1991)
19. Panagiotopoulos, P.D.: Hemivariational Inequalities, Applications in Mechanics and Engineering. Springer-Verlag, Berlin (1993)
20. Panagiotopoulos, P.D.: Hemivariational inequalitiy and fan-variational inequality. New applications and results. Atti. Sem. Mat. Fis. Univ. Modena **XLIII**, 159–191 (1995)
21. Xiao, Y.B., Huang, N.J.: Sub-supersolution method and extremal solutions for higher order quasi-linear elliptic hemi-variational inequalities. Nonlinear Anal. TMA **66**, 1739–1752 (2007)
22. Xiao, Y.B., Huang, N.J.: Generalized quasi-variational-like hemivariational inequalities. Nonlinear Anal. TMA **69**, 637–646 (2008)
23. Zeidler, E.: Nonlinear Functional Analysis and its Applications, vol. II. Springer-verlag, Berlin (1990)