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Constructing a DC decomposition for ordered median problems

Zvi Drezner · **Stefan Nickel**

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Abstract In this paper we show how to express ordered median problems as a difference between two convex functions (DC). Such an expression can be exploited in solving ordered median problems by using the special methodology available for DC optimization. The approach is demonstrated for solving ordered one median problems in the plane. Computational experiments demonstrated the effectiveness of the approach.

Keywords Location problems · DC optimization · Ordered median problem

1 Introduction

Continuous location has achieved an important degree of maturity. It is demonstrated by the large number of papers and research books published within this field. In addition, this development has also been recognized by the mathematical community by assigning the AMS code 90B85 to this area of research. Continuous location problems appear very often in economic models of distribution or logistics, in statistics when one tries to find an estimator from a data set or in pure optimization problems where one looks for optimizing a certain function. For a comprehensive overview the reader is referred to [2] or [11] . Despite the fact that many continuous location problems rely heavily on a common framework, specific solution approaches have been developed for each of the typical objective functions in location theory. To overcome this inflexibility and to work towards a unified approach to location theory models the so called Ordered Median Problem (OMP) was developed (see

Z. Drezner
Steven G. Mihaylo College of Business and Economics, California State University-Fullerton, Fullerton, CA 92834, USA

S. Nickel (✉)
Saarland University, 66123 Saarbrucken, Germany
e-mail: s.nickel@orl.uni-saarland.de

S. Nickel
Fraunhofer ITWM, Kaiserslautern, Germany

[9] and references therein). Ordered Median Problems represent as special cases many of the classical objective functions in location theory, including the Median, Cent-Dian, center and k-centra.

The 1-facility Ordered Median Problem in the plane can be formulated as follows: A vector of weights $\lambda_1, \lambda_2 \ldots \lambda_n$, where n is the number of demand points, is given. The distances between the demand points and the facility are sorted in a non-decreasing order. Note that the order depends on the location of the facility. The problem is to find a location for a facility that minimizes the weighted sum of distances where the distance to the closest point to the facility is multiplied by the weight λ_1, the distance to the second closest by λ_2, and so on. The distance to the farthest point is multiplied by λ_n.

Many location problems can be formulated as the Ordered One-Median Problem by selecting appropriate weights. For example, the vector for which all $\lambda_i = 1$ is the unweighted 1-median problem, the problem where $\lambda_n = 1$ and all others are equal to zero is the one center problem. Minimizing the range of distances is achieved by $\lambda_1 = -1$ and $\lambda_n = 1$ and all others are equal to zero. Minimizing the median of distances is achieved by $\lambda_{(n+1)/2} = 1$ for odd n and $\lambda_{n/2} = \lambda_{n/2+1} = 0.5$ for even n and all others are equal to zero. In a recent paper [5] the OMP approach is used to minimize the Gini coefficient of the Lorenz curve.

Solution methods for continuous OMPs so far have been mainly discretization results obtaining finite dominating sets (see [12]). Moreover, a linear programming approach for some OMPs was developed (see [8]). Recently [3] developed an efficient method for the OMP using the Big Triangle Small Triangle approach [1,4]. The objective function is convex if and only if $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ [9]. However, until now no proper embedding of the continuous OMP (as a global optimization problem) into the global optimization literature has been done.

This paper shows that the OMP belongs to the class of d.c. optimization problems [6]. It is pointed out in [7] that from the theoretical viewpoint the class of d.c. functions, i.e., functions that can be represented as difference between two convex functions, enjoys a remarkable stability with respect to operations frequently encountered in optimization. For example, the class of d.c. functions is closed under operations such as sum, multiplication, multiplication with a scalar (positive or negative), maximum and minimum of a finite number of functions, etc. Moreover, we know that every locally d.c. function, is also d.c. in the whole space. From this it can easily be deduced, for example, that every C^2-function (a function possessing a continuous second derivative) is d.c. However, Ordered Median Problems are not C^2 functions. At points where the order of distances is changed, the first derivative may not exist. For example, the function of the maximum distance to a set of points is continuous, but at points where the maximum distance is obtained for at least two points, the first derivative of the maximum distance may be discontinuous.

The main concern when using these properties is how to construct a d.c. representation of a function which is known to be a d.c. function but not given in d.c. form. This problem of finding appropriate d.c. representations is not yet solved for broad classes of d.c. functions [7]. We will present a constructive way of transforming a standard OMP into a d.c. problem. Moreover, we will prove the usefulness of the d.c. method by extensive numerical tests.

The rest of the paper is organized as follows: After restating some needed definitions and concepts we demonstrate how we derive a d.c. decomposition from the OMP. Different lower bounds for the planar single facility OMP as well as solution procedures are then developed in Sect. 3. Section 4 is devoted to extensive numerical experiments using the bounds from Sect. 3. The paper concludes with some discussion and an outlook to future research.

2 Deriving DC decomposition

The purpose of this paper is to show how every Ordered (One)-Median Problem can be expressed as a difference between two convex functions (DC). Once this is done, strong optimization techniques borrowed from DC-optimization can be employed to solve any Ordered Median Problem.

2.1 Notation

Let

n	be the number of demand points,
$X_i = (x_i, y_i)$	be the location of demand point i, $1 \le i \le n$,
$X = (x, y)$	be a point in the plane,
$d_i(X)$	be the distance between demand point i and point X,
$d_{(i)}(X)$	be the sorted vector of distances $(d_{(i)}(X) \le d_{(i+1)}(X))$,
$\lambda = \{\lambda_i\}$	be the vector of Ordered Median weights.

The objective function to be minimized, $F(\lambda, X)$, is:

$$F(\lambda, X) = \sum_{i=1}^{n} \lambda_i d_{(i)}(X) \tag{1}$$

The distances $d_i(X)$ can be Rectilinear, Euclidean, ℓ_p or any other convex distances in any dimensional space. The theory developed in this paper holds for any convex functions $d_i(X)$, not necessarily functions based on norms.

2.2 Derivation

Theorem 1.3 in [9] states

Theorem 1 *An Ordered Median function is convex if and only if $\lambda_i \ge 0$ and $\lambda_{i-1} \le \lambda_i$.*

Let us first assume $\lambda_i \ge 0$. Let $s_k = \sum_{i=1}^{k} \lambda_i$, also define $s_0 = 0$. Define $\alpha_k = s_k$ and $\beta_k = s_{k-1}$. It can be easily verified that

(i) $\lambda_k = \alpha_k - \beta_k$.
(ii) $0 \le \alpha_k \le \alpha_{k+1}$ and $0 \le \beta_k \le \beta_{k+1}$.
(iii) Therefore, $\sum_{i=1}^{n} \alpha_i d_{(i)}(X)$ and $\sum_{i=1}^{n} \beta_i d_{(i)}(X)$ are both convex functions by Theorem 1.
(iv) The Ordered Median function is thus expressed as a difference between two convex functions.

If there are negative λs, let $\lambda_{\min} = \min\{\lambda_i\}$. Define $\lambda'_i = \lambda_i - \lambda_{\min}$ and $\mu_i = -\lambda_{\min}$, then $\lambda_i = \lambda'_i - \mu_i$. The vector μ defines a convex function by Theorem 1, and the vector λ'_i is expressed as a difference between two convex functions as described above. The function defined by μ is added to the function defined by β yielding a convex function because a sum of two convex functions is convex. The result is a difference between two convex functions.

This actually accomplishes our goal. However, the vectors α and β can possibly be reduced to eliminate redundancy. In particular, let $\lambda_i = \alpha_i - \beta_i$ fulfilling $\alpha_i \ge 0, \beta_i \ge 0$ and $\alpha_i \le \alpha_{i+1}$ and $\beta_i \le \beta_{i+1}$. We wish to find a vector $\gamma \ge 0$ (with maximum possible values) such that $\alpha'_i = \alpha_i - \gamma_i$ and $\beta'_i = \beta_i - \gamma_i$ have the same properties.

For γ_1 we must have $\alpha_1 - \gamma_1 \geq 0$ and $\beta_1 - \gamma_1 \geq 0$, therefore $\gamma_1 = \min\{\alpha_1, \beta_1\}$. Note that $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$ must remain true. For each $k = 2, \ldots, n$ in order determine γ_k as follows. Four conditions must hold. $\alpha_k - \gamma_k \geq 0$; $\beta_k - \gamma_k \geq 0$; $\alpha_k - \gamma_k \geq \alpha_{k-1} - \gamma_{k-1} \geq 0$; $\beta_k - \gamma_k \geq \beta_{k-1} - \gamma_{k-1} \geq 0$. This leads to:

$$\gamma_k = \min\{\alpha_k, \beta_k, \alpha_k - \alpha_{k-1} + \gamma_{k-1}, \beta_k - \beta_{k-1} + \gamma_{k-1}\}$$

This condition can be simplified by observing that $\gamma_{k-1} \leq \alpha_{k-1}$ and thus $\alpha_k - \alpha_{k-1} + \gamma_{k-1} \leq \alpha_k$ and the same for β_k. Therefore,

$$\begin{aligned}\gamma_k &= \min\{\alpha_k - \alpha_{k-1} + \gamma_{k-1}, \beta_k - \beta_{k-1} + \gamma_{k-1}\} \\ &= \min\left\{\alpha_k - \alpha'_{k-1}, \beta_k - \beta'_{k-1}\right\}\end{aligned} \tag{2}$$

The following process finds the vectors α' and β'.

The Updating Process

1. Set $\alpha'_0 = \beta'_0 = 0$
2. For $k = 1, \ldots, n$ in order update:
3. $\gamma_k = \min\{\alpha_k - \alpha'_{k-1}, \beta_k - \beta'_{k-1}\}$
4. $\alpha'_k = \alpha_k - \gamma_k$
5. $\beta'_k = \beta_k - \gamma_k$

Lemma 1 $\alpha'_1 \geq 0$ *and* $\beta'_1 \geq 0$.

Proof Since $\gamma_1 \leq \alpha_1$, $\alpha'_1 = \alpha_1 - \gamma_1 \geq 0$ and the same for β'_1.

Lemma 2 $\alpha'_{i-1} \leq \alpha'_i$ *and* $\beta'_{i-1} \leq \beta'_i$.

Proof By (2): $\alpha'_i = \alpha_i - \gamma_i \geq \alpha_i - (\alpha_i - \alpha'_{i-1}) = \alpha'_{i-1}$ and the same for β.

Theorem 2 *Both* $F(\alpha', X)$ *and* $F(\beta', X)$ *are convex.*

Proof Follows Lemma 1, Lemma 2, and Theorem 1.

Theorem 3 *Every Ordered Median function can be expressed as a difference between two convex functions.*

Proof Follows $F(\lambda, X) = F(\alpha', X) - F(\beta', X)$ and Theorem 2.

2.3 Examples

2.3.1 Truncated mean

$\lambda = \{0, 0, 0, 1, 1, 1, 1, 0, 0, 0\}$.
$s = \{0, 0, 0, 1, 2, 3, 4, 4, 4, 4\}$
$\alpha = \{0, 0, 0, 1, 2, 3, 4, 4, 4, 4\}$
$\beta = \{0, 0, 0, 0, 1, 2, 3, 4, 4, 4\}$
and after updating
$\alpha' = \{0, 0, 0, 1, 1, 1, 1, 1, 1, 1\}$
$\beta' = \{0, 0, 0, 0, 0, 0, 0, 1, 1, 1\}$

2.3.2 Range

$\lambda = \{-1, 0, 0, 0, 0, 1\}$
After subtracting $\lambda_{\min}$
$\lambda' = \{0, 1, 1, 1, 1, 2\}$
$s = \{0, 1, 2, 3, 4, 6\}$
$\alpha = \{0, 1, 2, 3, 4, 6\}$
$\beta + \mu = \{1, 1, 2, 3, 4, 5\}$
and after updating
$\alpha' = \{0, 1, 1, 1, 1, 2\}$
$\beta' = \{1, 1, 1, 1, 1, 1\}$

3 Planar One Median Problems

The proposed DC decomposition is illustrated and tested for the solution of Ordered One Median Problems in the plane [3]. In [3] it was proposed to solve the Ordered One Median Problem in the plane by using the Big Triangle Small Triangle method [4]. Following is a short description of the BTST method.

3.1 The BTST approach

The method is described for a minimization problem. For maximization problems the role of the lower and upper bounds are reversed. A feasible region which consists of a finite number of convex polygons is given.

Phase 1: Each convex polygon is triangulated using the Delaunay triangulation. The vertices of the triangles are the demand points and the vertices of the convex polygon. The union of the triangulations is the initial set of triangles.
Phase 2: Calculate an upper bound, UB, and a lower bound, LB, for each triangle. Let the largest LB be $\overline{LB}$. Discard all triangles for which $UB \leq \overline{LB}(1+\epsilon)$.
Phase 3: Choose the triangle with the largest LB and divide it into four small triangles by connecting the centers of its sides. Calculate UB and LB for each triangle, and update the $\overline{LB}$ if necessary. The large triangle and all triangles for which $UB \geq \overline{LB}(1+\epsilon)$ are discarded.
Stopping Criterion: The branch and bound is terminated when there are no triangles left. The solution $\overline{LB}$ is within a relative accuracy of ϵ from the optimum.

Note that: (i) A lower bound in a triangle is the value of the objective function at any point in the triangle (such as the center of gravity). (ii) Since the triangulation is based on the demand points as vertices, no demand point is in the interior of a triangle. This is also true for all triangles generated in the process.

3.2 A DC based lower bound

The objective function is expressed by a difference of two convex functions $F(\lambda, X) = F_1(\alpha, X) - F_2(\beta, X)$. Let X_0 be the center of the triangle (unweighted center of gravity of the three vertices). The tangent plane, $G(\alpha, X)$, of the function $F_1(\alpha, X)$ at X_0 is constructed.

$$G(\alpha, X) = F_1(\alpha, X_0) + (x - x_0)\sum_{i=1}^{n}\alpha_i \frac{x_{(i)} - x_0}{d_{(i)}(X_0)} + (y - y_0)\sum_{i=1}^{n}\alpha_i \frac{y_{(i)} - y_0}{d_{(i)}(X_0)}. \tag{3}$$

Note that the order (i) in Eq. 3 is the order of the distances at X_0.

Since $F_1(\alpha, X)$ is convex, $G(\alpha, X) \le F_1(\alpha, X)$ and therefore $F(\lambda, X) = F_1(\alpha, X) - F_2(\beta, X) \ge G(\alpha, X) - F_2(\beta, X)$. The function $H(X) = G(\alpha, X) - F_2(\beta, X)$ is concave because $G(\alpha, X)$ is linear and $F_2(\beta, X)$ is convex. The function $H(X)$ obtains its minimum on one of the vertices of the triangle. Let the three vertices of the triangle be V_1, V_2, V_3, then $H(X) \ge \min_{k=1,2,3}\{H(V_k)\}$.

The DC lower bound is therefore

$$LB_{DC} = \min_{k=1,2,3}\{H(V_k)\} \tag{4}$$

3.3 A lower bound for a ratio

There are instances where the objective function is a ratio of an Ordered Median objective and another function $D(X)$. For example, the Gini coefficient objective [5] is a ratio of an Ordered Median objective and the sum of all distances. Other examples are the ratio between the maximum distance and the average distance, range and the average distance, truncated mean and the average distance. In such instances, when both the Ordered Median objective and the denominator are positive, a lower bound can be constructed by any lower bound on the Ordered Median objective divided by the maximum value of $D(X)$ in the triangle (if the Ordered Median objective is negative, then the minimum of $D(X)$ in the triangle is required). Such a lower bound was proposed in [5] for the solution of the minimization of the Gini coefficient. However, such a lower bound may not be that tight because the lower bound of the numerator may be calculated at one vertex of the triangle while the maximum of $D(X)$ may be calculated at another vertex. The lower bound LB_{DC} (4) is based on the function $H(X)$ which is a concave function. If $D(X)$ is convex, and both are positive, then the following theorem can be used to tighten such a lower bound.

Theorem 4 *Let $f(x) > 0$ be a concave function and $g(x) > 0$ be a convex function. Then, $\frac{f(x)}{g(x)}$ is quasi-concave.*

Proof Quasi concavity of the ratio means that for every $0 \le \lambda \le 1$ and two points x and y:

$$\frac{f(\lambda x + (1-\lambda)y)}{g(\lambda x + (1-\lambda)y)} \ge \min\left\{\frac{f(x)}{g(x)}, \frac{f(y)}{g(y)}\right\}.$$

Suppose that $\frac{f(x)}{g(x)} \le \frac{f(y)}{g(y)}$ and the same argument holds for the opposite case. It follows that $f(y) \ge \frac{f(x)g(y)}{g(x)}$. Therefore,

$$\begin{aligned}\frac{f(\lambda x + (1-\lambda)y)}{g(\lambda x + (1-\lambda)y)} &\ge \frac{\lambda f(x) + (1-\lambda)f(y)}{\lambda g(x) + (1-\lambda)g(y)}\\ &\ge \frac{\lambda f(x) + (1-\lambda)\frac{f(x)g(y)}{g(x)}}{\lambda g(x) + (1-\lambda)g(y)}\\ &= \frac{f(x)}{g(x)}\frac{\lambda + (1-\lambda)\frac{g(y)}{g(x)}}{\lambda + (1-\lambda)\frac{g(y)}{g(x)}}\\ &= \frac{f(x)}{g(x)}\end{aligned}$$

which proves the quasi-concavity of the ratio.

Since the ratio $\frac{H(X)}{D(X)}$ is quasi-concave, it attains its minimum in a triangle at a vertex of the triangle. Therefore, instead of finding the minimum value of $H(X)$ among the vertices of the triangle and dividing it by the maximum value of $D(X)$ in the triangle, the minimum value of $\frac{H(X)}{D(X)}$ among the three vertices is a tighter lower bound. This is usually a tighter lower bound because the numerator and the denominator are evaluated at the same vertex. We term this lower bound LB_{Rat}. Note that this approach cannot be implemented for the lower bounds suggested in Drezner and Nickel [3] because those lower bounds are not based on a concave function and Theorem 4 does not apply.

3.4 Saving calculation time of the DC lower bound

Most of the computer time in calculating the DC lower bound is consumed by sorting vectors of distances. The vectors of the distances to the center of the triangle and its three vertices require sorting. Run time was decreased by more than 50% by employing the following strategy. The distances vector to the center of the triangle is sorted. Once the order is determined, the three vectors of the distances to the vertices of the triangle are re-ordered by the same order before the sorting procedure is applied. Since the four vectors of distances are not much different from one another, the reordered vectors are "almost" sorted and the sort procedure is much faster. Note that all four vectors are sorted properly. Time is saved because three of the four sorts require very little effort. This simple ploy reduced the run time of the procedure by more than 50% compared with performing four sorting procedures on the original distance vectors.

4 Computational experiments

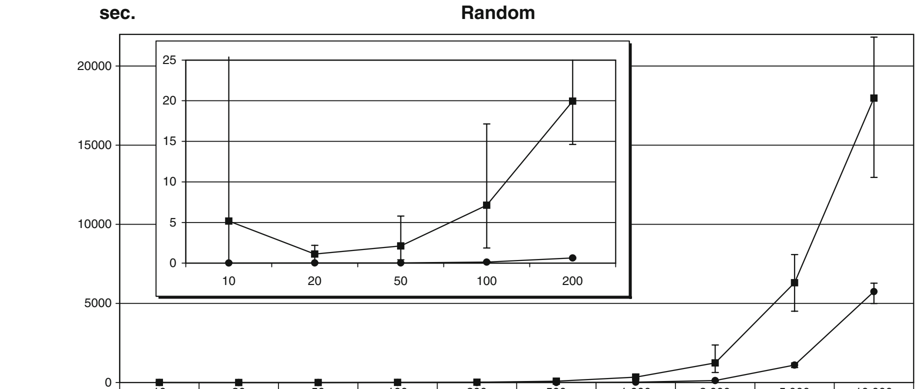
We compared the efficiency of the DC lower bound on the nine problems tested in [3] and the two problems tested in [5] with the results reported there. There are several lower bounds developed in [3]. Three variants of the best lower bound are used for comparison. They are based on the shortest possible distance δ_i between demand point i and any point in the triangle [1] and the longest possible distance Δ_i which is measured to one of the three vertices. These are defined as the vectors δ and Δ, respectively. For a point X in the triangle, $\delta_i \leq d_i(X) \leq \Delta_i$. The lower bound LB_3 is [3]:

$$LB_3 = \sum_{i=1}^{n} \left[\max\{\lambda_i, 0\}\delta_{(i)} + \min\{\lambda_i, 0\}\Delta_{(i)}\right] \tag{5}$$

The lower bounds LB_1 and LB_2 are special cases of LB_3. LB_1 is defined for $\lambda_i \geq 0$ when Δ_i is not required for calculating (5). Similarly, LB_2 is defined for $\lambda_i \leq 0$ when δ_i is not required for calculating (5).

Programs in Fortran[1] using double precision arithmetic were coded, compiled by the Intel 9.0 FORTRAN Compiler, and ran on a 2.8 GHz Pentium IV desk top computer with 256 MB RAM. The programs written for this paper required few modifications of the programs coded for the [3] paper. They were run on the same platform. Therefore, for comparison purposes it is sufficient to compare the run times recorded for the solution of the different problems.

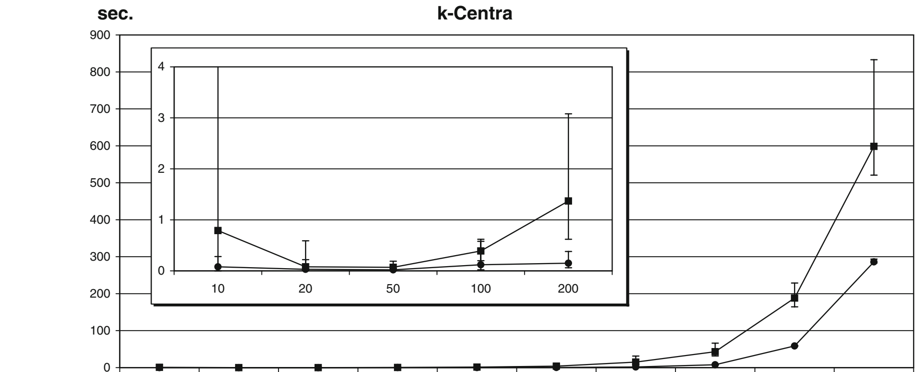
[1] We thank Atsuo Suzuki for his Fortran program that finds the triangulation based on [13] subroutines first developed in [10].



	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LB1, ε=10-5	5.18	1.12	2.12	7.13	19.95	85.56	340.2	1,237.97	6,306.33	17,971.63
LBDC, ε=10-8	0.02	0.03	0.03	0.13	0.64	4.52	23.65	127.3	1,103.10	5,748.85

n

Fig. 1 Comparing the lower bounds for Random λ



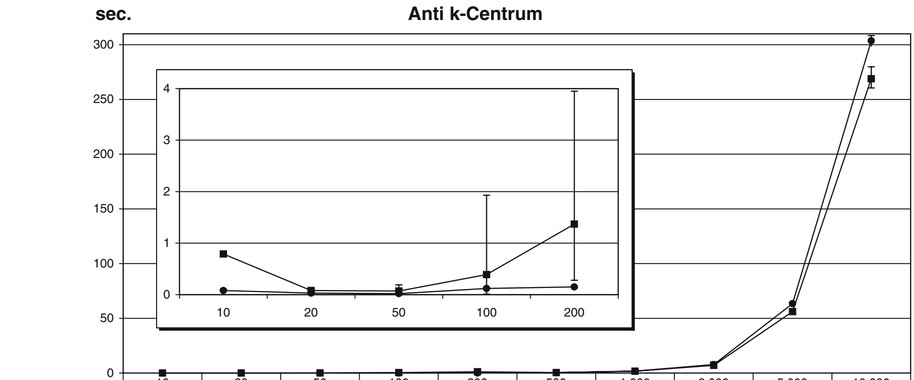
	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LB1, ε=10-4	0.79	0.08	0.07	0.39	1.37	3.99	15.04	43.04	188.13	598.44
LBDC, ε=10-8	0.08	0.03	0.02	0.12	0.15	0.41	1.71	7.81	58.56	286.12

n

Fig. 2 Comparing the lower bounds for k-centra

In Figs. 1–9 the performance of the algorithms on the nine problems suggested in [3] is reported. For each problem size we ran 10 randomly generated instances. We used a relative accuracy of at least $\epsilon = 10^{-8}$ except when the accuracy used in [3] was lower. In [3] many problems were solved to an accuracy as high as $\epsilon = 10^{-4}$. Lower accuracy required the storage of more than 500,000 triangles which is the capacity of the program. Therefore, these problems could not be solved to a lower accuracy. These problems could not be practically solved to a better accuracy by the lower bounds in [3]. When a problem is solvable to within an accuracy of $\epsilon = 10^{-8}$ by employing the DC bound while the lower bounds suggested in [3] failed to do so, the DC bound is superior for such problems regardless of the run times.

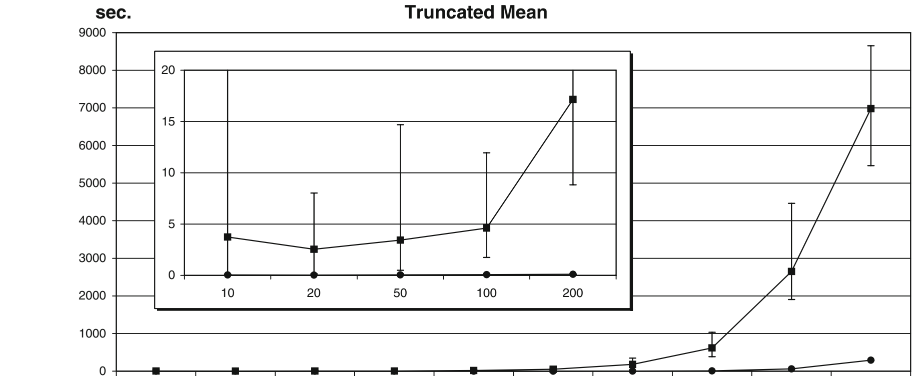
In Figs. 10–12 comparison of the performance of the algorithms for solving problems with the two objectives related to the Lorenz curve [5] is reported. The mean difference objective is an Ordered One Median Problem and is grouped with the nine problems in Figs. 1–9



	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LB2, ε=10-10	0	0.02	0.07	0.4	1.15	0.42	1.68	6.99	56.1	268.92
LBDC, ε=10-10	0	0	0.02	0.03	0.1	0.43	1.72	7.76	63.45	303.69

n

Fig. 3 Comparing the lower bounds for anti k-centrum



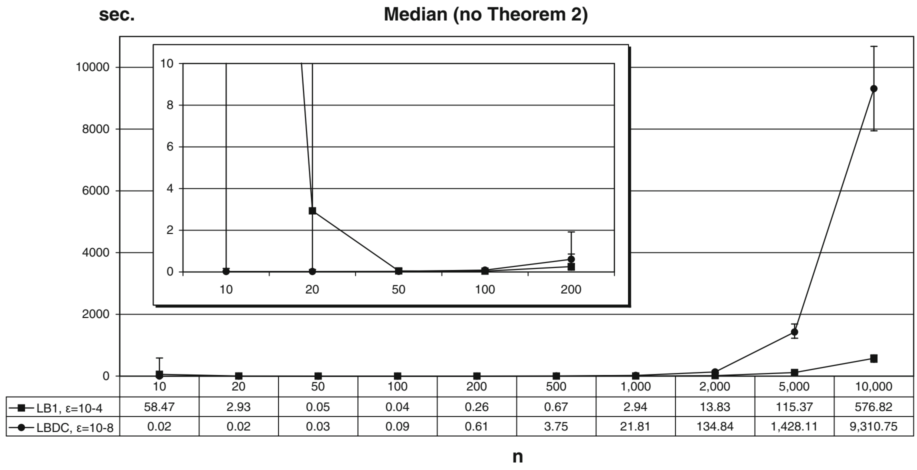
	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LB1, ε=10-4	3.74	2.55	3.44	4.61	17.15	51.96	180.61	617.45	2,650.98	6,981.73
LBDC, ε=10-8	0.04	0.03	0.05	0.07	0.11	0.52	1.87	8.37	61.03	291.43

n

Fig. 4 Comparing the lower bounds for truncated mean

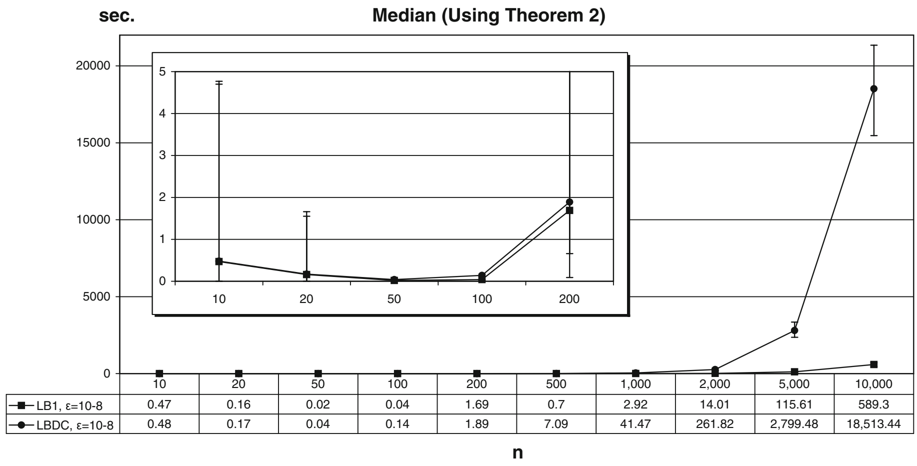
leading to a comparison between the performance of the lower bounds for ten ordered one median problems.

Consider solutions of $n = 10,000$ problems reported in Table 2. We observe that four problems were solved in about 5 min to a relative accuracy of $\epsilon = 10^{-8}$ or better. Only one of these problems was solved in a slightly shorter time by LB_2. The other three problems were originally solved only to an accuracy of $\epsilon = 10^{-4}$ and took much longer to solve. Three problems were solved to a better accuracy but required significantly longer run time. One problem (random) was solved to a better accuracy in a shorter time, and two problems required significantly longer time to achieve the same accuracy. Since being able to achieve better accuracy is an improvement, the DC bound performed better in solving seven of the ten problems, tied on one problem, and performed worse on two problems. If minimizing the Gini coefficient is included (Figs. 11–12), the DC lower bound performed better in solving eight of the eleven problems, tied in performance on one problem, and performed worse on



n	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LB1, ε=10-4	58.47	2.93	0.05	0.04	0.26	0.67	2.94	13.83	115.37	576.82
LBDC, ε=10-8	0.02	0.02	0.03	0.09	0.61	3.75	21.81	134.84	1,428.11	9,310.75

Fig. 5 Comparing the lower bounds for median without Theorem 2



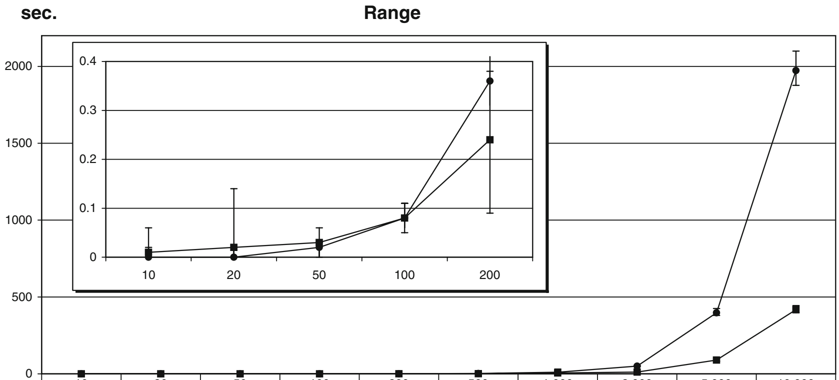
n	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LB1, ε=10-8	0.47	0.16	0.02	0.04	1.69	0.7	2.92	14.01	115.61	589.3
LBDC, ε=10-8	0.48	0.17	0.04	0.14	1.89	7.09	41.47	261.82	2,799.48	18,513.44

Fig. 6 Comparing the lower bounds for median with Theorem 2

two problems. Also, we observe that the run times by the DC lower bound are less variable than those reported in [5,3]. The range of times is hardly noticeable for the DC lower bound.

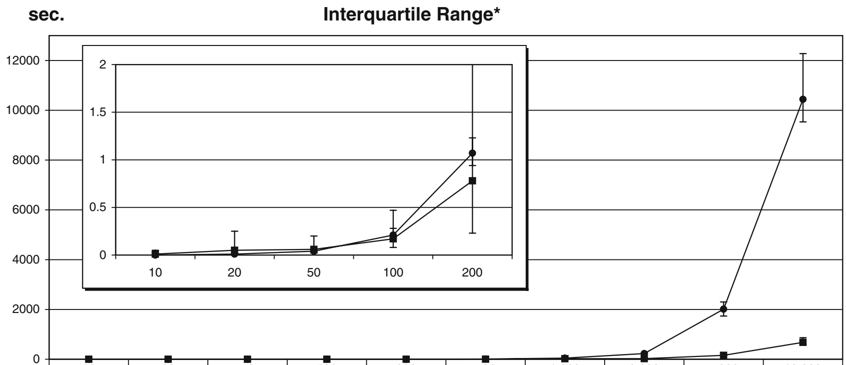
4.1 Analysis of the DC bound's performance

The DC bound performed consistently well but for some problems the lower bounds suggested in [3] performed better. We analyze the structure of the problems to identify the structures that lead to such differences in performance. We suspect that those differences stem from the possibility that a vector λ, that may have very few non-zero elements, is decomposed into two vectors α and β each having many non-zero elements. The difference in the values between the tangent plane $G(\alpha, X)$ and $F_1(\alpha, X)$ depends on the number of non-zero αs or their sum. We therefore evaluated some measures of the problems in order to be able to draw some conclusions. The following measures were calculated for $n = 10$, $n = 100$, $n = 1,000$, and



n	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LB3, ε=10-6	0.01	0.02	0.03	0.08	0.24	1.01	5.96	11.32	89.57	417.48
LBDC, ε=10-8	0	0	0.02	0.08	0.36	2.29	10.55	50.42	397.28	1,973.02

Fig. 7 Comparing the lower bounds for range



n	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LB3, ε=10-6	0.01	0.05	0.06	0.17	0.78	1.86	21.24	24.01	156.42	677.27
LBDC, ε=10-8	0	0.01	0.04	0.21	1.07	8.25	43.68	228.43	2,012.50	10,440.21

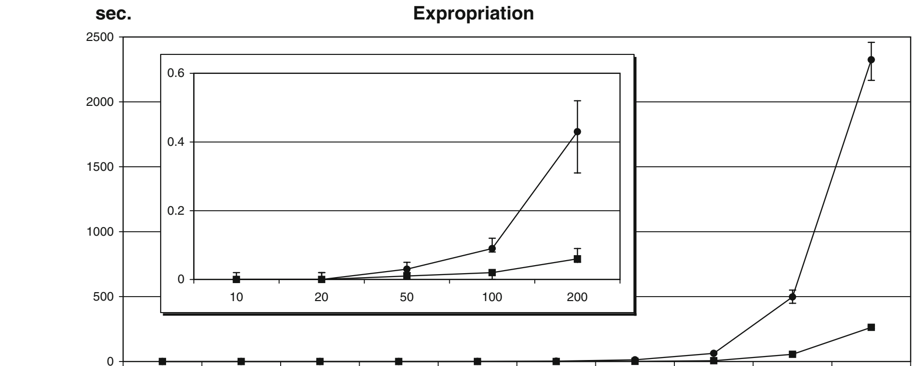
Fig. 8 Comparing the lower bounds for interquartile range

$n = 10,000$ problems:

$$\theta_1 = \sum_{i=1}^{n} |\lambda_i|, \quad \theta_2 = \sum_{i=1}^{n} (\alpha_i + \beta_i) \tag{6}$$

and their ratio. These measures are depicted in Table 1 for the ten Ordered One Median Problems tested in this paper.

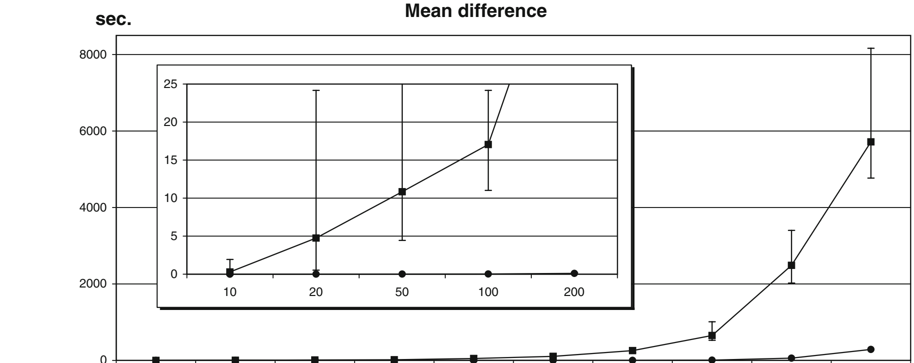
By examining Table 1 we conclude that the four problems with a low ratio of θ_2/θ_1 (less than 20 for $n = 10,000$) are the only ones that were solved in about 5 min for the $n = 10,000$ problems. All other problems (for $n = 10,000$) were solved in more than half an hour. Also, problems with a small value of θ_1 (one or two), and a large value of θ_2 require longer run time by the DC bound. However, in some cases a better accuracy can be achieved by employing the DC bound.



	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LB2, ε=10-10	0	0	0.01	0.02	0.06	0.28	1.24	6.06	55.85	263.4
LBDC, ε=10-10	0	0	0.03	0.09	0.43	2.91	12.93	63.2	497.59	2,324.41

n

Fig. 9 Comparing the lower bounds for expropriation



	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LB3, ε=10-4	0.29	4.75	10.84	17.05	49.65	105.45	255.98	652.95	2,487.53	5,715.25
LBDC, ε=10-8	0	0.01	0.02	0.03	0.11	0.47	1.83	8.02	59.97	286.84

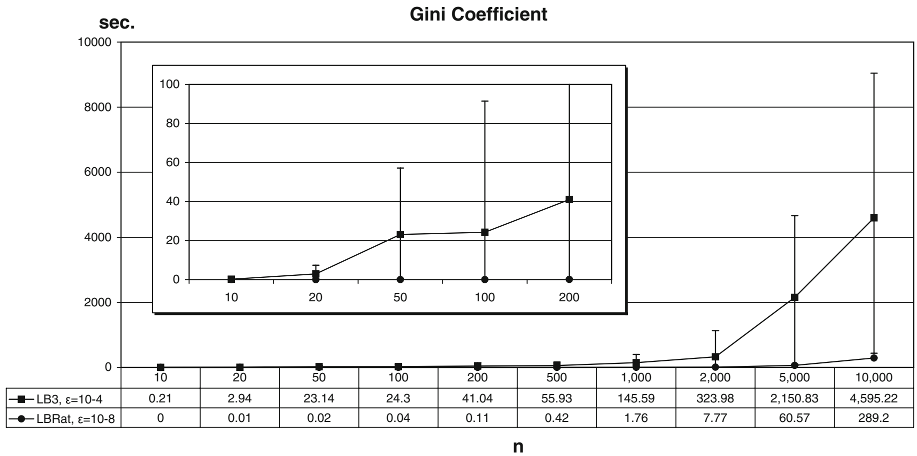
n

Fig. 10 Comparing the lower bounds for mean difference

4.2 Comparing performance of the DC bound on Lorenz problems

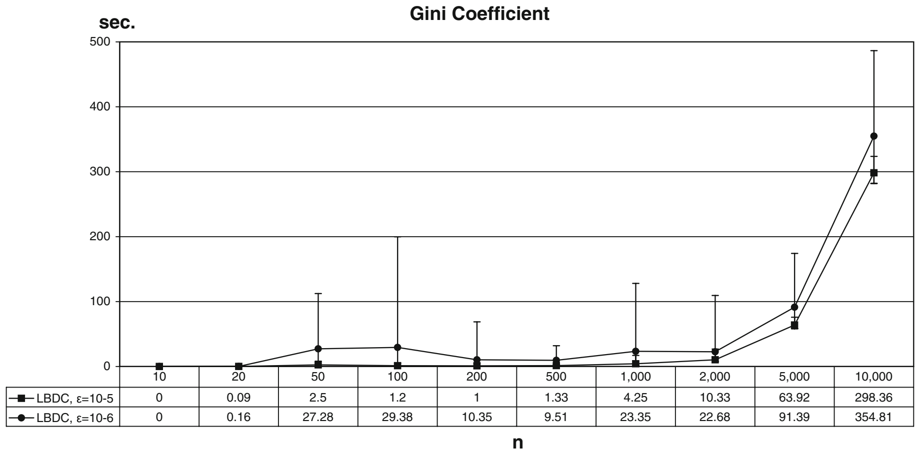
In [5] two problems were solved: (i) Minimizing the mean difference objective, which is the sum of all the differences between pairs of distances divided by n and (ii) minimizing the Gini coefficient which is the ratio between the mean difference objective and the sum of the distances. As is shown in [5], the mean difference objective is an Ordered One Median function with the weights $\lambda_i = \frac{2i-1}{n} - 1$. Therefore, LB_{DC} applies directly to solving this problem.

The Gini coefficient objective $Gini(X)$ is a ratio between an Ordered Median objective and the sum of distances which can be converted to a difference between convex functions divided by a convex function.



	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LB3, ε=10-4	0.21	2.94	23.14	24.3	41.04	55.93	145.59	323.98	2,150.83	4,595.22
LBRat, ε=10-8	0	0.01	0.02	0.04	0.11	0.42	1.76	7.77	60.57	289.2

Fig. 11 Comparing the lower bounds for Gini coefficient with LB_3 and LB_{Rat}



	10	20	50	100	200	500	1,000	2,000	5,000	10,000
LBDC, ε=10-5	0	0.09	2.5	1.2	1	1.33	4.25	10.33	63.92	298.36
LBDC, ε=10-6	0	0.16	27.28	29.38	10.35	9.51	23.35	22.68	91.39	354.81

Fig. 12 Comparing the lower bounds for Gini coefficient with LB_{DC}, $\varepsilon = 10^{-5}$ and LB_{DC}, $\varepsilon = 10^{-6}$

$$Gini(X) = \frac{\sum_{i=1}^{n} \left[\frac{2i-1}{n} - 1\right] d_{(i)}(X)}{\sum_{i=1}^{n} d_i(X)} \tag{7}$$

Therefore, as suggested in [5], LB_{DC} for the numerator divided by the maximum sum of distances to the three vertices is a lower bound for the Gini coefficient objective. However, by the analysis in Sect. 3.3 a better lower bound LB_{Rat} is suggested. We therefore compared the performance of the algorithms in [5] with both lower bounds.

Comparative results are depicted in Figs. 10–12. The mean difference objective was solved much faster to a much better accuracy (using $\epsilon = 10^{-8}$ rather than $\epsilon = 10^{-4}$ used in [5]). The Gini coefficient results where compared with LB_{Rat} using $\epsilon = 10^{-8}$ rather than $\epsilon = 10^{-4}$ used in [5]. For comparison, we also report the results of LB_{DC} for $\epsilon = 10^{-5}, 10^{-6}$. Run times are, again, much faster solving the problems to a better accuracy.

Table 1 Problems' measures

Problem name	$n = 10$			$n = 100$			$n = 1{,}000$			$n = 10{,}000$		
	θ_1	θ_2	θ_2/θ_1	θ_1	θ_2	θ_2/θ_1	θ_1	θ_2	θ_2/θ_1	θ_1	θ_2	θ_2/θ_1
Random	4.8	24	5.03	49.2	1,755	35.7	499	165,844	332.6	4,996	16,725,796	3,348
k-centra	5	5	1	10	10	1	100	100	1	1,000	1,000	1
Anti k-centrum	5	15	3	10	190	19	100	1,900	19	1,000	19,000	19
Truncated mean	6	10	1.67	60	100	1.67	600	1,000	1.67	6,000	10,000	1.67
Median (no Th2)	1	9	9	1	99	99	1	999	999	1	9,999	9,999
Median (using Th2)	1	5	5	1	50	50	1	500	500	1	5,000	5,000
Range	2	20	10	2	200	100	2	2,000	1,000	2	20,000	10,000
Inter-quartile	2	24	12	2	202	101	2	2,000	1,000	2	20,000	10,000
Expropriation	1	15	15	1	159	159	1	1,599	1,599	1	15,999	15,999
Mean difference	5	18	3.6	50	198	3.96	500	1,998	3.996	5,000	19,998	3.9996

Table 2 Comparison between average run times for $n = 10{,}000$ problems

Problem	LB_{DC}		$LB_{1,2,\mathrm{or}3}$	
	ϵ	Time (s)	ϵ	Time (s)
Random	10^{-8}	5,748.85	10^{-5}	17,971.63
k-centra	10^{-8}	286.12	10^{-4}	598.44
Anti k-centrum	10^{-10}	303.69	10^{-10}	268.92
Truncated mean	10^{-8}	291.43	10^{-4}	6,981.73
Median (no Th2)	10^{-8}	9,310.75	10^{-4}	576.82
Median (using Th2)	10^{-8}	18,513.84	10^{-8}	589.30
Range	10^{-8}	1,973.02	10^{-6}	417.48
Inter-quartile	10^{-8}	10,440.21	10^{-6}	677.27
Expropriation	10^{-10}	2,324.41	10^{-10}	263.40
Mean difference	10^{-8}	286.84	10^{-4}	5,715.25

5 Conclusions

We showed that every Ordered Median function can be expressed as a difference of two convex functions (DC). Therefore, general approaches for DC optimization can be applied to solving Ordered Median Problems.

This approach is illustrated for the solution of the Ordered One Median Problem in the plane. A general lower bound based on the DC decomposition is constructed for the values of the objective function in a triangle. The BTST approach [4] is then applied to optimally solve Ordered One Median Problems in the plane. Computational experiments demonstrated that the new lower bound is more effective than the lower bounds suggested in [3] for most tested problems. The new lower bound is especially effective when the ratio θ_2/θ_1 (see Eq. 6 and Table 2) is low.

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