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Stampacchia generalized vector quasi-equilibrium problem with set-valued mapping

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Abstract In this paper, the solvability of Stampacchia generalized vector quasi-equilibrium problem (in short, GVQEP) with set-valued mapping is studied. By using continuous selection theorem and fixed point theorems, some existence theorems for (GVQEP) are obtained without any monotonicity assumption. These theorems unify and improve some results in the recent references.

Keywords Generalized vector quasi-equilibrium problem · Set-valued mapping · Continuous selection · Fixed point · C-convexity

1 Introduction

Let E, Z be two topological vector spaces, $X \subseteq E$ a nonempty subset, $C \subseteq Z$ a convex cone with apex at the origin, and with nonempty interior, int$C \neq \emptyset$. Let $L(E, Z)$ denote the space of all continuous linear mappings from E into Z, and denote by $\langle \ell, x \rangle$ the value of $\ell \in L(E, Z)$ at $x \in E$. Let $T\colon X \to L(E, Z)$ be a given mapping. The weak vector variational inequality (in short, WVVI) consists in finding $\bar{x} \in X$ such that

$$\text{(WVVI)} \qquad \langle T\bar{x}, y - \bar{x} \rangle \notin -\text{int}C, \quad \forall y \in X.$$

The vector variational inequality was first introduced and studied by Giannessi [16] in the setting of finite dimensional Euclidean spaces. Later on, vector variational inequalities and vector complementarity problems in infinite dimensional spaces were studied by many authors (see Refs. [7,8,13,20,21,24,25,27,31], and references therein). Recently, Fang and Huang [11] considered the following strong vector variational inequality (in short, VVI): find $\bar{x} \in X$ such that

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$$\text{(VVI)} \qquad \langle T\bar{x}, y - \bar{x}\rangle \notin -C\backslash\{0\}, \quad \forall y \in X.$$

As a significant generalization of variational inequalities and complementarity problems, the equilibrium problem was proposed and studied by Blum and Oettli [5]. Recently, the equilibrium problem was extensively generalized to the vector mapping (see Refs. [1,2,4, 9,15,17,18,23,26], and references therein). Let $f\colon X \times X \to Z$ be a given mapping. The weak vector equilibrium problem (in short, WVEP) consists in finding $\bar{x} \in X$ such that

$$\text{(WVEP)} \qquad f(\bar{x}, y) \notin -\text{int}C, \quad \forall y \in X.$$

Let $K\colon X \to 2^X$ be a set-valued mapping. Lately, there have been increasing interests (for example, Fu [14], Wang et al. [29]) in considering the following Stampacchia vector quasi-equilibrium problem (in short, VQEP): find $\bar{x} \in X$ such that $\bar{x} \in K(\bar{x})$ and

$$\text{(VQEP)} \qquad f(\bar{x}, y) \notin -C\backslash\{0\}, \quad \forall y \in K(\bar{x}).$$

To the best of our knowledge, most of the studies are for (WVVI) or (WVEP), but just a few for (VVI) or (VQEP) in the literature.

In this paper, we will consider the following more general vector quasi-equilibrium problem.

Let Y be a nonempty subset of a Hausdorff topological vector space and $T\colon X \to 2^Y$, $f\colon X \times Y \times X \to 2^Z$ be two set-valued mappings. Moreover, let $C\colon X \to 2^Z$ be a set-valued mapping such that, for all $x \in X$, $C(x)$ is a nonempty convex cone with apex at the origin. We consider the following Stampacchia generalized vector quasi-equilibrium problem (in short,GVQEP): find $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in K(\bar{x})$ and

$$\text{(GVQEP)} \quad f(\bar{x}, \bar{y}, x) \nsubseteq -C(\bar{x})\backslash\{0\}, \quad \forall x \in K(\bar{x}).$$

The main purpose of this paper is to study the solvability of (GVQEP). Motivated by Refs. [19,28], we obtained some existence theorems for (GVQEP) without any monotonicity assumption by using continuous selection theorem and fixed point theorems.

2 Preliminaries

In this section, we shall recall some definitions and lemmas used in the sequel.

Definition 2.1 [3] Let X, Y be two topological spaces and $T\colon X \to 2^Y$ be a set-valued mapping.

(i) T is said to be upper semi-continuous (in short, $u.s.c.$) at $x \in X$ if, for any open set V containing $T(x)$, there exists an open set U containing x such that $T(t) \subseteq V$ for all $t \in U$; T is said to be $u.s.c.$ if it is $u.s.c.$ at every $x \in X$.
(ii) T is said to be lower semi-continuous (in short, $l.s.c.$) at $x \in X$ if, for any open set V with $T(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $T(t) \cap V \neq \emptyset$ for all $t \in U$; T is said to be l.s.c. if it is $l.s.c.$ at every $x \in X$.
(iii) T is said to be continuous if it is both $u.s.c.$ and $l.s.c.$ at the same time.

Lemma 2.1 [3] *Let X, Y be two topological spaces and $T\colon X \to 2^Y$ be a set-valued mapping.*

(i) *If T is $u.s.c.$ with closed values, then T is closed;*
(ii) *if T is closed and Y is compact, then T is $u.s.c.$;*

Definition 2.2 [30] Let X, Y be two topological spaces and $T: X \to 2^Y$ be a set-valued mapping. For each $y \in Y$, $T^{-1}(y) := \{x \in X: y \in T(x)\}$ is said to be the lower sections of T.

Let A be a subset of a topological vector space, we denote by CoA the convex hull of A.

Definition 2.3 Let X, Z be two topological vector spaces and $C: X \to 2^Z$ be a set-valued mapping such that, for every $x \in X$, $C(x)$ is a nonempty convex cone with apex at the origin. Let $F: X \times X \to 2^Z$ be a given set-valued mapping and $x \in X$ be a given point.

(i) $F(x, y)$ is called C-convex in $y \in X$ if, for all $y_1, y_2 \in X$ and $\lambda \in (0, 1)$, one has

$$F(x, \lambda y_1 + (1-\lambda)y_2) \subseteq \lambda F(x, y_1) + (1-\lambda)F(x, y_2) - C(x);$$

(ii) $F(x, y)$ is called properly C-quasiconvex in $y \in X$ if, for all $y_1, y_2 \in X$, $\lambda \in (0, 1)$ and for any $z \in F(x, \lambda y_1 + (1-\lambda)y_2)$, there exist some i and some $z_i \in F(x, y_i)$, such that $z \in z_i - C(x)$.

Remark 2.1 (1) If F is a single-valued mapping and $C(x) \equiv C$ (a fixed convex cone), $\forall x \in X$, the above properly C-quasiconvexity reduces to the conventional properly convexity introduced by Ferro ([12]); (2) A mapping may be C-convex and not properly C-quasiconvex, and conversely (see Ref. [12]); (3) By induction, if, for any fixed $x \in X$, $F(x, y)$ is C-convex in $y \in X$, then, for any finite set $\wedge = \{y_1, y_2, \ldots, y_n\} \subseteq X$ and for any $y = \sum_{i=1}^n t_i y_i \in Co\wedge$, one has

$$F(x, y) \subseteq \sum_{i=1}^n t_i F(x, y_i) - C(x);$$

Similarly, if, for any fixed $x \in X$, $F(x, y)$ is properly C-quasiconvex in $y \in X$, then, for any finite set $\wedge = \{y_1, y_2, \ldots, y_n\} \subseteq X$ and for any $y = \sum_{i=1}^n t_i y_i \in Co\wedge$, for any $z \in F(x, y)$, there must exist some i and some $z_i \in F(x, y_i)$ such that

$$z \in z_i - C(x).$$

The following lemmas are our important tools.

Lemma 2.2 [32] *Let X be a topological space and Y be a convex set of a topological vector space. Suppose a mapping $G: X \to 2^Y$ has open lower sections. Then, the mapping $F: X \to 2^Y$, defined by $F(x) = CoG(x)$ for all $x \in X$, has open lower sections.*

Lemma 2.3 [32] *Let X, Y be two topological spaces, and let $G: X \to 2^Y$ and $K: X \to 2^Y$ be two set-valued mappings with open lower sections. Then, the mapping $\theta: X \to 2^Y$, defined by $\theta(x) = G(x) \cap K(x)$ for all $x \in X$, has open lower sections.*

Theorem 2.1 [32, Continuous Selection Theorem] *Let X be a paracompact Hausdorff topological space and Y be a topological vector space. Suppose that $F: X \to 2^Y$ is a set-valued mapping with nonempty convex values and open lower sections. Then there exists a continuous mapping $f: X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.*

Theorem 2.2 [6, Browder Fixed-Point Theorem] *Let X be a nonempty compact convex subset of a Hausdorff topological vector space. Suppose that $H: X \to 2^X$ is a set-valued mapping with nonempty convex values and open lower sections. Then, H has a fixed point.*

Theorem 2.3 [10, Eilenberg–Montgomery Fixed Point Theorem] *Let X be a compact convex subset of a locally convex Hausdorff topological vector space and let $T: X \to 2^X$ be an upper semi-continuous set-valued mapping with nonempty closed acyclic values. Then, T has a fixed point in X.*

3 Main results

In this section, we shall present some existence theorems for GVQEP under some suitable assumptions of continuity and convexity by using continuous selection theorem and fixed point theorems.

Throughout this section, unless otherwise specified, we always assume that Z is a Hausdorff topological vector space, X, Y are two nonempty compact convex sets of two locally convex Hausdorff topological vector spaces, respectively.

Theorem 3.1 *Suppose that*

(i) $K: X \to 2^X$ *is a set-valued mapping with nonempty closed convex values and open lower sections;*
(ii) $T: X \to 2^Y$ *is u.s.c. with nonempty closed acyclic values;*
(iii) $C: X \to 2^Z$ *is a set-valued mapping such that, for each* $x \in X, C(x)$ *is a convex pointed cone with apex at the origin;*
(iv) $f: X \times Y \times X \to 2^Z$ *is a set-valued mapping with nonempty values, which satisfies the following conditions:*

   (a) *For all* $u \in X$, *the set* $\{(x, y) \in X \times Y: f(x, y, u) \subseteq -C(x)\backslash\{0\}\}$ *is open;*
   (b) *For all* $x \in X, y \in Y, f(x, y, x) \subseteq C(x)$*;*
   (c) *For all* $x \in X, y \in Y$, *the mapping* $f(x, y, u)$ *is* C*-convex or properly* C*-quasi-convex in* u.

Then, there exist $\bar{x} \in X$ *and* $\bar{y} \in T(\bar{x})$ *such that* $\bar{x} \in K(\bar{x})$ *and*

$$f(\bar{x}, \bar{y}, x) \nsubseteq -C(\bar{x})\backslash\{0\}, \quad \forall x \in K(\bar{x}).$$

Proof Define a set-valued mapping $P: X \times Y \to 2^X$ by

$$P(x, y) = \{u \in X: f(x, y, u) \subseteq -C(x)\backslash\{0\}\}, \quad \forall(x, y) \in X \times Y. \tag{3.1}$$

Then, the theorem will be proven if we can show that there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in K(\bar{x})$ and

$$K(\bar{x}) \cap P(\bar{x}, \bar{y}) = \emptyset. \tag{3.2}$$

Consider the set-valued mapping $G: X \times Y \to 2^X$ defined by

$$G(x, y) = K(x) \cap CoP(x, y), \quad \forall x \in X, y \in Y.$$

Now, we shall show that there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in K(\bar{x})$ and

$$G(\bar{x}, \bar{y}) = K(\bar{x}) \cap CoP(\bar{x}, \bar{y}) = \emptyset. \tag{3.3}$$

In particular,

$$K(\bar{x}) \cap P(\bar{x}, \bar{y}) = \emptyset.$$

i.e., (3.2) holds.

Indeed, let $U = \{(x, y) \in X \times Y: G(x, y) \neq \emptyset\}$.

(1) If $U = \emptyset$, then $\forall x \in X, y \in Y, \ G(x, y) = \emptyset$. Since X is compact and convex, and $K: X \to 2^X$ has nonempty convex values and open lower sections, it follows from the Browder Fixed Point Theorem that there exists a fixed point $\bar{x} \in K(\bar{x})$. Also, from the assumption (ii), we have $T(\bar{x}) \neq \emptyset$. By picking $\bar{y} \in T(\bar{x})$, then, we have $G(\bar{x}, \bar{y}) = \emptyset$. Hence, the assertion (3.3) holds in this particular case.

(2) If $U \neq \emptyset$.

By hypothesis (a), for all $u \in X$, the set

$$\{(x, y) \in X \times Y\colon f(x, y, u) \subseteq -C(x)\backslash\{0\}\} \tag{3.4}$$

is open, or equivalently the set $P^{-1}(u)$ is open for all $u \in X$. Hence, P has open lower sections. It follows from Lemma 2.2 that CoP has open lower sections. In addition, K has open lower sections by hypothesis (i), we can apply Lemma 2.3 to obtain that G has open lower sections. Notice that $U = \bigcup_{v\in X} G^{-1}(v)$, hence, U is open. Define a set-valued mapping $H\colon X \times Y \to 2^X$ by

$$H(x, y) = \begin{cases} G(x, y), & \text{if } (x, y) \in U, \\ K(x), & \text{otherwise.} \end{cases}$$

Then, for each $v \in X$, we have $H^{-1}(v) = G^{-1}(v) \cup (K^{-1}(v) \times Y)$, which is also open, i.e., H has open lower sections. Moreover, for all $x \in X$, $y \in Y$, $H(x, y)$ is nonempty and convex, it follows from the Continuous Selection Theorem that there exists a continuous selector $h\colon X \times Y \to X$ of H. Now, we consider the set-valued mapping $M\colon X \times Y \to 2^{X\times Y}$ given by

$$M(x, y)\colon= (h(x, y), T(x)), \quad \forall x \in X, y \in Y.$$

Then, by the assumptions, M has nonempty closed acyclic values. Further, M is closed since h is continuous and T is $u.s.c.$ with nonempty closed values (see Lemma 2.1(i)). Notice that $X \times Y$ is compact, it follows from Lemma 2.1(ii) that M is $u.s.c.$. Hence, by the Eilenberg–Montgomery Fixed Point Theorem, there exists a fixed point $(\bar{x}, \bar{y}) \in M(\bar{x}, \bar{y})$. Moreover, $(\bar{x}, \bar{y}) \notin U$. Suppose to the contrary that $(\bar{x}, \bar{y}) \in U$. Then

$$\bar{x} = h(\bar{x}, \bar{y}) \in H(\bar{x}, \bar{y}) = G(\bar{x}, \bar{y}) \subseteq CoP(\bar{x}, \bar{y})$$

i.e.,

$$\bar{x} \in CoP(\bar{x}, \bar{y}).$$

Hence, there exist $x_1, x_2, \ldots, x_n$ in X with $x_i \in P(\bar{x}, \bar{y})$, $i = 1, 2, \ldots, n$, and $t_i \geq 0$ with $\sum_{i=1}^n t_i = 1$ such that $\bar{x} = \sum_{i=1}^n t_i x_i$. Thus,

$$f(\bar{x}, \bar{y}, x_i) \subseteq -C(\bar{x})\backslash\{0\}, \quad i = 1, 2, \ldots, n. \tag{3.5}$$

If $f(\bar{x}, \bar{y}, \cdot)$ is C-convex, then

$$\begin{aligned} f(\bar{x}, \bar{y}, \bar{x}) &\subseteq \sum_{i=1}^n t_i f(\bar{x}, \bar{y}, x_i) - C(\bar{x}) \\ &\subseteq -C(\bar{x})\backslash\{0\} - C(\bar{x}) \\ &\subseteq -C(\bar{x})\backslash\{0\}. \end{aligned} \tag{3.6}$$

If $f(\bar{x}, \bar{y}, \cdot)$ is properly C-quasiconvex, then, for any $z \in f(\bar{x}, \bar{y}, \bar{x})$, there exist some i and some $z_i \in f(\bar{x}, \bar{y}, x_i)$ such that

$$\begin{aligned} z &\in z_i - C(\bar{x}) \\ &\subseteq f(\bar{x}, \bar{y}, x_i) - C(\bar{x}) \\ &\subseteq -C(\bar{x})\backslash\{0\} - C(\bar{x}) \\ &\subseteq -C(\bar{x})\backslash\{0\}. \end{aligned} \tag{3.7}$$

By the arbitrariness of z, we also have

$$f(\bar{x}, \bar{y}, \bar{x}) \subseteq -C(\bar{x})\backslash\{0\}. \tag{3.8}$$

Moreover, $f(\bar{x}, \bar{y}, \bar{x}) \subseteq C(\bar{x})$ by the hypothesis (b). Hence,

$$C(\bar{x}) \cap (-C(\bar{x})\backslash\{0\}) \neq \emptyset. \tag{3.9}$$

On the other hand, from the assumption (iii), we have $C(\bar{x})$ is a pointed cone with apex at the origin. Hence,

$$C(\bar{x}) \cap (-C(\bar{x})\backslash\{0\}) = \emptyset. \tag{3.10}$$

(3.9) contradicts with (3.10). Thus $(\bar{x}, \bar{y}) \notin U$. Therefore, $\bar{x} \in K(\bar{x})$, $\bar{y} \in T(\bar{x})$ and $G(\bar{x}, \bar{y}) = \emptyset$. So, the assertion (3.3) also holds in this case.

This completes the proof.

From Theorem 3.1, we can obtain the following result.

Corollary 3.1 *Let $C \subseteq Z$ be a nonempty convex pointed cone with apex at the origin. Suppose that*

(i) *$K: X \to 2^X$ is a set-valued mapping with nonempty closed convex values and open lower sections;*
(ii) *$T: X \to 2^Y$ is u.s.c. with nonempty closed acyclic values;*
(iii) *$f: X \times Y \times X \to Z$ is a single-valued mapping, which satisfies the following conditions:*

   (a) *For all $u \in X$, the set $\{(x, y) \in X \times Y: f(x, y, u) \in -C\backslash\{0\}\}$ is open;*
   (b) *For all $x \in X$, $y \in Y$, $f(x, y, x) \in C$;*
   (c) *For all $x \in X$, $y \in Y$, the mapping $f(x, y, u)$ is C-convex or properly C-quasiconvex in u.*

Then, there exist $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in K(\bar{x})$ and

$$f(\bar{x}, \bar{y}, x) \notin -C\backslash\{0\}, \quad \forall x \in K(\bar{x}).$$

Proof In Theorem 3.1, let

$$C(x) \equiv C, \quad \forall\, x \in X.$$

Then, Theorem 3.1 yields the conclusion.

Theorem 3.2 *Suppose that*

(i) *$K: X \to 2^X$ is a set-valued mapping with nonempty closed convex values and open lower sections;*
(ii) *$C: X \to 2^Z$ is a set-valued mapping such that, for each $x \in X$, $C(x)$ is a convex pointed cone with apex at the origin;*
(iii) *$F: X \times X \to 2^Z$ is a set-valued mapping with nonempty values, which satisfies the following conditions:*

   (a) *For all $y \in X$, the set $\{x \in X: F(x, y) \subseteq -C(x)\backslash\{0\}\}$ is open;*
   (b) *For all $x \in X$, $F(x, x) \subseteq C(x)$;*
   (c) *For all $x \in X$, the mapping $F(x, y)$ is C-convex or properly C-quasiconvex in y.*

Then, there exists $\bar{x} \in X$ such that $\bar{x} \in K(\bar{x})$ and

$$F(\bar{x}, x) \nsubseteq -C(\bar{x})\backslash\{0\}, \quad \forall x \in K(\bar{x}).$$

Proof In Theorem 3.1, let $Y = \{\bar{y}\}$ be a singleton set and let

$$T(x) \equiv \{\bar{y}\}, \quad \text{for all } x \in X,$$
$$f(x, \bar{y}, y) = F(x, y), \quad \text{for all } (x, y) \in X \times X.$$

Then, Theorem 3.1 yields the conclusion.

From Theorem 3.2, we can obtain the following result.

Corollary 3.2 *Let $C \subseteq Z$ be a nonempty convex pointed cone with apex at the origin. Suppose that*

(i) *$K: X \to 2^X$ is a set-valued mapping with nonempty closed convex values and open lower sections;*
(ii) *$F: X \times X \to 2^Z$ is a set-valued mapping with nonempty values, which satisfies the following conditions:*
   (a) *For all $y \in X$, the set $\{x \in X: F(x, y) \subseteq -C\backslash\{0\}\}$ is open;*
   (b) *For all $x \in X$, $F(x, x) \subseteq C$;*
   (c) *For all $x \in X$, the mapping $F(x, y)$ is C-convex or properly C-quasiconvex in y.*

Then, there exists $\bar{x} \in X$ such that $\bar{x} \in K(\bar{x})$ and

$$F(\bar{x}, x) \not\subseteq -C\backslash\{0\}, \quad \forall x \in K(\bar{x}).$$

Theorem 3.3 *Let X be a nonempty compact convex set in a locally convex Hausdorff topological vector space E. Let C be a nonempty convex cone with apex at the origin in a Hausdorff topological vector space Z. Suppose that*

(i) *$K: X \to 2^X$ is a set-valued mapping with nonempty closed convex values and open lower sections;*
(ii) *$T: X \to L(E, Z)$ is a single-valued mapping such that, for all $y \in X$, the set $\{x \in X: \langle Tx, y - x\rangle \in -C\backslash\{0\}\}$ is open.*

Then, there exists $\bar{x} \in X$ such that $\bar{x} \in K(\bar{x})$ and

$$\langle T\bar{x}, x - \bar{x}\rangle \notin -C\backslash\{0\}, \ \forall x \in K(\bar{x}).$$

Proof The proof is similar as that of Theorem 3.1 and so is omitted.

Remark 3.1 In the above Theorem 3.3, the cone C is not necessary pointed.

Corollary 3.3 *Let X, E, C, Z be the same as in Theorem 3.3. Suppose that $T: X \to L(E, Z)$ is a single-valued mapping such that, for all $y \in X$, the set $\{x \in X: \langle Tx, y - x\rangle \in -C\backslash\{0\}\}$ is open. Then, there exists $\bar{x} \in X$ such that*

$$\langle T\bar{x}, x - \bar{x}\rangle \notin -C\backslash\{0\}, \quad \forall x \in X.$$

Remark 3.2 If X, Z are two real Banach spaces, then, the above corollary 3.3 is the main result: Theorem 1 in Fang and Huang[11].

The following theorem is a generalization of the main result—Theorem 2 ([28], P. 757).

Theorem 3.4 *Suppose that*

(i) *$K: X \to 2^X$ is a set-valued mapping with nonempty closed convex values and open lower sections;*

(ii) $T\colon X \to 2^Y$ *is* $u.s.c.$ *with nonempty closed acyclic values;*
(iii) $g\colon X \times Y \times X \to R \cup \{\pm\infty\}$ *is a function such that* $g(x, y, u)$ *is* $l.s.c.$ *in* x, y *and* $\gamma-$*diagonally quasiconcave in* u (*see* [28, p. 754]).

Then, there exist $\bar{x} \in X$ *and* $\bar{y} \in T(\bar{x})$ *such that* $\bar{x} \in K(\bar{x})$ *and*

$$g(\bar{x}, \bar{y}, x) \leq \gamma, \quad \forall x \in K(\bar{x}).$$

Proof Let $Z = R \cup \{\pm\infty\}$, $C = R_+ = [0, +\infty)$. Let

$$C(x) = C = R_+, \quad \forall x \in X;$$

and

$$f(x, y, u) = \gamma - g(x, y, u), \quad \forall\, x, u \in X,\ y \in Y.$$

Obviously, $f(x, y, u)$ is $u.s.c.$ in x, y since $g(x, y, u)$ is $l.s.c.$ in x, y. Hence, for any fixed $u \in X$, the set $\{(x, y) \in X \times Y\colon f(x, y, u) \in -C\backslash\{0\}\} = \{(x, y) \in X \times Y\colon f(x, y, u) < 0\}$ is open. The remaining arguments are the same as that of Theorem 3.1 except for replacing $'\subseteq'$ with $'\in'$ in (3.1) and (3.4) and proving again that $(\bar{x}, \bar{y}) \notin U$.

Indeed, suppose to the contrary that $(\bar{x}, \bar{y}) \in U$. Then

$$\bar{x} = h(\bar{x}, \bar{y}) \in H(\bar{x}, \bar{y}) = G(\bar{x}, \bar{y}) \subseteq CoP(\bar{x}, \bar{y})$$

i.e.,

$$\bar{x} \in CoP(\bar{x}, \bar{y}).$$

Hence, there exist $x_1, x_2, \ldots, x_n$ in X with $x_i \in P(\bar{x}, \bar{y}),\ i = 1, 2, \ldots, n$, and $t_i \geq 0$ with $\sum_{i=1}^n t_i = 1$ such that $\bar{x} = \sum_{i=1}^n t_i x_i$. Thus,

$$f(\bar{x}, \bar{y}, x_i) \in -C(\bar{x})\backslash\{0\} = (-\infty, 0), \quad i = 1, 2, \ldots, n.$$

i.e.,

$$f(\bar{x}, \bar{y}, x_i) < 0, \quad i = 1, 2, \ldots, n. \tag{3.11}$$

On the other hand, since $g(\bar{x}, \bar{y}, u)$ is γ—diagonally quasiconcave in u, it follows that

$$\min_i g(\bar{x}, \bar{y}, x_i) \leq \gamma.$$

Thus, there must exists some i_0 such that

$$g(\bar{x}, \bar{y}, x_{i_0}) \leq \gamma.$$

Therefore,

$$f(\bar{x}, \bar{y}, x_{i_0}) = \gamma - g(\bar{x}, \bar{y}, x_{i_0}) \geq 0, \tag{3.12}$$

which contradicts with (3.11). Hence, $(\bar{x}, \bar{y}) \notin U$.

This completes the proof.

Remark 3.3 In the main result−Theorem 2 in Tian ([28], p. 757), except all the assumptions in the above Theorem 3.4, the condition " K is $u.s.c.$" is necessary. Hence, the above theorem 3.4 is a generalization of it.

Let $C \subseteq Z$ be a convex cone, and let Z^* be the topological dual space of Z. Denote by C^* the dual cone of C, i.e.,

$$C^* = \{\ell \in Z^* : \langle \ell, x \rangle \geq 0, \quad \forall x \in C\}.$$

and denote the quasi-interior of C^* by $C^\#$ (see Ref. [22]), i.e.,

$$C^\# = \{\ell \in Z^* : \langle \ell, x \rangle > 0, \quad \forall x \in C \backslash \{0\}\}.$$

Clearly, $C^\#$ is a convex subset of Z^*.

$C^\# \neq \emptyset$ if and only if C has a base (see Ref. [22]). If C is a closed convex pointed cone of a real separable normed space, by the Krein-Rutman Theorem, then $C^\# \neq \emptyset$ (see Ref. [22]).

Theorem 3.5 *Suppose that*

(i) $K: X \to 2^X$ *is a set-valued mapping with nonempty closed convex values and open lower sections;*
(ii) $T: X \to 2^Y$ *is u.s.c. with nonempty closed acyclic values;*
(iii) $C: X \to 2^Z$ *is a set-valued mapping such that, for all* $x \in X$, $C(x)$ *is a convex cone, and* $C^\#: X \to 2^{Z^*}$ *has continuous selection;*
(iv) *the bilinear form* $\langle \cdot, \cdot \rangle$ *is continuous on* $Z^* \times Z$*;*
(v) $f: X \times Y \times X \to 2^Z$ *has nonempty compact values, which satisfies the following conditions:*
    (a) *For all* $x \in X,\ y \in Y,\ f(x, y, x) \subseteq C(x)$*;*
    (b) *For all* $u \in X,\ f(x, y, u)$ *is u.s.c. in* x, y*;*
    (c) *For all* $x \in X, y \in Y,\ f(x, y, u)$ *is C-convex or properly C-quasiconvex in* u.

Then, there exist $\bar{x} \in X$ *and* $\bar{y} \in T(\bar{x})$ *such that* $\bar{x} \in K(\bar{x})$ *and*

$$f(\bar{x}, \bar{y}, x) \nsubseteq -C(\bar{x}) \backslash \{0\}, \quad \forall x \in K(\bar{x}).$$

Proof Since $C^\#$ has continuous selection, i.e., there exists a continuous mapping $\xi: X \to Z^*$ such that

$$\xi(x) \in C^\#(x), \quad \forall x \in X.$$

Let

$$\xi(x, z) = \xi(x)(z) = \langle \xi(x), z \rangle, \quad \forall\, x \in X, z \in Z;$$

and

$$\varphi(x, y, u) = \min \xi(x)[-f(x, y, u)] = -\max_{z \in f(x,y,u)} \xi(x, z), \quad \forall\, x, u \in X, z \in Z.$$

From the hypothesis, the function $\xi(\cdot, \cdot)$ is continuous. In addition, $f(x, y, u)$ is $u.s.c.$ in x, y and f has nonempty compact values. It follows from Proposition 21 [3, p. 119] that $\varphi(x, y, u)$ is $l.s.c.$ in x, y. Hence, if we can show that $\varphi(x, y, u)$ is 0−diagonally quasiconcave in u, then, it follows from Theorem 3.4 that there exists $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in K(\bar{x})$ and

$$\varphi(\bar{x}, \bar{y}, x) \leq 0, \quad \forall x \in K(\bar{x}).$$

i.e.,

$$\max_{z \in f(\bar{x}, \bar{y}, x)} \xi(\bar{x}, z) \geq 0, \quad \forall\, x \in K(\bar{x}). \tag{3.13}$$

We assert that

$$f(\bar{x}, \bar{y}, x) \not\subseteq -C(\bar{x})\backslash\{0\}, \quad \forall\, x \in K(\bar{x}). \tag{3.14}$$

Indeed, suppose to the contrary that there exists some $x_0 \in K(\bar{x})$ such that

$$f(\bar{x}, \bar{y}, x_0) \subseteq -C(\bar{x})\backslash\{0\}$$

Since $\xi(\bar{x}) \in C^{\#}(\bar{x})$, it follows that, for all $z \in f(\bar{x}, \bar{y}, x_0)$, $\xi(\bar{x}, z) < 0$. Hence,

$$\max_{z\in f(\bar{x},\bar{y},x_0)}\xi(\bar{x}, z) < 0.$$

which contradicts with (3.13). Therefore, (3.14) holds, which implies that the theorem is proven.

So, it remains to prove that $\varphi(x, y, u)$ is 0−diagonally quasiconcave in u. Suppose to the contrary that there exists a finite set $\wedge = \{x_1, x_2, \ldots, x_n\}$ and $t_i \geq 0$ with $\sum_{i=1}^{n} t_i = 1$ such that $x = \sum_{i=1}^{n} t_i x_i \in Co\wedge$ and $\min\limits_i \varphi(x, y, x_i) > 0$. It follows that

$$\max_{z\in f(x,y,x_i)}\xi(x, z) < 0, \quad i = 1, 2, \ldots, n.$$

Then, $\forall\, i$, we have

$$\xi(x, z_i) < 0, \quad \forall\, z_i \in f(x, y, x_i).$$

If $f(x, y, u)$ is C-convex in u, then

$$f(x, y, x) \subseteq \sum_{i=1}^{n} t_i f(x, y, x_i) - C(x).$$

Thus, for any $z \in f(x, y, x)$, there exists $z_i \in f(x, y, x_i)$, $i = 1, 2, \ldots, n$ and $c \in C(x)$ such that

$$z = t_1 z_1 + t_2 z_2 + \cdots + t_n z_n - c.$$

Notice that $\xi(x, z) = \xi(x)(z)$ is linear in z since $\xi(x) \in C^{\#}(x)$, hence,

$$\xi(x, z) = \sum_{i=1}^{n} t_i \xi(x, z_i) - \xi(x, c) < 0.$$

By the arbitrariness of z, we have

$$\min \xi(x) f(x, y, x) < 0. \tag{3.15}$$

If $f(x, y, u)$ is properly C-quasiconvex in u, then, for any $z \in f(x, y, x)$, there exists some i and some $z_i \in f(x, y, x_i)$ and $c \in C(x)$ such that $z = z_i - c$, Hence,

$$\xi(x, z) = \xi(x, z_i) - \xi(x, c) < 0.$$

By the arbitrariness of z, we also have

$$\min \xi(x) f(x, y, x) < 0. \tag{3.16}$$

On the other hand, since $f(x, y, x) \subseteq C(x)$ and $\xi(x) \in C^{\#}(x)$, it follows that, for any $z \in f(x, y, x)$, $\xi(x, z) \geq 0$. Hence,

$$\min \xi(x) f(x, y, x) \geq 0. \tag{3.17}$$

Both (3.15) and (3.16) contradicts with (3.17). Thus, $\varphi(x, y, u)$ is $0-$diagonally quasiconcave in u.

This completes the proof.

Remark 3.4 If $\forall x \in X$, $C^{\#}(x) \neq \emptyset$ and $\forall z^* \in Z^*$, the set $\{x \in X\colon z^* \in C^{\#}(x)\}$ is open in X, then, it follows from the Continuous Selection Theorem that the mapping $C^{\#}$ has a continuous selector.

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