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Vector optimization and variational-like inequalities

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Abstract In this paper, some properties of pseudoinvex functions are obtained. We study the equivalence between different solutions of the vector variational-like inequality problem. Some relations between vector variational-like inequalities and vector optimization problems for non-differentiable functions under generalized monotonicity are established.

Keywords Vector variational-like inequalities · Pseudoinvex functions · Pseudomonotone mappings · Vector optimization problems

Mathematics Subject Classification (2000) 26B25 · 49J52 · 90C30 · 49J40

1 Introduction

Weir and Mond [23] and Mohan and Neogy [21] have studied some basic properties of the preinvex functions and their role in optimization. They showed that under certain condition an invex function defined on an invex set is preinvex and a quasiinvex function defined on an invex set is prequasiinvex. Jabarootain and Zafarani [16] have devoted to the study of relationships between several kinds of generalized invexity of locally Lipschitz functions and generalized monotonicity of corresponding Clarke's subdifferentials. In particular, some necessary and sufficient conditions of being a locally Lipschitz function invex, quasiinvex or pseudoinvex have given in terms of momotonicity, quasimonotonicity and pseudomonotonicity of its Clarke's subdifferential, respectively. The concept of vector variational inequality was first introduced and studied by Giannessi [11]. Recently, Giannessi [12] has shown that the

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equivalence between efficient solutions of differentiable, convex optimization problem and solutions of a variational inequality of Minty type. He also proved the equivalence between weak efficient solutions of a differentiable convex optimization problem and solutions of a variational inequality of weak-Minty type.

By using a variational inequality, Yang et al. [28] obtained some results of existence of solution for non-smooth invex problems, which are generalizations of those obtained in [11] for differentiable convex problems. Yang et al. in [30] have pointed out that the concept of invexity plays exactly the same role in variational-like inequality problems as the classical convexity plays in variational inequality problems. Chen and Yang [4] and Chen [3] studied different kinds of vector variational inequality problems. Now the study of vector variational inequalities has become an important research for vector optimization problems. For details we refer to Daniilidis and Hadjisavvas [7] and [8], Giannessi [12,13] Giannessi and Maugeri [14], Giannessi and Maugeri [15], Konnov and Yao [18], Yang [24], Yang and Goh [26]. Yang et al. [29] have obtained some existence result of vector variational inequality problem and for differentiable non-convex vector optimization problems. In particular, they established some relations between a Minty vector variational inequality problems and vector optimization problems under pseudoconvexity or pseudomonotonicity of function or its differential, respectively. Lee and Lee [19] have shown vector variational inequality for non-differentiable convex vector optimization problems. Vector variational-like inequality problems and vector optimization problems have been studied by Mishra and Wang [20], Fang and Huang [10], Chiang [5], Santos, Medar and Lizana [22] and [6]. Vector variational-like inequality and non-smooth vector optimization problems have been studied by Jabarootian and Zafarani [17] in the case of invex functions. Yang and Yang [31] devoted to the study of vector variational-like inequality models in the case of differentiable functions. They obtained some important properties of differentiable pseudoinvex functions and established relations between vector variational-like inequality problems and vector optimization problems.

In this article, we obtain some important properties of pseudoinvex functions and study the equivalence between different solutions of the variational-like inequality problems. Furthermore, the relations between vector variational-like inequalities and vector optimization problems for non-differentiable functions are established. The paper is organized as follows. In Sects. 1, 2, some basic definitions and preliminary results are given. Some properties of pseudoinvex and quasiinvex functions are obtained in Sect. 3. The vector variational-like inequalities and their relations with vector optimization are studied in Sect. 4 and the weak versions of these results are presented in Sect. 5.

2 Preliminaries

Let X be a real Banach space endowed with a norm $\|.\|$ and X^* its dual space with a norm $\|.\|_*$. We denote by 2^{X^*} and $\langle ., .\rangle$ the family of all nonempty subsets of X^* and the dual pair between X and X^*, respectively. Let K be a X non-empty open subset of X, $\eta : X \times X \to X$ a vector-valued function. Throughout this paper, $f_i : K \to \mathbb{R}$ the components of $f : K \to \mathbb{R}^n$ are non-differentiable locally Lipschitz functions, $\{C(x) : x \in K\}$ is a family of convex and pointed cones of $\mathbb{R}^n$ such that $C(x) \subseteq \mathbb{R}^n_+ \setminus \{0\}, \forall x \in K$, with $\text{int}C(x) \neq \emptyset$.

Let C be a closed, convex and pointed cone with $\text{int}C \neq \emptyset$. In the sequel we adopt the following ordering relations:

$$x \geq_C y \Leftrightarrow x - y \in C; \quad x \not\geq_C y \Leftrightarrow x - y \notin C;$$
$$x >_C y \Leftrightarrow x - y \in \text{int}C; \quad x \not>_C y \Leftrightarrow x - y \notin \text{int}C.$$

Note that $0 \notin \text{int}C$.

Let X be a nonempty set, we shall denote by $\mathcal{F}(X)$ the family of all nonempty finite subsets of X. Let Y be a nonempty set, X a topological space and $F : Y \rightrightarrows X$ a set-valued mapping. Then F is said to be transfer closed-valued iff $\forall (y, x) \in Y \times X$ with $x \notin F(y)$, $\exists y' \in Y$, such that $x \notin \mathrm{cl}F(y')$. It is clear that this definition is equivalent to:

$$\bigcap_{y \in Y} F(y) = \bigcap_{y \in Y} \mathrm{cl}F(y).$$

If $B \subseteq Y$ and $A \subseteq X$, then we call $F : B \rightrightarrows A$ transfer closed-valued iff the multi-valued mapping $y \to F(y) \bigcap A$ is transfer closed-valued. When $X{=}Y$ and $A{=}B$, we call F transfer closed-valued on A. Let K be a convex subset of a vector space X. Then a mapping $F : K \rightrightarrows X$ is called a KKM mapping iff for each nonempty finite subset A of K, $\mathrm{conv}A \subseteq F(A)$, where $\mathrm{conv}A$ denotes the convex hull of A, and $F(A) = \cup\{F(x) : x \in A\}$.

The following generalized Fan's KKM theorem is due to Fakhar and Zafarani [9].

Lemma 2.1 [9]. *Let K be a nonempty and convex subset of a Hausdorff topological vector space X. Suppose that $\Gamma, \hat{\Gamma} : K \rightrightarrows K$ are two set-valued mappings such that the following conditions are satisfied:*

(A1) $\hat{\Gamma}(x) \subseteq \Gamma(x), \quad \forall x \in K,$
(A2) $\hat{\Gamma}$ *is a KKM map,*
(A3) $\forall A \in \mathcal{F}(K)$, Γ *is transfer closed-valued on* $\mathrm{conv}A$*,*
(A4) $\forall A \in \mathcal{F}(K)$, $cl_K(\bigcap_{x \in convA} \Gamma(x)) \cap convA = (\bigcap_{x \in convA} \Gamma(x)) \cap convA,$
(A5) *there is a nonempty compact convex set $B \subseteq K$, such that $cl_K(\bigcap_{x \in B} \Gamma(x))$ is compact.*

Then, $\bigcap_{x \in K} \Gamma(x) \neq \emptyset$.

Remark 2.1 When Γ is closed-valued, then conditions (A3)–(A4) are trivially satisfied.

Definition 2.1 A subset K of X is said to be invex with respect to $\eta : X \times X \to X$ if , for any $x, y \in K$ and $\lambda \in [0, 1]$, we have $y + \lambda\eta(x, y) \in K$.

Throughout this paper K stands for an invex subset of X.

Definition 2.2 The Clarke's generalized derivative of $f : K \to \mathbb{R}$ at x in direction $v \in X$ is defined by

$$f^o(x; v) = \limsup_{y \to x \ \lambda \to 0^+} \frac{f(y + \lambda v) - f(y)}{\lambda},$$

and the Clarke's generalized subdifferential of f at $x \in X$ is defined by

$$\partial f(x) = \{\xi \in X^* : \langle \xi, v\rangle \le f^o(x; v), \quad \forall v \in X\}.$$

Some important generalizations of convex functions, namely prequasiinvex functions and pseudoinvex functions were introduced by Weir and Mond [23] and Mohan and Neogy [21] for differentiable functions and by Jabarootian and Zafarani [16] for locally Lipschitz functions.

Definition 2.3 Let $f : K \to \mathbb{R}$. Then

(1) f is said to pseudoinvex with respect to η on K if for any $x, y \in K$ and any $\zeta \in \partial f(y)$ one has

$$\langle \zeta, \eta(x, y)\rangle \ge 0 \Rightarrow f(x) \ge f(y);$$

(2) f is said to strictly pseudoinvex with respect to η on K if for any $x, y \in K$ with $x \neq y$ and any $\zeta \in \partial f(y)$ one has

$$\langle \zeta, \eta(x, y) \rangle \geq 0 \Rightarrow f(x) > f(y);$$

(3) f is said to be prequasiinvex with respect to η on K if for any $x, y \in K, 0 \leq \lambda \leq 1$, one has

$$f(y + \lambda\eta(x, y)) \leq \max\{f(x), f(y)\};$$

(4) f is said to be semi-strictly prequasiinvex with respect to η on K if for any $x, y \in K, 0 < \lambda < 1$ with $f(x) \neq f(y)$, one has

$$f(y + \lambda\eta(x, y)) < \max\{f(x), f(y)\};$$

(5) f is said to quasiinvex with respect to η on K if for any $x, y \in K$ and any $\zeta \in \partial f(y)$ one has

$$f(x) \leq f(y) \Rightarrow \langle \zeta, \eta(x, y) \rangle \leq 0.$$

In the sequel we need the following notions of monotonicity.

Definition 2.4 Let $f : K \to \mathbb{R}$. Then

(1) ∂f is said to be quasimonotone on K with respect to η if for any $x, y \in K$ and any $\xi \in \partial f(x)$, $\zeta \in \partial f(y)$ one has

$$\langle \xi, \eta(y, x) \rangle > 0 \Rightarrow \langle \zeta, \eta(x, y) \rangle \leq 0;$$

(2) ∂f is said to be pseudomonotone on K with respect to η if for any $x, y \in K$ and any $\xi \in \partial f(x)$, $\zeta \in \partial f(y)$ one has

$$\langle \xi, \eta(y, x) \rangle \geq 0 \Rightarrow \langle \zeta, \eta(x, y) \rangle \leq 0.$$

It is trivial that in Definition 2.4, we have the following implications: (2) $\Rightarrow$ (1).

Mohan and Neogy [21] introduced Condition (C) defined as follows.

Condition (C). *Let $\eta : X \times X \to X$. We say the function η satisfies the condition (C) if, for any $x, y \in X$, $\lambda \in [0, 1]$,*

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y),$$
$$\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

Remark 2.2 Yang et al. [30] have shown that when $\eta : X \times X \to X$ satisfies Assumption (C), then

$$\eta(y + \lambda\eta(x, y), y) = \lambda\eta(x, y), \quad \lambda \in [0, 1].$$

3 Pseudoinvex and quasiinvex functions

In this section we obtain some relationships between different kinds invexity. We give first a non-differentiable version of Theorem 2.1 in [31]. Then we complete the implications cycles in Theorem 4.1 of [16].

Theorem 3.1 *Let $f : K \to \mathbb{R}$ be pseudoinvex with respect to η. Then f is semi-strictly prequasiinvex function on K.*

Proof With some modifications in the proof of Theorem 2.1 of [31], one can obtain the result. Hence the proof is omitted. □

In the following result we obtain a relation between quasiinvex and prequasiinvex functions.

Theorem 3.2 *Let $f : K \to \mathbb{R}$ and η satisfy Condition (C).*

(1) *If f is quasiinvex with respect to η, then f is prequasiinvex.*
(2) *Conversely, if f is prequasiinvex and the function $y \longmapsto \eta(x, y)$ is continuous, then f is quasiinvex.*

Proof

(1) follows from Theorem 4.1 of [16].
(2) Let $x, y \in K$ and $f(y) \leq f(x)$, then if $f(y) < f(x)$, the result follows from Lemma 4.1 of [16].

If $f(y) = f(x)$, since f is locally Lipschitz, there exists $\delta > 0$ such that f is Lipschitz on $M = \{z \in K : \|z - x\| < \delta\}$. Now, we have two cases;
Case(I): For all $z \in M$, we have $f(y){=}f(x) \leq f(z)$. Then the prequasiinvexity of f implies that

$$f(z + \lambda\eta(y, z)) \leq f(z), \quad for \|z - x\| < \delta.$$

If L is a Lipschitz constant of f near the point x, then for $\delta{>}0$ small enough, and for $\|z - x\| < \delta, 0 < \lambda < \delta$, one has

$$\begin{aligned}\frac{f(z + \lambda\eta(y, x)) - f(z)}{\lambda} &\leq \frac{f(z + \lambda\eta(y, z)) - f(z)}{\lambda} + L\|\eta(y, x) - \eta(y, z)\| \\ &\leq 0 + L\|\eta(y, x) - \eta(y, z)\|.\end{aligned}$$

By continuity of η with respect to the second argument, we have

$$f^o(x, \eta(y, x)) = \lim_{\delta \to 0} \sup_{\|z - x\| < \delta, 0 < \lambda < \delta} \frac{f(z + \lambda\eta(y, x)) - f(z)}{\lambda} \leq 0,$$

hence for any $\xi \in \partial f(x)$, we obtain $\langle \xi, \eta(y, x) \rangle \leq 0$.
Case(II): There exists $z \in M$ such that $f(z) < f(y) = f(x)$. Then by continuity of f and $\lambda \mapsto x + \lambda\eta(y, x)$, there exist $\lambda_0 \in [0, 1]$ and $z_0 = x + \lambda_0\eta(y, x) \in K$ such that

$$f(z_0) = \min_{\lambda \in [0,1]} f(x + \lambda\eta(y, x)).$$

Since $f(z_0) < f(x)$, we have $\lambda_0 \in (0, 1]$. From $f(z_0) < f(x)$ and Lemma 4.1 of [16], for any $\xi \in \partial f(x)$ we deduce

$$\langle \xi, \eta(z_0, x) \rangle \leq 0.$$

By Condition (C), $\eta(x + \lambda_0\eta(y, x), x) = \lambda_0\eta(y, x)$, hence the above inequality implies that $\langle \xi, \eta(y, x) \rangle \leq 0$. Thus, for any $\xi \in \partial f(x)$ we have

$$\langle \xi, \eta(y, x) \rangle \leq 0,$$

which completes the proof. □

Remark 3.1 When $f : K \to \mathbb{R}$ with respect to η is pseudoinvex, then by Theorem 3.1 f is semi-strictly prequasiinvex and hence Theorem 2.2 of [31] implies that f is prequasiinvex. Therefore, from Theorem 3.2, when η is continuous in the second argument and satisfies Condition (C), f is quasiinvex. Now from Theorem 4.1 of [16] we deduce that ∂f is quasimonotone.

Theorem 3.3 *Let $f : K \to \mathbb{R}$ be a semi-strictly prequasiinvex function with respect to η on K and $\eta(x, y) \neq 0$ whenever $x \neq y$. Then every strict local minimizer of the function f is also a global minimizer.*

Proof Suppose that there exists $\delta > 0$ such that $f(x_0) < f(y)$ whenever $y \in K$, $\|y - x_0\| < \delta$ and there exists $z \in K, z \neq x_0$ such that $f(z) \leq f(x_0)$. Consider $y_0 = x_0 + \lambda\eta(z, x_0)$ such that $0 < \lambda < \min\{\frac{\delta}{\|\eta(z,x_0)\|}, 1\}$, thus since $\|y_0 - x_0\| < \delta$ we deduce

$$f(x_0) < f(y_0) \tag{3.1}$$

and since f is a semi-strictly prequasiinvex function with respect to η on K

$$f(y_0) = f(x_0 + \lambda\eta(z, x_0)) < \max\{f(x_0), f(z)\} = f(x_0),$$

which contradicts (3.1). Then $x_0 \in K$ is a global minimizer f on K. □

By the same argument as in the above result, we can obtain the following theorem for the pseudoinvex functions.

Theorem 3.4 *Let $f : K \to \mathbb{R}$ be pseudoinvex with respect to η on K and $\eta(x, y) \neq 0$ whenever $x \neq y$. Then every strict local minimizer of the function f is also a global minimizer.*

4 Vector variational-like inequalities

Let $f : K \longrightarrow \mathbb{R}^n$ be a locally Lipschitz function. We consider the following vector-minimization problem (VP):

$$\min_{x \in K} {}_{C(y)} \; f(x).$$

Solving a (VP) means finding all the (weakly) efficient solutions, which are defined as follows.

Definition 4.1

(1) $y \in K$ is said to be an efficient solution (Pareto solution) of (VP) iff

$$f(y) - f(x) \notin C(y), \forall x \in K$$

(2) $y \in K$ is said to be a weak efficient solution (weak Pareto) of (VP) iff

$$f(y) - f(x) \notin \text{int}C(y), \forall x \in K.$$

The definition of Clarke's generalized derivative can be extended to a locally Lipschitz vector-valued function $f : K \longrightarrow \mathbb{R}^n$. In fact the Clarke's generalized derivative of f at x in direction v is

$$f^o(x; v) = f_1^o(x; v) \times f_2^o(x; v) \times \cdots \times f_n^o(x; v),$$

and the Clarke's generalized subdifferential of f at $x \in X$ is the set

$$\partial f(x) = \partial f_1(x) \times \partial f_2(x) \times \cdots \times \partial f_n(x).$$

Remark 4.1 Similar to the real-valued case, one can show that the set-valued mapping $\partial f: K \to X^{*n}$ of a function $f: K \to \mathbb{R}^n$ is $(\|.\| - w^*)$-u.s.c. (See; [2]).

We state now a vector version of Definition 2.4.

Definition 4.2 Let $f : K \to \mathbb{R}^n$. We say that ∂f, the Clarke's generalized subdifferential of f is:

(1) C-pseudomonotone with respect to η on K whenever, $x, y \in K$

$$\langle \partial f(x), \eta(y, x)\rangle \subseteq C(y) \Rightarrow \langle \partial f(y), \eta(x, y)\rangle \subseteq -C(y);$$

(2) C-strictly pseudomonotone with respect to η on K whenever, $x, y \in K$

$$\langle \partial f(x), \eta(y, x)\rangle \subseteq C(y) \Rightarrow \langle \partial f(y), \eta(x, y)\rangle \subseteq -\mathrm{int}C(y);$$

(3) C-quasimonotone with respect to η on K whenever, $x, y \in K$

$$\langle \partial f(x), \eta(y, x)\rangle \subseteq \mathrm{int}C(y) \Rightarrow \langle \partial f(y), \eta(x, y)\rangle \subseteq -C(y);$$

(4) C-strictly quasimonotone with respect to η on K whenever, $x, y \in K$

$$\langle \partial f(x), \eta(y, x)\rangle \subseteq \mathrm{int}C(y) \Rightarrow \langle \partial f(y), \eta(x, y)\rangle \subseteq -\mathrm{int}C(y);$$

(5) C-properly quasimonotone with respect to η on K, properly η-quasimonotone of Stampacchia type in the sense of [27], whenever, $\{x_1, x_2, \ldots, x_n\} \subseteq K$ for all $y \in \mathrm{conv}\{x_1, x_2, \ldots, x_n\}$, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$\langle \partial f(y), \eta(x_i, y)\rangle \subseteq C(y);$$

(6) C-weakly properly quasimonotone with respect to η on K whenever, $\{x_1, x_2, \ldots, x_n\} \subseteq K$ for all $y \in \mathrm{conv}\{x_1, x_2, \ldots, x_n\}$, there exists $i \in \{1, 2, \ldots, n\}$ such that

$$\langle \partial f(y), \eta(x_i, y)\rangle \nsubseteq -\mathrm{int}C(y).$$

Remark 4.2 In the case that we ordered $\mathbb{R}^n$ by the closed convex cone $\mathbb{R}^n_+$, and each components of $f: K \longrightarrow \mathbb{R}^n$ is pseudoinvex, η satisfies Condition (C) and the function $y \longmapsto \eta(x, y)$ is continuous, then by the same argument as in Remark 3.1, one can show that when f is pseudoinvex, then $\partial f : K \rightrightarrows X^{*n}$ is C-quasimonotone with respect to η on K. In general we have trivially the following implications:

$$\begin{array}{ccc} (2) & \Longrightarrow & (1) \\ \Downarrow & & \Downarrow \\ (4) & \Longrightarrow & (3). \end{array}$$

Moreover, when η is affine in the first argument and $\eta(x, x) = 0$, then ∂f is both C-properly quasimonotone and C- weakly properly quasimonotone with respect to η. The definition of properly quasimonotone for set-valued maps with values in the dual of an arbitrary topological vector spaces, properly η-quasimonotone of Minty type in the sense of [27], is due to Daniilidis and Hadjisavvas [8].

We consider now the following Minty vector variation-like inequality (MVLI): Find a $y \in K$ such that

$$\langle \partial f(x), \eta(x, y)\rangle \nsubseteq -C(y), \quad \forall x \in K.$$

When $\eta(x, y) = y - x$ and f is differentiable, (MVLI) becomes the following Minty vector variational inequality (MVLI), which was studied by Giannessi [12]:

Find a $y \in K$ such that

$$\langle \nabla f(x), y - x \rangle \notin -C(y), \quad \forall x \in K.$$

Definition 4.3 $\eta : X \times X \to X$ is called skew, if $\eta(x, y) + \eta(y, x) = 0$ for each $x, y \in X$.

By using the proof techniques of [19] and [31], we obtain the following result between relations of (MVLI) and (VP).

Theorem 4.1 *If $y \in K$ is a solution of (VP), $C = \mathbb{R}^n_+$ and η is skew, then y is a solution of (MVLI).*

Proof Let y be a solution of (VP). By contradiction, suppose that there exists an $x_0 \in K$, such that

$$\langle \partial f(x_0), \eta(x_0, y) \rangle \subseteq -C.$$

Since η is skew, hence

$$\langle \partial f(x_0), \eta(y, x_0) \rangle \subseteq C.$$

From the pseudoinvexity of f it follows that

$$f(y) - f(x_0) \in C, \quad x_0 \in K,$$

which contradicts the fact that y is a solution of (VP). □

Here, we consider the Stampacchia vector variational-like inequality (SVLI): Find a $y \in K$ such that

$$\langle \partial f(y), \eta(x, y) \rangle \nsubseteq -C(y), \quad \forall x \in K.$$

Theorem 4.2 *Let K be an invex set with respect to a skew function η satisfying Condition (C) such that ∂f be C-pseudomonotone with respect to η on K. If $y \in K$ is a solution of (SVLI), then y is a solution of (MVLI).*

Proof Let $y \in K$ be a solution of (SVLI) and by contradiction, suppose that y is not a solution of (MVLI), thus there exists $x_0 \in K$, such that

$$\langle \partial f(x_0), \eta(x_0, y) \rangle \subseteq -C(y).$$

Since η is skew, hence

$$\langle \partial f(x_0), \eta(y, x_0) \rangle \subseteq C(y).$$

By C-pseudomonotonicity of ∂f,

$$\langle \partial f(y), \eta(x_0, y) \rangle \subseteq -C(y),$$

thus, $y \in K$ is not a solution of (SVLI), which is a contradiction. □

Theorem 4.3 *Let f be quasiinvex with respect to η on K and $C = \mathbb{R}^n_+ \setminus \{0\}$. If $y \in K$ is a solution of (SVLI), then $y \in K$ is a solution of (VP).*

Proof Let $y \in K$ be a solution of (SVLI), by contradiction, suppose that there exist an $x_0 \in K$ such that

$$f(y) - f(x_0) \in C.$$

Quasiinvexity of f implies that

$$\langle \partial f(y), \eta(x_0, y) \rangle \subseteq -C,$$

which contradicts the fact that $y \in K$ is a solution of (SVLI). □

Now consider a perturbed vector variational-like inequality (PVLI): Find a $y \in K$ for which there exists $t_0 \in (0, 1)$ such that

$$\langle \partial f(y + t\eta(x, y)), \eta(x, y) \rangle \nsubseteq -C(y), \quad \forall x \in K, \forall t \in (0, t_0].$$

By using the proof techniques of Theorem 3.2 of [10] and Theorem 3.5 of [31], we obtain the following result between relations of (MVLI) and (PVLI).

Theorem 4.4 *Let K be invex with respect to a function η satisfying Condition (C). If $y \in K$ is a solution of (MVLI), then y is a solution of (PVLI). Conversely, if ∂f is C-pseudomonotone with respect with to η for $C = \mathbb{R}^n_+ \setminus \{0\}$ and y is a solution of (PVLI), then y is a solution of (MVLI).*

Proof Let y be a solution of (MVLI). The invexity of K with respect to η implies $x(t) := y + t\eta(x, y) \in K$, for any $x \in K, t \in (0, t_0], t_0 \in (0, 1)$. It follows that

$$\langle \partial f(y + t\eta(x, y)), \eta(y + t\eta(x, y), y) \rangle \nsubseteq -C(y), \quad \forall t \in (0, t_0].$$

Since $\eta(y + t\eta(x, y), y) = t\eta(x, y)$ and $t > 0$, it follows

$$\langle \partial f(y + t\eta(x, y)), \eta(x, y) \rangle \nsubseteq -C(y), \quad \forall t \in (0, t_0],$$

that implies (PVLI).

Let y be a solution of (PVLI): Find a $y \in K$ for which there exists $t_0 \in (0, 1)$ such that

$$\langle \partial f(y + t\eta(x, y)), \eta(x, y) \rangle \nsubseteq -C, \quad \forall x \in K, \quad \forall t \in (0, t_0].$$

Now suppose to the contrary that $y \in K$ is not a solution of (MVLI). Then there exists an $x_0 \in K$ such that

$$\langle \partial f(x_0), \eta(x_0, y) \rangle \subseteq -C.$$

The invexity of K respect to η implies $x(t) := y + t\eta(x_0, y) \in K, \forall t \in (0, t_0]$. From Condition C, $\eta(x_0, x(t)) = (1 - t)\eta(x_0, y)$. Thus we have

$$\langle \partial f(x_0), \eta(x_0, x(t)) \rangle \subseteq -C, \quad \forall t \in (0, t_0].$$

Since η is a skew function, we have

$$\langle \partial f(x_0), \eta(x(t), x_0) \rangle \subseteq C, \quad \forall t \in (0, t_0].$$

By C-pseudomonotonicity of ∂f, we deduce that

$$\langle \partial f(x(t)), \eta(x_0, x(t)) \rangle \subseteq -C.$$

Since $\eta(x_0, y + t\eta(x_0, y)) = (1 - t)\eta(x_0, y)$, hence

$$\langle \partial f(y + t\eta(x_0, y)), \eta(x_0, y) \rangle \subseteq -C, \quad \forall t \in (0, t_0].$$

which contradicts the fact y is a solution of (PVLI). □

Now we will obtain an existence result for (MVLI).

Theorem 4.5 *Let K be a nonempty convex subset of X, $f : K \longrightarrow \mathbb{R}^n$ and ∂f be C-pseudomonotone and C-properly quasimonotone with respect to η on K. Assume that the following conditions are satisfied:*

(1) *The set-valued mapping $\Gamma : K \rightrightarrows K$ defined by*

$$\Gamma(x) = \{y \in K : \langle \partial f(x), \eta(y, x)\rangle \nsubseteq C(y)\}$$

is closed valued;

(2) *η is skew;*

(3) *There are a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that for each $y \in K \setminus M$, there exists $x \in B$ such that*

$$\langle \partial f(x), \eta(y, x)\rangle \subseteq C(y).$$

Then (MVLI) has a solution.

Proof Define the set-valued mapping $\hat{\Gamma} : K \rightrightarrows K$ by

$$\hat{\Gamma}(x) := \{y \in K : \langle \partial f(y), \eta(x, y)\rangle \nsubseteq -C(y)\}.$$

for each $x \in K$. Note that $\hat{\Gamma}(x)$ is nonempty for each $x \in K$, since by virtue of condition(2), it contains x. The proof of theorem is divided into the following steps.

(a) $\hat{\Gamma}$ is a KKM mapping on K. Since if $\hat{\Gamma}$ is not a KKM mapping, then there exists $\{x_1, x_2, \ldots, x_n\} \subset K, t_i \geq 0, i = 1, 2, \ldots, n$ with $\Sigma_{i=1}^n t_i = 1$ such that $y = \Sigma_{i=1}^n t_i x_i \notin \cup_{i=1}^n \hat{\Gamma}(x_i)$. Thus for any $i = 1, 2, \ldots, n$ and $\zeta \in \partial f(y)$ we have

$$\langle \zeta, \eta(x_i, y)\rangle \in -C(y),$$

which contradicts the C-proper quasimonotonicity of ∂f. Hence, $\hat{\Gamma}$ is a KKM mapping.

(b) Since ∂f is C-pseudomonotone with respect to η on K, we have $\hat{\Gamma}(x) \subseteq \Gamma(x)$ for each $x \in K$.

(c) Hence Γ is also a KKM mapping. Thus all of the conditions of Lemma 2.1 are fulfilled by mappings $\hat{\Gamma}$ and Γ. Therefore,

$$\bigcap_{x \in K} \Gamma(x) \neq \emptyset.$$

Hence, there exists $y \in K$ such that for any $\xi \in \partial f(x)$, we have

$$\langle \partial f(x), \eta(y, x)\rangle \nsubseteq C(y), \quad \forall x \in K.$$

Since η is skew, we obtain

$$\langle \partial f(x), \eta(x, y)\rangle \nsubseteq -C(y), \quad \forall x \in K.$$

Therefore, (MVLI) has a solution. □

Example 4.1 Let $X = \mathbb{R}$, $C(x) = \mathbb{R}^3_+ \setminus \{0\}$, $K = [-2, 2)$ and the function $f : K \to \mathbb{R}^3$ be defined as $f(x) = (f_1(x), f_2(x), f_3(x))$

$$f_1(x) := \begin{cases} \frac{1}{2}x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

$$f_2(x) := \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

$$f_3(x) := \begin{cases} \frac{1}{3}x^3 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The Clarke subdifferential mapping of f is

$$\partial f(x) := \begin{cases} (\frac{1}{2}, 1, x^2) & \text{if } x > 0, \\ \{(t, l, 0) : t \in [0, \frac{1}{2}], l \in [0, 1]\} & \text{if } x = 0, \\ (0, 0, 0) & \text{if } x < 0. \end{cases}$$

Let $\eta : X \times X \to X$ be defined as $\eta(x, y) := \alpha(x - y)$, such that $0 < \alpha < 1$. Then, η is affine and continuous in the both arguments and skew,

$$\langle \partial f(x), \eta(y, x) \rangle = \begin{cases} \{(\frac{\alpha(y-x)}{2}, \alpha(y - x), \alpha x^2(y - x))\} & \text{if } x > 0, \\ \{(\alpha t y, \alpha l y, 0) : 0 \leq t \leq \frac{1}{2}, l \in [0, 1]\} & \text{if } x = 0, \\ \{(0, 0, 0))\} & \text{if } x < 0. \end{cases}$$

and

$$\langle \partial f(y), \eta(x, y) \rangle = \begin{cases} \{(\frac{\alpha(x-y)}{2}, \alpha(x - y), \alpha y^2(x - y))\} & \text{if } y > 0, \\ \{(\alpha t x, \alpha l x, 0) : 0 \leq t \leq \frac{1}{2}, l \in [0, 1]\} & \text{if } y = 0, \\ \{(0, 0, 0)\} & \text{if } y < 0. \end{cases}$$

There are a nonempty compact set $M = [-2, 0]$ and a nonempty compact convex set $B = \{0\} \subset K$ such that for each $y \in K \setminus M$, there exists $x = 0 \in B$ such that

$$\langle \partial f(0), \eta(y, 0) \rangle = \{(\alpha t y, \alpha l y, 0)\} \subseteq C(y).$$

Furthermore, ∂f is C-pseudomonotone and C-properly quasimonotone with respect to η on K. Therefore, by Theorem 4.5, $y = 0$ is a solution of (MVLI).

5 Weak vector variational-like inequalities

In this section, we consider the Stampacchia weak vector variational-like inequality (SWVLI): Find a $y \in K$ such that

$$\langle \partial f(y), \eta(x, y) \rangle \not\subseteq -\text{int}C(y), \quad \forall x \in K.$$

The Minty weak vector variational-like inequality (MWVLI): Find a $y \in K$ such that

$$\langle \partial f(x), \eta(y, x) \rangle \not\subseteq \text{int}C(y), \quad \forall x \in K.$$

Theorem 5.1 *Let $-f$ is quasiinvex function with respect to skew function η on K and $C = \mathbb{R}^n_+ \setminus \{0\}$. If y is a solution of (MVLI), then y is a solution of (SWVLI).*

Proof Let $y \in K$ be a solution of (MVLI), thus

$$\langle \partial f(x), \eta(x, y) \rangle \not\subseteq -C,$$

since η is skew

$$\langle \partial f(x), -\eta(y, x) \rangle = \langle \partial(-f)(x), \eta(y, x) \rangle \not\subseteq -C.$$

Theorem 4.1 of [16] implies that $\partial(-f) = -\partial f$ is C-quasimonotone with respect to η on K, hence

$$\langle \partial(-f)(y), \eta(x, y)\rangle \not\subseteq \mathrm{int}C.$$

Therefore, $\langle \partial f(y), \eta(x, y)\rangle \not\subseteq -\mathrm{int}C$, thus $y \in K$ is a solution of (SWVLI). □

Theorem 5.2 *Let $f : K \to \mathbb{R}^n$ be pseudoinvex with respect to η satisfying Condition (C), the function $y \mapsto \eta(x, y)$ be continuous and $C = \mathbb{R}^n_+ \setminus \{0\}$. If $y \in K$ is a weak solution of (VP), then $y \in K$ is a solution of (SWVLI).*

Proof Let y be a weak solution of (VP). For each $x \in K$, $t \in [0, 1]$, since K is invex with respect to η, $x(t) := y + t\eta(x, y) \in K$, thus there exists $i \in \{1, 2, \ldots, n\}$ such that $f_i(y) \leq f_i(x(t))$. By the proof of part (2) of Theorem 3.2, for each $\zeta_t \in \partial f_i(x(t))$ we have

$$\langle \zeta_t, \eta(y, x(t))\rangle \leq 0.$$

From Condition (C), we have $\eta(y, x(t)) = -t\eta(x, y)$, hence

$$-t(\langle \zeta_t, \eta(x, y)\rangle) \leq 0,$$

and therefore,

$$\langle \zeta_t, \eta(x, y)\rangle \geq 0.$$

Since ∂f_i is locally bounded at y, hence, there exists a neighborhood of y and a constant $k > 0$ such that for each z in this neighborhood and $\xi \in \partial f_i(z)$ we have $\|\xi\| \leq k$. Since $x(t) \to y$ when $t \to 0^+$, thus for $t > 0$ small enough $\|\zeta_t\| \leq k$. Without loss of generality, we may assume that $\zeta_t \to \zeta_i'$ in w*-topology. Since the set-valued mapping $z \mapsto \partial f_i(z)$ has a closed graph, thus for each $x \in K$ there exists $\zeta_i' \in \partial f_i(y)$ such that

$$\langle \zeta_i', \eta(x, y)\rangle \geq 0.$$

Hence,

$$\langle \partial f(y), \eta(x, y)\rangle \not\subseteq -\mathrm{int}C,$$

which shows that y is a solution of (SWVLI). □

Theorem 5.3 *Let K be an invex set with respect to η satisfying Condition (C), the function $y \mapsto \eta(x, y)$ be continuous, $C = \mathbb{R}^n \setminus \{0\}$ and f be a pseudoinvex function on K. If y is a solution of (SVLI), then $y \in K$ is a weak solution of (VP).*

Proof By contradiction, suppose that there exists an x_0 such that

$$f(y) - f(x_0) \in \mathrm{int}C.$$

Clearly, by part (2) of Theorem (3.2) for any $\zeta \in \partial f(y)$ we have

$$\langle \zeta, \eta(x_0, y)\rangle \leq 0.$$

Thus

$$\langle \partial f(y), \eta(x_0, y)\rangle \subseteq -C.$$

which is a contradiction to the fact that y is a solution of (SVLI). □

Theorem 5.4 *Let $f : K \to \mathbb{R}^n$ be strictly pseudoinvex with respect to η and $C = \mathbb{R}^n \setminus \{0\}$. If $y \in K$ is a solution of (SWVLI), then y is a solution of (MWVLI). Conversely, assume that η satisfies Condition (C). If $y \in K$ is a solution of (MWVLI), then y is a solution of (SWVLI).*

Proof Suppose that $y \in K$ is a solution of (SWVLI), but not a solution of (MWVLI). Then, there exists $x_0 \in K$ such that

$$\langle \partial f(x_0), \eta(y, x_0)\rangle \subseteq \mathrm{int}C.$$

Since by Lemma 5.1 of [16], ∂f is C- strictly pseudomonotone, we have

$$\langle \partial f(y), \eta(x_0, y)\rangle \subseteq -\mathrm{int}C,$$

that is

$$\langle \zeta, \eta(x_0, y)\rangle \in -\mathrm{int}C, \quad \forall \zeta \in \partial f(y).$$

which contradicts the fact y is a solution of (SWVLI).

Now, let $y \in K$ be a solution of (MWVLI). Consider any $x \in K$ and any $t \in (0, 1]$, since K is invex with respect to η, $x(t) := y + t\eta(x, y) \in K$. As $y \in K$ is a solution of (MWVLI), there exists $\zeta_t \in \partial f(x(t))$ such that

$$\langle \zeta_t, \eta(y, x(t))\rangle \notin \mathrm{int}C.$$

By Condition (C), $\eta(y, x(t)) = -t\eta(x, y)$, we have

$$-t(\langle \zeta_t, \eta(x, y)\rangle) \notin \mathrm{int}C.$$

We note that $\mathbb{R}^n \setminus \mathrm{int}C(y)$ is a closed cone, therefore, we have

$$\langle \zeta_t, \eta(x, y)\rangle \notin -\mathrm{int}C.$$

Since ∂f is locally bounded at y, hence, there exists a neighborhood of y and a constant $k>0$ such that for each z in this neighborhood and $\xi \in \partial f(z)$ we have $\|\xi\| \leq k$. Since $x(t) \to y$ when $t \to 0^+$, thus for $t > 0$ small enough, $\|\zeta_t\| \leq k$. Without loss of generality, we may assume that $\zeta_t \to \xi$ in w*-topology. Since the set-valued mapping $z \mapsto \partial f(z)$ has a closed graph, thus for each $x \in K$ there exists $\zeta \in \partial f(y)$ such that

$$\langle \zeta, \eta(x, y)\rangle \notin -\mathrm{int}C.$$

Hence, $y \in K$ is a solution of (SWVLI). □

Theorem 5.5 *Let K be a nonempty convex subset of X, $f : K \longrightarrow \mathbb{R}^n$ and ∂f be C-quasimonotone and C-properly quasimonotone. Assume that the following conditions are satisfied:*

(1) *η is continuous in the first argument.*
(2) *There are a nonempty compact set $M \subset K$, and a nonempty compact convex set $B \subset K$ such that for each $y \in K \setminus M$, there exists $x \in B$ such that*

$$\langle \partial f(x), \eta(y, x)\rangle \subseteq intC(y).$$

(3) *The set-valued mapping $W : K \rightrightarrows \mathbb{R}^n$ defined by $W(y) = \mathbb{R}^n \setminus intC(y)$ has closed graph.*

Then (MWVLI) has a solution.

Proof Define the set-valued mappings $\Gamma, \hat{\Gamma} : K \rightrightarrows K$ by

$$\Gamma(x) = \{y \in K : \langle \partial f(x), \eta(y, x)\rangle \not\subseteq \text{int}C(y)\},$$
$$\hat{\Gamma}(x) := \{y \in K : \langle \partial f(y), \eta(x, y)\rangle \not\subseteq -C(y)\}$$

for each $x \in K$. Note that $\Gamma(x)$ and $\hat{\Gamma}(x)$ are nonempty for each $x \in K$, since by virtue of condition(1) they contain x. The proof of theorem is divided into the following four steps.

(a) By the same argument as that of the first part of the proof of Theorem 4.5, $\hat{\Gamma}$ is a KKM mapping on K.
(b) By virtue of C-quasimonotonicity of f, $\hat{\Gamma}(x) \subseteq \Gamma(x)$ for each $x \in K$.
(c) Γ is closed valued. Let $\{x_n\}$ be a sequence in $\Gamma(x)$ which converges to $x_0 \in K$. Since $x_n \in \Gamma(x)$ there exists $\xi_n \in \partial f(x)$ satisfying

$$z_n = \langle \xi_n, \eta(x_n, x)\rangle \notin \text{int}C(x_n).$$

Therefore, $z_n \in W(x_n)$ and hence $(x_n, z_n) \in G_r(W)$. Since $\partial f(x)$ is w*-compact, $\{\xi_n\}$ has a convergent subnet in $\partial f(y)$. Let $\{\xi_m\}$ be such a subnet of $\{\xi_n\}$ that w*-converges to $\xi_0 \in \partial f(x)$. By continuity of η, $\{\eta(x_m, x)\}$ is a convergent net. Hence, it is norm bounded for $m \geq m_0$ and therefore, by Proposition 2.3 of [5], we have

$$z_0 = \lim_m z_m = \langle \xi_0, \eta(x_0, x)\rangle.$$

Since $G_r(W)$ is closed, then $(x_0, z_0) \in G_r(W)$ and hence,

$$\langle \xi_0, \eta(x_0, x)\rangle \notin \text{int}C(x_0).$$

Thus, $x_0 \in \Gamma(x)$, this means that $\Gamma(x)$ is closed.
(d) Thus all of the conditions of Lemma 2.1 are fulfilled by mappings Γ and $\hat{\Gamma}$. Hence,

$$\bigcap_{x \in K} \Gamma(x) \neq \emptyset.$$

Therefore, there exists $y \in K$ such that

$$\langle \partial f(x), \eta(y, x)\rangle \not\subseteq \text{int}C(y), \quad \forall x \in K.$$

Thus (MWVLI) has a solution. □

Remark 5.1 When each of the components of f is quasiinvex, then by Theorem 4.1 of [16], ∂f is C-quasimonotone for $C = \mathbb{R}^n_+ \setminus \{0\}$. Hence, from Theorem 5.5 we can obtain a solution of (MWVLI) in this case.

As a consequence of Theorems 5.4 and 5.5, we obtain an existence result for weak (VP).

Corollary 5.1 *In Theorem 5.5, if $C = \mathbb{R}^n_+ \setminus \{0\}$ and f is strictly pseudoinvex with respect to η satisfying Condition (C), then the weak (VP) has a solution.*

Proof From Lemma 5.1 in [16] ∂f is strictly pseudomonotone and therefore it is quasimonotone. Hence, Theorem 5.5 implies that (MWVLI) has a solution $y \in K$ and from Theorem 5.4, y is a solution of (SWVLI). Now if y is not a weak solution of (VP), then there exists an $x_0 \in K$ such that

$$f(y) - f(x_0) \in \text{int}C.$$

From strict pseudoinvexity of f, we have

$$\langle \partial f(y), \eta(x_0, y)\rangle \subseteq -\text{int}C,$$

which contradicts with the fact that y is a solution of (SWVLI). □

Example 5.1 Let $X = \mathbb{R}$, $K = [-2, 2)$, $C(x) = \mathbb{R}_+ \setminus \{0\}$ for each $x \in K$ and the function $f : K \to \mathbb{R}$ be defined as $f(x) = \sqrt{|x|}$. The Clarke subdifferential mapping of the function f is

$$\partial f(x) := \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } x > 0, \\ (-\infty, +\infty) & \text{if } x = 0, \\ \frac{-1}{2\sqrt{-x}} & \text{if } x < 0. \end{cases}$$

The function $\eta : X \times X \to X$ is defined as

$$\eta(x, y) := \begin{cases} x - y & \text{if } x \geq 0, y \geq 0 \quad \text{or} \quad x \leq 0, y \leq 0, \\ -y & \text{if } x \leq 0, y > 0 \quad \text{or} \quad x > 0, y \leq 0. \end{cases}$$

The function η is continuous in the first argument and satisfies Condition (C) [25].

$$\langle \partial f(y), \eta(x, y)\rangle = \begin{cases} \{\frac{(x-y)}{2\sqrt{y}}\} & \text{if } x \geq 0, y > 0, \\ \{\frac{(y-x)}{2\sqrt{-y}}\} & \text{if } x \leq 0, y < 0, \\ \{\frac{y}{2\sqrt{-y}}\} & \text{if } x \geq 0, y < 0, \\ \{\frac{-y}{2\sqrt{y}}\} & \text{if } x \leq 0, y > 0, \\ \{tx : t \in \mathbb{R}\} & \text{if } x \geq 0, y = 0, \\ \{0\} & \text{if } x < 0, y = 0. \end{cases}$$

and

$$\langle \partial f(x), \eta(y, x)\rangle = \begin{cases} \{\frac{(y-x)}{2\sqrt{x}}\} & \text{if } x > 0, y > 0, \\ \{\frac{(x-y)}{2\sqrt{-x}}\} & \text{if } x < 0, y < 0, \\ \{\frac{x}{2\sqrt{-x}}\} & \text{if } x < 0, y \geq 0, \\ \{\frac{-x}{2\sqrt{x}}\} & \text{if } x > 0, y \leq 0, \\ \{ty : t \in \mathbb{R}\} & \text{if } x = 0, y > 0, \\ \{0\} & \text{if } x = 0, y < 0. \end{cases}$$

There are a nonempty compact set $M = [-2, 1]$ and a nonempty compact convex set $B = \{1\} \subset K$ such that for each $y \in K \setminus M$, there exists $x = 1 \in B$ such that

$$\langle \partial f(1), \eta(y, 1)\rangle = \{\frac{y-1}{2}\} \subseteq \text{int}C(y).$$

Moreover ∂f is C-properly quasimonotone. Now by Theorem 5.5, $y = 0$ is a solution of (SWVLI). The function f is strictly pseudoinvex, since for every $x, y \in K, x \neq y$ and $y^* \in \partial f(y)$

$$\langle y^*, \eta(x, y)\rangle \geq 0 \Rightarrow f(x) > f(y).$$

Therefore, by Corollary 5.1, the weak (VP) has $y = 0$ as a solution.

We state and prove now the following generalized vector version of Minty's lemma.

Lemma 5.1 *Let K be convex subset of X and ∂f be C-strictly quasimonotone. Assume that the following conditions are satisfied:*

(1) *η is affine in the first argument and continuous in the second argument .*

(2) *For all $x \in K$, $\eta(x, x) = 0$.*

Then, (SWVLI) and (MWVLI) are equivalent.

Proof Suppose that $y \in K$ is solution of (SWVLI), thus

$$\langle \partial f(y), \eta(x, y)\rangle \nsubseteq -\mathrm{int}C(y), \quad \forall x \in K.$$

Since ∂f is C- strictly quasimonotone ,

$$\langle \partial f(x), \eta(y, x)\rangle \nsubseteq \mathrm{int}C(y), \quad \forall x \in K.$$

Therefore, y is a solution of (MWVLI).

Conversely, suppose that we can find $y \in K$ such that

$$\langle \partial f(x), \eta(y, x)\rangle \nsubseteq \mathrm{int}C(y)\forall x \in K. \tag{5.1}$$

We consider $x_t = y + t(x - y)$ for $t \in (0, 1)$ and replacing x by x_t in (4.1), then

$$\langle \partial f(x_t), \eta(y, x_t)\rangle \nsubseteq \mathrm{int}C(y). \tag{5.2}$$

By virtue of conditions (1) and (2), we obtain for each $\xi_t \in \partial f(x_t)$

$$0 = \langle \xi_t, \eta(x_t, x_t)\rangle = t\langle \xi_t, \eta(x, x_t)\rangle + (1 - t)\langle \xi_t, \eta(y, x_t)\rangle. \tag{5.3}$$

Hence, from (5.2) and (5.3), we deduce that

$$\langle \partial f(x_t), \eta(x, x_t)\rangle \nsubseteq -\mathrm{int}C(y).$$

Consequently for each $t \in [0, 1]$, there exists $\zeta_t \in \partial f(x_t)$ such that

$$\langle \zeta_t, \eta(x, x_t)\rangle \notin -\mathrm{int}C(y).$$

Since η is continuous in the second argument, as $t \to 0^+$, we have $\eta(x, x_t) \to \eta(x, y)$ and the set $\{\eta(x, x_t) : t \in [0, 1]\}$ is bounded for small enough t. Furthermore, since ∂f is norm-w* u.s.c. and w*-compact valued, there exists a $\zeta \in \partial f(y)$ and a subnet $\{\zeta_s\}$ of $\{\zeta_t\}$ such that ζ_s converges to ζ in w*-topology of X^{*n}. Thus Proposition 2.3 of [5] implies that

$$\langle \zeta_s, \eta(x, x_s)\rangle \to \langle \zeta, \eta(x, y)\rangle.$$

But, $\langle \zeta_s, \eta(x, x_s)\rangle \in \mathbb{R}^n \setminus -\mathrm{int}C(y)$, which is a closed set, hence $\langle \zeta, \eta(x, y)\rangle \in \mathbb{R}^n \setminus -\mathrm{int}C(y)$. Therefore, y is a solution of (SWVLI). □

Remark 5.2 In the case that ∂f is C-strictly quasimonotone and η is continuous, affine in the first argument and $\eta(x, x) = 0$ for all $x \in K$, then by Lemma 5.1 and from Theorem 5.5 we obtain a solution for (SWVLI). When the function f is invex, Jabarootian and Zafarani[17] obtained a necessary and sufficient condition for the coincidence between weak solution of (VP) and solution of (MWVLI). In finite dimensional case, a similar result has been obtained by Ansari and Yao [1].

When ∂f is only C-weakly properly quasimonotone, we can obtain the following existence result for (SWVLI).

Theorem 5.6 *Let K be a nonempty closed convex subset of a separable Banach space X and $f : K \to \mathbb{R}^n$. Assume that ∂f is C-weakly properly quasimonotone and the following conditions are satisfied:*

(1) *η is continuous in the second argument.*
(2) *There are a nonempty compact set $M \subset K$, and a nonempty compact convex set $B \subset K$ such that for each $y \in K \setminus M$, there exists $x \in B$ such that*

$$\langle \partial f(y), \eta(x, y) \rangle \subseteq -intC(y).$$

(3) *The set-valued mapping $W : K \rightrightarrows \mathbb{R}^n$ defined by $W(y) = \mathbb{R}^n \setminus -intC(y)$ has closed graph.*

Then (SWVLI) has a solution.

Proof Define the set-valued mappings $\Gamma = \hat{\Gamma} : K \rightrightarrows K$ by

$$\Gamma(x) := \{y \in K : \langle \partial f(y), \eta(x, y) \rangle \nsubseteq -\text{int}C(y)\},$$

for each $x \in K$. By the same argument as that of the first part of the proof of Theorem 4.5, Γ is a KKM mapping.

Next we prov e that $\Gamma(x)$ is closed for each $x \in K$. Let $\{y_m\}$ be a sequence of $\Gamma(x)$ which converges to some $y \in K$. Then there exists $\zeta_m^i \in \partial f_i(y_m),\ i = 1, \ldots, n$, such that

$$(\langle \zeta_m^1, \eta(x, y_m) \rangle, \ldots, \langle \zeta_m^n, \eta(x, y_m) \rangle \in W(y_m). \tag{5.4}$$

Since f_i is locally Lipschitz, there exists a neighborhood $N(y)$ of y and $L > 0$ such that for any $u, v \in N(y)$,

$$|f_i(u) - f_i(v)| \leq L\|u - v\|, \quad i = 1, \ldots, n.$$

Hence for any $u \in N(y)$ and any $\zeta^i \in \partial f_i(u)$, we have $\|\zeta^i\| \leq L$. So by w*-metrizability of the unit ball of X^{*n} we can assume that ζ_m^i w*-converges to ζ^i for each $i = 1, \ldots, n$. Since the set-valued mapping $u \mapsto \partial f_i(u)$ is closed and $\zeta_m^i \in \partial f_i(y_m)$, hence $\zeta^i \in \partial f_i(y),\ i = 1, \ldots, n$. As η is continuous in the second argument and $\eta(x, y_m)$ converges to $\eta(x, y)$, therefore $\langle \zeta_m^i, \eta(x, y_m) \rangle$ converges to $\langle \zeta^i, \eta(x, y) \rangle$ by Proposition 2.3 of [5]. From (5.4) and closedness of the graph of W,

$$(\langle \zeta^1, \eta(x, y) \rangle, \ldots, \langle \zeta^n, \eta(x, y) \rangle \in W(y) = \mathbb{R}^n \setminus -\text{int}C(y).$$

Thus $y \in \Gamma(x)$ and hence is closed.

Now Lemma 2.1 implies that, $\cap_{x \in K} \Gamma(x) \neq \emptyset$. Hence any point y in this intersection is a solution of (SWVLI). □

Corollary 5.2 *In Theorem 5.6, if $C = \mathbb{R}^n_+ \setminus \{0\}$ and f is pseudoinvex, then the weak (VP) has a solution.*

Proof Suppose that $y \in K$ is a solution of (SWVLI) and is not a weak solution of (VP). Hence, there exists an $x_0 \in K$ such that

$$f(y) - f(x_0) \in \text{int}C.$$

From pseudoinvexity of f, we have

$$\langle \partial f(y), \eta(x_0, y) \rangle \subseteq -\text{int}C,$$

which contradicts with the fact that y is a solution of (SWVLI). □

Example 5.2 Let $X = \mathbb{R}$, $C(x) = \mathbb{R}^2_+ \backslash \{0\}$ for each $x \in K$, $K = [-4, 4)$ and $\eta : X \times X \to X$ be defined as $\eta(x, y) = \alpha(x - y)$ for each $x, y \in K$ such that $0 < \alpha < 1$. The function η is affine and continuous in both arguments and $\eta(x, x) = 0$, for each $x \in K$. If $f : K \to \mathbb{R}^2$ be defined as $f(x) = (f_1(x), f_2(x))$

$$f_1(x) := \begin{cases} \sqrt{x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

and

$$f_2(x) := \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

then

$$\partial f(x) = \begin{cases} (\frac{1}{2\sqrt{x}}, 1) & \text{if } x > 0, \\ \{(t, l) : t \geq 0, l \in [0, 1]\} & \text{if } x = 0, \\ (0, 0) & \text{if } x < 0. \end{cases}$$

Furthermore,

$$\langle \partial f(x), \eta(y, x) \rangle = \begin{cases} \{(\frac{\alpha(y-x)}{2\sqrt{x}}, \alpha(y - x))\} & \text{if } x > 0, \\ \{(\alpha t y, \alpha l y) : t \geq 0, l \in [0, 1]\} & \text{if } x = 0, \\ \{(0, 0\} & \text{if } x < 0. \end{cases}$$

and

$$\langle \partial f(y), \eta(x, y) \rangle = \begin{cases} \{(\frac{\alpha(x-y)}{2\sqrt{y}}, \alpha(x - y))\} & \text{if } y > 0, \\ \{(\alpha t x, \alpha l x) : t \geq 0, l \in [0, 1]\} & \text{if } y = 0, \\ \{(0, 0)\} & \text{if } y < 0. \end{cases}$$

One can show that ∂f is C-weakly properly quasimonotone. Moreover, there are a nonempty compact set $M = [-4, 0]$ and a nonempty compact convex set $B = \{0\} \subset K$ such that for each $y \in K \setminus M$, there exists $x = 0 \in B$ such that

$$\langle \partial f(y), \eta(0, y) \rangle = \{(\frac{-\alpha y}{2\sqrt{y}}, -\alpha y)\} \subseteq -\text{int}C(y).$$

Hence, Theorem 5.6 implies that $y = 0$ is a solution of (SWVLI). If

$$\langle \partial f(x), \eta(y, x) \rangle \subseteq \text{int}C(y)$$

then,

$$\langle \partial f(y), \eta(x, y) \rangle \subseteq -\text{int}C(y),$$

therefore, ∂f is C-strictly quasimonotone with respect to η on K. Since η is continuous by Lemma 5.1, $y = 0$ is also a solution for (MWVLI). Now as f is pseudoinvex, Corollary 5.2 implies that the weak (VP) has $y = 0$ as a solution.

6 Conclusion

Giannessi [12] has shown that the equivalence between efficient solutions of differentiable convex optimization problem and solutions of a variational inequality of Minty type. Yang and Yang [31] have established the relations between vector variational-like inequality problems and vector optimization problems for differentiable pseudoinvex and quasiinvex functions. Here, we show that similar relations between solutions of vector variational-like inequalities and solutions of vector optimization problems for non-differentiable pseudoinvex and quasiinvex functions also hold.

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References

1. Ansari, Q.H., Yao, J.C.: On non-differentiable and nonconvex vector optimization problems. J. Optim. Theory Appl. **106**, 475–488 (2000)
2. Clarke, F.H., Stern, R.J., Ledyaev, Y.S., Wolenski, P.R.: Nonsmooth Analysis and Control Theory. Springer, New York (1998)
3. Chen, G.Y.: Existence of solutions for a vector variational inequality. An extension of Hartman-Stampacchia theorem. J. Optim. Theory Appl. **74**, 445–456 (1992)
4. Chen, G.Y., Yang, X.Q.: The vector complementarity problem and its equivalence with the weak minimal element in ordered spaces. J. Math. Anal. Appl. **153**, 136–158 (1990)
5. Chiang, Y.: Semicontinuous mapping in t. v. s. with applications to mixed vector variational like inequalities. J. Global Optim. **32**, 467–486 (2005)
6. Chinaie, M., Jabarootian T., Rezaie, M., Zafarani, J.: Minty's lemma and vector variational-like inequalities. J. Global Optim. **40**, 463–473 (2008)
7. Daniilidis, A., Hadjisavvas, N.: Existence theorems for vector variational inequalities. Bull. Austral. Math. Soc. **54**, 473–481 (1996)
8. Daniilidis, A., Hadjisavvas, N.: On generalized cyclically monotone operators and proper quasimonotonicity. Optimization **47**, 123–135 (2000)
9. Fakhar, M., Zafarani, J.: Generalized vector equilibrium problems for pseudomonotone bifunctions. J. Optim. Theory Appl. **126**, 109–124 (2005)
10. Fang, Y.P., Huang, N.J.: On minty vector prevariational inequalities and vector optimization problems. Manuscript (2005)
11. Giannessi, F.: Theorems of the alternative quadratic programs and complementarity problems. In: Cottle, R.W., Giannessi, F., Lions, J.L. (eds.) Variational Inequalities and Complementarity Problems, pp. 151–186. Wiley, Chichester (1980)
12. Giannessi, F.: On Minty Variational Principle. New Trends in Mathematical Programming. Kluwer Academic Publishers, Dordrecht, Netherlands (1997)
13. Giannessi, F. (ed.): Vector Variational Inequalities and Vector Equilibria. Kluwer Academic Publishers, Dordrecht (2000)
14. Giannessi, F., Maugeri, A.: Variational Inequalities and Network Equilibrium Problems. Plenum Press, New York (1995)
15. Giannessi, F., Maugeri, A.: Variational Analysis and Applications. Non-convex Optimization and Its Applications. Vol. 79, Springer, New York (2005)
16. Jabarootian, T., Zafarani, J.: Generalized invariant monotonicity and invexity of nondifferentiable functions. J. Global. Optim. **36**, 537–564 (2004)
17. Jabarootian, T., Zafarani, J.: Generalized vector variational-like inequalities. J. Optim. Theory Appl. **136**, 15–30 (2008)
18. Konnov, I.V., Yao, J.C.: On the generalized vector variational inequality problem. J. Math. Anal. Appl. **206**, 42–58 (1997)
19. Lee, G.M., Lee, K.B.: Vector variational inequalities for non-differentiable convex vector optimization problems. J. Global Optim. **32**, 597–612 (2005)
20. Mishra, S.K., Wang, S.Y.: Vector variational-like inequalities and non-smooth vector optimization problems. Nonlinear Anal. **64**, 1939–1945 (2005)
21. Mohan, S.R., Neogy, S.K.: On invex sets and priinvex functions. J. Math. Anal. Appl. **189**, 901–908 (1995)
22. Santos, L.B., Medar, M.R., Lizana, A.R.: Some relations between variational-like inequalities and efficient solutions of certain non-smooth optimizations problems. Int. J. Math. Math. Sci. Art. ID 26808, 16 pp (2006)
23. Weir, T., Mond, B.: Preinvex functions in multiple-objective optimization. J. Math. Anal. Appl. **136**, 29–38 (1988)

24. Yang, X.Q.: On vector variational inequalities: application to vector equilibria. J. Optim. Theory Appl. **95**, 729–734 (1997)
25. Yang, X.M.: Generalized convexity in optimization. Ph.D. Dissertation (2002)
26. Yang, X.Q., Goh, C.J.: On vector variational inequality with application to vector traffic equilibria. J. Optim. Theory Appl. **95**, 431–443 (1997)
27. Zhao, Y., Xia, Z.: Existence results for systems of vector variational-like inequalities. Non-Linear Anal:Real World Appl. **8**, 1370–1378 (2007)
28. Yang, X.M., Yang, X.Q., Teo, K.L.: Generalized invexity and generalized invariant monotonicity. J. Optim. Theory Appl. **117**, 607–625 (2003)
29. Yang, X.M., Yang, X.Q., Teo, K.L.: Some remarks on the minty vector variational inequality. J. Optim. Theory Appl. **121**, 193–201 (2004)
30. Yang, X.M., Yang, X.Q., Teo, K.L.: Criteria for generalized invex monotonicities. Eur. J. Oper. Res. **164**, 115–119 (2005)
31. Yang, X.M., Yang, X.Q.: Vector variational-like inequality with pseudoinvexity. Optimization **55**, 157–170 (2006)