J Glob Optim (2008) 41:1–13
DOI 10.1007/s10898-007-9205-6

Lipschitz behavior of convex semi-infinite optimization problems: a variational approach

Maria J. Cánovas · Abderrahim Hantoute · Marco A. López · Juan Parra

Received: 3 October 2006 / Accepted: 22 June 2007 / Published online: 31 July 2007

Abstract In this paper we make use of subdifferential calculus and other variational techniques, traced out from [Ioffe, A.D.: Metric regularity and subdifferential calculus. Uspekhi Mat. Nauk 55, **3**(333), 103–162; Engligh translation Math. Surveys **55**, 501–558 (2000); Ioffe, A.D.: On rubustness of the regularity property of maps. Control cybernet **32**, 543–554 (2003)], to derive different expressions for the Lipschitz modulus of the optimal set mapping of canonically perturbed convex semi-infinite optimization problems. In order to apply this background for obtaining the modulus of metric regularity of the associated inverse multifunction, we have to analyze the stable behavior of this inverse mapping. In our semi-infinite framework this analysis entails some specific technical difficulties. We also provide a new expression of a global variational nature for the referred regularity modulus.

Keywords Convex semi-infinite programming · Metric regularity · Optimal set · Lipschitz modulus

Mathematics Subject Classification 90C34 · 49J53 · 90C25 · 90C31

M. J. Cánovas · A. Hantoute · J. Parra (✉)
Operations Research Center, Miguel Hernández University of Elche,
03202, Elche, Alicante, Spain
e-mail: parra@umh.es

M. J. Cánovas
e-mail: canovas@umh.es

A. Hantoute
e-mail: ahantoute@umh.es

M. A. López
Department of Statistics and Operations Research, University of Alicante,
03071, Alicante, Spain
e-mail: marco.antonio@ua.es

1 Introduction

This paper is in the line of some recent developments devoted to use variational tools to quantify the Lipschitz behavior of optimization problems. We appeal to *Ekeland's variational principle* and some subdifferential calculus related to the metric regularity property, traced out from Ioffe [7,8], to quantify the Lipschitz behavior of the optimal solutions set in convex semi-infinite optimization. Several authors have devoted a notable effort to analyze different notions and general tools related to metric regularity. See, for instance, Azé, Corvellec, and Lucchetti [1], Dontchev, Lewis and Rockafellar [3], Henrion and Klatte [6], Ioffe [7,8], and Klatte and Kummer [9], among others. See also Mordukhovich [10] and Rockafellar and Wets [12] for a comprehensive overview on variational analysis.

Our aim is to quantify the Lipschitz behavior of the optimal set of the canonically perturbed convex programming problem, in $\mathbb{R}^n$,

$$\begin{aligned} P(c,b):\ &\operatorname{Inf}\ f(x)+\langle c,x\rangle \\ &\text{s.t. } g_t(x)\le b_t,\quad t\in T, \end{aligned} \tag{1}$$

where $x\in\mathbb{R}^n$ is the vector of decision variables, $c\in\mathbb{R}^n$, $\langle .,.\rangle$ represents the usual inner product in $\mathbb{R}^n$, T is a compact metric index space (this assumption covers the case in which T is a finite set), $b\in C(T,\mathbb{R})$, i.e., $t\mapsto b_t$ is continuous on T, and $f, g_t:\mathbb{R}^n\to\mathbb{R}$, $t\in T$, are given convex functions such that $t\mapsto g_t(x)$ is continuous on T for each $x\in\mathbb{R}^n$. In this case [11, Thm. 10.7] ensures that $(t,x)\mapsto g_t(x)$ is continuous on $T\times\mathbb{R}^n$. In our setting the pair $(c,b)\in\mathbb{R}^n\times C(T,\mathbb{R})$ is regarded as the parameter to be perturbed. Sometimes we appeal to the constraint system of $P(c,b)$, which will be denoted by $\sigma(b)$; i.e.,

$$\sigma(b):=\{g_t(x)\le b_t,\quad t\in T\}.$$

The parameter space $\mathbb{R}^n\times C(T,\mathbb{R})$ is endowed with the norm

$$\|(c,b)\|:=\max\{\|c\|,\|b\|_\infty\}, \tag{2}$$

where $\mathbb{R}^n$ is equipped with any given norm $\|\cdot\|$ and $\|b\|_\infty:=\max_{t\in T}|b_t|$. The corresponding dual norm in $\mathbb{R}^n$ is given by $\|u\|_*:=\max\{\langle u,x\rangle\mid\|x\|\le 1\}$, and d_* denotes the related distance.

Associated with the parametric family of problems $P(c,b)$, we consider the *feasible set mapping*, $\mathcal{F}:C(T,\mathbb{R})\rightrightarrows\mathbb{R}^n$, which assigns to each $b\in C(T,\mathbb{R})$ the feasible set of $\sigma(b)$,

$$\mathcal{F}(b):=\left\{x\in\mathbb{R}^n\mid g_t(x)\le b_t,\quad t\in T\right\},$$

and the *optimal set mapping*, $\mathcal{F}^*:\mathbb{R}^n\times C(T,\mathbb{R})\rightrightarrows\mathbb{R}^n$, assigning to each parameter $(c,b)\in\mathbb{R}^n\times C(T,\mathbb{R})$ the *optimal set* of $P(c,b)$; i.e.,

$$\mathcal{F}^*(c,b):=\arg\min\{f(x)+\langle c,x\rangle\mid x\in\mathcal{F}(b)\}.$$

We also consider the inverse multifunction

$$\mathcal{G}^*:=\left(\mathcal{F}^*\right)^{-1},$$

given by

$$(c,b)\in\mathcal{G}^*(x)\Leftrightarrow x\in\mathcal{F}^*(c,b).$$

Our analysis is focused on the *metric regularity* of $\mathcal{G}^*$ at a given point $\left(\overline{x},\left(\overline{c},\overline{b}\right)\right)\in\operatorname{gph}(\mathcal{G}^*)$ (the graph of $\mathcal{G}^*$), that is the existence of a constant $\kappa\ge 0$ and associated neighborhoods U of $\overline{x}$ and V of $\left(\overline{c},\overline{b}\right)$ such that

$$d\left(x, \mathcal{F}^*(c,b)\right) \leq \kappa d(c,b), \mathcal{G}^*(x)), \tag{3}$$

for all $x \in U$ and all $(c,b) \in V$, where, as usual, $d(x, \emptyset) = +\infty$.

Lipschitz properties of a set-valued mapping are known to be equivalent to certain regularity notions of its inverse (see for instance [9] and references therein). In particular the *metric regularity* of $\mathcal{G}^*$ at $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) \in \text{gph}\,(\mathcal{G}^*)$ is equivalent to the so-called *Aubin property* (also called *pseudo-Lipschitz*) of $\mathcal{F}^*$ at $\left((\overline{c}, \overline{b}), \overline{x}\right) \in \text{gph}\,(\mathcal{F}^*)$. Moreover, in our context of problems (1) the Aubin property of $\mathcal{F}^*$ at $\left((\overline{c}, \overline{b}), \overline{x}\right)$ turns out to be equivalent to the *strong Lipschitz stability* of $\mathcal{F}^*$ at this point (see Lemma 5 in [2]); that is, to *local single-valuedness* and *Lipschitz continuity* of $\mathcal{F}^*$ near $\left(\bar{c}, \bar{b}\right)$.

Specifically, this paper is devoted to characterize the infimum of (Lipschitz) constants κ satisfying (3) for some associated neighborhoods U and V, which is called *modulus- or rate- of metric regularity* (also *regularity modulus*) of $\mathcal{G}^*$ at $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) \in \text{gph}\,(\mathcal{G}^*)$, and it is denoted by $\text{reg}\,\mathcal{G}^*\left(\bar{x} \mid \left(\bar{c}, \bar{b}\right)\right)$. The relevance of this notion is emphasized through the fact that the distance in the right hand side of (3) is usually easier to compute or estimate than the left hand side distance. We use the convention $\text{reg}\,\mathcal{G}^*\left(\bar{x} \mid \left(\bar{c}, \bar{b}\right)\right) = +\infty$ when $\mathcal{G}^*$ is not metrically regular at $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right)$. Having the previous comments in mind, it is clear that this regularity modulus coincides with the *Lipschitz modulus* of $\mathcal{F}^*$ at the nominal parameter, i.e.,

$$\text{reg}\,\mathcal{G}^*\left(\bar{x} \mid \left(\bar{c}, \bar{b}\right)\right) = \text{lip}\,\mathcal{F}^*\left(\bar{c}, \bar{b}\right) := \limsup_{\substack{(c,b),(\widetilde{c},\widetilde{b}) \to (\overline{c},\overline{b}) \\ (c,b) \neq (\widetilde{c},\widetilde{b})}} \frac{\left\| x(c,b) - x(\widetilde{c}, \widetilde{b}) \right\|}{\left\| (c,b) - (\widetilde{c}, \widetilde{b}) \right\|}, \tag{4}$$

where $x(c,b)$ represents the unique optimal solution of $P(c,b)$ (i.e., $\mathcal{F}^*(c,b) = \{x(c,b))\}$ for (c,b) close enough to $(\overline{c}, \overline{b})$). Here "lim sup" is understood, as usual, as the supremum of all possible "sequential $\limsup_{r\to+\infty}$".

At this point we summarize the structure of the paper. Section 2 presents some preliminary results, including the version of Ekeland's variational principle used in the paper and some concepts from subdifferential calculus. Section 3 analyzes some stability properties of $\mathcal{G}^*$ in order to guarantee the applicability of Theorem 1 in Sect. 2 (traced out from Ioffe [8]). In Sect. 4 we apply this theorem and also provide new expressions for the regularity modulus by using the referred variational tools.

2 Preliminaries

In this section we provide further notation and some preliminary results. Given $\emptyset \neq X \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by cone (X) the *conical convex hull* of X. It is assumed that cone (X) always contains the zero-vector, 0_k, and so $\text{cone}(\emptyset) = \{0_k\}$. If y is a point in any metric space, we denote by $B_\delta(y)$ the open ball centered at y with radius δ, whereas the corresponding closed ball is represented by $\overline{B}_\delta(y)$.

Given a consistent system $\sigma\,(b)$, for any $x \in \mathcal{F}\,(b)$, we consider

$$T_b(x) := \{t \in T \mid g_t(x) = b_t\} \text{ and } A_b(x) := \text{cone}\left(\cup_{t\in T_b(x)} \left(-\partial g_t(x)\right)\right),$$

where, for all t, $\partial g_t(x)$ represents the ordinary subdifferential of the convex function g_t at x.

Our system $\sigma\,(b)$ satisfies the *Slater constraint qualification* (SCQ) if $T_b(x^0)$ is empty for some feasible point x^0, in which case x^0 is called a Slater point of $\sigma\,(b)$.

At this moment we recall the well-known *Karush-Kuhn-Tucker* optimality conditions:

Lemma 1 (see [5, Ch. 7]) *Let* $(c, b) \in \mathbb{R}^n \times C(T, \mathbb{R})$ *and* $x \in \mathcal{F}(b)$. *If*

$$(c + \partial f(x)) \cap A_b(x) \neq \emptyset$$

then $x \in \mathcal{F}^*(c, b)$. *The converse holds when* $\sigma(b)$ *satisfies the SCQ.*

It is obvious from (3) that the metric regularity of $\mathcal{G}^*$ at $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) \in \text{gph}\,(\mathcal{G}^*)$ entails SCQ for $\sigma(\overline{b})$. In [2], a sufficient condition for the metric regularity of $\mathcal{G}^*$ is provided in terms of the nominal problem's data. Throughout the paper we appeal to the following result, already announced in Sect. 1:

Proposition 1 [2, Thm. 10 and Lem. 5] *For the convex program* (1), $\mathcal{G}^*$ *is metrically regular at* $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) \in \text{gph}\,(\mathcal{G}^*)$ *if and only if* $\mathcal{F}^*$ *is single-valued and Lipschitz continuous in a neighborhood of* $\left(\bar{c}, \bar{b}\right)$.

The following proposition states Ekeland's variational principle in the way it is used in the paper.

Proposition 2 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous function bounded from below, and let* x *and* $\varepsilon > 0$ *be such that* $f(x) < \inf_{\mathbb{R}^n} f + \varepsilon$. *Then, for every* $\delta > 0$, *there exists a point* $z \in B_{\varepsilon/\delta}(x)$ *with* $f(z) \leq f(x)$ *and*

$$f(z) < f(y) + \delta \|y - z\|, \text{ for every } y \neq z,$$

i.e., z *is a strict global minimum of the perturbed function* $f + \delta \|\cdot - z\|$.

We shall use the following notions of general derivatives:

Definition 1 Consider a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a point $z \in \mathbb{R}^n$ with $f(z)$ finite.

(a) [4] The *strong slope* of f at $z \in \mathbb{R}^n$ is given by

$$|\nabla f|(z) := \limsup_{\substack{y \to z \\ y \neq z}} \frac{(f(z) - f(y))^+}{\|z - y\|},$$

where $\alpha^+ := \max\{\alpha, 0\}$ is the positive part of α.

(b) [12, Defs. 8.1 and 8.3, and Exa. 8.4] Given $w \in \mathbb{R}^n$, the *(lower) subderivative* of f at z for w is defined by

$$df(z)(w) := \liminf_{\tau \searrow 0,\ w' \to w} \frac{f\left(z + \tau w'\right) - f(z)}{\tau},$$

and a vector $v \in \mathbb{R}^n$ is called a *regular subgradient* of f at z, written $v \in \widehat{\partial} f(z)$, if

$$df(z)(w) \geq \langle v, w \rangle \text{ for all } w \in \mathbb{R}^n.$$

The regular subdifferential $\widehat{\partial} f(z)$ is a closed convex subset. If, in addition, f is lower semicontinuous, then, according to [7, Prop. 3 in p. 546], we have

$$\inf_{z \in U} |\nabla f|(z) = \inf_{z \in U} d_*(0_n, \widehat{\partial} f(z)), \tag{5}$$

for any open set $U \subset \mathbb{R}^n$. Finally, if f is a proper convex function, then, according to [12, Prop. 8.12], $\widehat{\partial} f(z)$ coincides with the ordinary subdifferential set in convex analysis, denoted by $\partial f(z)$,

$$\widehat{\partial} f(z) = \partial f(z) := \{v \in \mathbb{R}^n \mid f(x) \geq f(z) + \langle v, x - z \rangle \text{ for all } x\}.$$

The next key result will lead to different expressions of the regularity modulus of $\mathcal{G}^*$, as we show in Sect. 4.

Theorem 1 [8, Thm. 2.2] *Let Y be a Banach space and let $F : \mathbb{R}^n \rightrightarrows Y$ be a set-valued mapping with a nonempty closed graph. Let $(\overline{x}, \overline{y}) \in \operatorname{gph}(F)$ and assume that the functions*

$$\psi_y := d\,(y, F(\cdot))$$

are lower semicontinuous for all y in a neighborhood of $\overline{y}$. Then

$$\operatorname{reg} F\,(\overline{x} \mid \overline{y}) = \limsup_{\substack{(x,y)\to(\overline{x},\overline{y}) \\ y\notin F(x)}} \left(\left|\nabla\psi_y\right|(x)\right)^{-1} = \limsup_{\substack{(x,y)\to(\overline{x},\overline{y}) \\ y\notin F(x)}} \left(d_*(0_n, \widehat{\partial}\psi_y(x))\right)^{-1}.$$

The second equality comes straightforwardly from (5).

3 Stability of $\mathcal{G}^*$

In order to apply Theorem 1 to $\mathcal{G}^*$ at a given point $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right)$ of its graph, we have to solve some technical problems to guarantee the fulfillment of the hypotheses of that theorem. This leads us to analyze the behavior of $\mathcal{G}^*$ in relation to some stability properties.

The following example shows that, in general, $\operatorname{gph}(\mathcal{G}^*)$ is not closed.

Example 1 Consider the linear optimization problem, in $\mathbb{R}$,

$$\begin{aligned} P(c, b) : \ & \mathit{Inf}\ cx \\ & s.t.\ tx \le b_t, \quad t \in T := [0, 1]\,. \end{aligned}$$

For each $r = 2, 3, \ldots$, let $x^r = -1$, $c^r = -1$ and b^r be the piecewise affine function determined by $b_0^r = 1/r$, $b_{1/r}^r = -1/r$, $b_{2/r}^r = 0$, and $b_1^r = 0$. Obviously $\{(x^r, (c^r, b^r))\}_{r\ge 2}$ converges to $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) := (-1, (-1, 0_T))$, where 0_T denotes the zero function on T. The reader can check that, for all r, $\mathcal{F}(b^r) =]-\infty, -1]$ and, consequently, $\mathcal{F}^*(c^r, b^r) = \{-1\}$; i.e., $(x^r, (c^r, b^r)) \in \operatorname{gph}(\mathcal{G}^*)$. However, $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) \notin \operatorname{gph}(\mathcal{G}^*)$, since $\mathcal{F}^*\left(\overline{c}, \overline{b}\right) = \{0\}$.

The main fact behind Example 1 is that the "limit constraint system" $\sigma\left(\overline{b}\right)$ does not satisfy SCQ. Observe that $\sigma(b^r)$ does satisfy SCQ for all r. In this section we show that a closed graph may be obtained by intersecting the images of $\mathcal{G}^*$ with an appropriate neighborhood, V, of $\left(\overline{c}, \overline{b}\right)$. In order to apply Theorem 1, the choice of this neighborhood turns out to be crucial for establishing, not only the closedness of $\operatorname{gph}(\mathcal{G}^*(\cdot) \cap V)$, but also the lower semicontinuity of the associated distance functions $d\,((c, b), \mathcal{G}^*(\cdot) \cap V)$ for (c, b) close to $\left(\overline{c}, \overline{b}\right)$.

From now on assume that $\sigma\left(\overline{b}\right)$ satisfies SCQ, and consider $x^0 \in \mathbb{R}^n$ and $\rho > 0$ such that

$$g_t\left(x^0\right) \le \overline{b}_t - 2\rho \text{ for all } t \in T.$$

We define

$$W := \left\{b \in C\,(T, \mathbb{R}) \mid b_t \ge g_t\left(x^0\right) + \rho \text{ for all } t \in T\right\}. \tag{6}$$

Note that, W is a closed neighborhood of $\overline{b}$ containing $\overline{B}_\rho\left(\overline{b}\right)$. Define

$$V := \mathbb{R}^n \times W, \tag{7}$$

and introduce the mapping $\mathcal{G}_V^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times C\,(T, \mathbb{R})$ given by

$$\mathcal{G}_V^*(x) := \mathcal{G}^*(x) \cap V. \tag{8}$$

Sometimes along the paper we appeal to the following technical lemma:

Lemma 2 *Let* $\{(x^r, (c^r, b^r))\}_{r\in\mathbb{N}} \subset \mathrm{gph}(\mathcal{G}_V^*)$ *be a sequence such that* $\{x^r\}$ *and* $\{c^r\}$ *converge, respectively, to certain* x *and* c *in* $\mathbb{R}^n$. *For each* $r \in \mathbb{N}$ *let us write* (*according to KKT conditions and taking Caratheodory's theorem into account*)

$$c^r + u^r = \sum_{i=1}^{n} \lambda_i^r u_i^r, \tag{9}$$

where, for all $r \in \mathbb{N}$ *and all* $i \in \{1, \ldots, n\}$,

$$u^r \in \partial f\left(x^r\right),\ u_i^r \in -\partial g_{t_i^r}\left(x^r\right) \text{ for some } t_i^r \in T_{b^r}\left(x^r\right), \text{ and } \lambda_i^r \geq 0. \tag{10}$$

Then there exists a subsequence of r*'s* (*denoted as the original sequence for the sake of simplicity*) *such that*

$$u^r \to u,\ u_i^r \to u_i, t_i^r \to t_i, \text{ and } \lambda_i^r \to \lambda_i, i = 1, \ldots, n, \tag{11}$$

for certain

$$u \in \partial f(x),\ u_i \in -\partial g_{t_i}(x),\ t_i \in T, \text{ and } \lambda_i \geq 0, \tag{12}$$

verifying

$$c + u = \sum_{i=1}^{n} \lambda_i u_i. \tag{13}$$

If, moreover, $\{b^r\}$ *converges to a certain* $b \in W$, *then*

$$t_i \in T_b(x) \text{ for } i = 1, \ldots, n, \text{ and } (c, b) \in \mathcal{G}_V^*(x). \tag{14}$$

Remark before the proof. Since $b^r \in W$, the system $\sigma(b^r)$ satisfies SCQ.

Proof Along the proof, all the subsequences (corresponding to a successive filtering) will be indexed by $r \in \mathbb{N}$. The compactness of T entails, for some subsequence of r's, $t_i^r \to t_i$, for some $t_i \in T$, $i = 1, \ldots, n$. From [11, Thms. 23.2 and 23.4], all the subdifferentials involved are compact (note that f and g_t's in (1) are finite convex functions on $\mathbb{R}^n$). Thus, taking the continuity of the function $(t, x) \longmapsto g_t(x)$ into account, [11, Thm. 24.5] yields, for a new subsequence of r's, $u^r \to u$ and $u_i^r \to u_i$ for some $u \in \partial f(x)$, $u_i \in -\partial g_{t_i}(x)$, $i = 1, \ldots, n$.

Next we see that $\left\{\sum_{i=1}^n \lambda_i^r\right\}_{r\in\mathbb{N}}$ must be bounded (which constitutes a Gauvin's type result). Otherwise, set $\gamma_r := \sum_{i=1}^n \lambda_i^r$. Since each $\left\{\lambda_i^r/\gamma_r\right\}_{r\in\mathbb{N}}$ is bounded, we may assume (by taking suitable subsequences) $\lambda_i^r/\gamma_r \to \beta_i \geq 0$, for $i = 1, \ldots, n$, with $\sum_{i=1}^n \beta_i = 1$. Then, dividing both sides of (9) by γ_r and letting $r \to +\infty$, we obtain

$$0_n = \Sigma_{i=1}^n \beta_i u_i. \tag{15}$$

Moreover, from (6) and (10) we have, for each r, recalling $g_{t_i^r}(x^r) = b_{t_i^r}^r$,

$$\left\langle u_i^r, x^0 - x^r\right\rangle \leq g_{t_i^r}\left(x^0\right) - g_{t_i^r}\left(x^r\right) \leq -\rho,$$

and then, letting $r \to +\infty$,

$$\left\langle u_i, x^0 - x\right\rangle \leq -\rho.$$

Consequently, appealing to (15),

$$0 = \left\langle \Sigma_{i=1}^n \beta_i u_i, x^0 - x\right\rangle \leq \left(\Sigma_{i=1}^n \beta_i\right)(-\rho) = -\rho,$$

which is clearly a contradiction.

Once we have established the boundedness of $\left\{\sum_{i=1}^{n}\lambda_i^r\right\}_{r\in\mathbb{N}}$, we may assume, for a suitable subsequence of r's, that each $\left\{\lambda_i^r\right\}_{r\in\mathbb{N}}$ converges to certain $\lambda_i \geq 0$, for $i = 1, \ldots, n$, and (13) holds.

Moreover, if $\{b^r\}$ converges to $b \in C(T, \mathbb{R})$, then (14) follows from the fact that $t_i^r \in T_{b^r}(x^r)$ for all r, together with (13). □

Next we introduce the distance functions $f_{c,b} : \mathbb{R}^n \to [0, +\infty]$, where $(c, b) \in V$, given by

$$f_{c,b}(x) := d\left((c, b), \mathcal{G}_V^*(x)\right). \tag{16}$$

The following theorem enables us to apply Theorem 1 to $\mathcal{G}_V^*$.

Theorem 2 *Let* $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) \in \mathrm{gph}(\mathcal{G}^*)$ *be such that* $g_t\left(x^0\right) \leq \overline{b}_t - 2\rho$ *for all* $t \in T$ *and certain* $x^0 \in \mathbb{R}^n$ *and* $\rho > 0$. *Let* V *be defined as in* (7)*, and consider* $\mathcal{G}_V^*$ *given by* (8). *Then*

(i) $\mathrm{gph}(\mathcal{G}_V^*)$ *is closed;*
(ii) $f_{c,b}$ *is finite-valued and lower semicontinuous on* $\mathbb{R}^n$ *for all* $(c, b) \in V$.

Proof (i) Let $\{(x^r, (c^r, b^r))\}_{r\in\mathbb{N}} \subset \mathrm{gph}(\mathcal{G}_V^*)$ converging to $(x, (c, b)) \in \mathbb{R}^n \times V$ (recall that V is closed). Then, Lemma 2 entails $(x, (c, b)) \in \mathrm{gph}(\mathcal{G}_V^*)$.

(ii) To begin with, note that, for every $x \in \mathbb{R}^n$, $(\widehat{c}, \widehat{b}) \in \mathcal{G}_V^*(x)$, with

$$\widehat{c} \in -\partial f(x) \text{ and } \widehat{b}_t := \max\left\{g_t(x), g_t\left(x^0\right) + \rho\right\}, t \in T.$$

Then, $f_{c,b}(x) \leq d((c, b), (\widehat{c}_n, \widehat{b})) < +\infty$.

Assume, reasoning by contradiction, that, for some $(c, b) \in V$, $f_{c,b}$ fails to be lower semicontinuous at certain $\widetilde{x} \in \mathbb{R}^n$. Then, there will exist positive scalars α and β and a certain sequence $\{x^r\}_{r\in\mathbb{N}} \subset \mathbb{R}^n$ converging to $\widetilde{x}$ such that

$$f_{c,b}(x^r) < \alpha < \beta < f_{c,b}(\widetilde{x}), \text{ for all } r \in \mathbb{N}.$$

Thus, for each r, there exists $(c^r, b^r) \in \mathcal{G}_V^*(x^r)$ satisfying

$$f_{c,b}(x^r) \leq d((c, b), (c^r, b^r)) < \alpha. \tag{17}$$

We may assume w.l.o.g. that $\{c^r\}_{r\in\mathbb{N}}$ converges to some $\widetilde{c}$ since $\|c^r - c\| < \alpha$ for all $r \in \mathbb{N}$.

We proceed by constructing a suitable $\widetilde{b} \in W$ such that

$$(\widetilde{c}, \widetilde{b}) \in \mathcal{G}_V^*(\widetilde{x}) \text{ and } d((c, b), (\widetilde{c}, \widetilde{b})) \leq \beta, \tag{18}$$

leading to the contradiction

$$f_{c,b}(\widetilde{x}) \leq d((c, b), (\widetilde{c}, \widetilde{b})) \leq \beta < f_{c,b}(\widetilde{x}).$$

To do this, we apply Lemma 2 to the sequence $\{(x^r, (c^r, b^r))\}_{r\in\mathbb{N}} \subset \mathrm{gph}(\mathcal{G}_V^*)$, and consider the associated $T_{b^r}(x^r) \ni t_i^r \to t_i \in T$, for $i = 1, \ldots, n$, as well as

$$u \in \partial f(\widetilde{x}), \ u_i \in -\partial g_{t_i}(\widetilde{x}), \quad \text{and} \quad \lambda_i \geq 0, \ i = 1, \ldots, n,$$

such that

$$\widetilde{c} + u = \sum_{i=1}^{n} \lambda_i u_i. \tag{19}$$

Next we show that

$$\left|b_{t_i} - g_{t_i}(\widetilde{x})\right| \leq \alpha, \quad \text{for } i = 1, \ldots, n. \tag{20}$$

In fact, we have, for each $i = 1, \ldots, n$ and each $r \in \mathbb{N}$, and due to (17),

$$\left|b_{t_i^r} - g_{t_i^r}\left(x^r\right)\right| = \left|b_{t_i^r} - b_{t_i^r}^r\right| \leq \left\|b - b^r\right\|_\infty \\ \leq d((c, b), (c^r, b^r)) < \alpha.$$

Letting $r \to \infty$, taking the continuity of the function $(t, x) \longmapsto g_t(x)$ into account, we obtain (20).

Next, we apply Urysohn's lemma to conclude the existence of $\varphi \in C(T, [0, 1])$ such that

$$\varphi(t) = \begin{cases} 1 \text{ if } t \in \{t_1, \ldots, t_n\}, \\ 0 \text{ if } |b_t - g_t(\widetilde{x})| \geq \beta. \end{cases}$$

In the case when the latter set is empty we may take $\varphi \equiv 1$. Note that, according to (20), $\{t_1, \ldots, t_n\}$ and $\{t \in T : |b_t - g_t(\widetilde{x})| \geq \beta\}$ are closed disjoint subsets of T.

Define, for each $t \in T$,

$$\widetilde{b}_t := (1 - \varphi(t)) \max\{b_t, g_t(\widetilde{x})\} + \varphi(t) \max\left\{g_t\left(x^0\right) + \rho, g_t(\widetilde{x})\right\}. \tag{21}$$

Next we check, in three steps, that (18) holds.

Step 1 $(\widetilde{c}, \widetilde{b}) \in \mathcal{G}^*(\widetilde{x})$. From (21) we observe that $\widetilde{b}_t \geq g_t(\widetilde{x})$ for all $t \in T$; i.e., $\widetilde{x} \in \mathcal{F}(\widetilde{b})$. Moreover, we can prove that

$$\widetilde{b}_{t_i} = \max\left\{g_{t_i}\left(x^0\right) + \rho, g_{t_i}(\widetilde{x})\right\} = g_{t_i}(\widetilde{x}), \text{ for } i = 1, \ldots, n.$$

To see this, observe that

$$g_{t_i^r}\left(x^r\right) = b_{t_i^r}^r \geq g_{t_i^r}\left(x^0\right) + \rho, \quad \text{for } i = 1, \ldots, n \text{ and all } r,$$

because, for each r, $(c^r, b^r) \in \mathcal{G}_V^*(x^r)$ and, in particular, $b^r \in W$ (see (6)). Then, letting $r \to \infty$ we get

$$g_{t_i}(\widetilde{x}) \geq g_{t_i}\left(x^0\right) + \rho, i = 1, \ldots, n.$$

In other words we have checked that $t_i \in T_{\widetilde{b}}(\widetilde{x})$, for all i, and (19) provides the KKT optimality conditions yielding $(\widetilde{c}, \widetilde{b}) \in \mathcal{G}^*(\widetilde{x})$.

Step 2 Next we prove that $\widetilde{b} \in W$ and, hence, $(\widetilde{c}, \widetilde{b}) \in \mathcal{G}_V^*(\widetilde{x})$. Actually, from (21) we have, for all t,

$$\widetilde{b}_t \geq (1 - \varphi(t)) b_t + \varphi(t)\left(g_t\left(x^0\right) + \rho\right) \geq g_t\left(x^0\right) + \rho,$$

because $b \in W$.

Step 3 Finally we prove the other statement in (18); i.e., $d((c, b), (\widetilde{c}, \widetilde{b})) \leq \beta$. Since $\|c^r - c\| < \alpha < \beta$ for every r, and then $\|\widetilde{c} - c\| \leq \beta$, we only have to prove $\left\|\widetilde{b} - b\right\|_\infty \leq \beta$. For each $t \in T$ we have

$$\left|\widetilde{b}_t - b_t\right| \leq (1 - \varphi(t)) |\max\{b_t, g_t(\widetilde{x})\} - b_t| \\ + \varphi(t) \left|\max\left\{g_t\left(x^0\right) + \rho, g_t(\widetilde{x})\right\} - b_t\right|. \tag{22}$$

Now let us see that

$$|\max\{b_t, g_t(\widetilde{x})\} - b_t| \leq \alpha < \beta \text{ for all } t \in T. \tag{23}$$

To see this, fix $t \in T$ in the non-trivial case $g_t(\widetilde{x}) > b_t$. Then, we have $g_t(x^r) > b_t$ for r large enough. Thus,

$$b_t^r \geq g_t\left(x^r\right) > b_t,$$

and then,

$$\left|g_t\left(x^r\right)-b_t\right| \le \left|b_t^r-b_t\right| \le \left\|b^r-b\right\|_\infty < \alpha$$

(recall (17)). Letting $r \to \infty$ we obtain (23).

In order to finally conclude that $\left\|\widetilde{b}-b\right\|_\infty \le \beta$ via (22), we only have to see that, for t in the non-trivial case $\varphi(t) > 0$, we have

$$\left|\max\left\{g_t\left(x^0\right)+\rho, g_t\left(\widetilde{x}\right)\right\}-b_t\right| < \beta.$$

If $g_t(\widetilde{x}) \ge g_t\left(x^0\right)+\rho$, the aimed inequality is a consequence of the fact that $\varphi(t) > 0$ (recall the definition of φ). Otherwise, we have

$$g_t(\widetilde{x}) < g_t\left(x^0\right)+\rho \le b_t$$

(recall that $b \in W$), and then

$$\left|g_t\left(x^0\right)+\rho-b_t\right| \le \left|g_t(\widetilde{x})-b_t\right| < \beta,$$

again due to $\varphi(t) > 0$. □

Remark 1 In the finite case (T finite), the lower semicontinuity property of $f_{c,b}$ in the previous theorem is a consequence of (i) and [12, Prop 5.11(a)].

The following example shows that, roughly speaking, the set of parameters for which some $x \in \mathbb{R}^n$ is optimal may shrink abruptly when perturbing x, i.e., the mapping $\mathcal{G}_V^*$ may fail to be (Berge) lower semicontinuous. In fact, we also show that $f_{c,b}$ may fail to be upper semicontinuous even at a point in which $\mathcal{G}^*$ is metrically regular (and, then, SCQ holds at the corresponding parameter).

Example 2 Consider the convex optimization problem, in $\mathbb{R}^2$, endowed with the Euclidean norm:

$$\begin{aligned} P(c,b): \mathit{Inf}\ & c_1x_1+c_2x_2 \\ s.t.\ & |x_1|-x_2 \le b. \end{aligned}$$

First we show that $\mathcal{G}^*$ is metrically regular at the nominal point $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) = (0_2, ((0,1),0)) \in \mathrm{gph}(\mathcal{G}^*)$. In fact, the reader can easily check that

$$\mathcal{F}^*(c,b) = \{(0,-b)\}, \quad \text{if } \|c-\overline{c}\| < \frac{1}{\sqrt{2}},$$

and thus, $\mathcal{F}^*$ is strongly by Lipschitz stable at $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right)$ and

$$\mathrm{reg}\,\mathcal{G}^*\left(\overline{x} \mid \left(\overline{c}, \overline{b}\right)\right) = \mathrm{lip}\,\mathcal{F}^*\left(\bar{c}, \bar{b}\right) = 1,$$

according to (4).

Nevertheless, we show that, for V in (7), $\mathcal{G}_V^*$ is not lower semicontinuous at $\overline{x}$ and $f_{c,b}$ is not upper semicontinuous at $\overline{x}$. Take any $\rho > 0$ and $x^0 = (0, 2\rho)$. Then,

$$V = \mathbb{R}^2 \times [-\rho, +\infty[,$$

according to (6) and (7). The reader can easily check that

$$\mathcal{G}^*(x_1,x_2) = \begin{cases} \{((-\alpha,\alpha), x_1-x_2) : \alpha \ge 0\}, & \text{if } x_1 > 0; \\ \{((\alpha,\alpha), -x_1-x_2) : \alpha \ge 0\}, & \text{if } x_1 < 0; \\ \{((\alpha_1,\alpha_2), -x_2) : \alpha_2 \ge |\alpha_1|\}, & \text{if } x_1 = 0. \end{cases}$$

Therefore,

$$\mathcal{G}_V^*(x_1, x_2) = \begin{cases} \mathcal{G}^*(x_1, x_2), & \text{if } |x_1| - x_2 \geq -\rho; \\ \varnothing & \text{if } |x_1| - x_2 < -\rho; \end{cases}$$

and, for any $(c, b) \in \mathbb{R}^2 \times \mathbb{R}$ such that $\|c - \overline{c}\| < 1/\sqrt{2}$ and any $x = (x_1, x_2)$ such that $|x_1| - x_2 \geq -\rho$, we have

$$f_{c,b}(x) = d\left((c, b), \mathcal{G}_V^*\right) = d\left((c, b), \mathcal{G}^*(x)\right)$$
$$= \begin{cases} \max\left\{\frac{|c_1 + c_2|}{\sqrt{2}}, |x_1 - x_2 - b|\right\}, & \text{if } x_1 > 0; \\ \max\left\{\frac{|c_1 - c_2|}{\sqrt{2}}, |-x_1 - x_2 - b|\right\}, & \text{if } x_1 < 0; \\ |-x_2 - b|, & \text{if } x_1 = 0. \end{cases}$$

So, for instance, $((0, 1), 0) \in \mathcal{G}_V^*(0_2)$ cannot be approached by any sequence $\{(c^r, b^r)\}$ with $(c^r, b^r) \in \mathcal{G}_V^*(1/r, 0)$ and, therefore, neither $\mathcal{G}_V^*$ nor $\mathcal{G}^*$ are lower semicontinuous at $\overline{x} = 0_2$ (both mappings coincide in a neighborhood of 0_2). Moreover, whenever $\frac{|c_1 + c_2|}{\sqrt{2}} > |-b|$ and $\|c - \overline{c}\| < 1/\sqrt{2}$, we have

$$\lim_{\substack{x \to 0_2, \\ x_1 > 0}} f_{c,b}(x) = \frac{|c_1 + c_2|}{\sqrt{2}} > |-b| = f_{c,b}(0_2),$$

and therefore $f_{c,b}$ is not upper semicontinuous at 0_2.

4 Regularity modulus of $\mathcal{G}^*$

We start by observing that intersecting the images of $\mathcal{G}^*$ with V has no influence on the regularity modulus at the nominal point.

Proposition 3 *Assume that $\mathcal{G}^*$ is metrically regular at $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) \in \operatorname{gph}(\mathcal{G}^*)$. Then $\mathcal{G}_V^*$ also has this property and*

$$\operatorname{reg} \mathcal{G}_V^*\left(\overline{x} \mid \left(\overline{c}, \overline{b}\right)\right) = \operatorname{reg} \mathcal{G}^*\left(\overline{x} \mid \left(\overline{c}, \overline{b}\right)\right).$$

Proof Observe that $\left(\mathcal{G}_V^*\right)^{-1} = \mathcal{F}_{|V}^*$, which is given by

$$\mathcal{F}_{|V}^*(c, b) = \begin{cases} \mathcal{F}^*(c, b), & \text{if } (c, b) \in V \\ \varnothing, & \text{otherwise.} \end{cases}$$

So, since V is a neighborhood of $\left(\overline{c}, \overline{b}\right)$, the strong Lipschitz stability of $\mathcal{F}^*$ at $\overline{x}$ (see Sect. 1) yields

$$\operatorname{reg} \mathcal{G}^*\left(\bar{x} \mid \left(\bar{c}, \bar{b}\right)\right) = \operatorname{lip} \mathcal{F}^*\left(\bar{c}, \bar{b}\right) = \operatorname{lip} \mathcal{F}_{|V}^*\left(\bar{c}, \bar{b}\right) = \operatorname{reg} \mathcal{G}_V^*\left(\bar{x} \mid \left(\bar{c}, \bar{b}\right)\right).$$

□

Theorem 3 *Assume that $\mathcal{G}^*$ is metrically regular at $\left(\overline{x}, \left(\overline{c}, \overline{b}\right)\right) \in \operatorname{gph}(\mathcal{G}^*)$, and take V as defined in (7). For each $(c, b) \in V$, let $f_{c,b}$ be given by (16). Then we have*

$$\begin{aligned}
\operatorname{reg} \mathcal{G}^* \left(\bar{x} \mid (\bar{c}, \bar{b})\right) &= \limsup_{\substack{(z,c,b)\to(\bar{x},\bar{c},\bar{b}) \\ f_{c,b}(z)>0}} \left(\left|\nabla f_{c,b}\right|(z)\right)^{-1} \\
&= \limsup_{\substack{(z,c,b)\to(\bar{x},\bar{c},\bar{b}) \\ f_{c,b}(z)>0}} \left(d_*(0_n, \widehat{\partial} f_{c,b}(z))\right)^{-1} \\
&= \limsup_{\substack{(z,c,b)\to(\bar{x},\bar{c},\bar{b}) \\ f_{c,b}(z)>0}} \left(\sup_{y\neq z} \frac{f_{c,b}(z) - f_{c,b}(y)}{\|z-y\|}\right)^{-1} \\
&= \limsup_{\substack{(z,c,b)\to(\bar{x},\bar{c},\bar{b}) \\ f_{c,b}(z)\searrow 0}} \left(\sup_{y\neq z} \frac{f_{c,b}(z) - f_{c,b}(y)}{\|z-y\|}\right)^{-1}.
\end{aligned}$$

Proof First of all, recall that the metric regularity assumption ensures that $\sigma\left(\overline{b}\right)$ satisfies SCQ and, then, the existence of V as a neighborhood of $\left(\overline{c}, \overline{b}\right)$ is guaranteed.

The first two equalities are consequences of Theorem 1, applied to $\mathcal{G}_V^*$, together with Proposition 3 and Theorem 2.

For (c, b) close enough to $(\overline{c}, \overline{b})$, we have $f_{c,b}(y) = 0$ provided that $\mathcal{F}^*(c, b) = \{y\}$. Then, if $f_{c,b}(z) > 0$, we can write

$$\left|\nabla f_{c,b}\right|(z) \leq \sup_{y\neq z} \frac{f_{c,b}(z) - f_{c,b}(y)}{\|z-y\|},$$

and consequently

$$\begin{aligned}
\operatorname{reg} \mathcal{G}^* \left(\bar{x} \mid (\bar{c}, \bar{b})\right) &\geq \limsup_{\substack{(z,c,b)\to(\bar{x},\bar{c},\bar{b}) \\ f_{c,b}(z)>0}} \left(\sup_{y\neq z} \frac{f_{c,b}(z) - f_{c,b}(y)}{\|z-y\|}\right)^{-1} \\
&\geq \limsup_{\substack{(z,c,b)\to(\bar{x},\bar{c},\bar{b}) \\ f_{c,b}(z)\searrow 0}} \left(\sup_{y\neq z} \frac{f_{c,b}(z) - f_{c,b}(y)}{\|z-y\|}\right)^{-1} =: \kappa.
\end{aligned}$$

So, we only have to prove (take Proposition 3 into account)

$$\operatorname{reg} \mathcal{G}_V^* \left(\bar{x} \mid (\bar{c}, \bar{b})\right) \leq \kappa. \tag{24}$$

According to Theorem 2(ii) the functions $f_{c,b} : \mathbb{R}^n \to \mathbb{R}$ are lower semicontinuous for all $(c, b) \in V$, whereas for some neighborhood V_0 of $(\bar{c}, \bar{b})$, $\mathcal{F}^*$ is single-valued and (Lipschitz) continuous on V_0.

For the sake of simplicity, given $z \in \mathbb{R}^n$ and $(c, b) \in V$, we set

$$\Delta(z, c, b) := \sup_{y\neq z} \frac{f_{c,b}(z) - f_{c,b}(y)}{\|z-y\|}.$$

In order to prove (24) we proceed by contradiction and we assume the existence of $\varepsilon > 0$ such that $\operatorname{reg} \mathcal{G}_V^* \left(\bar{x} \mid (\bar{c}, \bar{b})\right) > \kappa + \varepsilon$. From the definition of regularity modulus, there would exist sequences $x^r \to \bar{x}$ and $(c^r, b^r) \to (\bar{c}, \bar{b})$ such that

$$d(x^r, \mathcal{F}^*(c^r, b^r)) > (\kappa + \varepsilon)\, d((c^r, b^r), \mathcal{G}_V^*(x^r)). \tag{25}$$

Since $(c^r, b^r) \to (\bar{c}, \bar{b})$ we can assume that $\{(c^r, b^r)\}_{r\in\mathbb{N}} \subset V \cap V_0$ and, so, there will exist a sequence $\{y^r\}_{r\in\mathbb{N}}$ such that $\mathcal{F}^*(c^r, b^r) = \{y^r\}$, $r = 1, \ldots$, and $y^r \to \overline{x}$. Then, (25) is rewritten as

$$\alpha_r := \left\| x^r - y^r \right\| > (\kappa + \varepsilon)\, f_{c^r,b^r}(x^r), \quad r = 1, 2, \ldots; \tag{26}$$

hence

$$f_{c^r,b^r}(x^r) < \inf_{\mathbb{R}^n} f_{c^r,b^r} + (\kappa + \varepsilon)^{-1} \alpha_r, \quad r = 1, 2, \ldots.$$

Applying Proposition 2 (remember Theorem 2), with the role of "ε" and "δ" in that proposition being played by $(\kappa + \varepsilon)^{-1} \alpha_r$ and $(\kappa + \varepsilon)^{-1}$, respectively, there will exist $z^r \in \mathbb{R}^n$ such that $\|x^r - z^r\| < \alpha_r$ (hence $z^r \neq y^r$), and

$$0 < f_{c^r,b^r}(z^r) \leq f_{c^r,b^r}(x^r), \tag{27}$$

at the same time that

$$f_{c^r,b^r}(z^r) < f_{c^r,b^r}(y) + (\kappa + \varepsilon)^{-1} \left\| y - z^r \right\|, \quad \text{for all } y \in \mathbb{R}^n,\ y \neq z^r. \tag{28}$$

The strict inequality in (27) comes from the fact that $f_{c^r,b^r}(x) = 0$ if and only if $(c^r, b^r) \in \mathcal{G}_V^*(x)$ (here Theorem 2(i) applies to ensure that $\mathcal{G}^*(x) \cap V$ is closed); in other words, if and only if $(c^r, b^r) \in \mathcal{G}^*(x)$ or, equivalently, if and only if $x \in \mathcal{F}^*(c^r, b^r)$.

Now combining (26) and (27),

$$0 < f_{c^r,b^r}(z^r) \leq f_{c^r,b^r}(x^r) < (\kappa + \varepsilon)^{-1} \alpha_r, \tag{29}$$

meanwhile (28) yields

$$\frac{f_{c^r,b^r}(z^r) - f_{c^r,b^r}(y)}{\|z^r - y\|} < (\kappa + \varepsilon)^{-1}, \quad \text{for all } y \in \mathbb{R}^n,\ y \neq z^r. \tag{30}$$

Observe that $\{z^r\}_{r\in\mathbb{N}}$ converges to $\overline{x}$, and $f_{c^r,b^r}(z^r)$ converges to 0 from (29), since both $\{x^r\}_{r\in\mathbb{N}}$ and $\{y^r\}_{r\in\mathbb{N}}$ converge to $\overline{x}$. Therefore, by (30),

$$(\kappa + \varepsilon)^{-1} \geq \Delta(z^r, c^r, b^r) \geq \frac{f_{c^r,b^r}(z^r) - f_{c^r,b^r}(y^r)}{\|z^r - y^r\|} = \frac{f_{c^r,b^r}(z^r)}{\|z^r - y^r\|} > 0,$$

and so

$$\kappa + \varepsilon \leq \Delta(z^r, c^r, b^r)^{-1}, \quad r = 1, 2, \ldots,$$

which entails the contradiction

$$\begin{aligned} \kappa + \varepsilon &\leq \limsup_{r\to\infty} \Delta(z^r, c^r, b^r)^{-1} \\ &\leq \limsup_{\substack{(z,c,b)\to(\bar{x},\bar{c},\bar{b}) \\ f_{c,b}(z)\searrow 0}} \left(\sup_{y\neq z} \frac{f_{c,b}(z) - f_{c,b}(y)}{\|z - y\|} \right)^{-1} = \kappa. \end{aligned} \tag{31}$$

□

Acknowledgements This research has been partially supported by grants MTM2005-08572-C03 (01-02) and MTM 2006-27491-E, from MEC (Spain) and FEDER (E.U.), and ACOMP06/117-203, and ACOMP/2007/247–292, from Generalitat Valenciana (Spain).

References

1. Azé, D., Corvellec, J.-N., Lucchetti, R.E.: Variational pairs and applications to stability in nonsmooth analysis. Nonlinear Anal. **49**, 643–670 (2002)
2. Cánovas, M.J., Klatte, D., López, M.A., Parra, J.: Metric regularity in convex semi-infinite optimization under canonical perturbations, SIAM J. Optim. (in press)
3. Dontchev, A.L., Lewis, A.S., Rockafellar, R.T.: The radius of metric regularity. Trans. Amer. Math. Society **355**, 493–517 (2003)
4. De Giorgi, E., Marino, A., Tosques, M.: Problemi di evoluzione in spazi metrici e curve di massima pendenza. Atti Acad. Nat. Lincei, Rend, Cl. Sci. Fiz. Mat. Natur. **68**, 180–187 (1980)
5. Goberna, M.A., López, M.A.: Linear Semi-Infinite Optimization. John Wiley & Sons, Chichester (UK) (1998)
6. Henrion, R., Klatte, D.: Metric regularity of the feasible set mapping in semi-infinite optimization. Appl. Math. Optimiz. **30**, 103–109 (1994)
7. Ioffe, A.D.: Metric regularity and subdifferential calculus. Uspekhi Mat. Nauk 55, **3** (333), 103–162; English translation Math. Surveys **55**, 501–558 (2000)
8. Ioffe, A.D.: On rubustness of the regularity property of maps. Control Cybernet. **32**, 543–554 (2003)
9. Klatte, D., Kummer, B.: Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications. Kluwer Academic Publ., Dordrecht (2002)
10. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation (I,II). Springer-Verlag, Berlin (2006)
11. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton, NJ (1970)
12. Rockafellar, R.T., Wets, R.J.-B.: Variational Analysis. Springer-Verlag, Berlin (1997)