J Glob Optim (2008) 40:29–39
DOI 10.1007/s10898-007-9194-5

Regularity results for evolutionary nonlinear variational and quasi-variational inequalities with applications to dynamic equilibrium problems

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Received: 30 May 2007 / Accepted: 9 June 2007 / Published online: 17 July 2007

Abstract The aim of this paper is to obtain the continuity of solutions to time-dependent nonlinear variational and quasi-variational inequalities which express many dynamic equilibrium problems. To prove our results, we make use of Minty's Lemma and of the notion of the Mosco's convergence.

Keywords Traffic equilibrium problems · Time-dependent variational and quasi-variational inequalities · Continuity of solutions · Mosco's convergence

1 Introduction

The paper develops the works of Refs. [1,2] regarding the continuity of solutions to evolutionary variational and quasi-variational inequalities, showing that the analogous continuity results hold also for nonlinear evolutionary variational and quasi-variational inequalities. The Minty's Lemma for variational inequalities and the notion of the Mosco's convergence play an important role in the attainment of these results.

The outline of the paper is as follows: we first provide the theoretical foundations and the historical development of the theory of time-dependent variational and quasi-variational inequalities which equivalently express dynamic equilibrium problems, in Sect. 2. In Sect. 3, we introduce a nonlinear and strongly monotone variational inequality and we prove that the unique solution is continuous with respect to time under the assumption that the data are continuous (see Theorem 3.2). In Sect. 4, we consider a nonlinear and strongly monotone quasi-variational inequality and also in this case we are able to prove the continuity of the solution (see Theorem 4.2).

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2 The dynamic model

Let us consider the model of a dynamic traffic network. It is represented by a graph $G = [N, L]$, where N is the set of nodes and L is the set of directed links between the nodes. Let R_r be a path consisting of a sequence of links which connect an Origin-Destination (O/D) pair of nodes. Let m be the number of the paths in the network. Let $\mathcal{W}$ denote the set of the O/D pairs with typical O/D pair w_j, $|\mathcal{W}| = l$ and $m > l$. The set of paths connecting the O/D pair w_j is represented by $\mathcal{R}_j$ and the entire set of paths in the network by $\mathcal{R}$. The topology of the network is described by the pair-link incidence matrix $\Phi = (\varphi_{j,r})$, where $\varphi_{j,r}$ is 1 if path $R_r \in \mathcal{R}_j$ and 0 otherwise. The flow vector is a time-dependent flow trajectory $F : [0, T] \to \mathbb{R}^m_+$, while the topology remains fixed. Each component $F_r(t)$ of $F(t)$ gives the flow trajectory $F : [0, T] \to \mathbb{R}^m_+$ which has to satisfy almost everywhere on $[0, T]$ the capacity constraints $\lambda(t) \le F(t) \le \mu(t)$ and the traffic conservation law $\Phi F(t) = \rho(t)$, where the bounds $\lambda < \mu$ and the demand $\rho = (\rho_j)_{w_j \in \mathcal{W}}$ are given. We assume that λ and μ belong to $L^2([0, T], \mathbb{R}^m_+)$ and that ρ lies in $L^2([0, T], \mathbb{R}^m_+)$. Assuming in addition that $\Phi\lambda(t) \le \rho(t) \le \Phi\mu(t)$ a.e. in $[0, T]$, we obtain that the set of feasible flows

$$\mathbf{K} = \Big\{ F \in L^2([0, T], \mathbb{R}^m) : \lambda(t) \le F(t) \le \mu(t), \quad \Phi F(t) = \rho(t), \text{ a.e. in } [0, T] \Big\}$$

is nonempty, as it is shown in Ref. [7]. Clearly $\mathbf{K}$ is a convex, closed, bounded subset of $L^2([0, T], \mathbb{R}^m_+)$. Let $C : [0, T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be the cost trajectory. The equilibrium condition is given by a generalized version of Wardrop's condition (see Refs. [5,6]), namely:

Definition 1 A flow $H \in \mathbf{K}$ is a user traffic equilibrium flow if $\forall w_j \in \mathcal{W}$, $\forall q, s \in \mathcal{R}_j$ and a.e. in $[0, T]$ it results:

$$C_q(t, H(t)) > C_s(t, H(t)) \Longrightarrow H_q(t) = \lambda_q(t) \quad or \quad H_s(t) = \mu_s(t). \tag{1}$$

The overall flow pattern obtained according with condition (1) fits very well in the framework of the theory of variational inequalities. In fact, in Refs. [5,6] the following result has been proved:

Theorem 1 *A flow $H \in \mathbf{K}$ is an equilibrium pattern if and only if it satisfies the following evolutionary variational inequality:*

$$\int_0^T \langle C(t, H(t)), F(t) - H(t) \rangle \mathrm{d}t \ge 0, \quad \forall F \in \mathbf{K}. \tag{2}$$

We recall a sufficient condition in terms of the operator $C(t, F)$ (see Ref. [8]).

Theorem 2 *Let $C(t, F) : [0, T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be a vector-function measurable in t, continuous in F and such that for each $F \in \mathbf{K}$ it results*

$$\|C(t, F(t))\|_m \le A(t)\|F(t)\|_m + B(t), \quad \text{a.e. in } [0, T], \tag{3}$$

with $B \in L^2([0, T], \mathbb{R}_+)$ and $A \in L^\infty([0, T], \mathbb{R}_+)$, and for each $H, F \in \mathbf{K}$ it results

$$\langle C(t, H(t)) - C(t, F(t)), H(t) - F(t) \rangle \ge 0, \quad \text{a.e. in } [0, T]. \tag{4}$$

Let $\lambda, \mu \in L^2([0, T], \mathbb{R}^m_+)$ and let $\rho \in L^2([0, T], \mathbb{R}^l_+)$ be vector-functions. Then, the variational inequality (2) admits solutions.

It is well known that if C is in addition strongly monotone, namely for all $F, H \in \mathbf{K}$ there exists $\nu > 0$ such that

$$\langle C(t, F(t)) - C(t, H(t)), F(t) - H(t) \rangle \ge \nu \|F - H\|^2_{\mathbb{R}^m}, \quad \text{a.e. in } [0, T]. \tag{5}$$

the solution to the evolutionary variational inequality (2) is unique.

We remark that problem (2)(see Ref. [9]) is also equivalent to the following one:
Find $H \in \mathbf{K}$ such that

$$\langle C(t, H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0, T], \tag{6}$$

where

$$\mathbf{K}(t) = \Big\{F(t) \in \mathbb{R}^m : \lambda(t) \leq F(t) \leq \mu(t),\ \Phi F(t) = \rho(t)\Big\},$$

for a.e. $t \in [0, T]$.

Moreover, it is easy to see that if the path cost vector is linear with respect to the path flow vector, i.e. $C(t, H(t)) = A(t)H(t) + B(t)$, where $A = (A_{rs})_{r,s=1,2,\ldots,m} : [0, T] \to \mathbb{R}^{m\times m}_+$ is a bounded nonnegative definite matrix-function, that is,

$$\exists M > 0 : \ \|A(t)\|_{m\times m} = \Big(\sum_{r,s=1}^{m} A^2_{rs}(t)\Big)^{\frac{1}{2}} \leq M, \quad \text{a.e. in } [0, T], \tag{7}$$

$$\langle A(t)F(t), F(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \quad \text{a.e. in } [0, T], \tag{8}$$

and $B \in L^2([0, T], \mathbb{R}^m_+)$, then there exists some solution to the variational inequality (6). Moreover, if A is a positive definite matrix-function, namely

$$\exists \nu > 0 : \ \langle A(t)F(t), F(t)\rangle \geq \nu\|F(t)\|^2_m, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0, T], \tag{9}$$

the solution to the evolutionary variational inequality is unique.

Now, let us introduce the time-dependent elastic problem which arose whenever travel demands are not only dependent on the time but also on the equilibrium distribution. In fact, it is clear that travel demands are influenced by the evaluation of the amount of traffic flows on the paths, namely by the forecasted equilibrium solutions. In particular, we assume that the travel demand ρ depends on the equilibrium solutions $H(t)$ in the average sense, namely that $\rho(H) = \frac{1}{T}\int_0^T \rho(t, H(\tau))\mathrm{d}\tau$.

Let $\lambda, \mu : [0, T] \to \mathbb{R}^m_+$, let $\rho : [0, T] \times \mathbb{R}^m_+ \to \mathbb{R}^l_+$, let $D \subseteq L^2([0, T], \mathbb{R}^m_+)$ be a non-empty, compact and convex subset and let $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m_+)}$ be a set-valued mapping, defined by

$$\mathbf{K}(H) = \Big\{F \in L^2([0, T], \mathbb{R}^m_+) : \ \lambda(t) \leq F(t) \leq \mu(t) \quad \text{a.e. in } [0, T],$$

$$\Phi F(t) = \frac{1}{T}\int_0^T \rho(t, H(\tau))d\tau \quad \text{a.e. in } [0, T]\Big\}.$$

Then the quasi-variational inequality that models the traffic equilibrium problem in the elastic case is the following:
Find $H \in \mathbf{K}$ such that

$$\int_0^T \langle C(t, H(t)), F(t) - H(t)\rangle \mathrm{d}t \geq 0, \quad \forall F \in \mathbf{K}(H). \tag{10}$$

Regarding the existence of solutions, let us recall the following general result (see Ref. [14]):

Theorem 3 *Let X be a topological linear locally convex Hausdorff space and let $E \subset X$ be a convex, compact and nonempty subset. Let $C : E \to X'$ be an upper semicontinuous multivalued mapping with $C(H)$, $H \in E$, convex, compact and nonempty and let $\mathbf{K} : E \to E$ be a closed lower semi continuous multivalued mapping with $\mathbf{K}(H)$, $H \in E$, convex, compact and nonempty and let $\varphi : E \to \mathbb{R}$ a convex lower semicontinuous function. Then, there exists $H^* \in C$ such that:*

a $H^* \in \mathbf{K}(H^*)$,
b *there exists* $F^* \in C(H^*)$ *for which*

$$\langle H - H^*, F^* \rangle + \varphi(H) - \varphi(H^*) \geq 0, \quad \forall H \in \mathbf{K}(H^*).$$

A consequence of this result is a sufficient condition showed in Ref. [11], Theorem 3, and in the following it is presented.

Theorem 4 *Let $C : [0, T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be an operator verifying the following conditions:*

$$C(t, F) \text{ is measurable in } t, \forall F \in \mathbb{R}^m_+, \text{ continuous in } F, \text{ a.e. in } [0, T],$$

$$\exists \gamma \in L^2([0, T]) : \|C(t, F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m_+, \ a.e.\ in\ [0, T],$$

and

$$\exists \nu > 0 : \langle C(t, H) - C(t, F), H - F \rangle \geq \nu \|H - F\|^2_m, \quad \forall H, F \in \mathbb{R}^m_+, \ a.e.\ in\ [0, T].$$

Let $\lambda, \mu \in L^2([0, T], \mathbb{R}^m_+)$ be vector-functions and let $\rho \in L^2([0, T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ be an operator verifying the following conditions

$$\exists \psi \in L^1([0, T]) : \|\rho(t, F)\|_l \leq \psi(t) + \|F\|^2_m, \quad \forall F \in \mathbb{R}^m_+, \ a.e.\ in\ [0, T],$$

$$\exists \nu \in L^2([0, T]) : \|\rho(t, H) - \rho(t, F)\|_l \leq \nu(t)\|H - F\|_m, \quad \forall H, F \in \mathbb{R}^m_+, \ a.e.\ in\ [0, T].$$

Then, the evolutionary quasi-variational inequality

$$H \in \mathbf{K}(H) : \int_0^T \langle C(t, H(t)), F(t) - H(t) \rangle dt \geq 0, \quad \forall F \in \mathbf{K}(H),$$

admits a solution.

We set

$$\mathbf{K}(t, H) = \left\{ F(t) \in \mathbb{R}^m : \lambda(t) \leq F(t) \leq \mu(t), \ \Phi F(t) = \frac{1}{T} \int_0^T \rho(t, H(\tau)) \mathrm{d}\tau \right\},$$

for a.e. $t \in [0, T]$, and we observe that problem (10) (see Ref. [9]) is equivalent to the following one:

Find $H \in \mathbf{K}(H)$ such that

$$\langle C(t, H(t)), F(t) - H(t) \rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \ \text{a.e. in } [0, T]. \tag{11}$$

3 Continuity for evolutionary nonlinear variational inequalities

In this section, we will extend the theorem of continuity for solutions to linear evolutionary variational inequalities proved in Ref. [1] assuming, now, that the cost operator is nonlinear.

We recall that if the cost $C(t, F(t))$ is a linear operator with respect to the flows, namely it results

$$C(t, F(t)) = A(t)F(t) + B(t),$$

for each $t \in [0, T]$, where $A : [0, T] \to \mathbb{R}_+^{m\times m}$ and $B : [0, T] \to \mathbb{R}_+^m$ are two functions, we obtained the following continuity result (see Theorem 3.2 in Ref. [1])

Theorem 5 *Let $A \in C([0, T], \mathbb{R}_+^{m\times m})$ be a positive defined matrix-function and let $B \in C([0, T], \mathbb{R}_+^m)$ be a vector-function. Suppose that $\lambda, \mu \in C([0, T], \mathbb{R}_+^m)$ and $\rho \in C([0, T], \mathbb{R}_+^l)$. Then, the evolutionary variational inequality*

$$H \in \mathbf{K} : \langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0, T], \tag{12}$$

admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0, T], \mathbb{R}_+^m)$. Moreover, the estimate

$$\|H_1 - H_2\|_{C([0,T],\mathbb{R}_+^m)} \leq \frac{1}{\nu}\|B_1 - B_2\|_{C([0,T],\mathbb{R}_+^m)}$$

holds, where ν is the constant of positive definition of matrix A.

This result also holds if the matrix $A \in C([0, T], \mathbb{R}_+^{m\times m})$ is supposed degenerate, namely A satisfies the following condition

$$\langle A(t)F(t), F(t)\rangle \geq \nu(t)\|F(t)\|_m^2, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0, T],$$

where $\nu \in L^\infty([0, T], \mathbb{R}_0^+)$ is such that

$$\nexists I \subseteq [0, T],\ \mu(I) > 0 : \ \nu(t) = 0, \quad \forall t \in I,$$

as Theorem 3.2 shows in Ref. [2].

Before proving our results, we recall the concept of Mosco's convergence.

Definition 2 (see Ref. [10]) Let $(V, \|\cdot\|)$ be an Hilbert space and $\mathbf{K} \subset V$ a closed, nonempty, convex set. A sequence of nonempty, closed, convex sets $\mathbf{K}_n$ converges to $\mathbf{K}$ as $n \to +\infty$, i.e. $\mathbf{K}_n \to \mathbf{K}$, if and only if

(M1) for any $H \in \mathbf{K}$, there exists a sequence $\{H_n\}_{n\in\mathbb{N}}$ strongly converging to H in V such that H_n lies in $\mathbf{K}_n$ for all $n \in \mathbb{N}$,

(M2) for any subsequence $\{H_{k_n}\}_{n\in\mathbb{N}}$ weakly converging to H in V, such that H_{k_n} lies in $\mathbf{K}_{k_n}$ for all $n \in \mathbb{N}$, then the weak limit H belongs to $\mathbf{K}$.

Definition 3 (see Ref. [12]) A sequence of operators $A_n : \mathbf{K}_n \to V'$ converges to an operator $A : \mathbf{K} \to V'$ if

$$\|A_n H_n - A_n F_n\|_* \leq M\|H_n - F_n\|, \quad \forall H_n, F_n \in \mathbf{K}_n, \tag{13}$$

$$\langle A_n H_n - A_n F_n, H_n - F_n\rangle \geq \nu\|H_n - F_n\|^2, \quad \forall H_n, F_n \in \mathbf{K}_n, \tag{14}$$

hold with fixed constants $M, \nu > 0$ and

(M3) the sequence $\{A_n H_n\}_{n\in\mathbb{N}}$ strongly converges to AH in V', for any sequence $\{H_n\}_{n\in\mathbb{N}}$, such that H_n lies in $\mathbf{K}_n$ for all $n \in \mathbb{N}$, strongly converging to $H \in \mathbf{K}$.

In (13) $\|\cdot\|_*$ is the norm in the dual space of V.

We remak that the following Lemma holds.

Lemma 1 *Let $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$, let $\rho \in C([0,T], \mathbb{R}^l_+)$ and let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence such that $t_n \to t \in [0,T]$, as $n \to +\infty$. Then, the sets sequence*

$$\mathbf{K}(t_n) = \Big\{F(t_n) \in \mathbb{R}^m : \lambda(t_n) \le F(t_n) \le \mu(t_n),\ \Phi F(t_n) = \rho(t_n)\Big\},$$

$\forall n \in \mathbb{N}$, converges to

$$\mathbf{K}(t) = \Big\{F(t) \in \mathbb{R}^m : \lambda(t) \le F(t) \le \mu(t),\ \Phi F(t) = \rho(t)\Big\},$$

as $n \to +\infty$, in Mosco's sense.

Proof See proof of Theorem 3.2 in Ref. [1]. □

Remark 1 Theorem 5 still holds true if $\mathbf{K}(t), t \in [0,T]$, is a family of sets satisfying the following condition

(M) $\mathbf{K}(t), t \in [0,T]$, is a family of nonempty convex, closed sets of $\mathbb{R}^m$ such that $\mathbf{K}(t_n)$ converges to $\mathbf{K}(t)$ in Mosco's sense, for each sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$, with $t_n \to t$, as $n \to +\infty$,

as it is shown in Ref. [3], Theorem 3.2.

In the following, let us consider a nonlinear cost operator

$$C : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m,$$

and let us study under which assumptions the continuity of the solution to the next evolutionary variational inequality with respect to the time can be ensured:

Find $H \in \mathbf{K}$ such that

$$\langle C(t, H(t)), F(t) - H(t)\rangle \ge 0, \quad \forall F(t) \in \mathbf{K}(t),\ \text{a.e. in } [0,T], \tag{15}$$

where the family $\mathbf{K}(t)$ satisfies condition (M).

From here onward, let us assume that the operator $C : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ verifies the following assumptions:

$$\|C(t, F(t))\|_m \le A(t)\|F(t)\|_m + B(t), \quad \forall F(t) \in \mathbf{K}(t),\ \text{in } [0,T], \tag{16}$$

with $B \in L^2([0,T])$ and $A \in L^\infty([0,T])$, and it results

$$\langle C(t, F_1(t)) - C(t, F_2(t)), F_1(t) - F_2(t)\rangle \ge \nu\|F_1(t) - F_2(t)\|_m^2, \tag{17}$$

for each $F_1(t), F_2(t) \in \mathbf{K}(t)$ and in $[0,T]$.

In the proof of the continuity result the following important Lemma (see Ref. [13], Lemma 2.2), regarding Minty variational inequality, will be used.

Lemma 2 *Let $C(t, F) : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be a vector-function measurable in t, continuous in F and satisfying conditions (16) and (17). Let $\lambda, \mu \in L^2([0,T], \mathbb{R}^m_+)$ and let $\rho \in L^2([0,T], \mathbb{R}^l_+)$ be vector-functions. Then, the time-dependent variational inequality (15) is equivalent to*

$$H \in \mathbf{K} : \langle C(t, F(t)), F(t) - H(t)\rangle \ge 0, \quad \forall F(t) \in \mathbf{K}(t), \quad a.e.\ in\ [0,T].$$

Now, we can generalize Theorem 5 for nonlinear time-dependent variational inequalities, showing the following result, that is the main result of this work.

Theorem 6 *Let* $C \in C([0, T] \times \mathbb{R}^m_+, \mathbb{R}^m_+)$ *be an operator verifying conditions* (16) *and* (17). *Let* $\mathbf{K}(t), t \in [0, T]$*, be a family of sets satisfying condition* (M). *Then, the evolutionary variational inequality*

$$H \in \mathbf{K}: \ \langle C(t, H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \ \text{in } [0, T]. \tag{18}$$

admits a unique solution $H \in \mathbf{K}$ *such that* $H \in C([0, T], \mathbb{R}^m_+)$.

Proof Taking into account Theorem 2 and condition (17), it results that (18) admits a unique solution $H(t) \in \mathbf{K}(t)$, for $t \in [0, T]$.

Let $t \in [0, T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0, T]$ be a sequence, with $t_n \to t$. Our statement is equivalent to say that the unique solution $H(t_n)$, for $n \in \mathbb{N}$, to the following variational inequality

$$H(t_n) \in \mathbf{K}(t_n): \ \langle C(t_n, H(t_n)), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \tag{19}$$

converges strongly, as $n \to +\infty$, to the solution $H(t)$ to the limit problem

$$H(t) \in \mathbf{K}(t): \ \langle C(t, H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t), \tag{20}$$

namely

$$\lim_{n\to+\infty} H(t_n) = H(t) \quad \text{in } \mathbb{R}^m.$$

For the solution $H(t) \in \mathbf{K}(t)$ to (20), we use the properties of the Mosco's convergence of $\{\mathbf{K}(t_n)\}_{n\in\mathbb{N}}$ to $\mathbf{K}(t)$, as $n \to +\infty$. Then, it is possible to choose a sequence $\{G(t_n)\}_{n\in\mathbb{N}}$, with $G(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$, such that,

$$\lim_{n\to+\infty} G(t_n) = H(t) \quad \text{in } \mathbb{R}^m$$

and, by virtue of the assumption on the operator C, we obtain

$$\lim_{n\to+\infty} C(t_n, G(t_n)) = C(t, H(t)) \quad \text{in } \mathbb{R}^m.$$

Setting $F(t_n) = G(t_n)$ in (19), we have

$$\langle C(t_n, H(t_n)), G(t_n) - H(t_n)\rangle \geq 0, \tag{21}$$

and using the strong monotonicity of the operator C, it results

$$\langle C(t_n, H(t_n)) - C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle \geq \nu \|H(t_n) - G(t_n)\|^2_m.$$

From (21) we derive that

$$\begin{aligned}\langle C(t_n, H(t_n)) - C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle &= \langle C(t_n, H(t_n)), H(t_n) - G(t_n)\rangle \\ -\langle C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle &\leq -\langle C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle,\end{aligned}$$

then

$$\begin{aligned}\nu \|H(t_n) - G(t_n)\|^2_m &\leq -\langle C(t_n, G(t_n)), H(t_n) - G(t_n)\rangle \\ &\leq \|C(t_n, G(t_n))\|_m \|H(t_n) - G(t_n)\|_m,\end{aligned}$$

that is

$$\nu \|H(t_n) - G(t_n)\|_m \leq \|C(t_n, G(t_n))\|_m.$$

Hence, one deduces

$$\begin{aligned}\|H(t_n)\|_m &\leq \|H(t_n) - G(t_n)\|_m + \|G(t_n)\|_m \\ &\leq \frac{\|C(t_n, G(t_n))\|_m}{\nu} + \|G(t_n)\|_m.\end{aligned}$$

Since $\{C(t_n, G(t_n))\}_{n\in\mathbb{N}}$ is a convergent sequence then it is bounded, i.e.

$$\exists h \in \mathbb{R}_+ : \ \|C(t_n, G(t_n))\|_m \leq h, \quad \forall n \in \mathbb{N},$$

for the same reason, the sequence $\{G(t_n)\}_{n\in\mathbb{N}}$ is bounded, i.e.

$$\exists k \in \mathbb{R}_+ : \ \|G(t_n)\|_m \leq k, \quad \forall n \in \mathbb{N}.$$

From those conditions, it follows

$$\|H(t_n)\|_m \leq c, \quad \forall n \in \mathbb{N},$$

where the constant c is independent on n. Hence there exists a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$ converging in $\mathbb{R}^m$ to an element $\widetilde{H}(t) \in \mathbb{R}^m$, and thus

$$\lim_{n\to+\infty} H(t_{k_n}) = \widetilde{H}(t).$$

Moreover, taking into account the second condition of the Mosco's convergence, we have

$$\widetilde{H}(t) \in \mathbf{K}(t).$$

By virtue of the Mosco's convergence, it follows

$$\forall F(t) \in \mathbf{K}(t), \ \exists F(t_n) \in \mathbf{K}(t_n) : \ \lim_{n\to+\infty} F(t_n) = F(t), \ \text{in } \mathbb{R}^m,$$

and from the assumption on the operator C, we get

$$\lim_{n\in\mathbb{N}} C(t_n, F(t_n)) = C(t, F(t)), \ \text{in } \mathbb{R}^m.$$

Now, we consider the following variational inequality

$$\langle C(t_{k_n}, F(t_{k_n})), F(t_{k_n}) - H(t_{k_n})\rangle \geq 0,$$

and passing to the limit as $n \to +\infty$, we obtain

$$\widetilde{H}(t) \in \mathbf{K}(t) : \ \langle C(t, F(t)), F(t) - \widetilde{H}(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t),$$

which, again by Lemma 2 and the uniqueness of the solution to (20), implies

$$\widetilde{H}(t) = H(t).$$

Then it follows that every subsequence of $\{H(t_n)\}_{n\in\mathbb{N}}$ converges to the same limit $\widetilde{H}(t)$ and hence

$$\lim_{n\to+\infty} H(t_n) = H(t).$$

Remark 2 Taking into account Lemma 1, it results that Theorem 6 holds if K is the typical constraints set of dynamical equilibrium problems (see [4], namely)

$$\mathbf{K} = \Big\{F \in L^2([0, T], \mathbb{R}^m_+) : \ \lambda(t) \leq F(t) \leq \mu(t), \ \Phi F(t) = \rho(t), \ \text{ in } [0, T]\Big\},$$

where $\lambda, \mu \in C([0, T], \mathbb{R}^m_+)$ and $\rho \in C([0, T], \mathbb{R}^l_+)$, then the solutions to time-dependent traffic equilibrium problems are continuous.

4 Continuity for nonlinear evolutionary quasi-variational inequalities

In this section, we will prove a continuity result for solutions to evolutionary nonlinear quasi-variational inequalities.

We remark that the set as in (11) fulfils conditions of the Mosco's convergence, in particular the following lemma holds (see proof of Theorem 4.1 in Ref. [1]):

Lemma 3 *Let $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$ be and let $\rho \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ be such that*

$$\exists \psi \in L^1([0,T], \mathbb{R}^m_+) : \|\rho(t,F)\|_l \leq \psi(t) + \|F\|_m^2,$$

and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence such that $t_n \to t \in [0,T]$, as $n \to +\infty$. Then, the sequence of sets

$$\mathbf{K}(t_n, H) = \left\{ F(t_n) \in \mathbb{R}^m : \lambda(t_n) \leq F(t_n) \leq \mu(t_n), \quad \Phi F(t_n) = \frac{1}{T}\int_0^T \rho(t_n, H(\tau))\mathrm{d}\tau \right\},$$

$\forall n \in \mathbb{N}$ *converges to*

$$\mathbf{K}(t, H) = \left\{ F(t) \in \mathbb{R}^m : \lambda(t) \leq F(t) \leq \mu(t), \quad \Phi F(t) = \frac{1}{T}\int_0^T \rho(t, H(\tau))\mathrm{d}\tau \right\},$$

as $n \to +\infty$, in Mosco's sense.

Now, we recall that if the cost $C(t, F(t))$ is a linear operator with respect to the flows, namely

$$C(t, F(t)) = A(t)F(t) + B(t),$$

for a.e. $t \in [0,T]$, where $A : [0,T] \to \mathbb{R}^{m\times m}_+$ and $B : [0,T] \to \mathbb{R}^m_+$ are two functions, we proved the following continuity result (see Theorem 4.1 in Ref. [1])

Theorem 7 *Let $A \in C([0,T], \mathbb{R}^{m\times m}_+)$ be a positive definite matrix-function and let $B \in C([0,T], \mathbb{R}^m_+)$ be a vector-function. Let $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$ be and let $\rho \in C([0,T] \times \mathbb{R}^m_+, \mathbb{R}^l_+)$ be such that*

$$\exists \psi \in C([0,T]) : \ \|\rho(t,F)\|_l \leq \psi(t) + \|F\|_m^2, \quad \forall F \in \mathbb{R}^m_+, \ in \ [0,T],$$

$$\exists \nu \in C([0,T]) : \ \|\rho(t,F_1) - \rho(t,F_2)\|_l \leq \nu(t)\|F_1 - F_2\|_m, \quad \forall F \in \mathbb{R}^m_+, \ in \ [0,T],$$

Then, the evolutionary quasi-variational inequality

$$H \in \mathbf{K}(H) : \ \langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t,H), \ \text{in } [0,T],$$

admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0,T], \mathbb{R}^m_+)$.

Remark 3 Theorem 7 still holds true if $\mathbf{K}$ is a set-valued mapping satisfying the following condition

(MM) $\mathbf{K} : D \to 2^{L^2([0,T],\mathbb{R}^m_+)}$ is lower semicontinuous with $\mathbf{K}(t,H), t \in [0,T], H \in D$ nonempty, convex and compact of $\mathbb{R}^m$ such that $\mathbf{K}(t_n, H)$ converges to $\mathbf{K}(t,H)$ in Mosco's sense, for each sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$, with $t_n \to t \in [0,T]$, as $n \to +\infty$.

From here onward, let us consider a nonlinear cost operator

$$C : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m,$$

and let us study the continuity of the solution to the following evolutionary quasi-variational inequality

Find $H \in \mathbf{K}(H)$ such that

$$\langle C(t, H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0, T], \tag{22}$$

where the set-valued mapping $\mathbf{K}$ satisfies condition (MM).

With little few modifications of Minty's Lemma (see Ref. [13], Lemma 2.2), we can prove this result.

Lemma 4 *Let $C : [0, T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be an operator satisfying the following conditions:*

$C(t, F)$ is measurable in t, $\forall F \in \mathbb{R}^m_+$, continuous in F, a.e. in $[0, T]$,

$$\exists \gamma \in L^2([0, T]) : \|C(t, F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m_+, \ a.e. \ in \ [0, T],$$

and

$$\exists \nu > 0 : \langle C(t, F_1) - C(t, F_2), F_1 - F_2\rangle \geq \nu\|F_1 - F_2\|^2_m, \quad \forall F_1, F_2 \in \mathbb{R}^m_+, \ a.e. \ in \ [0, T].$$

Let $\mathbf{K}$ be a set-valued mapping satisfying condition (MM). *Then, the time-dependent variational inequality* (22) *is equivalent to*

$$H \in \mathbf{K}(H) : \langle C(t, F(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \text{ a.e. in } [0, T].$$

Now, we can prove the continuity result.

Theorem 8 *Let $C \in C([0, T] \times \mathbb{R}^m_+, \mathbb{R}^m_+)$ be an operator, verifying the following conditions*

$$\exists \gamma \in C([0, T]) : \|C(t, F)\|_m \leq \gamma(t) + \|F\|_m, \quad \forall F \in \mathbb{R}^m_+, \ in \ [0, T], \tag{23}$$

and

$$\exists \nu > 0 : \langle C(t, F_1) - C(t, F_2), F_1 - F_2\rangle \geq \nu\|F_1 - F_2\|^2, \quad \forall F_1, F_2 \in \mathbb{R}^m_+, \ in \ [0, T]. \tag{24}$$

Let $\mathbf{K}$ be a set-valued mapping satisfying condition (MM). *Then, the evolutionary variational inequality*

$$H \in \mathbf{K}(H) : \langle C(t, H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \ in \ [0, T], \tag{25}$$

admits a solution $H \in \mathbf{K}(H)$ such that $H \in C([0, T], \mathbb{R}^m_+)$.

Proof From the assumption that C is a continuous operator, it follows that C satisfies all conditions of Theorem 4. Then, the existence of $H(t)$ for $t \in [0, T]$ is guaranteed. Moreover, the assumption (24) ensures that the solution $H(t)$ is unique in the set $\mathbf{K}(t, H)$.

Now, let $t \in [0, T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0, T]$ be a sequence, with $t_n \to t$. The thesis is equivalent to tell that the unique solution $H(t_n)$, for $n \in \mathbb{N}$, to

$$H(t_n) \in \mathbf{K}(t_n, H) : \langle C(t_n, H(t_n)), F(t_n) - H(t_n)\rangle \geq 0, \quad \forall F(t_n) \in \mathbf{K}(t_n, H), \tag{26}$$

converges strongly to the solution $H(t)$ to the limit problem

$$H(t) \in \mathbf{K}(t, H) : \langle C(t, H(t)), F(t) - H(t)\rangle \geq 0, \quad \forall F(t) \in \mathbf{K}(t, H), \tag{27}$$

that is

$$\lim_{n\to+\infty} H(t_n) = H(t) \quad \text{in } \mathbb{R}^m.$$

With the same technique used in the proof of Theorem 6, we obtain that the solution to evolutionary quasi-variational inequality (25) is continuous in $[0, T]$.

Remark 4 Theorem 8, by virtue of Lemma 3, holds if the set-valued mapping $\mathbf{K}$ is defined by

$$\mathbf{K}(H) = \Bigg\{ F \in L^2([0,T], \mathbb{R}^m_+) : \lambda(t) \le F(t) \le \mu(t),$$

$$\Phi F(t) = \int_0^T \rho(t, H(\tau))\mathrm{d}\tau, \text{ in } [0,T] \Bigg\},$$

with $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$ and $\rho \in C([0,T], \mathbb{R}^l_+)$, so the solutions to time-dependent elastic traffic equilibrium problems turn out to be continuous.

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