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Minty's lemma and vector variational-like inequalities

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Abstract In this paper, we consider two vector versions of Minty's Lemma and obtain existence theorems for three kinds of vector variational-like inequalities. The results presented in this paper are extension and improvement of the corresponding results of other authors.

Keywords Fan's KKM theorem · Minty's Lemma · Vector variational-like inequality · Hausdorff metric

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1 Introduction

Since Giannessi [9] introduced the vector variational inequality (VVI) in finite dimensional Euclidian space, many authors have intensively studied (VVI) and its various extensions. Several authors have investigated relationships between (VVI) and vector optimization problems, vector complementarity problem. For details we refer to Chen [3], Chen and Yang [4], Daniilidis and Hadjisavvas [6], Giannessi [10,11], Giannessi and Maugeri [12], Giannessi and Maugeri [13], Huang and Fang [14], Konnov and Yao [18], Yang [22], Yang and Goh [23], and Zeng and Yao [24] and reference therein. The vector variational-like inequalities (VVLI), a generalization of (VVI) was studied by Ansari, Siddiqi and Yao [1], Chiang [5], Fang and Huang [8], Jabarootian and Zafarani [16], Lin [20], Yang [22]. Minty's Lemma has

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been shown to be an important tool in the variational field including variational inequality problems, obstacle problems, confined plasmas, free boundary problems, elasticity problems and stochastic optimal control problems when the operator is monotone and the domain is convex; see Baiocchi and Capelo [2] and Giannessi [10]. Lee and Lee [19] obtained a vector version of Minty's lemma using Nadler's result [21] and with their result they considered two kinds of vector variational-like inequalities for set-valued mappings under certain pseudomonotonicity condition and certain new hemicontinuity condition, respectively. Motivated by the above works, we first obtain two vector versions of Minty's Lemma and deduce existence theorems for the solvability of three classes of vector variational-like inequalities in normed spaces. In fact we prove the solvability results for these classes of generalized vector variational-like inequalities under certain pseudomonotonicity assumptions. We also prove the solvability of these classes of generalized vector variational-like inequalities without monotonicity assumptions.

2 Preliminaries

Let X and Y be two normed spaces and let $L(X, Y)$ denote the family of all continuous linear operators from X into Y equipped with the uniform convergence norm. When Y is the set R of real numbers, $L(X, Y)$ is the usual dual space X^* of X. For any $x \in X$ and any $u \in L(X, Y)$, we shall write the value $u(x)$ as $\langle u, x\rangle$. We suppose throughout this paper that K is a nonempty and convex subset of X, $T : K \rightrightarrows L(X, Y)$ is a set-valued mapping, $\eta : K \times K \longrightarrow X$ and $f : K \times K \longrightarrow Y$ are functions, and $\{C(x) : x \in K\}$ is a family of closed, convex and pointed cones of Y.

Let C be a closed, convex and pointed cone with $\text{int}C \neq \emptyset$. Then a partial order $\leq_C$ in Y is defined as for $y_1, y_2 \in Y$

$$y_1 \leq_C y_2 \Leftrightarrow y_2 - y_1 \in C. \tag{1}$$

Note that $C \neq Y$ iff $0 \notin \text{intC}$.

The purpose of this article is to prove the existence of solutions to the following three kinds of vector variational-like inequalities:

Problem (1): Find $x_0 \in K$ such that

$$\langle T(y), \eta(y, x_0)\rangle + f(y, x_0) \nsubseteq -\text{int}C(x_0), \ \forall y \in K.$$

Problem (2): Find $x_0 \in K$ such that

$$\langle T(x_0), \eta(y, x_0)\rangle + f(y, x_0) \nsubseteq -\text{int}C(x_0), \quad \forall y \in K.$$

Problem (3): Find $x_0 \in K$ such that

$$\langle T(y), \eta(x_0, y)\rangle + f(x_0, y) \nsubseteq \text{int}C(x_0), \quad \forall y \in K.$$

We recall the following concepts and results which are essential in the sequel.

Definition 2.1 A set-valued mapping $T : K \rightrightarrows L(X, Y)$ is said to be

(1) η-f pseudomonotone-type (I) if for each $x, y \in K$,

$$\langle T(x), \eta(y, x)\rangle + f(y, x) \nsubseteq -\text{int}C(x) \Longrightarrow \langle T(y), \eta(y, x)\rangle + f(y, x) \nsubseteq -\text{int}C(x).$$

(2) η-f pseudomonotone-type (II) if for each $x, y \in K$,

$$\langle T(x), \eta(y,x)\rangle + f(y,x) \nsubseteq -\mathrm{int}C(x) \Longrightarrow \langle T(y), \eta(x,y)\rangle + f(x,y) \nsubseteq \mathrm{int}C(x).$$

Definition 2.2 A set-valued mapping $F : K \rightrightarrows Y$ is said to be C-convex where C is a convex cone in Y if for all $x, y \in K$ and $t \in [0, 1]$, we have

$$(1-t)F(x) + tF(y) \subseteq F((1-t)x + ty) + C.$$

Lemma 2.1 [3]. *Let (Y, C) be an ordered topological vector space with a closed, convex and pointed cone C with $intC \neq \emptyset$. Then $\forall x, y \in Y$, one has*

(1) $y - x \in intC$ *and* $y \notin intC \Rightarrow x \notin intC$.
(2) $y - x \in C$ *and* $y \notin intC \Rightarrow x \notin intC$.
(3) $y - x \in -intC$ *and* $y \notin -intC \Rightarrow x \notin -intC$.
(4) $y - x \in -C$ *and* $y \notin -intC \Rightarrow x \notin -intC$.

Lemma 2.2 [21]. *Let $(X, \|.\|)$ be a normed space and H be a Hausdorff metric on the collection $CB(X)$ of all nonempty, closed and bounded subsets of X, induced by a metric d in terms of $d(u,v) = \|u - v\|$, which is defined by*

$$H(U,V) = \max(\sup_{u\in U}\inf_{v\in V}\|u-v\|, \sup_{v\in V}\inf_{u\in U}\|u-v\|),$$

for U and V in $CB(X)$. If U and V are compact sets in X, then for each $u \in U$, there exists $v \in V$ such that $\|u - v\| \leq H(U,V)$.

Definition 2.3 Let X and Y be normed spaces. A set-valued mapping $T : K \rightrightarrows L(X,Y)$ with compact values is said to be H-hemicontinuous on K if for every $x, y \in K$, the mapping $t \to H(T(x+t(y-x)), T(x))$ is continuous at 0^+, where H is the Hausdorff metric defined on $CB(L(X,Y))$.

Let X be a nonempty set, we shall denote by $\mathcal{F}(X)$ the family of all nonempty finite subsets of X. Let Y be a nonempty set, X a topological space and $F : Y \rightrightarrows X$ a set-valued mapping. Then F is said to be transfer closed-valued iff $\forall (y,x) \in Y \times X$ with $x \notin F(y)$, $\exists y' \in Y$, such that $x \notin \mathrm{cl}F(y')$. It is clear that this definition is equivalent to:

$$\bigcap_{y\in Y} F(y) = \bigcap_{y\in Y} \mathrm{cl}F(y).$$

If $B \subseteq Y$ and $A \subseteq X$, then we call $F : B \rightrightarrows A$ transfer closed-valued iff the multi-valued mapping $y \to F(y) \bigcap A$ is transfer closed-valued. When $X = Y$ and $A = B$, we call F transfer closed-valued on A. Let K be a convex subset of a vector space X. Then a mapping $F : K \rightrightarrows X$ is called a KKM mapping iff for each nonempty finite subset A of K, $\mathrm{conv}A \subset F(A)$, where $\mathrm{conv}A$ denotes the convex hull of A, and $F(A) = \cup\{F(x) : x \in A\}$.

Lemma 2.3 [7]. *Let K be a nonempty and convex subset of a Hausdorff t.v.s. X. Suppose that Γ,*
$\hat{\Gamma} : K \rightrightarrows K$ are two set-valued mappings such that the following conditions are satisfied:

(A1) $\hat{\Gamma}(x) \subseteq \Gamma(x), \quad \forall x \in K$,
(A2) $\hat{\Gamma}$ *is a KKM map,*
(A3) $\forall A \in$
$\mathcal{F}(K), \quad \Gamma$ *is transfer closed-valued on conv$A$,*
(A4) $\forall A \in \mathcal{F}(K)$, $cl_K(\bigcap_{x\in convA}\Gamma(x)) \cap convA = (\bigcap_{x\in convA}\Gamma(x)) \cap convA$,

(A5) *there is a nonempty compact convex set* $B \subseteq K$*, such that* $cl_K(\bigcap_{x\in B}\Gamma(x))$ *is compact.*

Then, $\bigcap_{x\in K}\Gamma(x) \neq \emptyset$*.*

Remark 2.1 When Γ is closed-valued, then conditions (A3)–(A4) are trivially satisfied.

3 Vector variational-like inequalities with monotonicity

In this section, we prove the solvability of (VVLI) with monotone set-valued mappings.

In order to establish an existence result for problem (II), we state and prove the following generalized vector version of Minty's lemma first.

Lemma 3.1 *Let* X *and* Y *be two normed spaces. Assume that* $T : K \rightrightarrows L(X, Y)$ *is* η-f *pseudomonotone type(I) and H-hemicontinuous with compact values. If*

(1) *for any fixed* $x, y, z \in K$*, the set-valued mapping* $y \rightrightarrows \langle T(z), \eta(y, x)\rangle + f(y, x)$ *is* $C(x)$*-convex and*
(2) *for each* $x, y \in K$, $\langle T(y), \eta(x, x)\rangle + f(x, x) \subseteq -C(x)$*,*

then Problems (I) and (II) are equivalent.

Proof Since T is η-f pseudomonotone type(I), therefore any solution of Problem (II) is also a solution for Problem(I).

Conversely, suppose that we can find $x_0 \in K$, such that for each $y \in K$

$$\langle T(y), \eta(y, x_0)\rangle + f(y, x_0) \nsubseteq -\text{int}C(x_0).$$

We consider $y_t = x_0 + t(y - x_0) \in K$ for $t \in (0, 1)$. Replacing y by y_t in the above inequality, we deduce

$$\langle T(y_t), \eta(y_t, x_0)\rangle + f(y_t, x_0) \nsubseteq -\text{int}C(x_0). \tag{2}$$

By condition (1), we have

$$\begin{aligned} &t[\langle T(y_t), \eta(y, x_0)\rangle + f(y, x_0)] + (1-t)[\langle T(y_t), \eta(x_0, x_0)\rangle + f(x_0, x_0)] \\ &\subseteq \langle T(y_t), \eta(y_t, x_0)\rangle + f(y_t, x_0) + C(x_0). \end{aligned} \tag{3}$$

From (1) and Lemma 2.1, we have

$$\langle T(y_t), \eta(y_t, x_0)\rangle + f(y_t, x_0) + C(x_0) \nsubseteq -\text{int}C(x_0). \tag{4}$$

Hence, (2), (3) and condition (2) imply that

$$\langle T(y_t), \eta(y, x_0)\rangle + f(y, x_0) \nsubseteq -\text{int}C(x_0). \tag{5}$$

Since $T(y_t)$ and $T(x_0)$ are compact, from Lemma 2.2 it follows that for each fixed $v_t \in T(y_t)$ there exists $u_t \in T(x_0)$ such that

$$\|v_t - u_t\| \leq H(T(y_t), T(x_0)).$$

As $T(x_0)$ is compact, without loss of generality, we may suppose that $u_t \to u_0 \in T(x_0)$ as $t \to 0^+$. Since T is H-hemicontinuous, $H(T(y_t), T(x_0)) \to 0$ as $t \to 0^+$. Thus one has

$$\|v_t - u_0\| \leq \|v_t - u_t\| + \|u_t - u_0\| \leq H(T(y_t), T(x_0)) + \|u_t - u_0\|,$$

as $t \to 0^+$. Therefore, letting $t \to 0^+$, we obtain

$$\|\langle (v_t - u_0), \eta(y, x_0)\rangle\| \leq \|v_t - u_0\|\|\eta(y, x_0)\| \to 0.$$

Since $Y \setminus -\text{int}C(x_0)$ is closed, hence from (4) we deduce that

$$\langle u_0, \eta(y, x_0)\rangle + f(y, x_0) \notin -\text{int}C(x_0).$$

Thus,

$$\langle T(x_0), \eta(y, x_0)\rangle + f(y, x_0) \nsubseteq -\text{int}C(x_0).$$

Remark 3.1 Lemma 3.1 generalizes Lemma 2.1 of Ref. [14]. It also improves Lemma 2.3 of [24] if we replace their mapping ToA, by our mapping T.

Theorem 3.1 *Assume that $T : K \rightrightarrows L(X, Y)$ is η-f pseudomonotone type(I), H-hemicontinuous and compact valued. Suppose that the following conditions are satisfied:*

(1) *The set-valued mapping $W : K \rightrightarrows Y$ defined by $W(x) = Y \setminus -intC(x)$ is $w \times \tau$-closed, where w is the weak topology of X.*
(2) *f and η are weak-norm continuous in the second argument.*
(3) *For each $x, y \in K$,*
 $\langle T(y), \eta(x, x)\rangle + f(x, x) \subseteq -C(x)$ and $\langle T(x), \eta(x, x)\rangle + f(x, x) = \{0\}$.
(4) *For any fixed $x, y, z \in K$, the set-valued mapping*
 $y \rightrightarrows \langle T(z), \eta(y, x)\rangle + f(y, x)$ is $C(x)$-convex.
(5) *There exist a nonempty weak compact set $M \subset K$ and a nonempty weak compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there is $y \in B$ such that*

$$\langle T(y), \eta(y, x)\rangle + f(y, x) \subseteq -intC(x).$$

Then Problem (II) holds.

Proof We show that for each $y \in K$, the set

$$\Gamma(y) = \{x \in K : \langle T(y), \eta(y, x)\rangle + f(y, x) \nsubseteq -\text{int}C(x)\}$$

is weakly closed. Let $\{x_\beta\}$ be a net in $\Gamma(y)$ weakly convergent to $x_0 \in K$. Since $x_\beta \in \Gamma(y)$ there exists $v_\beta \in T(y)$ satisfying

$$z_\beta = \langle v_\beta, \eta(y, x_\beta)\rangle + f(y, x_\beta) \notin -\text{int}C(x_\beta),$$

then $z_\beta \in W(x_\beta)$ and hence $(x_\beta, z_\beta) \in G_r(W)$. Since $T(y)$ is compact, $\{v_\beta\}$ has a convergent subnet in $T(y)$. Let $\{v_\lambda\}$ be a subnet of $\{v_\beta\}$ that converges to $v_0 \in T(y)$. By continuity of η, $\{\eta(y, x_\lambda)\}$ is a convergent net with norm. Hence, there exists λ_0 such that the set $\{\eta(y, x_\lambda) : \lambda \geq \lambda_0\}$ is norm bounded and therefore, by Proposition 2.3 of Ref. [5] and continuity of f in the second argument, we have

$$z_0 = \lim_{\lambda \geq \lambda_0} z_\lambda = \langle v_0, \eta(y, x_0)\rangle + f(y, x_0).$$

Since $G_r(W)$ is $w \times \tau$-closed, then $(x_0, z_0) \in G_r(W)$ and hence,

$$\langle v_0, \eta(y, x_0)\rangle + f(y, x_0) \notin -\text{int}C(x_0).$$

Thus, $x_0 \in \Gamma(y)$, this means $\Gamma(y)$ is weakly closed. Now, for each $y \in K$, we define the set-valued mapping $\hat{\Gamma} : K \rightrightarrows K$ by

$$\hat{\Gamma}(y) := \{x \in K : \langle T(x), \eta(y, x)\rangle + f(y, x) \nsubseteq -\text{int}C(x)\}.$$

We show that $\hat{\Gamma}$ is a KKM mapping. Since if $\hat{\Gamma}$ is not a KKM mapping, then there exists $\{x_1, x_2, \ldots, x_n\} \subset K$, $t_i \geq 0$, $i = 1, 2, \ldots, n$ with $\Sigma_{i=1}^n t_i = 1$ such that $x = \Sigma_{i=1}^n t_i x_i \notin \cup_{i=1}^n \hat{\Gamma}(x_i)$. Thus for any $i = 1, 2, \ldots, n$, we have

$$\langle T(x), \eta(x_i, x)\rangle + f(x_i, x) \subseteq -\text{int}C(x),$$

therefore, we deduce

$$\Sigma_{i=1}^n t_i \langle T(x), \eta(x_i, x)\rangle + \Sigma_{i=1}^n t_i f(x_i, x) \subseteq -\text{int}C(x). \tag{6}$$

On the other hand by (iv),

$$\langle T(x), \eta(x, x)\rangle + f(x, x) - \Sigma_{i=1}^n t_i [\langle T(x), \eta(x_i, x)\rangle + f(x_i, x)] \subseteq -C(x). \tag{7}$$

Thus by (5), (6) and the second part of condition (3), we have

$$\langle T(x), \eta(x, x)\rangle + f(x, x) = \{0\} \subseteqq -\text{int}C(x), \tag{8}$$

which contradicts $C(x) \neq Y$. Hence, $\hat{\Gamma}$ is a KKM mapping. Since T is η-f pseudomonotone type (I), we have $\hat{\Gamma}(y) \subseteq \Gamma(y)$ for each $y \in K$. Thus all of the conditions of Lemma 2.3 are fulfilled by the mappings $\hat{\Gamma}$ and Γ. Therefore,

$$\bigcap_{y \in K} \Gamma(y) \neq \emptyset.$$

Hence, Problem (I) holds. Since Lemma 3.1 implies the equivalence between Problem (I) and (II), the result follows.

Remark 3.2 Theorem 3.1 generalizes Theorem 2.1 of Ref. [14]. It also improves Theorem 2.1 of [24] if we replace their mapping ToA, by our mapping T.

Corollary 3.1 *Let K be a nonempty convex subset of a reflexive Banach space X with $0 \in K$ and Y be a normed space. Assume that $T : K \rightrightarrows L(X, Y)$ is η-f pseudomonotone type(I) and H-hemicontinuous with compact values. Suppose that the conditions (1)–(4) of Theorem 3.1 are satisfied and there exists some $r > 0$ such that*

$$\langle T(x), \eta(0, x)\rangle + f(0, x) \subseteq -intC(x), \quad x \in K \quad with\ \|x\| = r. \tag{9}$$

Then Problem (II) holds.

Proof Let $B_r = \{x \in X : \|x\| \leq r\}$. By Theorem 3.1, there exists $x_r \in K \cap B_r$ such that

$$\langle T(x_r), \eta(y, x_r)\rangle + f(y, x_r) \nsubseteq -\text{int}C(x_r), \quad \forall y \in K \cap B_r. \tag{10}$$

Putting $y = 0$ in the above inequality, one has

$$\langle T(x_r), \eta(0, x_r)\rangle + f(0, x_r) \nsubseteq -\text{int}C(x_r). \tag{11}$$

Combining (8) and (10), we deduce that $\|x_r\| < r$. For any $z \in K$, choose $t \in (0, 1)$ small enough such that $z_t = (1 - t)x_r + tz \in K \cap B_r$, hence from (9), one has

$$\langle T(x_r), \eta(z_t, x_r)\rangle + f(z_t, x_r) \nsubseteq -\text{int}C(x_r). \tag{12}$$

Condition (4) implies that

$$\begin{aligned} &t[\langle T(x_r), \eta(z, x_r)\rangle + f(z, x_r)] + (1 - t)[\langle T(x_r), \eta(x_r, x_r)\rangle + f(x_r, x_r)] \\ &\quad \subseteqq \langle T(x_r), \eta(z_t, x_r)\rangle + f(z_t, x_r) + C(x_r). \end{aligned} \tag{13}$$

Then from (11), (12), the second part of condition (3) and Lemma 2.1, we deduce

$$\langle T(x_r), \eta(z, x_r)\rangle + f(z, x_r) \nsubseteq -\mathrm{int}C(x_r), \quad \forall z \in K. \tag{14}$$

This completes the proof.

Remark 3.3 Corollary 3.1 improves Theorem 2.1 of Ref. [24] in many aspects if we replace their mapping ToA, by our mapping T.

Theorem 3.2 *Assume that T: $K \rightrightarrows L(X, Y)$ is η-f pseudomonotone type(I) and H-hemicontinuous with compact values. Suppose that the following conditions are satisfied:*

(1) *The set-valued mapping $W : K \rightrightarrows Y$ defined by $W(x) = Y \setminus -intC(x)$ is closed.*
(2) *f and η are continuous in the second argument.*
(3) *For each $x, y \in K$,*
 $\langle T(y), \eta(x, x)\rangle + f(x, x) \subseteq -C(x)$ and $\langle T(x), \eta(x, x)\rangle + f(x, x) = \{0\}$.
(4) *For any fixed $x, y, z \in K$, the set-valued mapping*
 $y \rightrightarrows \langle T(z), \eta(y, x)\rangle + f(y, x)$ is $C(x)$-convex.
(5) *There exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there is $y \in B$ such that*

$$\langle T(y), \eta(y, x)\rangle + f(y, x) \subseteq -intC(x).$$

Then Problem (II) holds.

Proof By a similar proof as that of Theorem 3.1, one can deduce the result.

Remark 3.4 Theorem 3.2 is a generalized version of Corollary 2.1 of Ref. [24].

In the following we will establish another vector version of Minty's Lemma.

Lemma 3.2 *Let X and Y be two normed spaces. Assume that $T : K \rightrightarrows L(X, Y)$ is η-f pseudomonotone type(II) and H-hemicontinuous with compact values. If*

(1) *for each $x, y \in K$, $\langle T(y), \eta(y, y)\rangle + f(y, y) \subseteq C(x)$,*
(2) *for each $x, y, z \in K$, the set-valued mapping*
 $y \rightrightarrows \langle T(z), \eta(y, z)\rangle + f(y, z)$ is C(x)-convex and
(3) *f and η are continuous,*

then, Problems (II) and (III) are equivalent.

Proof Since T is η-f pseudomonotone type(II), therefore any solution of Problem (II) is also a solution for Problem(III).

Conversely, suppose that we can find $x_0 \in K$, such that for each $y \in K$

$$\langle T(y), \eta(x_0, y)\rangle + f(x_0, y) \nsubseteq \mathrm{int}C(x_0).$$

We consider $y_t = x_0 + t(y - x_0) \in K$ for $t \in (0, 1)$. Replacing y by y_t in the above inequality, we deduce

$$\langle T(y_t), \eta(x_0, y_t)\rangle + f(x_0, y_t) \nsubseteq \mathrm{int}C(x_0). \tag{15}$$

From condition (1), we obtain that

$$\langle T(y_t), \eta(y_t, y_t)\rangle + f(y_t, y_t) \subsetneqq C(x_0). \tag{16}$$

Hence, condition (2) and (15) imply that

$$t[\langle T(y_t), \eta(y, y_t)\rangle + f(y, y_t)] + (1-t)[\langle T(y_t), \eta(x_0, y_t)\rangle + f(x_0, y_t)] \subsetneqq C(x_0). \tag{17}$$

Therefore, from (14) and (16) and Lemma 1.1, we have

$$\langle T(y_t), \eta(y, y_t)\rangle + f(y, y_t) \nsubseteq -\text{int}C(x_0). \tag{18}$$

Now since $T(y_t)$ and $T(x_0)$ are compact, and T is H-hemicontinuous, by the same argument as that of the proof of Lemma 3.1, for each $v_t \in T(y_t)$ we can find $u_0 \in T(x_0)$ such that $v_t \to u_0 \in T(x_0)$ as $t \to 0^+$. By continuity of η and f in the second argument, $\eta(y, y_t) \to \eta(y, x_0)$ and $f(y, y_t) \to f(y, x_0)$ as $t \to 0^+$, respectively. Furthermore, $\{\eta(y, y_t)\}$ is bounded for a sufficiently small $t > 0$. Thus by Proposition 2.3 of [5]

$$\langle v_t, \eta(y, y_t)\rangle + f(y, y_t) \to \langle u_0, \eta(y, x_0)\rangle + f(y, x_0) \tag{19}$$

as $t \to 0^+$. Since $Y \setminus -\text{int}C(x_0)$ is closed, hence from (17) and (18) we deduce that

$$\langle u_0, \eta(y, x_0)\rangle + f(y, x_0) \notin -\text{int}C(x_0).$$

Therefore,

$$\langle T(x_0), \eta(y, x_0)\rangle + f(y, x_0) \nsubseteq -\text{int}C(x_0), \quad \forall y \in K.$$

Remark 3.5 If for each $x, y, z \in K$, the mapping $y \longmapsto \langle T(z), \eta(y, x)\rangle + f(y, x)$ is affine, then condition (2) is satisfied. Hence, the above Lemma improves Theorem 3.1 of Ref. [17] and therefore Theorem 2.3 of Ref. [19], if we replace their mapping ToA, by our mapping T. In the proof of Lemma 3.2 in condition (3), the continuity of f and η in the second argument is sufficient. Lemma 3.2 is also a vector version of Lemmas 6.1 and 6.2 of Ref. [15].

Theorem 3.3 *Let X and Y be normed spaces. Assume that all of the conditions of Lemma* 3.2 *are satisfied and*

(1) *The set-valued mapping $W : K \rightrightarrows Y$ defined by $W(x) = Y \setminus intC(x)$ is closed.*
(2) *There exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there is $y \in B$ such that*

$$\langle T(y), \eta(x, y)\rangle + f(x, y) \subseteq intC(x).$$

Then Problem (*II*) *holds.*

Proof For each $y \in K$, we define the set-valued mapping $\hat{\Gamma} : K \rightrightarrows K$ by

$$\hat{\Gamma}(y) := \{x \in K : \langle T(x), \eta(y, x)\rangle + f(y, x) \nsubseteq -\text{int}C(x)\}.$$

We show that $\hat{\Gamma}$ is a KKM mapping. Since if $\hat{\Gamma}$ is not a KKM mapping, then there exists $\{x_1, x_2, \dots, x_n\} \subset K$, $t_i \geq 0$, $i = 1, 2, \dots, n$ with $\Sigma_{i=1}^n t_i = 1$ such that $x = \Sigma_{i=1}^n t_i x_i \notin \cup_{i=1}^n \hat{\Gamma}(x_i)$. Thus for any $i = 1, 2, \dots, n$, we have

$$\langle T(x), \eta(x_i, x)\rangle + f(x_i, x) \subseteq -\text{int}C(x),$$

therefore, we deduce

$$\Sigma_{i=1}^n t_i \langle T(x), \eta(x_i, x)\rangle + \Sigma_{i=1}^n t_i f(x_i, x) \subseteq -\text{int}C(x). \tag{20}$$

and by condition (2) of Lemma 3.2, we have

$$\langle T(x), \eta(x, x)\rangle + f(x, x) - \Sigma_{i=1}^{n} t_i[\langle T(x), \eta(x_i, x)\rangle + f(x_i, x)] \subseteq -C(x), \tag{21}$$

From (19), (20) and condition (1) of Lemma 3.2, we obtain

$$\langle T(x), \eta(x, x)\rangle + f(x, x) \subseteq C(x) \cap -\text{int}C(x), \tag{22}$$

which is a contradiction. Hence, $\hat{\Gamma}$ is a KKM mapping.

Now for each $y \in K$, we define the set-valued mapping $\Gamma : K \rightrightarrows K$ by

$$\Gamma(y) = \{x \in K : \langle T(y), \eta(x, y)\rangle + f(x, y) \nsubseteq \text{int}C(x)\}.$$

Since T is η-f pseudomonotone type (II), we have $\hat{\Gamma}(y) \subseteq \Gamma(y)$ for each $y \in K$. We will show that for each $y \in K$, $\Gamma(y)$ is closed. Let $\{x_n\}$ be a sequence in $\Gamma(y)$ convergent to $x_0 \in K$. Since $x_n \in \Gamma(y)$ there exists $v_n \in T(y)$ satisfying

$$z_n = \langle v_n, \eta(x_n, y)\rangle + f(x_n, y) \notin \text{int}C(x_n)$$

Therefore, $z_n \in W(x_n)$ and hence $(x_n, z_n) \in G_r(W)$. Since $T(y)$ is compact, $\{v_n\}$ has a convergent subsequence in $T(y)$. Let $\{v_m\}$ be such a subsequence of $\{v_n\}$ that converges to $v_0 \in T(y)$. By continuity of η, $\{\eta(x_m, y)\}$ is a convergent sequence. Hence, it is norm bounded and therefore, by Proposition 2.3 of [5] and continuity of f, we have

$$z_0 = \lim_{m} z_m = \langle v_0, \eta(x_0, y))\rangle + f(x_0, y).$$

Since $G_r(W)$ is closed, then $(x_0, z_0) \in G_r(W)$ and hence,

$$\langle v_0, \eta(x_0, y))\rangle + f(x_0, y)) \notin \text{int}C(x_0).$$

Thus, $x_0 \in \Gamma(y)$, this means $\Gamma(y)$ is closed. Thus all of the conditions of Lemma 2.3 are fulfilled by the mappings $\hat{\Gamma}$ and Γ. Therefore,

$$\bigcap_{y \in K} \Gamma(y) \neq \emptyset.$$

Hence, Problem (III) holds and from Lemma 3.2, Problem (II) is deduced.

Remark 3.6 The above result improves Theorem 3.2 of Ref. [17] and Theorem 3.2 of Ref. [19], if we replace their mapping ToA, by our mapping T. Theorem 3.3 is also a vector version of Theorem 6.2 of Ref. [16].

4 Vector variational-like inequalities without monotonicity

In this section, some existence results for vector variational-like inequality problem without any monotonicity are obtained. We suppose that $\{C(x) : x \in K\}$ is a family of closed and convex cones in Y.

Theorem 4.1 *Assume that the conditions* (*iii*)–(*iv*) *of Theorem 3.1 are satisfied and the set-valued mapping* $\Gamma : K \rightrightarrows K$ *defined by*

$$\Gamma(y) = \{x \in K : \langle T(x), \eta(y, x)\rangle + f(y, x) \nsubseteq -intC(x)\}$$

is weakly closed valued. If there exist a nonempty weak compact set $M \subset K$ *and a nonempty weak compact convex set* $B \subset K$ *such that for each* $x \in K \setminus M$, *there is* $y \in B$ *such that* $x \notin \Gamma(y)$, *then Problem* (*II*) *holds.*

Proof By the same argument as that of the second part of the proof of Theorem 3.1, one can deduce that $\Gamma = \hat{\Gamma}$ is a KKM mapping. Hence the result follows from Lemma 2.3.

Remark 4.1 Theorem 4.1 generalizes Theorem 2.2 of Ref. [14] in many aspects. It also improves Theorem 3.1 of Ref. [24] if we replace their mapping ToA, by our mapping T.

Corollary 4.1 *Let X and Y be normed spaces and let $T : K \rightrightarrows L(X, Y)$ be a set-valued mapping. Assume that the following conditions are satisfied:*

(1) *For each $y \in K$, the set-valued mapping defined by*

$$\Gamma(y) = \{x \in K : \langle T(x), \eta(y, x)\rangle + f(y, x) \nsubseteq -intC(x)\}$$

is closed valued

(2) *For each $x, y, z \in K$, the set-valued mapping $y \rightrightarrows \langle T(z), \eta(y, y)\rangle + f(y, y)$ is C(x)-convex in the first argument.*

(3) *For each $x, y \in K$, $\langle T(x), \eta(y, y)\rangle + f(y, y) \subseteq C(x)$,*

(4) *There exist a nonempty compact set $M \subset K$ and a nonempty compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there is $y \in B$ such that $x \notin \Gamma(y)$.*

Then Problem (II) holds.

Proof By the same argument as that of the first part of the proof of Theorem 3.3, one can deduce that $\Gamma = \hat{\Gamma}$ is a KKM mapping. Hence the result follows from Lemma 2.3.

Remark 4.2 Corollary 4.1 generalizes Theorem 2.1 of Ref. [14]. It also improves Theorem 3.2 of Ref. [24] if we replace their mapping ToA, by our mapping T.

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