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Commutators of fractional integral operators on Vanishing-Morrey spaces

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Abstract In this note we prove a sufficient condition for commutators of fractional integral operators to belong to Vanishing Morrey spaces $VL^{p,\lambda}$. In doing this we use an extension on Morrey spaces of an inequality by Fefferman and Stein concerning the sharp maximal function and the fractional maximal function and related Morrey inequalities.

Keywords Vanishing mean oscillation functions · Commutators · Singular integral operators · Morrey Spaces

Mathematics Subject Classification (2000) Primary 42B20, 43A15, 32A37. Secondary 46E35, 35J15

1 Introduction

In this paper we are concerned with a sufficient condition for commutators of fractional integral operators to belong to Vanishing Morrey spaces. Key ingredients are the use of an extension to Morrey spaces of an inequality by Fefferman and Stein concerning the maximal and the sharp maximal function (see Lemma 2), the fractional maximal function used by Muckenhoupt and Wheeden in the note [10] and the related properties.

This suitable subspace $VL^{p,\lambda}$ of the classical Morrey Spaces $L^{p,\lambda}$ was introduced by Vitanza in [12] and applied there to obtain a regularity result for elliptic partial differential equations. Later in [13] Vitanza proved an existence theorem for a Dirichlet problem, under weaker assumptions then those introduced by Miranda in [9], and a $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to Vanishing Morrey spaces depending on the dimension.

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We hope to continue the study and to characterize the subspace of BMO of Vanishing Mean Oscillation functions b, in terms of commutators

$$\big[b, K\big](f) = b(x)\,\big(K\,f\big)(x) - I_\alpha\,\big(b\,f\big)(x),$$

where K is the fractional integral operator, as in 1976 Coifman et al. [4] have characterized the John–Nirenberg space (BMO) of Bounded Mean Oscillation functions b in terms of commutators:

$$\big[b, K\big](f) = b(x)\,\big(K\,f\big)(x) - K\,\big(b\,f\big)(x)$$

between the Riesz transform K and b locally integrable function b on $\mathbb{R}^n$.

We finally observe that in the sequel the letter c will be used to denote various constants which not depend on the functions. The various uses of the letter do not, however, all denote the same constant.

2 Definitions and preliminary tools

Let $B = B(x, \rho)$ be a ball in $\mathbb{R}^n$ of radius ρ centered at the point x.

Definition 1 Given $f \in L^1_{loc}(\mathbb{R}^n)$ let us set

$$M\,f\,(x) = \sup_{B \supset x} \frac{1}{|\,B\,|} \int_B |\,f(y)\,|\,d\,y \quad \text{for } a.a.x \in \mathbb{R}^n$$

M is the *Hardy Littlewood Maximal Operator*.

The following *Sharp Maximal Function* is one of its variants

$$f^{\#}(x) = \sup_{B \supset x} \frac{1}{|\,B\,|} \int_B |\,f(y) - f_B|\,d\,y \quad \text{for } a.a.x \in \mathbb{R}^n.$$

Let us define the *FractionalMaximalFunction*, used by Muchkenhoupt and Wheeden in their relevant results contained in [10].

Definition 2 Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $0 < \eta < 1$, we set

$$M_\eta\,f\,(x) = \sup_{B \supset x} \frac{1}{|\,B\,|^{1-\eta}} \int_B |\,f(y)\,|\,d\,y \quad \text{for } a.a.x \in \mathbb{R}^n.$$

Definition 3 Given $f \in L^1(\mathbb{R}^n)$ and $0 < \alpha < n$ let us set

$$I_\alpha\,f\,(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|\,x - y\,|^{n-\alpha}}\,d\,y \quad a.e. \in \mathbb{R}^n$$

the *FractionalIntegralOperator* of order α.

Definition 4 *(see [8])* Let $1 \le p < \infty, 0 \le \lambda < n$. A measurable function $f \in L^p(\mathbb{R}^n)$ belongs to the Morrey class $L^{p,\lambda}(\mathbb{R}^n)$ if

$$\|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\, \rho > 0} \frac{1}{\rho^\lambda} \int\limits_{B(x,\rho)} |\,f(y)\,|^p dy < +\infty.$$

Definition 5 *(see, e.g. [12,13])* Let $1 \leq p < \infty, 0 \leq \lambda < n$. We say that $f \in L^{p,\lambda}(\mathbb{R}^n)$ belongs to the Vanishing Morrey space $VL^{p,\lambda}(\mathbb{R}^n)$ if setting

$$\zeta^p(r) = \sup_{x\in\mathbb{R}^n,\, \rho\leq r} \frac{1}{\rho^\lambda} \int\limits_{B(x,\rho)} |f(y)|^p dy$$

we have

$$\lim_{r\to 0} \zeta(r) = 0.$$

In a similar way we obtain the definition of $VL^{p,\lambda}(X)$, $X \subset \mathbb{R}^n$ open set having sufficiently smooth boundary, replacing $\mathbb{R}^n$ by X and the ball $B(x,\rho)$ by $B(x,\rho) \cap X$.

Definition 6 Let f be a locally integrable function defined on $\mathbb{R}^n$. We say that f is in the space BMO($\mathbb{R}^n$) (see [7]) if

$$\|f\|_* \equiv \sup_{x_0\in\mathbb{R}^n,\rho>0} \frac{1}{|B(x_0,\rho)|} \int\limits_{B(x_0,\rho)} |f(y) - f_B| dy < \infty,$$

where

$$f_B = \frac{1}{|B|} \int\limits_B f(y) dy.$$

Let $f \in$ BMO($\mathbb{R}^{n+1}$) and $r > 0$. We define the VMO modulus of f by the rule

$$\eta(r) = \sup_{\rho\leq r} \frac{1}{|B_\rho|} \int\limits_{B_\rho} |f(y) - f_{B_\rho}| dy$$

where B_ρ is a generic ball having radius ρ.

BMO is a Banach space with the norm $\|f\|_* = \sup_{r>0} \eta(r)$.

Definition 7 *(see [11])* We say that a function $f \in$ BMO($\mathbb{R}^n$) is in the Sarason class VMO($\mathbb{R}^n$) if

$$\lim_{r\to 0} \eta(r) = 0.$$

Lemma 1 (see [5], Lemma 2) *Let I_β be a fractional integral operator of order β, $0 < \beta < n$, $1 < r, t < p < \frac{n}{\beta}$, $0 < \lambda < n - \beta p$ and $b \in BMO(\mathbb{R}^n)$.*

Then there exists a constant $c \geq 0$ independent of b and f such that

$$\left([b, I_\beta](f)\right)^\#(x) \leq c\,\|b\|_* \left\{\left(M\,|I_\beta f|^r\right)^{\frac{1}{r}}(x) + \left(M_{\frac{\beta t}{n}}\,|f|^t\right)^{\frac{1}{t}}(x)\right\}$$

for a. a. $x \in \mathbb{R}^n$ and every $f \in L^{p,\lambda}(\mathbb{R}^n)$.

The next Lemma is a generalization of an inequality by Fefferman and Stein contained in [6] (p. 153) and is proved in [5], Lemma 3.

Lemma 2 *Let $1 < p < +\infty$ and $0 < \lambda < n$. Then there exists a nonnegative constant c independent of f such that*

$$\|Mf\|_{p,\lambda} \leq c\,\|f^\#\|_{p,\lambda}$$

for every $f \in L^{p,\lambda}(\mathbb{R}^n)$.

3 Results

As a consequence of a celebrated result by Adams (Theorem 3.15 in [1]), we have the following estimate of the Fractional Integral Operator in Morrey Spaces.

Theorem 1 *Let* $0 < \alpha < n, 1 < p < \frac{n}{\alpha}, 0 < \lambda < n - \alpha p$. *Set* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$ *and* $\mu = \frac{\lambda\cdot(n-\lambda)}{n-\lambda-\alpha p}$ $(i.e., \frac{\lambda}{p} = \frac{\mu}{q})$. *There exists a constant* $c > 0$ *independent of* f *such that*

$$\|I_{\frac{\alpha n}{n-\lambda}} f\|_{q,\mu} \le c \cdot \|f\|_{p,\lambda}, \quad \forall f \in L^{p,\lambda}(\mathbb{R}^n).$$

Proof At first we observe that the relation between p, λ, q, μ is $\frac{\mu}{\lambda} = \frac{q}{p}$ and, because of

$$\mu = \frac{\lambda(n-\lambda)}{n-\lambda-\alpha p}$$

we have

$$q = \frac{p}{\lambda}\left[\frac{\lambda(n-\lambda)}{n-\lambda-\alpha p}\right] = \frac{(n-\lambda)p}{n-\lambda-\alpha p}.$$

Then

$$\frac{1}{q} = \frac{n-\lambda-\alpha p}{(n-\lambda)p}$$

or equivalently

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}.$$

The estimate is obtained as follows. We recall that in [2] Theorem 2 is proved

$$\|I_\beta f\|_{q,\mu} \le c \cdot \|f\|_{p_1,\mu}, \tag{3.1}$$

where

$$\frac{1}{q} = \frac{1}{p_1} - \frac{\beta}{n-\mu}$$

we also have that

$$L^{p,\lambda} \subset L^{p_1,\mu}, \quad \frac{n-\lambda}{p} \le \frac{n-\mu}{p_1}$$

then

$$\|f\|_{p_1,\mu} \le c\|f\|_{p,\lambda}. \tag{3.2}$$

Combining the estimates (3.1) and (3.2), we obtain

$$\|I_\beta f\|_{q,\mu} \le c \cdot \|f\|_{p_1,\mu} \le c\|f\|_{p,\lambda},$$

where

$$\begin{aligned}\frac{1}{p_1} - \frac{1}{q} &\geq \left(\frac{n-\lambda}{n-\mu}\right) \cdot \frac{1}{p} - \frac{1}{q}\\ &= \frac{1}{n-\mu} \cdot \left(\frac{n-\lambda}{p} - \frac{n-\mu}{q}\right)\\ &= \frac{1}{n-\mu} \cdot \left(\frac{n}{p} - \frac{\lambda}{q} - \frac{n}{q} + \frac{\mu}{q}\right)\\ &= \frac{1}{n-\mu} \cdot \left(\frac{n}{p} - \frac{n}{q}\right)\\ &= \frac{n}{n-\mu} \cdot \left(\frac{1}{p} - \frac{1}{q}\right)\\ &= \frac{n}{n-\mu} \cdot \left(\frac{\alpha}{n-\lambda}\right)\\ &= \frac{1}{n-\mu} \cdot \left(\frac{\alpha n}{n-\lambda}\right).\end{aligned}$$

Then we obtain the conclusion being

$$\beta = \frac{\alpha n}{n-\lambda}.$$

We point out that $\beta < n$ because the hypotheses $\lambda < n - \alpha\, p$ imply $\lambda < n - \alpha$, then we have $\frac{\alpha}{n-\lambda} < 1$.

Theorem 2 *Let* $0 < \alpha < n, 1 < p < +\infty, 0 < \lambda < n - \alpha p$; *then for* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$ *and* $\mu = \frac{\lambda \cdot q}{p}$ *there exists* $c > 0$ *independent on* f *such that*

$$\|M_{\frac{\alpha}{n-\lambda}} f\|_{q,\mu} \leq c \cdot \|f\|_{p,\lambda}, \quad \forall f \in L^{p,\lambda}(\mathbb{R}^n).$$

Proof From Lemma 4 in [5] if $1 < p < +\infty, 0 < \lambda < n, 0 < \eta < \left(1 - \frac{\lambda}{n}\right)\frac{1}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{n\,\eta}{n-\lambda}$, there exists $c > 0$:

$$\|M_\eta f\|_{q,\lambda} \leq c\,\|f\|_{p,\lambda}, \quad \forall f \in L^{p,\lambda}(\mathbb{R}^n),$$

then we have

$$\|M_\eta f\|_{q,\mu} \leq c\,\|f\|_{p_1,\mu}, \quad \frac{1}{p_1} = \frac{1}{q} + \frac{n\,\eta}{n-\mu}.$$

Because of

$$\|f\|_{p_1,\mu} \leq \|f\|_{p,\lambda}, \quad \frac{1}{p_1} = \frac{n-\lambda}{n-\mu} \cdot \frac{1}{p}$$

from the above two inequalities we obtain

$$\|M_\eta f\|_{q,\mu} \leq c\,\|f\|_{p,\lambda},$$

where

$$\left(\frac{n-\lambda}{n-\mu}\right) \cdot \frac{1}{p} - \frac{1}{q} = \frac{n\,\eta}{n-\mu}, \quad \frac{1}{q} = \left(\frac{n-\lambda}{n-\mu}\right) \cdot \frac{1}{p} - \frac{n\,\eta}{n-\mu},$$

$$\frac{1}{q} = \left(\frac{n-\mu+\mu-\lambda}{n-\mu}\right) \cdot \frac{1}{p} - \frac{n\eta}{n-\mu},$$

$$\frac{1}{q} = \frac{1}{p} - \left[\frac{n\,p\,\eta - (\mu - \lambda)}{(n-\mu)\cdot p}\right].$$

We have to prove now that

$$\left[\frac{n\,p\,\eta - (\mu - \lambda)}{(n-\mu)\cdot p}\right] = \frac{\alpha}{n-\lambda}.$$

It is true because, if $\eta = \frac{\alpha}{n-\lambda}$, by a direct computation we have:

$$\left[\frac{n\eta p - (\mu - \lambda)}{(n-\mu)\cdot p}\right] = \frac{\left(\frac{n\,p\,\alpha}{n-\lambda}\right) - (\mu-\lambda)}{(n-\mu)\cdot p}$$

and

$$\frac{\left(\frac{n\,p\,\alpha}{n-\lambda}\right) - (\mu-\lambda)}{(n-\mu)\cdot p} = \frac{\alpha}{n-\lambda}$$

if and only if

$$\begin{aligned}
&\left(\frac{n\,p\,\alpha}{n-\lambda}\right)\cdot(n-\lambda) - (\mu-\lambda)\cdot(n-\lambda) = \alpha\,p(n-\mu)\\
&\Leftrightarrow n\,p\,\alpha - \mu\,(n-\lambda) + \lambda(n-\lambda) = \alpha\,p\,n - \alpha p\,\mu\\
&\Leftrightarrow \lambda\cdot(n-\lambda) = \mu\cdot\left[n - \lambda - \alpha\,p\right]
\end{aligned}$$

or equivalently

$$\mu = \frac{\lambda\,(n-\lambda)}{n - \lambda - \alpha\,p}.$$

Theorem 3 *Let* $0 < \alpha < n, 1 < p < \frac{n}{\alpha}, 0 < \lambda < n - \sqrt{n\alpha\,p}, q > 0 : \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$ *and* $\mu = \frac{\lambda\cdot(n-\lambda)}{n-\lambda-\alpha p}$ *(i.e.,* $\frac{\lambda}{p} = \frac{\mu}{q}$*).*

Let b be a function in $VMO(\mathbb{R}^n)$*. Then there exists a fractional integral operator* I_γ *of order* $0 < \gamma < n$ *such that the commutator* $[b, I_\gamma]$ *is bounded from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{q,\mu}(\mathbb{R}^n)$*.*

Proof Let us consider two real numbers t, r later specified, according to the restriction that will be neessary for them. From Lemma 2 we have

$$\|[b, I_\beta](f)\|_{q,\mu} \le \|M([b, I_\beta](f))\|_{q,\mu} \le c\cdot\|([b, I_\beta](f))^{\#}\|_{q,\mu}$$

applying Lemma 1 for $\beta = \frac{\alpha n}{n-\lambda}$, $1 < r, t < p$ and also $t < \frac{n-\lambda}{\alpha}$, we have

$$\begin{aligned}
&\le c\cdot\|b\|_*\cdot\left\{\|M(|I_\beta f|^r)^{\frac{1}{r}}\|_{q,\mu} + \|M_{\frac{\beta t}{n}}(|f|^t)^{\frac{1}{t}}\|_{q,\mu}\right\}\\
&= c\cdot\|b\|_*\cdot\left\{\|M(|I_{\frac{\alpha n}{n-\lambda}} f|^r)^{\frac{1}{r}}\|_{q,\mu} + \|M_{\frac{\alpha t}{n-\lambda}}(|f|^t)^{\frac{1}{t}}\|_{q,\mu}\right\}. \qquad (3.3)
\end{aligned}$$

Let us verify that β satisfies the conditions of Lemma 1:

(1) $0 < \beta < n, or : 0 < \frac{n\alpha}{n-\lambda} < n \Leftrightarrow \alpha < n-\lambda, true\,because{:}\ \lambda < (n-\alpha p) < n-\alpha;$
(2) $p < \frac{n}{\beta}, \quad or : p < \frac{n}{\left(\frac{\alpha n}{n-\lambda}\right)} = \frac{n-\lambda}{\alpha},\ true\,because : \lambda < n - \alpha p;$
(3) $\lambda < n - \beta\,p, or : \beta < \frac{n-\lambda}{p} \Leftrightarrow \frac{n\alpha}{n-\lambda} < \frac{n-\lambda}{p} \Leftrightarrow n\alpha p < (n-\lambda)^2\, or : \lambda < n - \sqrt{n\alpha\,p}.$

Are then verified for β the conditions of Lemma 1. Since we suppose

$$\lambda < n - \sqrt{n\alpha\, p}$$

we have $\lambda < n - \alpha\, p$, it implies $p < \frac{n-\lambda}{\alpha}$. Therefore

$$t < p \quad \text{and} \quad t < \frac{n-\lambda}{\alpha}$$

are both satisfied if $t < p$. Hence we obtain that

$$\|M(|I_{\frac{\alpha n}{n-\lambda}} f|^r)^{\frac{1}{r}}\|_{q,\mu} = \|M(|I_{\frac{\alpha n}{n-\lambda}} f|)^r\|^{\frac{1}{r}}_{\frac{q}{r},\mu}$$

from a result by Chiarenza and Frasca contained in [2]

$$\leq \||I_{\frac{\alpha n}{n-\lambda}} f|^r\|^{\frac{1}{r}}_{\frac{q}{r},\mu} = \| I_{\frac{\alpha n}{n-\lambda}} f\|_{q,\mu} \leq \|f\|_{p,\lambda}$$

the last inequality is true applying Theorem 1 above.

Thus we have

$$\|M(|I_{\frac{\alpha n}{n-\lambda}} f|^r)^{\frac{1}{r}}\|_{q,\mu} \leq \|f\|_{p,\lambda}.$$

The second term in (3.3) can be estimate as follows

$$\|M_{\frac{\alpha t}{n-\lambda}}(|f|^t)^{\frac{1}{t}}\|_{q,\mu} = \|M_{\frac{\alpha t}{n-\lambda}}(|f|^t)\|^{\frac{1}{t}}_{\frac{q}{t},\mu}$$

because of $\frac{\alpha\, t}{n-\lambda} < \left(1 - \frac{\lambda}{n}\right)\frac{1}{\left(\frac{p}{t}\right)}$, we apply Theorem 2 and obtain

$$\leq c \cdot \|(|f|^t)\|^{\frac{1}{t}}_{\frac{p}{t},\lambda} = c\,\|f\|_{p,\lambda}.$$

Then we have proved that

$$\|\,[b, I_{\frac{\alpha n}{n-\lambda}}]\,(f)\,\|_{L^{q,\mu}(\mathbb{R}^n)} \leq c \cdot \|b\|_* \cdot \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$$

and the proof is complete, being $\gamma = \beta$.

Theorem 4 *Let* $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \sqrt{n\alpha\, p}$, $q > 0 : \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$ *and* $\mu = \frac{\lambda\cdot(n-\lambda)}{n-\lambda-\alpha p}$ *(i.e.,* $\frac{\lambda}{p} = \frac{\mu}{q}$*). Suppose* b *in VMO* $(\mathbb{R}^n)$ *and* $f \in L^{p,\lambda}(\mathbb{R}^n)$.

Then $\exists\, \rho_0 > 0 : \forall R < \rho_0$ *we have* $[b, I_{\frac{\alpha n}{n-\lambda}}]\, f \in VL^{q,\mu}(Q_R)$.

Proof Let B be a generic ball in $\mathbb{R}^n$, from Theorem 3 we have that

$$\left(\sup_{x\in B,\, \rho>0} \frac{1}{\rho^\mu} \int_{B(x,\rho)\cap B} |\,[b, I_{\frac{\alpha n}{n-\lambda}}](f)|^q(y)dy\right)^{\frac{1}{q}}$$

$$\leq \left(\sup_{x\in\mathbb{R}^n,\, \rho>0} \frac{1}{\rho^\mu} \int_{B(x,\rho)} |\,[b, I_{\frac{\alpha n}{n-\lambda}}](f)|^q(y)dy\right)^{\frac{1}{q}}$$

$$\equiv \|\,[b, I_{\frac{\alpha n}{n-\lambda}}]\,(f)\,\|_{L^{q,\mu}(\mathbb{R}^n)} \leq c \cdot \|b\|_* \cdot \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$$

or

$$\|[b, I_{\frac{\alpha n}{n-\lambda}}](f)\,|\|_{L^{q,\mu}(B)} \leq c \cdot \|b\|_* \cdot \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \qquad \forall B.$$

We observe, likewise Theorem 2.13 in [3], that for any $\epsilon > 0 \exists\, \rho_0 > 0$ such that for any generic ball $B_R = B(x, R)$ with radius R such that $0 < R < \rho_0$,

$$\|[b, I_{\frac{\alpha n}{n-\lambda}}](f)\|\|_{L^{q,\mu}(B_R)} \leq c \cdot \epsilon \cdot \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$$

then

$$\sup_{x \in B_R,\, 0<\rho<diam B_R} \frac{1}{\rho^\mu} \int\limits_{B(x,\rho)\cap B_R} \Big(|\,[b, I_{\frac{\alpha n}{n-\lambda}}](f)|\Big)^q (y) dy \leq c \cdot \epsilon, \quad \forall \epsilon > 0.$$

Because of we are interested in $\lim_{r \to 0} \zeta(r)$, let us now consider only $r < \mathrm{diam} B_R$, then

$$\begin{aligned} \zeta^q(r) \equiv & \sup_{x \in B_R,\, \rho<r} \frac{1}{\rho^\mu} \int\limits_{B(x,\rho)\cap B_R} |[b, I_{\frac{\alpha n}{n-\lambda}}](f)|^q(y)\, dy \\ \leq & \sup_{x \in B_R,\, \rho<diam B_R} \frac{1}{\rho^\mu} \int\limits_{B(x,\rho)\cap B_R} |[b, I_{\frac{\alpha n}{n-\lambda}}](f)\,|^q(y)\, dy \leq c \cdot \epsilon, \quad \forall \epsilon > 0 \end{aligned}$$

it follows

$$\zeta^q(r) \leq c \cdot \epsilon,\ \forall r < \mathrm{diam} B_R, \quad \forall \epsilon > 0$$

then

$$\lim_{r \to 0} \zeta(r) = 0$$

we have proved that

$$[b, I_{\frac{\alpha n}{n-\lambda}}]\, f \in VL^{q,\mu}(B_R), \quad \forall R < \rho_0$$

and the conclusion follows.

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