J Glob Optim
DOI 10.1007/s10898-007-9170-0

On the solution existence of pseudomonotone variational inequalities

B. T. Kien · J.-C. Yao · N. D. Yen

Received: 22 September 2006 / Accepted: 25 May 2007

Abstract As shown by N. Thanh Hao (submitted data), the solution existence results established by F. Facchinei and J.-S. Pang [(vols. I, II, Springer, Berlin, 2003) Prop. 2.2.3 and Theorem 2.3.4] for variational inequalities in general and for pseudomonotone variational inequalities in particular, are very useful for studying the range of applicability of the Tikhonov regularization method. This paper proposes some extensions of these results of (Finite-Dimensional Variational Inequalities and Complementarity Problems, vols. I, II, Springer, 2003) to the case of generalized variational inequalities and of variational inequalities in infinite-dimensional reflexive Banach spaces. Various examples are given to analyze in detail the obtained results.

Keywords Variational inequality · Generalized variational inequality · Pseudomonotone operator · Solution existence · Degree theory

1 Introduction

Variational inequality (VI, for brevity), generalized variational inequality (GVI), and quasi-variational inequality (QVI) have been recognized as suitable mathematical models for dealing with many problems arising in different fields, such as optimization theory, game

B.T. Kien – on leave from the Hanoi University of Civil Engineering.

B.T. Kien · J.-C. Yao (✉)
Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung 804,
Taiwan
e-mail: yaojc@math.nsysu.edu.tw

B.T. Kien
e-mail: btkien@gmail.com

N. D. Yen
Institute of Mathematics, Vietnamese Academy of Science and Technology, 18 Hoang Quoc Viet,
Hanoi 10307, Vietnam
e-mail: ndyen@math.ac.vn

theory, economic equilibrium, mechanics, etc. In the last four decades, since the time of the celebrated Hartman–Stampacchia theorem (see [9,11]), solution existence of VIs, GVIs, QVIs, and other related problems has become a basic research topic which continues to attract attention of researchers in applied mathematics (see for instance [1,3–5,12,17–19], and the references therein). Difficult questions do exist in the field (see, e.g., [16,20]).

Recently, in the two-volume book [7] dedicated entirely to finite-dimensional VIs, Facchinei and Pang have used the degree theory to obtain the following existence theorems for VIs.

Theorem 1.1 ([7, Vol. I, p. 146]) *Let $K \subset \mathbb{R}^n$ be a closed convex set and $F : K \to \mathbb{R}^n$ be a continuous mapping. Consider the following statements:*

(a) *There exists a reference point $x^{\text{ref}} \in K$ such that the set*

$$L_< := \{x \in K : \langle F(x), x - x^{\text{ref}} \rangle < 0\}, \tag{1.1}$$

where $\langle , \rangle$ denotes the scalar product, is bounded (possibly empty).

(b) *There exist a bounded open set $\Omega \subset \mathbb{R}^n$ and a vector $x^{\text{ref}} \in \Omega \cap K$ such that*

$$\langle F(x), x - x^{\text{ref}} \rangle \geq 0 \quad \forall x \in K \cap \partial\Omega, \tag{1.2}$$

where $\partial\Omega$ denotes the boundary of Ω.

(c) *The variational inequality problem* $\mathrm{VI}(K, F)$*, which consists of finding an $x \in K$ such that*

$$\langle F(x), y - x \rangle \geq 0 \quad \forall y \in K$$

has a solution.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *Moreover, if the set*

$$L_\leq := \{x \in K : \langle F(x), x - x^{\text{ref}} \rangle \leq 0\} \tag{1.3}$$

is bounded, then the solution set $\mathrm{SOL}(K, F)$ *of* $\mathrm{VI}(K, F)$ *is nonempty and compact.*

Theorem 1.2 ([7, Vol. I, p. 158]) *Let $K \subset \mathbb{R}^n$ be closed convex and $F : K \to \mathbb{R}^n$ be continuous. Assume that F is a pseudomonotone operator, that is the implication*

$$\langle F(y), x - y \rangle \geq 0 \implies \langle F(x), x - y \rangle \geq 0$$

is valid for all $x, y \in K$. Then the statements (a), (b), *and* (c) *in the above theorem are equivalent.*

The concept of pseudomonotone operator was proposed by Karamardian [10].

It is clear that the implications (b) $\Rightarrow$ (c) and (a) $\Rightarrow$ (c) in Theorem 1.1 give sufficient conditions for the solution existence of the problem $\mathrm{VI}(K, F)$. Meanwhile, Theorem 1.2 shows that if F is a pseudomonotone operator then the solution existence of the problem $\mathrm{VI}(K, F)$ can be *characterized* via the conditions (a) and (b), which are now equivalent. From the proof given in [7, Vol. I] it follows that the conclusion of Theorem 1.1 is valid if in the statement (b) one requires that $\Omega \subset \mathbb{R}^n$ is an *open ball*.

As it has been noted in [7, Vol. I, p. 237], the result recalled in Theorem 1.2 gave for the first time a necessary and sufficient condition for a pseudomonotone VI on a general closed convex set to have a solution. Moreover, conditions in Theorem 1.2 are different from ordinary coercivity conditions which are often used to guarantee the solution existence of VIs on unbounded sets. For the coercivity conditions we refer the reader to [3,5,8].

Solution existence theorems for variational inequalities without monotonicity have played key role in the Tikhonov regularization method. Based on solution existence results of the dual variational inequalities and the initial variational inequality, Konnov et al. [13,14] proved the convergence of the Tikhonov regularization method for a class of nonmonotone variational inequalities.

Using Theorem 1.1, Qi [15] has shown that the union of the solution sets of the regularized problems is nonempty and bounded. In particular, N. Thanh Hao (submitted data) has solved in affirmative the question of Facchinei and Pang [7, Vol. II, Remark 12.2.4, p. 1129]: *Whether the conclusion of the convergence theorem of the Tikhonov regularization method* (see [7, Vol. II, Theorem 12.2.3, p. 1128]) *will remain valid if instead of monotone VIs one considers pseudomonotone VIs*?

In order to study the question of Facchinei and Pang in a broader context, one may wish to extend Theorems 1.1 and 1.2 to the case of finite-dimensional GVIs and the case of infinite-dimensional VIs (and GVIs). Our aim in this paper is to obtain such extensions.

The paper is organized as follows. In Sect. 2, from Theorems 1.1 and 1.2 we derive existence theorems for finite-dimensional GVIs. In the same section, we also construct several examples to analyze the relations between the statements (a), (b), and (c) in Theorem 1.1. In Sect. 3, we show that Theorem 1.2 can be extended for VIs and GVIs in infinite-dimensional reflexive Banach spaces, while the implications (b) $\Rightarrow$ (c) and (a) $\Rightarrow$ (c) in Theorem 1.1 are no longer valid if instead of $\mathbb{R}^n$ one considers an infinite-dimensional Hilbert space. (Note that the solution existence theorem for GVIs given in Sect. 3 does not encompass those given in Sect. 2.)

We now recall some standard definitions and notation which will be used in the sequel.

Let X be a reflexive Banach space over the reals, $K \subset X$ a nonempty closed convex set, $\Phi : K \rightrightarrows X^*$ a multifunction from K into the dual space X^* (which is equipped with the weak* topology).

The *generalized variational inequality* defined by K and Φ, denoted by GVI(K, Φ), is the problem of finding a point $x \in K$ such that

$$\exists x^* \in \Phi(x), \qquad \langle x^*, y - x\rangle \geq 0 \quad \forall y \in K. \tag{1.4}$$

Here $\langle , \rangle$ denotes the canonical pairing between X^* and X. The set of all $x \in K$ satisfying (1.4) is denoted by SOL(K, Φ). If $\Phi(x) = \{F(x)\}$ for all $x \in K$, where $F : K \to X^*$ is a single-valued map, then the problem GVI(K, Φ) is called a *variational inequality* and the abbreviation VI(f, K) is used instead of GVI(K, Φ).

If for any $x, y \in K$ and $x^* \in \Phi(x)$, $y^* \in \Phi(y)$ one has $\langle x^* - y^*, x - y\rangle \geq 0$, then one says that Φ is a *monotone operator*. If for any $x, y \in K$ and $x^* \in \Phi(x)$, $y^* \in \Phi(y)$ the implication

$$\langle y^*, x - y\rangle \geq 0 \implies \langle x^*, x - y\rangle \geq 0$$

is valid, then one says that Φ is a *pseudomonotone operator*. For the case $X = \mathbb{R}^n$, the dual space X^* is identified with X and the pairing between X^* and X just the scalar product in $\mathbb{R}^n$. With this convention, we see that the the notion of pseudomonotone operator given here is in full agreement with the one described (for single-valued maps) in Theorem 1.2.

It is clear that monotonicity implies pseudomonotonicity. The converse implication is not true in general (take, for instance, $K = \mathbb{R}$ and $F(x) = x^2 + 1$ for all $x \in K$).

One says that $\Phi : K \rightrightarrows X^*$ is a lower semicontinuous multifunction if $\Phi(x) \neq \emptyset$ for all $x \in K$ and for any $x \in K$, for any open set $W \subset X^*$ satisfying $\Phi(x) \cap W \neq \emptyset$, there exists an open neighborhood U of x such that $\Phi(y) \cap W \neq \emptyset$ for all $y \in U \cap K$. If for any open set $W \subset X^*$ satisfying $\Phi(x) \subset W$ there exists an open neighborhood U of x such that $\Phi(y) \subset W$ for all $y \in U \cap K$, then Φ is said to be an upper semicontinuous multifunction.

2 Finite-dimensional GVIs

The following solution existence theorem for finite-dimensional GVIs is an extension of Theorem 1.1.

Theorem 2.1 *Let $K \subset \mathbb{R}^n$ be a closed convex set and $\Phi\colon K \rightrightarrows \mathbb{R}^n$ be a lower semicontinuous multifunction with nonempty closed convex values. Consider the following statements:*

(a) *There exists $x^{\rm ref} \in K$ such that the set*

$$L_{<}(\Phi, x^{\rm ref}) := \left\{x \in K : \inf_{x^* \in \Phi(x)} \langle x^*, x - x^{\rm ref}\rangle < 0\right\} \tag{2.1}$$

is bounded (possibly empty).

(b) *There exists an open ball $\Omega \subset \mathbb{R}^n$ and a vector $x^{\rm ref} \in \Omega \cap K$ such that*

$$\inf_{x^* \in \Phi(x)} \langle x^*, x - x^{\rm ref}\rangle \geq 0 \quad \forall x \in K \cap \partial\Omega, \tag{2.2}$$

where $\partial\Omega$ denotes the boundary of Ω.

(c) *The generalized variational inequality* $\mathrm{GVI}(K, \Phi)$ *has a solution.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *Moreover, if there exists $x^{\rm ref} \in K$ such that the set*

$$L_{\leq}(\Phi, x^{\rm ref}) := \left\{x \in K : \inf_{x^* \in \Phi(x)} \langle x^*, x - x^{\rm ref}\rangle \leq 0\right\}$$

is bounded, then the solution set $\mathrm{SOL}(K, \Phi)$ *is nonempty and bounded.*

Proof Since the multifunction Φ is lower semicontinuous and has nonempty closed convex values, by Michael's selection theorem (see for instance [21, p. 466]) it admits a continuous selection; that is there exists a continuous mapping $F\colon K \to \mathbb{R}^n$ such that $F(x) \in \Phi(x)$ for every $x \in K$.

If (a) holds, then there exists an open ball, denoted by Ω such that

$$L_{<}(\Phi, x^{\rm ref}) \cup \left\{x^{\rm ref}\right\} \subset \Omega.$$

Combining the obvious property $\partial\Omega \cap L_{<}(\Phi, x^{\rm ref}) = \emptyset$ with (2.1) yields (2.2). We have shown that (a) implies (b).

Suppose now that (b) is valid. Then we have

$$\langle F(x), x - x^{\rm ref}\rangle \geq \inf_{x^* \in \Phi(x)} \langle x^*, x - x^{\rm ref}\rangle \geq 0 \quad \forall x \in K \cap \partial\Omega.$$

Applying Theorem 1.1 we get $\mathrm{SOL}(K, F) \neq \emptyset$. For any $x \in \mathrm{SOL}(K, F)$, if we choose $x^* = F(x)$ then

$$\langle x^*, y - x\rangle \geq 0 \quad \forall y \in K.$$

It follows that $\emptyset \neq \mathrm{SOL}(K, F) \subset \mathrm{SOL}(K, \Phi)$. Thus (b) implies (c).

If (c) is valid, then as $x^{\rm ref}$ we choose any vector from $\mathrm{SOL}(K, \Phi)$. Since $L_{<}(\Phi, x^{\rm ref}) = \emptyset$, (a) holds.

Finally, suppose that there is some $x^{\text{ref}} \in K$ such that the set $L_{\le}(\Phi, x^{\text{ref}})$ is bounded. Then $\text{SOL}(K, \Phi)$ is nonempty by virtue of the implication (a) $\Rightarrow$ (c). To prove that $\text{SOL}(K, \Phi)$ is bounded, it suffices to show that $\text{SOL}(K, \Phi) \subset L_{\le}(\Phi, x^{\text{ref}})$. Let $x \in \text{SOL}(K, \Phi)$. Substituting $y = x^{\text{ref}}$ into the inequality in (1.4) gives $\langle x^*, x - x^{\text{ref}}\rangle \le 0$. Then we have $\inf_{x^* \in \Phi(x)} \langle x^*, x - x^{\text{ref}}\rangle \le 0$, i.e., $x \in L_{\le}(\Phi, x^{\text{ref}})$. □

Theorem 2.2 *Let* $K \subset \mathbb{R}^n$ *be closed convex and* $\Phi : K \rightrightarrows \mathbb{R}^n$ *be a lower semicontinuous multifunction with nonempty closed convex values. Assume that* Φ *is a pseudomonotone operator. Then the statements* (a), (b), *and* (c) *in Theorem* 2.1 *are equivalent.*

Proof By Theorem 2.1, (a) $\Rightarrow$ (b) $\Rightarrow$ (c). So it suffices to prove that (c) $\Rightarrow$ (a). Assuming (c), we take any $x^{\text{ref}} \in \text{SOL}(K, \Phi)$. By (2.1), there exists $x^* \in \Phi(x^{\text{ref}})$ satisfying

$$\langle x^*, y - x^{\text{ref}}\rangle \ge 0 \quad \forall y \in K.$$

Therefore, by the pseudomonotonicity of Φ, for any $y \in K$ and $y^* \in \Phi(y)$, one has $\langle y^*, y - x^{\text{ref}}\rangle \ge 0$. It follows that

$$\inf_{y^* \in \Phi(y)} \langle y^*, y - x^{\text{ref}}\rangle \ge 0 \quad \forall y \in K;$$

hence $L_{<}(\Phi, x^{\text{ref}}) = \emptyset$ and (a) is valid. □

Remark 2.1 Except for the argument involving Michael's selection theorem, in the above proofs we have followed closely the arguments used in [7, Vol. I] for proving the results in Theorems 1.1 and 1.2.

Let us consider several useful illustrative examples. The next example shows that the reverse of the implication (b) $\Rightarrow$ (c) in Theorem 1.1 is not true in general.

Example 2.1 (*(c)* $\not\Rightarrow$ *(b)*) Let $K = [0, +\infty) \subset \mathbb{R}$, $F(x) = -x$ (or $F(x) = -x^2$) for all $x \in K$. It is easy to see that $\text{SOL}(K, F) = \{0\}$. Let $\Omega \subset \mathbb{R}$ be a bounded open set such that there exists a point $x^{\text{ref}} \in \Omega \cap K$. By the formula of K, we infer that there must exist some $\overline{x} \in \partial\Omega$ such that $\overline{x} > 0$ and $\overline{x} - x^{\text{ref}} > 0$. Then $\overline{x} \in \partial\Omega \cap K$ and we have $\langle F(\overline{x}), \overline{x} - x^{\text{ref}}\rangle < 0$. This shows that (1.2) fails to hold for the given pair $\{\Omega, x^{\text{ref}}\}$. Thus, the property (c) is valid for this problem $\text{VI}(K, F)$, while (b) is violated.

The reverse of the implication (a) $\Rightarrow$ (b) in Theorem 1.1 is also false in general.

Example 2.2 (*(b)* $\not\Rightarrow$ *(a)*) Let $K = \mathbb{R}$, $F(x) = -x(x-1)$ for all $x \in \mathbb{R}$. Taking $\Omega = (-1, 1)$ and $x^{\text{ref}} = 0$ we see at once that (1.2) holds. It is a simple matter to show that, for any $x^{\text{ref}} \in \mathbb{R}$, the set $L_{<}$ defined by (1.1) is unbounded. Thus the property (a) in Theorem 1.1 does not hold for this problem $\text{VI}(K, F)$, while (b) is valid.

We have seen that property (a) (resp., property (b)) in Theorem 1.1 is a *sufficient* but *not a necessary condition* for the solution existence of the problem $\text{VI}(K, F)$.

Remark 2.2 Concerning the inclusion $L_{<} \subset L_{\le}$, it is worthy to stress that the topological closure of $L_{<}$ can be a proper subset of $L_{\le}$. Indeed, consider the problem $\text{VI}(K, F)$ described in Example 2.2 observe that, for $x^{\text{ref}} = 0$, $L_{<} = (1, +\infty)$, while $L_{\le} = \{0\} \cup [1, +\infty)$.

As Theorems 2.1 and 2.2 can be considered as "set-valued extensions" of Theorems 1.1 and 1.2, Examples 2.1 and 2.2 show that the implications (b) $\Rightarrow$ (c) and (a) $\Rightarrow$ (b) in Theorem 2.1 are not reversible in general. Remark 2.2 says that the topological closure of the set $L_{<}(\Phi, x^{\text{ref}})$ (see Theorem 2.1) can be a proper subset of $L_{\le}(\Phi, x^{\text{ref}})$.

Remark 2.3 In the formulations of Theorems 2.1 and 2.2 instead of assuming that "$\Phi : K \rightrightarrows \mathbb{R}^n$ *is a lower semicontinuous multifunction with nonempty closed convex values*" one can assume that "$\Phi : K \rightrightarrows \mathbb{R}^n$ *is a multifunction with admits a continuous selection* $F : K \to \mathbb{R}^n$." The subsequent examples show that this weaker assumption significantly enlarges the class of problems to which Theorems 2.1 and 2.2 can be applied to.

Example 2.3 Let $K = [0, +\infty)$, $K_1 = [-1, +\infty)$, $\Phi : \mathbb{R} \rightrightarrows \mathbb{R}$ is the subdifferential mapping in the sense of convex analysis of the convex function $\varphi(x) = |x|$; that is

$$\Phi(x) = \begin{cases} \{1\} & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0. \end{cases}$$

Note that $F(x) \equiv 1$ is a continuous selection of the restriction of Φ on K. Applying the refined version of Theorem 2.1 described in Remark 2.3 to $\mathrm{VI}(K, \Phi)$, we conclude that $\mathrm{SOL}(K, \Phi) \neq \emptyset$. Since the restriction of Φ on K_1 does not have any continuous selection, the refined version of Theorem 2.1 is not applicable to $\mathrm{VI}(K_1, \Phi)$. Observe that $\mathrm{SOL}(K, \Phi) = \mathrm{SOL}(K_1, \Phi) = \{0\}$.

Example 2.4 Let $K = \mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_i \geq 0,\ i = 1, 2\}$, $K_1 = [-1, +\infty) \times [-1, +\infty)$, $\Phi : \mathbb{R} \rightrightarrows \mathbb{R}$ is the subdifferential mapping in the sense of convex analysis of $\varphi(x) = |x_1| + |x_2|$, $x = (x_1, x_2 \in \mathbb{R}$. Since

$$\Phi(x) = \begin{cases} \{1\} & \text{if } x_1 > 0, x_2 > 0, \\ [-1, 1] \times \{1\} & \text{if } x_1 = 0, x_2 > 0, \\ \{1\} \times [-1, 1] & \text{if } x_1 > 0, x_2 = 0, \\ [-1, 1] \times [-1, 1] & \text{if } x = 0, \\ (-1, -1) & \text{if } x_1 < 0, x_2 < 0, \\ [-1, 1] \times \{-1\} & \text{if } x_1 = 0, x_2 < 0, \\ \{-1\} \times [-1, 1] & \text{if } x_1 < 0, x_2 = 0, \end{cases}$$

$F(x) \equiv (1, 1)$ is a continuous selection of the restriction of Φ on K. The refined version of Theorem 2.1 asserts that $\mathrm{SOL}(K, \Phi) \neq \emptyset$. However, since Φ on K_1 does not have any continuous selection, the refined version of Theorem 2.1 is not applicable to $\mathrm{VI}(K_1, \Phi)$. Observe that $\mathrm{SOL}(K, \Phi) = \mathrm{SOL}(K_1, \Phi) = \{(0, 0)\}$.

The following result is a version of Theorem 1.1 to the case of GVIs with upper semicontinuous operators.

Theorem 2.3 *Let $K \subset \mathbb{R}^n$ be a closed convex set and $\Phi\colon K \rightrightarrows \mathbb{R}^n$ be a upper semicontinuous multifunction with nonempty compact convex values. Consider the following statements:*

(a) *There exists $x^{\mathrm{ref}} \in K$ such that the set*

$$L_{\leq}(\Phi, x^{\mathrm{ref}}) := \left\{ x \in K : \inf_{x^* \in \Phi(x)} \langle x^*, x - x^{\mathrm{ref}} \rangle \leq 0 \right\} \tag{2.3}$$

is bounded (possibly empty).

(b) *There exists an open ball $\Omega \subset \mathbb{R}^n$ and a vector $x^{\mathrm{ref}} \in \Omega \cap K$ such that*

$$\inf_{x^* \in \Phi(x)} \langle x^*, x - x^{\mathrm{ref}} \rangle > 0 \quad \forall x \in K \cap \partial\Omega, \tag{2.4}$$

where $\partial\Omega$ denotes the boundary of Ω.

(c) *The generalized variational inequality* GVI(K, Φ) *has a solution.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *Moreover, the solution set* SOL(K, Φ) *is nonempty and bounded.*

Proof The implication $(a) \Rightarrow (b)$ is similar with the proof of Theorem 2.1.

For the proof of implication (b) $\Rightarrow$ (c) we will use the following approximate selection theorem due to A. Cellina.

Lemma 2.1 ([2, Theorem 1, p. 84]) *Let X and Y be Banach space, $M \subset X$ and $T : M \to 2^Y$ be an u.s.c multifunction with closed and convex values. Then for each $\epsilon > 0$ there exists a continuous map $f_\epsilon : M \to Y$ such that*

$$f_\epsilon(x) \in T((x + \epsilon B_X) \cap M) + \epsilon B_Y, \tag{2.5}$$

where B_X and B_Y are open unit balls of X and Y, respectively.

Denote by $\{\phi_\epsilon\}_{\epsilon>0}$ the family of approximate selections of Φ satisfying the conclusion of Lemma 2.1. We now claim that there exists a $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0]$ one has

$$\langle \phi_\epsilon(x), x - x^{\text{ref}} \rangle \geq 0 \quad \forall x \in K \cap \partial\Omega. \tag{2.6}$$

Suppose that the assertion is false. Then there exist sequences $\epsilon_n \to 0$ and $x_n \in K \cap \partial\Omega$ such that

$$\langle \phi_{\epsilon_n}(x_n), x_n - x^{\text{ref}} \rangle < 0. \tag{2.7}$$

By Lemma 2.1, there exist $y_n \in K$ and $z_n \in \Phi(y_n)$ such that $\|y_n - x_n\| < \epsilon_n$ and $\|\phi_{\epsilon_n}(x_n) - z_n\| < \epsilon_n$. By the compactness of $K \cap \partial\Omega$ we can assume that $x_n \to x_0 \in K \cap \partial\Omega$. This implies that $y_n \to x_0$. Since $\Phi(x_0)$ is a compact set and Φ is u.s.c. at x_0, $z_n \to z_0 \in \Phi(x_0)$. Hence $\phi_{\epsilon_n}(x_n) \to z_0$. By letting $n \to \infty$ we obtain from (2.7) that $\langle z_0, x_0 - x^{\text{ref}} \rangle \leq 0$. It follows that

$$\inf_{x^* \in \Phi(x_0)} \langle x^*, x_0 - x^{\text{ref}} \rangle \leq 0$$

for $x_0 \in K \cap \partial\Omega$. This contradicts (2.4) and so the assertion is obtained.

We now consider VI$(K, \hat{\phi}_{\epsilon_n})$, where $\epsilon_n = 1/n$ and $\hat{\phi}_n := \phi_{\epsilon_n}$. Then $\hat{\phi}_n$ satisfies condition (b) of Theorem 1.1. By this theorem, there exists $x_n \in K \cap \Omega$ such that

$$\langle \hat{\phi}_n(x_n), x - x_n \rangle \geq 0 \quad \forall x \in K. \tag{2.8}$$

By the compactness of $K \cap \bar{\Omega}$ we can assume that $x_n \to x_0$. Using the similar arguments as the above we get $\hat{\phi}_n(x_n) \to z_0 \in \Phi(x_0)$. By letting $n \to \infty$ we obtain from (2.6) that

$$\langle z_0, x - x_0 \rangle \geq 0 \quad \forall x \in K.$$

The implication $(b) \Rightarrow (c)$ follows. Since SOL$(K, \Phi) \subset L_{\leq}(\Phi, x^{\text{ref}})$, SOL$(K, \Phi)$ is bounded. The proof is complete. □

3 Infinite-dimensional VIs and GVIs

Theorem 2.2 can be extended to the case of VIs in reflexive Banach spaces as follows.

Theorem 3.1 *Let X be a real reflexive Banach space and $K \subset X$ be a closed convex set. Assume that $F : K \to X^*$ is a pseudomonotone operator which is continuous on finite dimensional subspaces of X. Then the following statements are equivalent:*

(a) *There exists a reference point* $x^{\text{ref}} \in K$ *such that the set*

$$L_{<}(F, x^{\text{ref}}) := \{x \in K : \langle F(x), x - x^{\text{ref}} \rangle < 0\}$$

is bounded (possibly empty);

(b) *There exist an open ball* Ω *and a vector* $x^{\text{ref}} \in \Omega \cap K$ *such that*

$$\langle F(x), x - x^{\text{ref}} \rangle \geq 0 \quad \forall x \in K \cap \partial\Omega; \tag{3.1}$$

(a) *The problem* VI(K, F) *has a solution.*
Besides, if there exists a vector $x^{\text{ref}} \in K$ *such that the set*

$$L_{\leq}(F, x^{\text{ref}}) := \left\{x \in K : \langle F(x), x - x^{\text{ref}} \rangle \leq 0\right\}$$

is bounded, then the solution set SOL(K, F) *is nonempty and bounded.*

In order to prove this theorem we shall need the following generalization of the Minty lemma.

Lemma 3.1 (See [17, p. 693] and [19, p. 509]) *Let* K *be a closed convex subset in a real reflexive Banach space* X, $F : K \to X^*$ *be a pseudomonotone operator. Assume that* F *is hemicontinuous; that is for every pair of points* $x, y \in K$ *the function*

$$t \mapsto \langle F(tx + (1-t)y), x - y \rangle, \quad 0 \leq t \leq 1,$$

is continuous. Then $x \in K$ *is a solution of* VI(K, F) *if and only*

$$\langle F(y), y - x \rangle \geq 0 \quad \forall y \in K.$$

It is clear that the operator $F : K \to X^*$ is hemicontinuous whenever it is continuous on finite dimensional subspaces of X.

Pooof of Theorem 3.1 The implication (a) ⇒ (b) can be proved similarly as the corresponding assertion in Theorem 2.1 In order to prove the implication (b) ⇒ (c) we will use the method of proving solution existence theorems for monotone VIs in [11]. Suppose that there exist an open ball $\Omega \subset X$ and a vector $x^{\text{ref}} \in \Omega \cap K$ such that (3.1) is satisfied. For each $x \in K$, we put

$$Q(x) = \{y \in K \cap \bar{\Omega} : \langle F(x), x - y \rangle \geq 0\} \tag{3.2}$$

and notice that $Q(x)$ is a weakly closed subset of $K \cap \bar{\Omega}$. We will show that the family $\{Q(x)\}_{x \in K}$ has the finite intersection property. In fact, given a finite sequence $x^1, x^2, \ldots, x^m$ of vectors in K we denote by L the linear subspace of X generated by the vectors $x^1, x^2, \ldots, x^m, x^{\text{ref}}$. Let $K_L = K \cap L$, $\Omega_L = \Omega \cap L$, and let $\partial_L \Omega_L$ stand for the boundary of Ω_L in the induced topology of L. Then $\partial_L \Omega_L = (\partial\Omega) \cap L$. Consider the map $F_L : K_L \to L^*$ defined by

$$\langle F_L(x), y \rangle = \langle F(x), y \rangle \quad \forall y \in L. \tag{3.3}$$

From (3.1) and (3.3) all the conditions stated in the statement (b) of Theorem 1.2, where $(K_L, F_L, \Omega_L, x^{\text{ref}})$ plays the role of the $(K, F, \Omega, x^{\text{ref}})$, are fulfilled. Hence there exists a vector $u_L \in \Omega_L$ such that

$$\langle F(u_L), y - u_L \rangle \geq 0 \quad \forall y \in K_L.$$

By Lemma 3.1 and by the pseudomonotonicity of F, from the last property we deduce that

$$\langle F(y), y - u_L \rangle \geq 0 \quad \forall y \in K_L.$$

In particular,

$$\langle F(x^i), x^i - u_L \rangle \geq 0 \quad \forall i = 1, 2, \ldots, m.$$

Hence

$$u_L \in \bigcap_{i=1}^{m} Q(x^i).$$

We have shown that the family $\{Q(x)\}_{x \in K}$ has the finite intersection property. This and the weak compactness of $K \cap \bar{\Omega}$ imply

$$\bigcap_{x \in K} Q(x) \neq \emptyset.$$

Hence there exists a vector $\overline{u} \in K \cap \bar{\Omega}$ such that

$$\langle F(x), x - \overline{u} \rangle \geq 0 \quad \forall x \in K.$$

Applying Lemma 3.1 once more, we get

$$\langle F(\overline{u}), x - \overline{u} \rangle \geq 0 \quad \forall x \in K;$$

hence $\overline{u} \in \mathrm{SOL}(K, F)$.

The assertion $(c) \Rightarrow (a)$ can be proved similarly as the corresponding assertion in Theorem 2.2. The proof is complete. □

Comparing Theorem 1.1 with Theorem 3.1, one finds that the latter needs a stronger assumption: F is a pseudomonotone operator. Hence one can raise the following very natural question: *Whether the conclusion of Theorem 3.1 remains valid if the assumption on the pseudomonotonicity of F is omitted?*

The next example gives a *negative answer* for this question. It shows that without the pseudomonotonicity assumption on F, neither one of the implications (b) $\Rightarrow$ (c) and (a) $\Rightarrow$ (c) in Theorem 3.1 is valid. We will see that there exists a problem of the form $\mathrm{VI}(K, F)$ with a continuous mapping F which has no solutions, but for which we can find a point $x^{\mathrm{ref}} \in K$ and an open ball Ω containing x^{ref} such that the conditions (1.1) and (1.2) are both satisfied.

Example 3.1 Let $X = H$, where H is an infinite-dimensional Hilbert space with the inner product $\langle , \rangle$. We identify the dual space X^* with H and put $B_H = \{x \in H : \|x\| \leq 1\}$. According to a result of J. Dugundji (see for instance [6, p. 66]), there exists a continuous (in norm) mapping $G \colon B_H \to B_H$ which has no fixed points. Define

$$F(x) = x - G(x) \quad \forall x \in B_H$$

and put $K = B_H$. For the given pair $\{K, F\}$, property (1.2) is satisfied if we choose $x^{\mathrm{ref}} = 0$ and $\Omega = \{x \in H \colon \|x\| < 1\}$. To see this, it suffices to observe that

$$\langle F(x), x - x^{\mathrm{ref}} \rangle = \|x\|^2 - \langle G(x), x \rangle \geq 1 - \|G(x)\| \|x\| \geq 0$$

whenever $x \in \partial\Omega \cap K = \{x \in H : \|x\| = 1\}$. We have $\mathrm{SOL}(K, F) = \emptyset$. Indeed, if there exists $x \in \mathrm{SOL}(K, F)$ then are two possibilities: $\|x\| < 1$, or $\|x\| = 1$. If $\|x\| < 1$, then x is an interior point of K. This implies $F(x) = 0$, which is impossible because G has no fixed points. If $\|x\| = 1$, then the condition

$$\langle F(x), y - x \rangle \geq 0 \quad \forall y \in K = B_H$$

yields $F(x) = \lambda(-x)$ for some $\lambda \geq 0$. Hence

$$1 \geq \|G(x)\| = (1+\lambda)\|x\| = 1 + \lambda.$$

Thus $\lambda = 0$ and we obtain $F(x) = 0$; i.e., x is a fixed point of G. This contradicts the choice of G. Observe also that property (1.1) is satisfied for any choice of $x^{\text{ref}} \in K$ because K is a bounded set.

Using Theorem 3.1 and the arguments for proving Theorem 2.1 we can establish the following result.

Theorem 3.2 *Let X be a real reflexive Banach space and $K \subset X$ be a closed convex set. Let $\Phi : K \to 2^{X^*}$ a lower semicontinuous multifunction with nonempty closed convex values. Assume that Φ is a pseudomonotone operator. Then the following statements are equivalent:*

(a) *There exists $x^{\text{ref}} \in K$ such that the set $L_<(\Phi, x^{\text{ref}})$ defined as in* (2.1) *is bounded (possibly empty);*
(b) *There exists an open ball Ω and a vector $x^{\text{ref}} \in \Omega \cap K$ such that the condition* (2.2) *is satisfied;*
(c) *Problem* GVI(F, K) *has a solution.*

In the formulation of Theorem 3.2 instead of assuming that *"Φ is a lower semicontinuous multifunction with nonempty closed convex values"* one can assume that *"Φ is a multifunction with admits a continuous selection."*

Let us end this section with a remark about VIs and GVIs with quasimonotone operators. By definition, a multifunction $\Phi : K \rightrightarrows X^*$ from a closed convex set K of a Banach space X into the dual space X^*. If for any $x, y \in K$ and $x^* \in \Phi(x)$, $y^* \in \Phi(y)$ the implication

$$\langle y^*, x - y\rangle > 0 \implies \langle x^*, x - y\rangle \geq 0$$

is valid, then one says that Φ is a *quasimonotone operator*. Clearly, if Φ is pseudomonotone, then it is quasimonotone. *The conclusion of Theorems 1.2, 2.2, 3.1, and 3.2 is no longer valid if instead of problems with a pseudomonotone operators one considers problems quasimonotone operators.*

Example 3.2 The conclusion of Theorems 1.2, 2.2, 3.1, and 3.2 is no longer valid if instead of the problem with a pseudomonotone operator one considers a problem with a quasimonotone operator. To see this, it suffices to put $K = [0, \infty)$, $F(x) = -x^2$, and $\Phi(x) = \{F(x)\}$ for all $x \in K$. It is a simple matter to verify that F, hence Φ, is a quasimonotone operator. We have SOL$(K, F) = \{0\}$, while there does not exist any $x^{\text{ref}} \in K$ such that $L_< := \{x \in K : F(x).(x - x^{\text{ref}}) < 0\}$ is a bounded set (possibly empty).

Acknowledgements This research was partially supported by a grant from the National Science Council of Taiwan, R.O.C.

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