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Optimality and duality for nonsmooth multiobjective fractional programming with mixed constraints

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Abstract We consider nonsmooth multiobjective fractional programming problems with inequality and equality constraints. We establish the necessary and sufficient optimality conditions under various generalized invexity assumptions. In addition, we formulate a mixed dual problem corresponding to primal problem, and discuss weak, strong and strict converse duality theorems.

Keywords Multiobjective fractional problems · Efficient solution · Mixed duality · Nonsmooth analysis

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1 Introduction

The term multiobjective programming is used to denote a type of optimization problems where two or more objectives are to be minimized subject to certain constraints. Many approaches for multiobjective programming problems have been explored in considerable details, see for example [1,2,13,15,20]. Multiobjective fractional programming refers to a multiobjective problem where the objective functions are quotients, $\frac{f_i(x)}{g_i(x)}$. Multiobjective fractional programming has been widely reviewed by many authors, [3,4,10–12,16]. We point out in all references cited above, they did not consider the equality constraints.

In recent, optimality conditions and the duality for multiobjective fractional programming have been studied under kinds of generalized convexity and some results had been obtained. Invexity is a generalization of the convexity property that extends the sufficiency

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of Kuhn–Tucker conditions and duality theory of convex programs to a more general class of optimization problems.

A generalization of invexity for locally Lipschitz functions was introduced in [8,21,23]. It has been noted in [22] that invexity is not suitable for problems with equality constraints. Since the Kuhn–Tucker multipliers associated with these conditions are not necessarily nonnegative. So a new notion of infine function was defined and shown in [22] to be an adequate tool for equality constraints.

Recently, Nobakhtian [19] considered a nonsmooth multiobjective problem with mixed constraints (equality and inequality). Then by utilizing infine functions that are appropriate for optimization problems with equality constraints, optimality conditions and duality results are obtained without requiring of linearity of equality constraints.

The aim of this paper is to utilize the concept of infine functions for nonsmooth multiobjective fractional programming problems subject to mixed constraints to obtain optimality conditions and duality results.

We consider a multiobjective fractional problem with equality and inequality constraints. Then we obtain optimality conditions. In the equality constraints infineness [22] plays an important role. We also introduce a mixed dual model for nonsmooth fractional multiobjective programming problems which unifies the Mond–Weir, Wolfe and parameter dual models. Infineness for the equality constraints also plays a role in some of the duality results.

The article is organized as follows. Section 2 presents notations and definitions. In Sect. 3 optimality conditions are derived for fractional problems involving equality and inequality constraints. Section 4 presents a mixed dual model and some duality results.

2 Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. Throughout this paper, the following convention for vectors in $\mathbb{R}^n$ will be followed

$x > y$ if and only if $x_i > y_i, i = 1, \dots, n$,
$x \geqq y$ if and only if $x_i \geq y_i, i = 1, \dots, n$,
$x \geq y$ if and only if $x_i \geq y_i, i = 1, \dots, n$; but $x \neq y$.

Definition 2.1 [6,7] The generalized Clarke directional derivative of a locally Lipschitz function f at x in the direction d is defined by

$$f^c(x; d) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}.$$

The Clarke generalized subgradient of a locally Lipschitz function f at x is defined by

$$\partial_c f(x) := \{\xi \in \mathbb{R}^n : f^c(x; d) \geq \langle \xi, d \rangle \quad \forall d \in \mathbb{R}^n\}.$$

We consider the following multiobjective fractional problem:

$$(MF) \quad \min \left(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_m(x)}{g_m(x)} \right)$$
$$s.t.$$
$$x \in S = \{h(x) = (h_1(x), \dots, h_p(x)) \leqq 0,$$
$$r(x) = (r_1(x), \dots, r_q(x)) = 0, x \in X\},$$

where X is an open subset of $\mathbb{R}^n$, $f_i : X \to \mathbb{R}, g_i : X \to \mathbb{R}, h_j : X \to \mathbb{R}, r_j : X \to \mathbb{R}$ are locally Lipschitz functions and $f_i(x) \geq 0$, and $g_i(x) > 0, i = 1, \ldots, m$. The index sets are $M = \{1, 2, \ldots, m\}$, $J = \{1, \ldots, p\}$ and $Q = \{1, 2, \ldots, q\}$. We denote the feasible set $\{x \in \mathbb{R}^n \mid h_j(x) \leq 0, j = 1, \ldots, p, r_j(x) = 0, j = 1, \ldots, q\}$ by F_P. Let $I(x^*) = \{j \in J \mid h_j(x^*) = 0\}$ denote the index set of active constraints at given point x^*. We also denote $g_i, i \in A$ by g_A.

Definition 2.2 [9] A point $\bar{x} \in F_p$ is said to be an efficient solution of the minimum problem (MF) if there exists no $x \in F_P$ such that $f_i(x) < f_i(\bar{x})$ for some $i \in M$ and $f_j(x) \leq f_j(\bar{x})$ for all $j \in M$.

Let $f : X \to \mathbb{R}$ be locally Lipschitz at a given point $x^* \in X$ and $\eta : X \times X \to \mathbb{R}^n$.

Definition 2.3 The function f is said to be infine on X at x^* if for any $x \in X$ and any $\xi \in \partial_c f(x^*)$, there exists $\eta(x, x^*)$ such that

$$f(x) - f(x^*) = \langle \xi, \eta(x, x^*) \rangle.$$

Several sufficient conditions for infineness were given in [22]. Now let us recall a characterization of infine function on $\mathbb{R}^n$, which is taken from [22].

Lemma 2.4 [22] *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a locally Lipschitz function. Then* f *is infine on* $\mathbb{R}^n$ *at* $x_0 \in \mathbb{R}^n$ *if and only if inclusion* $0 \in \partial_c f(x^*)$ *implies that* f *is constant on* $\mathbb{R}^n$.

Following [22] we give an example of a nondifferentiable infine function.

Example 2.5 Let $x \in \mathbb{R}$, $x^* = 0$ and

$$f(x) = \begin{cases} x, & x \geq 0, \\ 2x, & x < 0. \end{cases}$$

Then $\partial_c f(x^*) = [1, 2]$. Since $0 \notin \partial_c f(x^*)$, by Lemma 2.4, f is infine on R at x^*.

We now give an example of a class of differentiable infine functions whose elements are not necessarily linear functions.

Example 2.6 Let U be an open subset of $\mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}$, is of class C^1 and is pseudolinear at $x_0 \in U$, in the sense of [5] (i.e. for any $x \in \mathbb{R}^n$, there exists a function $p(x, x_0) > 0$ such that $f(x) - f(x_0) = p(x, x_0).(x - x_0)\nabla f(x_0)$, then f is infine on U at x_0 with $\eta(x, x_0) = p(x, x_0)(x - x_0)$ for each $x \in \mathbb{R}^n$.

For example, the following differentiable infine function is not a linear function:

$$f(x) = x + x^3, \quad x \in \mathbb{R}.$$

Definition 2.7 The function f is said to be invex with respect to η on X at x^* if for any $x \in X$ and $\xi \in \partial_c f(x^*)$ there exists $\eta(x, x^*)$ such that

$$f(x) - f(x^*) \geq \langle \xi, \eta(x, x^*) \rangle.$$

If in the above definition, we have strict inequality for any $x \neq x^*$, then we say that f is strictly invex on X at x^*.

Definition 2.8 The function f is said to be pseudoinvex on X at x^* if for any $x \in X$ and $\xi \in \partial_c f(x^*)$ there exists $\eta(x, x^*)$ such that $\langle \xi, \eta(x, x^*)\rangle \geq 0$ implies $f(x) \geq f(x^*)$.

If in the above definition, the inequality satisfied as

$$\langle \xi, \eta(x, x^*)\rangle \geq 0 \to f(x) > f(x^*), \quad \forall x \neq x^*,$$

then we say that f is strictly pseudoinvex on X at x^*.

Definition 2.9 The function f is said to be quasiinvex at x^* if for any $x \in X$ and $\xi \in \partial_c f(x^*)$ there exists $\eta(x, x^*)$ such that $f(x) \leq f(x^*)$ implies $\langle \xi, \eta(x, x^*)\rangle \leq 0$.

3 Necessary and sufficient optimality conditions

In this section, we establish optimality conditions for a feasible solution x^* of (MF). For each $u = (u_1, \ldots, u_m) \in \mathbb{R}^m_+$, where $\mathbb{R}^m_+$ denotes the positive orthant of $\mathbb{R}^m$, we consider

$$\begin{aligned} (MF_u) \quad & \min(f_1(x) - u_1 g_1(x), \ldots, f_m(x) - u_m g_m(x)) \\ s.t. \quad & \\ & h_j(x) \leq 0, \ j \in J \\ & r_j(x) = 0, \ j \in Q, x \in X. \end{aligned}$$

Using the following lemma we can find the Karush–Kuhn–Tucker type necessary optimality criterion for the problem (MF).

Lemma 3.1 *If x^* is an efficient solution for (MF) then x^* is an efficient solution for (MF_{u^*}), where $u_i^* = \phi_i(x^*) = \frac{f_i(x^*)}{g_i(x^*)}, i \in M$.*

Note that the quotient of two η-invex functions is not necessarily an η-invex function. For example $f(x) = 1$ and $g(x) = x^2 + 1$ are invex functions with respect to $\eta(x, y) = x - y$. It is easy to verify that $h(x) = \frac{f(x)}{g(x)} = \frac{1}{x^2+1}$ is not η-invex on any point in the open interval $(-\sqrt{1/3}, \sqrt{1/3})$. Therefor in our setting the fractional programming is not the same as non fractional programming case.

Theorem 3.2 *Let x^* be an efficient solution for (MF), and let (MF_{u^*}) satisfies the suitable constraint qualification* [14] *at x^*. Then there exist $\tau^* \in \mathbb{R}^m_+$, $\mu^* \in \mathbb{R}^p_+$, $\nu^* \in \mathbb{R}^q$ such that*

$$0 \in \sum_{i=1}^{m} \tau_i^* \left(\partial_c(f_i(x^*) - u_i^* g_i(x^*))\right) + \sum_{j=1}^{p} \mu_j^* \partial_c h_j(x^*) + \sum_{j=1}^{q} \nu_j^* \partial_c r_j(x^*)$$

$$\mu_j^* h_j(x^*) = 0, \mu_j^* \geq 0, \ j \in J, \sum_{i=1}^{m} \tau_i^* = 1, \tau_i^* \geq 0. \qquad (I)$$

Proof Let x^* be an efficient solution of (MF). Then by Lemma 3.1, x^* is an efficient solution of (MF_{u^*}). Now by Theorem 3.2 in [19] there exist $\tau^* \in \mathbb{R}^m_+$, $\mu^* \in \mathbb{R}^p_+$ and $\nu^* \in \mathbb{R}^q$ such that

$$0 \in \sum_{i=1}^{m} \tau_i^* \partial_c (f_i - u_i^* g_i)(x^*) + \sum_{j=1}^{p} \mu_j^* \partial_c h_j(x^*) + \sum_{j=1}^{q} \nu_j^* \partial_c r_j(x^*)$$
$$\mu_j^* h_j(x^*) = 0, \quad \mu_j^* \geq 0, j \in J,$$
$$\sum_{i=1}^{m} \tau_i^* = 1, \quad \tau_i^* \geq 0, \quad i \in M.$$

Then the proof is complete. □

Theorem 3.3 *Suppose that there exists a feasible solution x^* for (MF) and scalars $\tau_i^* > 0$, $i \in M$, $\mu_j^* \geq 0$, $j \in J$, and $\nu^* \in \mathbb{R}^q$ such that $(x^*, \tau^*, \mu^*, \nu^*)$ satisfies (I). If $f_i's$, $-g_i's$, $i \in M$, and $h_j's$ are invex with respect to η at x^* and $r_j's$ are infine with respect to η at x^*, then x^* is an efficient solution of (MF).*

Proof From (I), there exist

$$\xi_i \in \partial_c f_i(x^*), \quad \eta_i \in \partial_c(g_i)(x^*), \quad \alpha_j \in \partial_c h_j(x^*), \quad \beta_j \in \partial_c r_j(x^*),$$

such that

$$\sum_{i=1}^{m} \tau_i^* (\xi_i - \phi_i(x^*)\eta_i) + \sum_{j=1}^{p} \mu_j^* \alpha_j + \sum_{j=1}^{q} \nu_j^* \beta_j = 0. \tag{1}$$

If x^* is not an efficient solution for problem (MF), then there exists $x \in F_p$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)}, \quad i = 1, \ldots, m,$$
$$\frac{f_k(x)}{g_k(x)} < \frac{f_k(x^*)}{g_k(x^*)}, \quad \text{for some } k \in M.$$

That is

$$f_i(x) - \phi_i(x^*)g_i(x) \leq f_i(x^*) - \phi_i(x^*)g_i(x^*), \quad i = 1, \ldots, m$$
$$f_k(x) - \phi_k(x^*)g_k(x) < f_k(x^*) - \phi_k(x^*)g_k(x^*).$$

Thus, we have

$$\sum_{i=1}^{m} \tau_i^* [f_i(x^*) - \phi_i(x^*)g_i(x^*)] > \sum_{i=1}^{m} \tau_i^* [f_i(x) - \phi_i(x^*)g_i(x)]. \tag{2}$$

Since $\mu_j^* h_j(x) \leq 0 = \mu_j^* h_j(x^*)$, $j \in J$ and $r_j(x^*) = r_j(x)$, $j \in Q$, we have

$$\sum_{j=1}^{p} \mu_j^* h_j(x) + \sum_{j=1}^{q} \nu_j^* r_j(x) \leq \sum_{j=1}^{p} \mu_j^* h_j(x^*) + \sum_{j=1}^{q} \nu_j^* r_j(x^*). \tag{3}$$

Equations (2) and (3) yield

$$\sum_{i=1}^{m} \tau_i^* [f_i(x) - \phi_i(x^*)g_i(x)] + \sum_{j=1}^{p} \mu_j^* h_j(x) + \sum_{j=1}^{q} \nu_j^* r_j(x)$$
$$< \sum_{i=1}^{m} \tau_i^* [f_i(x^*) - \phi_i(x^*)g_i(x^*)] + \sum_{j=1}^{p} \mu_j^* h_j(x^*) + \sum_{j=1}^{q} \nu_j^* r_j(x^*). \tag{4}$$

From invexity assumptions of $f_i's, -g_i's, h_i's$ and infineness of $r_j's$ we find from (1)

$$\sum \tau_i^*[f_i(x) - \phi_i(x^*)g_i(x)] + \sum_{j=1}^{p} \mu_j^* h_j(x) + \sum_{j=1}^{n} \nu_j^* r_j(x)$$
$$\geq \sum \tau_i^*[f_i(x^*) - \phi_i(x^*)g_i(x^*)] + \sum_{j=1}^{p} \mu_j^* h_j(x^*) + \sum_{j=1}^{q} \nu_j^* r_j(x^*),$$

which contradicts (4). This completes the proof. □

Theorem 3.4 *Suppose that there exist a feasible solution x^* for (MF) and scalars $\tau_i^* \geq 0$, $i \in M, \mu_j^* \geq 0, j \in J$, and $\nu^* \in \mathbb{R}^q$ such that $(x^*, \tau^*, \mu^*, \nu^*)$ satisfies (I). If one of the $f_i's, -g_i's$, or $h_j's$ are strictly invex while the others are invex with respect to η at x^* and $r_i's$ are infine with respect to η at x^*, then x^* is an efficient solution of (MF).*

Proof From (I), there exist

$$\xi_i \in \partial_c f_i(x^*), \quad \eta_i \in \partial_c(g_i)(x^*), \quad \alpha_j \in \partial_c h_j(x^*), \quad \beta_j \in \partial_c r_j(x^*),$$

such that

$$\sum_{i=1}^{m} \tau_i^*(\xi_i - \phi_i(x^*)\eta_i) + \sum_{j=1}^{p} \mu_j^* \alpha_j + \sum_{j=1}^{q} \nu_j^* \beta_j = 0. \tag{5}$$

If x^* is not an efficient solution for problem (MF), then there exists $x \in F_p$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)}, \quad i = 1, \ldots, m,$$
$$\frac{f_k(x)}{g_k(x)} < \frac{f_k(x^*)}{g_k(x^*)}, \quad \text{for some } k \in M.$$

That is

$$f_i(x) - \phi_i(x^*)g_i(x) \leq f_i(x^*) - \phi_i(x^*)g_i(x^*), \quad i = 1, \ldots, m,$$
$$f_k(x) - \phi_k(x^*)g_k(x) < f_k(x^*) - \phi_k(x^*)g_k(x^*).$$

Thus, we have

$$\sum_{i=1}^{m} \tau_i^*[f_i(x^*) - \phi_i(x^*)g_i(x^*)] \geq \sum_{i=1}^{m} \tau_i^*[f_i(x) - \phi_i(x^*)g_i(x)]. \tag{6}$$

Since $\mu_j^* h_j(x) \leq 0 = \mu_j^* h_j(x^*), \quad j \in J$ and $r_j(x^*) = r_j(x), \quad j \in Q$, we have

$$\sum_{j=1}^{p} \mu_j^* h_j(x) + \sum_{j=1}^{q} \nu_j^* r_j(x) \leq \sum_{j=1}^{p} \mu_j^* h_j(x^*) + \sum_{j=1}^{q} \nu_j^* r_j(x^*). \tag{7}$$

Equations (6) and (7) yield

$$\sum_{i=1}^{m} \tau_i^*[f_i(x) - \phi_i(x^*)g_i(x)] + \sum_{j=1}^{p} \mu_j^* h_j(x) + \sum_{j=1}^{q} \nu_j^* r_j(x)$$
$$\leq \sum_{i=1}^{m} \tau_i^*[f_i(x^*) - \phi_i(x^*)g_i(x^*)] + \sum_{j=1}^{p} \mu_j^* h_j(x^*) + \sum_{j=1}^{q} \nu_j^* r_j(x^*). \tag{8}$$

From strictly invexity assumptions of one of $f_i's$, $-g_i's$, $h_j's$ and infineness of $r_j's$ we find from (5)

$$\sum_{i=1}^{m}\tau_i^*[f_i(x)-\phi_i(x^*)g_i(x)]+\sum_{j=1}^{p}\mu_j^*h_j(x)+\sum_{j=1}^{n}\nu_j^*r_j(x)$$
$$>\sum_{i=1}^{m}\tau_i^*[f_i(x^*)-\phi_i(x^*)g_i(x^*)]+\sum_{j=1}^{p}\mu_j^*h_j(x^*)+\sum_{j=1}^{q}\nu_j^*r_j(x^*),$$

which contradicts (8). This completes the proof. □

4 Duality

In optimization theory, there are many type of duals for a given mathematical programming problem. Recently, the mixed dual has been considered for various optimization problems [17,18,26,27]. In this section, we introduce the mixed duality problem for problem (MF), and prove weak, strong and strict converse duality theorems. In the sequel, we will see that the Wolf, Mond–Weir and parameter type duals follows as special cases of this duality. The following dual problem is said to be a mixed type dual problem (MD).

$$(MD)\quad \max\left(\frac{f_1(y)+\sum_{j\in J_1}\mu_jh_j(y)+\sum_{j\in K_1}\nu_jr_j(y)}{g_1(y)},\ldots,\frac{f_m(y)+\sum_{j\in J_1}\mu_jh_j(y)+\sum_{j\in K_1}\nu_jr_j(y)}{g_m(y)}\right)$$

$s.t.$

$$0\in\sum_{i=1}^{m}g_i(y)(\partial_c(\tau_if_i)(y)+\tau_i\sum_{j\in J_1}\partial_c\mu_jh_j(y)+\tau_i\sum_{j\in K_1}\partial_c\nu_jr_j(y))$$
$$-\sum_{i=1}^{m}\Big(f_i(y)+\sum_{j\in J_1}\mu_jh_j(y)+\sum_{j\in K_1}\nu_jr_j(y)\Big)\partial_c(\tau_ig_i)(y)$$
$$+\sum_{j\in J_2}\partial_c\mu_jh_j(y)+\sum_{j\in K_2}\partial_c\nu_jr_j(y),\qquad (*)$$
$$\mu_jh_j(y)\ge 0,\quad j\in J_2$$
$$\nu_jr_j(y)=0,\quad j\in K_2$$
$$\sum_{i=1}^{m}\tau_i=1,\quad \tau_i\ge 0,\quad i\in M,\quad \mu_j\ge 0,\quad j\in J,$$

where J_1 is a subset of $J=\{1,2,\ldots,p\}$, $J_2=J\backslash J_1$ and K_1 is a subset of Q and $K_2=Q\backslash K_1$. We denote F_D the set of all feasible solutions of problem (MD). Throughout this section we assume that for each $i\in M$

$$f_i(y)+\sum_{j\in J_1}\mu_jh_j(y)+\sum_{j\in K_1}\nu_jr_j(y)\ge 0,\quad g_i(y)>0.$$

In the following, we shall prove the weak, strong and strict converse duality theorems.

Theorem 4.1 *Let x be feasible solution for problem* (MF) *and* (y, τ, μ, ν) *be feasible solution for problem* (MD). *If* $\tau_i > 0$ *and all of the functions* $f_i(.)$, $-g_i(.)$, *and* $h_{J_1}(.)$ *are invex at y, and* $\nu_j r_j$, $j \in K_1$ *are infine at y, and* $\nu_j r_j$, $j \in K_2$, $\mu_j h_j(y)$, $j \in J_2$ *are quasiinvex with respect to* η *at y, then the following can not hold:*

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(y) + \sum_{j\in J_1} \mu_j h_j(y) + \sum_{j\in K_1} \nu_j r_j(y)}{g_i(y)}, \quad \textit{for some} \quad i \in M,$$
$$\frac{f_j(x)}{g_j(x)} \leq \frac{f_j(y) + \sum_{j\in J_1} \mu_j h_j(y) + \sum_{j\in K_1} \nu_j r_j(y)}{g_j(y)}, \quad \forall j \neq i.$$

Proof It follows from $(*)$ that there exist,

$$\xi_i \in \partial_c(\tau_i f_i)(y), \quad \eta_i \in \partial_c(\tau_i g_i)(y), \quad \alpha_{J_1} \in \sum_{j\in J_1} \partial_c \mu_j h_j(y),$$
$$\alpha_{J_2} \in \sum_{j\in J_2} \partial_c \mu_j h_j(y), \quad \beta_{K_1} \in \sum_{j\in K_1} \partial_c \nu_j r_j(y), \quad \beta_{K_2} \in \sum_{j\in K_2} \partial_c \nu_j r_j(y),$$

such that

$$\sum_{i=1}^{m} g_i(y)(\xi_i + \tau_i \alpha_{J_1} + \tau_i \beta_{K_1}) - \sum_{i=1}^{m} [f_i(y) + \sum_{j\in J_1} \mu_j h_j(y)$$
$$+ \sum_{j\in K_1} \nu_j r_j(y)](\eta_i) + \alpha_{J_2} + \beta_{K_2} = 0. \quad (II)$$

Suppose contrary to the result of the theorem that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(y) + \sum_{j\in J_1} \mu_j h_j(y) + \sum_{j\in K_1} \nu_j r_j(y)}{g_i(y)}, \quad \text{for some} \quad i \in M,$$
$$\frac{f_j(x)}{g_j(x)} \leq \frac{f_j(y) + \sum_{j\in J_1} \mu_j h_j(y) + \sum_{j\in K_1} \nu_j r_j(y)}{g_j(y)}, \quad \forall j \in M.$$

Therefore

$$\sum_{i=1}^{m} \tau_i f_i(x) g_i(y) - \sum_{i=1}^{m} \tau_i g_i(x) \left(f_i(y) + \sum_{j\in J_1} \mu_j h_j(y) + \sum_{j\in K_1} \nu_j r_j(y) \right) < 0.$$

Since $g_i(y) > 0$ and $\mu_j h_j(x) \leq 0$, $\nu_j r_j(x) = 0$, we have

$$\sum_{i=1}^{m} \left[\tau_i f_i(x) + \tau_i \sum_{j\in J_1} \mu_j h_j(x) + \tau_i \sum_{j\in K_1} \nu_j r_j(x) \right] g_i(y)$$
$$- \sum_{i=1}^{m} \left[f_i(y) + \sum_{j\in J_1} \mu_j h_j(y) + \sum_{j\in K_1} \nu_j r_j(y) \right] \tau_i g_i(x) < 0.$$

Thus

$$\sum_{i=1}^{m}\left\{[\tau_i f_i(x)-\tau_i f_i(y)]g_i(y)+\left(\sum_{j\in J_1}\mu_j h_j(x)\right.\right.$$
$$\left.\left.-\sum_{j\in J_1}\mu_j h_j(y)+\sum_{j\in K_1}\nu_j r_j(x)-\sum_{j\in K_1}\nu_j r_j(y)\right)\tau_i g_i(y)\right\}$$
$$-\sum_{i=1}^{m}\left(f_i(y)+\sum_{j\in J_1}\mu_j h_j(y)+\sum_{j\in K_1}\nu_j r_j(y)\right)(\tau_i g_i(x)-\tau_i g_i(y))<0.$$

Based on the assumptions we have

$$\left\langle\sum_{i=1}^{m}(\xi_i+\tau_i\alpha_{J_1}+\tau_i\beta_{K_1})g_i(y)-\sum_{i=1}^{m}\left(f_i(y)+\sum_{j\in J_1}\mu_j h_j(y)\right.\right.$$
$$\left.\left.+\sum_{j\in K_1}\nu_j r_j(y)\right)(\eta_i),\eta(x,y)\right\rangle<0. \tag{9}$$

Since x and (y,τ,μ,ν) are feasible for problem (MF) and (MD), respectively, we have

$$\mu_j h_j(x)\le\mu_j h_j(y),\quad \forall j\in J_2, \tag{10}$$
$$\nu_j r_j(x)=\nu_j r_j(y),\quad \forall j\in K_2. \tag{11}$$

Using the quasiconvexity of $\mu_j h_j(x)$ and $\nu_j r_j(x)$ at y, we yield

$$\langle\alpha_{J_2},\eta(x,y)\rangle\le 0,\quad \langle\beta_{K_2},\eta(x,y)\rangle\le 0. \tag{12}$$

From (9) – (12), we have

$$\left\langle\sum_{i=1}^{m}(\xi_i+\tau_i\alpha_{J_1}+\tau_i\beta_{K_1})g_i(y)-\sum_{i=1}^{m}\left(f_i(y)+\sum_{j\in J_1}\mu_j h_j(y)\right.\right.$$
$$\left.\left.+\sum_{j\in K_1}\nu_j r_j(y)\right)(\eta_i)+\alpha_{K_2}+\beta_{K_2},\quad \eta(x,y)\right\rangle<0,$$

which is a contradiction to (II). Hence the proof is complete. □

Theorem 4.2 *Let x^* be a feasible for (MF) at which the suitable constraint qualification* [14] *is satisfied. Then there exist $\tau^*\in\mathbb{R}^m$, $\mu^*\in\mathbb{R}^p$, $\nu^*\in\mathbb{R}^q$, such that $(x^*,\tau^*,\mu^*,\nu^*)\in F_D$. If also the hypotheses of Theorem 4.1 hold, then (x^*,τ^*,μ^*,ν^*) is an efficient solution of problem (MD).*

Proof Since x^* is feasible for (MF) and the appropriate constraint qualification is met at x^*, by Theorem 3.2, there exist $\bar\tau\in\mathbb{R}^m_+$, $\bar\mu\in\mathbb{R}^p$, $\bar\nu\in\mathbb{R}^q$ such that

$$0\in\sum_{i=1}^{m}\bar\tau_i\partial_c(f_i(x^*)-\phi_i(x^*)g_i(x^*))+\sum_{j=1}^{p}\bar\mu_j\partial_c h_j(x^*)+\sum_{j=1}^{q}\bar\nu_j\partial_c r_j(x^*)$$
$$\bar\mu_j h_j(x^*)=0,\quad j=1,\cdots,p,\quad \mu_j\ge 0,\quad \sum_{i=1}^{m}\bar\tau_i=1,\quad \bar\tau_i\ge 0.$$

Since $g_i(x^*) > 0$, we have

$$0 \in \sum_{i=1}^{m} g_i(x^*)(\partial_c \tau_i^* f_i(x^*) - f_i(x^*)(\partial_c \tau_i^* g_i(x^*)) + \sum_{i=1}^{p} \bar{\mu_i} \partial_c h_j(x^*) + \sum_{j=1}^{q} \bar{\nu_j} \partial_c r_j(x^*),$$

where $\bar{\tau}_i = g_i(x^*)\tau_i{}^*$.
Then from (I), we know that

$$\bar{\mu_j} h_j(x^*) = 0, \quad \forall j \in J.$$

Now, let

$$g_i(x^*)\tau_i^* \mu_i^* = \bar{\mu_i}, i \in J_1, \quad \mu_i^* = \bar{\mu_i}, i \in J_2, \quad \mu^* = (\mu_{J_1}^*, \mu_{J_2}^*).$$
$$g_i(x^*)\tau_i^* \nu_i^* = \bar{\nu_i}, i \in K_1, \quad \nu_i^* = \bar{\nu_i}, i \in K_2, \quad \nu^* = (\nu_{K_1}^*, \nu_{K_2}^*).$$

Thus we obtain

$$0 \in \sum_{i=1}^{m} g_i(x^*)\left(\partial_c(\tau_i^* f_i(x^*) + \tau_i^* \sum_{J_1} \partial_c \mu_j^* h_j(x^*) + \tau_i^* \sum_{K_1} \partial_c \nu_j^* r_j(x^*)\right) - \sum_{i=1}^{m}\left(f_i(x^*) + \sum_{j \in J_1} \mu_j^* h_j(x^*) + \sum_{j \in K_1} \nu_j^* r_j(x^*)\right)\partial_c(\tau_i^* g_i)(x^*) + \sum_{j \in J_2} \partial_c(\mu_j^* h_j)(x^*) + \sum_{j \in K_2} \partial_c(\nu_j^* r_j)(x^*).$$

Hence $(x^*, \tau^*, \mu^*, \nu^*)$ is a feasible solution of problem (MD). Furthermore,

$$\frac{f_i(x^*)}{g_i(x^*)} = \frac{f_i(x^*) + \mu_{J_1}^*{}^t h_{J_1}(x^*) + \nu_{K_1}^*{}^t r_{K_1}(x^*)}{g_i(x^*)}, \quad \forall i \in M.$$

Thus, optimality of $(x^*, \tau^*, \mu^*, \nu^*)$ for (MD) follows from Theorem 4.1. □

Theorem 4.3 *Let x^* and $(y^*, \tau^*, \mu^*, \nu^*)$ be optimal solution of (MF) and (MD), respectively. If in addition the hypotheses of Theorem 4.2 hold, then $x^* = y^*$.*

Proof Suppose to the contrary that $x^* \neq y^*$. From Theorem 4.2, we know that there exist $\bar{\tau} \in \mathbb{R}_+^m$ and $\bar{\mu} \in \mathbb{R}^p, \bar{\nu} \in \mathbb{R}^q$ such that $(x^*, \bar{\tau}, \bar{\mu}, \bar{\nu})$ is an optimal solution of (MD) with the optimal value,

$$\left(\frac{f_1(x^*)}{g_1(x^*)}, \ldots, \frac{f_m(x^*)}{g_m(x^*)}\right) = \left(\frac{f_1(x^*) + \bar{\mu_{J_1}}^t h_{J_1}(x^*) + \bar{\nu_{K_1}}^t r_{K_1}(x^*)}{g_1(x^*)}, \ldots, \frac{f_m(x^*) + \bar{\mu_{J_1}}^t h_{J_1}(x^*) + \bar{\nu_{K_1}}^t r_{K_1}(x^*)}{g_m(x^*)}\right)$$

Now, proceeding as in the proof of Theorem 4.1, we obtain the following inequality:

$$\frac{f_i(x^*)}{g_i(x^*)} > \frac{f_i(y^*) + \mu_{J_1}^*{}^t h_{J_1}(y^*) + \nu_{K_1}^*{}^t r_{K_1}(y^*)}{g_i(y^*)}.$$

This contradicts the fact that

$$\frac{f_i(x^*)}{g_i(x^*)} = \frac{f_i(y^*) + \bar{\mu_{J_1}}^t h_{J_1}(y^*) + \bar{\nu_{K_1}}^t r_{K_1}(y^*)}{g_i(y^*)}.$$

Therefore, we conclude that $x^* = y^*$. □

Special cases:
(1) If $J_1 = \emptyset$, $K_1 = \emptyset$ in problem (MD), the mixed dual problem (MD) is replaced with the Mond–Weir type [25] problem (D_1):

$$\begin{aligned}
(D_1)\quad & \max \left(\frac{f_1(y)}{g_1(y)}, \ldots, \frac{f_m(y)}{g_m(y)}\right) \\
& s.t. \\
& 0 \in \sum_{i=1}^m g_i(y)\partial_c(\tau_i f_i)(y) - \sum_{i=1}^m (f_i(y))\partial_c(\tau_i g_i)(y) \\
& + \sum_{j\in J} \partial_c \mu_j h_j(y) + \sum_{j\in Q} \partial_c \nu_j r_j(y), \\
& \mu_j h_j(y) \geq 0, \mu_j \geq 0, \quad j \in J, \nu_j r_j(y) \geq 0, \quad j \in Q, \\
& \sum_{i=1}^m \tau_i = 1, \quad \tau_i \geq 0, \quad i \in M.
\end{aligned}$$

(2) If $J_2 = \emptyset$, $K_2 = \emptyset$ in problem (MD), the mixed dual problem (MD) is replaced to the Wolf type [24] problem (D_2):

$$\begin{aligned}
(D_2)\quad & \max \left(\frac{f_1(y) + \sum_{j\in J} \mu_j h_j(y) + \sum_{j\in Q} \nu_j r_j(y)}{g_1(y)}, \ldots, \right. \\
& \left. \frac{f_m(y) + \sum_{j\in J} \mu_j h_j(y) + \sum_{j\in Q} \nu_j r_j(y)}{g_m(y)}\right) \\
& s.t. \\
& 0 \in \sum_{i=1}^m g_i(y)\left(\partial_c(\tau_i f_i)(y) + \tau_i \sum_{j\in J} \partial_c \mu_j h_j(y) + \tau_i \sum_{j\in Q} \partial_c \nu_j r_j(y)\right) \\
& - \sum_{i=1}^m \left(f_i(y) + \sum_{j\in J} \mu_j h_j(y) + \sum_{j\in Q} \nu_j r_j(y)\right)\partial_c(\tau_i g_i)(y), \\
& \sum_{i=1}^m \tau_i = 1, \quad \tau_i \geq 0, \quad i \in M, \quad \mu_j \geq 0, \quad j \in J.
\end{aligned}$$

(3) If $J_2 = \emptyset$, $K_2 = \emptyset$, we can deduce parametric dual problem by taking

$$\lambda_i = \frac{f_i(y) + \sum_{j\in J} \mu_j h_j(y) + \sum_{j\in Q} \nu_j r_j(y)}{g_i(y)}, \quad i \in M,$$

(D_3) max λ

$$0 \in \sum_{i=1}^{m} \tau_i \partial_c(f_i)(y) + \tau_i \sum_{j\in J} \partial_c \mu_j h_j(y) + \tau_i \sum_{j\in Q} \partial_c \nu_j r_j(y)$$

$$-\sum_{i=1}^{m} \lambda_i \partial_c(\tau_i g_i)(y),$$

$$f_i(y) - \lambda_i g_i(y) + \sum_{j\in J} \mu_j h_j(y) + \sum_{j\in Q} \nu_j r_j(y) \geq 0, \quad \forall i \in M,$$

$$\sum_{i=1}^{m} \tau_i = 1, \quad \tau_i, \geq 0, \quad i \in M, \quad \mu_j \geq 0, \quad j \in J.$$

Remark 4.4 The duality model (MD) for nonfractional differentiable programming is similar to the duality model in [2].

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