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Existence theorems of systems of variational inclusion problems with applications

Lai-Jiu Lin · Song Yu Wang · Chih-Sheng Chuang

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Abstract In this paper, we study the existence theorems of systems of variational inclusions problems. As consequences of our results, we study existence theorems of systems of generalized vector quasi-equilibrium problems, mathematical program with systems of variational inclusion constraints, bilevel problem with systems of constraints.

Keywords Upper semicontinuous (lower semicontinuous) multivalued map · Systems of variational inclusions problem · Bilevel problem · Ideal minimal point · Efficient point

1 Introduction

Let I be any index set. For each $i \in I$, let Z_i be a real topological vector space (in short t.v.s.), X_i and Y_i be nonempty closed convex subsets of locally convex t.v.s. E_i and V_i, respectively. Let $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $A_i : X \times Y \multimap X_i$, $T_i : X \multimap Y_i$, $G_i : X \times Y \times Y_i \multimap Z_i$, $C_i : X \multimap Z_i$ be multivalued maps. Throughout this paper, we use these notations unless otherwise specified. Recently, Lin [1] studied the following systems of variational inclusion problems:
(**SVIP1**) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $0 \in G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$.

In [1], Lin established the existence theorems of (SVIP1), he also gave some applications. For detail, one can refer to [1].

Let E be a t.v.s., X be a nonempty subset of E and $f : X \times X \to \mathbb{R}$ be a function with $f(x, x) \geq 0$ for all $x \in X$, then the scalar equilibrium problem in the sense of Blum and Oettli [2] is to find $\bar{x} \in X$ such that $f(\bar{x}, y) \geq 0$ for all $y \in X$. The equilibrium problem contains optimization problems, fixed point problems, saddle point problems, complementary problems, and Ekeland's variational problems as special cases [2–5]. This problem was

L.-J. Lin (✉)· S. Y. Wang · C.-S. Chuang
Department of Mathematics,
National Changhua University of Education, Changhua 50058, Taiwan
e-mail: maljlin@cc.ncue.edu.tw

extensively investigated and generalized to the vector equilibrium problem for single valued or multivalued mappings [3,6–10] and references therein.

In this paper, we apply an existence theorem of (SVIP1) to study systems of generalized quasi-variational disclusions problem:

(**SVDP**) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $0 \notin G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$.

(SVDP) contains the following problems as special cases:

(**SVIP2**) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) \subseteq H_i(\bar{x}, \bar{y})$ for all $v_i \in T_i(\bar{x})$, where $H_i : X \times Y \multimap Z_i$ is a multivalued map.

(**SVIP3**) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) \subseteq G_i(\bar{x}, \bar{y}, \bar{y}_i) + C_i(\bar{x})$ for all $v_i \in T_i(\bar{x})$.

(**SVIP4**) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$, $F_i(\bar{x}, \bar{y}) \subseteq C_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) \subseteq G_i(\bar{x}, \bar{y}, \bar{y}_i) + C_i(\bar{x})$ for all $v_i \in T_i(\bar{x})$, where $F_i : X \times Y \multimap Z_i$ and $S_i : X \multimap Z_i$ are multivalued maps.

(**SVIP5**) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $H_i(\bar{x}, \bar{y}) \subseteq G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$.

(**SVIP6**) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, \bar{y}_i) \subseteq G_i(\bar{x}, \bar{y}, v_i) - C_i(\bar{x})$ for all $v_i \in T_i(\bar{x})$.

If we let $H_i(x, y) = Z_i \setminus (-int\ C_i(x))$ or $H_i(x, y) = C_i(x)$, we have the following systems of generalized vector quasi-equilibrium problem.

(**SEP1**) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) \cap (-int\ C_i(\bar{x})) = \emptyset$ for all $v_i \in T_i(\bar{x})$.

(**SEP2**) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) \subseteq C_i(\bar{x})$ for all $v_i \in T_i(\bar{x})$.

Lin and Tan [11,12] studied (SVIP3) for the case that I is a singleton, and $C_i(x) = C_i$ is a convex cone for each $x \in X$. But in (SVIP2) and (SVIP3), $C_i(x)$ is not assumed to be a cone.

Lin and Hsu [13], and Lin et al. [3,8,9] studied (SEP1) and (SEP2) when $C_i(x)$ is a cone for each $x \in X$. But in (SEP1) and (SEP2), $C_i(x)$ is not assumed to be a cone.

Luc and Tan [5], Tan [14], Lin and Tan [11,12] studied (SVIP6) when $C_i(x) = C_i$ is a cone for all $x \in X$.

If we assume that $IMin(G_i(x, y, y_i)/C_i(x)) \neq \emptyset$ for all $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$, then (SVIP4) will be reduced to the problem:

(**SQOP1**) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $F_i(\bar{x}, \bar{y}) \subseteq C_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, \bar{y}_i) \cap IMin(G_i(\bar{x}, \bar{y}, T_i(\bar{x}_i))/C_i(\bar{x})) \neq \emptyset$, where $C_i : X \multimap Z_i$ is a closed multivalued map such that for each $x \in X$, $C_i(x)$ is a nonempty closed convex cone.

If we let $H_i(x, y) = \{0\}$ for all $(x, y) \in X \times Y$ and $i \in I$, then (SVIP5) will be reduced to (SVIP1).

If $F_i : X \times Y \multimap Z_i$, $S_i : X \multimap X_i$, Z_0 is a real t.v.s. and C_0 is a proper closed convex cone in Z_0 and $f : X \times Y \multimap Z_0$. We also study the following bilevel problem.

(**BLEP1**) $Min(h(x, y)/C_0) \neq \emptyset$, $x \in X$, $y = (y_i)_{i\in I}$ such that for each $i \in I$, $y_i \in T_i(x)$, $x_i \in S_i(x)$, $F_i(x, y) \subseteq C_i(x)$, and $G_i(x, y, y_i) \cap IMin(G_i(x, y, T_i(x))$ $/C_i(x)) \neq \emptyset$.

If $Z_i = \mathbb{R}$ and $C_i(x) = [0, \infty)$ for all $i \in I$, and $Z_0 = \mathbb{R}$, and $C_0 = [0, \infty)$, and F_i and G_i are single valued functions, then (BLEP1) will be reduced to the following bilevel problem:

(**BLEP2**) $Min(h(x, y)/C_0) \neq \emptyset$, $x \in X$, $y = (y_i)_{i\in I} \in Y$ such that for each $i \in I$, $y_i \in T_i(x)$, $x_i \in S_i(x)$, $F_i(x, y) \geq 0$, and y_i is a solution of the problem $Min_{v_i \in T_i(x)} G_i(x, y, v_i)$.

Lin and Hsu [13] studied (BLEP2).

If $G_i(x, y, y_i) \geq 0$ for all $x \in X$ and $y = (y_i)_{i\in I} \in Y$, then (BLEP2) will be reduced to the following mathematical program with systems of equilibrium constraints:
(**MPEC**) $Min\ h(x, y)$, $x \in X$, $y = (y_i)_{i\in I} \in Y$ such that for each $i \in I$, $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x, y) \geq 0$, and $G_i(x, y, v_i) \geq 0$ for all $v_i \in T_i(x)$.

Lin and Still [15], Lin [16], Lin and Hsu [13] studied MPEC, but our approach is different from [15] and [16].

In this paper, we apply the existence theorem of systems of variational inclusion problems (SVIP1) in [1] to study the existence theorems of systems of variational disclusions problems (SVDP), systems of variational inclusions problems (SVIP2-6). Our approach to study (SVIP3) and (SVIP6) are much simple than Theorem 3.6 in Lin et al. [11,12]. Our results cannot be obtained from Lin et al. [11,12]. We establish the equivalent relations between (SVIP1), (SVIP5) and (SVDP) under some conditions. We also study existence theorems (SEP1) and (SEP2). As application of our results, we study (BLEP1) and (BLEP2). We also study (SVIP3) and (SVIP6) and (BLEP1) with different approach for the case $A_i(x, y) = S_i(x) = X_i$ for all $(x, y) \in X \times Y$ and $i \in I$.

Recently, Lin and Liu [9], Lin et al. [8] used existence theorems of abstract economy to study (SEP1) and (SEP2), and gave applications. In this paper, we apply systems of variational disclusion problems to study (SEP1), (SEP2), (BLEP1), and (BLEP2). Our results on (BLEP1) is different from Corollary 5.3 in [13], Corollary 3.1 in [16], Theorem 4.6 in [16], and Corollary 3 in [15].

2 Preliminaries

Let X and Y be topological spaces (in short t.s.), $T : X \multimap Y$ be a multivalued map. T is said to be upper semicontinuous (in short u.s.c.), respectively, lower semicontinuous (in short l.s.c.) at $x \in X$, if for every open set U in Y with $T(x) \subseteq U$ (resp. $T(x) \cap U \neq \emptyset$), there exists an open neighborhood $V(x)$ of x such that $T(x') \subseteq U$ (resp. $T(x') \cap U \neq \emptyset$) for all $x' \in V(x)$; T is said to be u.s.c. (resp. l.s.c.) on X if T is u.s.c. (resp. l.s.c.) at every point of X; T is continuous at x if T is u.s.c. and l.s.c. at x ; T is compact if there exists a compact set K such that $T(X) \subseteq K$; T is closed if $Gr(T) = \{(x, y) : \ y \in T(x), x \in X\}$ is a closed set.

Let Z be a real t.v.s. with a pointed cone C and A be a nonempty subset of Z. (i) $x \in A$ is said to be an ideal minimal (resp. ideal maximal) point of A with respect to C if $y - x \in C$ (resp. $x - y \in C$) for every $y \in A$. The set of ideal minimal point of A is denoted by $IMin(A/C)$. The set of ideal maximal point of A is denoted by $IMax(A/C)$. (ii) $x \in A$ is said to be an efficient point of A w.r.t. C if there is no $y \in A$ such that $x - y \in C \setminus \{0\}$. The set of efficient point of A is denoted by $Min(A/C)$.

Theorem 2.1 *[17] Let I be any index set and let X_i be a nonempty convex subset of a t.v.s. E_i, $X = \prod_{i\in I} X_i$. For each $i \in I$, let $P_i, Q_i : X \multimap X_i$ be multivalued maps satisfying the following conditions:*

(i) For each $x \in X$, $coP_i(x) \subseteq Q_i(x)$;
(ii) For each $x = (x_i)_{i\in I} \in X$, $x_i \notin Q_i(x)$;
(iii) For each $y_i \in X_i$, $P_i^{-1}(y_i)$ is open; and

(iv) *There exist a nonempty compact subset K of X and a compact convex subset D_i of X_i for all $i \in I$ such that for each $x \in X \setminus K$, there exist $j \in I$ and $y_j \in D_j$ such that $x \in P_j^{-1}(y_j)$.*

Then there exists $\bar{x} \in X$ such that $P_i(\bar{x}) = \emptyset$ for all $i \in I$.

Throughout this paper, we assume that all topological spaces are Hausdorff.

3 Existence results for a solution of systems of generalized quasi-variational inclusions problems

The following theorem is a variant of Theorem 3.1 [1], its proof is essentially the same as in Theorem 3.1 [1].

Theorem 3.1 *[1] For each $i \in I$, suppose that*

(i) *A_i is a compact u.s.c. multivalued map with nonempty closed convex values;*
(ii) *T_i is a compact continuous multivalued map with nonempty closed convex values;*
(iii) *G_i is a closed multivalued map with nonempty values and for each $x \in X$, $Q_i(x) = \{y_i \in T_i(x)$: $0 \in G_i(x, y, v_i)$ for all $v_i \in T_i(x)$ and for $y = (y_i)_{i\in I} \in Y\}$ is a convex set;*
(iv) *For each $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ is $\{0\}$-quasiconvex-like [1] and $0 \in G_i(x, y, y_i)$.*

Then these exists $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $0 \in G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$.

Remark 3.1 In Theorem 3.1, the condition "for each $(x, y) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ is $\{0\}$-quasiconvex" is replaced by "for each $x \in X$, $Q_i(x)$ is convex," where $Q_i(x)$ is defined as in (iii).

As a consequence of systems of generalized quasi-variational inclusions problems, we have the following existence theorem of systems of generalized quasi-variational disclusion problem.

Theorem 3.2 *Suppose that conditions (i) and (ii) of Theorem 3.1 are satisfied. For each $i \in I$, suppose that*

(iii) *G_i is a multivalued map with open graph, $G_i(x, y, v_i) \neq Z_i$ for all $(x, y, v_i) \in X \times Y \times Y_i$, and for each $x \in X$, $Q_i(x) = \{y_i \in T_i(x)$: $0 \notin G_i(x, y, v_i)$ for all $v_i \in T_i(x)$ and for $y = (y_i)_{i\in I} \in Y\}$ is a convex set;*
(iv) *For each $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ is $\{0\}$-quasiconvex and $0 \notin G_i(x, y, y_i)$.*

Then there exists $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $0 \notin G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$.

Proof Let $H_i : X \times Y \times Y_i$ be defined by $H_i(x, y, v_i) = Z_i \setminus G_i(x, y, v_i)$ for all $(x, y, v_i) \in X \times Y \times Y_i$. By (iii), H_i is a closed multivalued map with nonempty values and for each $x \in X$, $Q_i(x) = \{y_i \in T_i(x) : 0 \in H_i(x, y, v_i)$ for all $v_i \in T_i(x)$ and for $y = (y_i)_{i\in I} \in Y\}$ is convex. By (iv), for each $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$, $v_i \multimap H_i(x, y, v_i)$ is $\{0\}$-quasiconvex-like and $0 \in H_i(x, y, y_i)$. Then by Theorem 3.1 there exists $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $0 \notin G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$. □

The following two lemmas are essential tools in this paper.

Lemma 3.1 *Let X and Y be topological spaces, $H : X \multimap Y$ be a multivalued map with open graph and $M : X \multimap Y$ be a l.s.c. multivalued map, then $(H + M) : X \multimap Y$, defined by $(H + M)(x) = H(x) + M(x)$ for each $x \in X$, is a multivalued map with open graph.*

Proof Let $(x, y) \in \overline{[Gr(H + M)]^c}$. Then these exists a net $\{(x_\alpha, y_\alpha)\}_{\alpha\in\Lambda}$ in $[Gr(H+M)]^c$ such that $(x_\alpha, y_\alpha) \to (x, y)$. Then $y_\alpha \notin H(x_\alpha) + M(x_\alpha)$ for all $\alpha \in \Lambda$. We want to show that $y \notin H(x) + M(x)$. Suppose that $y \in H(x) + M(x)$, then these exist $u \in H(x)$ and $v \in M(x)$ such that $y = u + v$. Since M is l.s.c. and $x_\alpha \to x$, these exists a net $\{v_\alpha\}_{\alpha\in\Lambda}$ such that $v_\alpha \in M(x_\alpha)$ for all $\alpha \in \Lambda$ and $v_\alpha \to v$. We see $y_\alpha - v_\alpha \in Y \setminus H(x_\alpha)$.

Let $F : X \multimap Y$ be defined by $F(x) = Y \setminus H(x)$. By assumption, F has closed graph. Hence, $u = y - v \in F(x) = Y \setminus H(x)$. Therefore, $u = y - v \notin H(x)$. This leads to a contradiction. This shows that $y \notin H(x) + M(x)$. Hence $(x, y) \in [Gr(H + M)]^c$ and $[Gr(H + M)]^c$ is a closed set. Therefore, $H + M$ has open graph. □

Lemma 3.2 *Let X and Y be topological spaces, $G : X \multimap Y$ be an u.s.c. multivalued map with nonempty compact values and $M : X \multimap Y$ be a closed multivalued map, then the map $(G + M) : X \multimap Y$, defined by $(G + M)(x) = G(x) + M(x)$ for each $x \in X$, is a closed map.*

Proof Let $(y, z) \in \overline{Gr(G + M)}$. Then there exists a net $\{(y_\alpha, z_\alpha)\}_{\alpha\in\Lambda}$ in $Gr(G + M)$ such that $(y_\alpha, z_\alpha) \to (y, z)$. One has $z_\alpha \in M(y_\alpha) + G(y_\alpha)$ and there exists $u_\alpha \in M(y_\alpha)$, $v_\alpha \in G(y_\alpha)$ such that $z_\alpha = u_\alpha + v_\alpha$. Let $K = \{y_\alpha : \alpha \in \Lambda\} \cup \{y\}$. Then K is a compact set in X. Since $G : X \multimap Y$ is an u.s.c. multivalued map with nonempty compact values, $G(K)$ is a compact set. Then $\{v_\alpha\}_{\alpha\in\Lambda}$ has a subnet $\{v_{\alpha_\lambda}\}_{\alpha_\lambda\in\Lambda}$ such that $v_{\alpha_\lambda} \to v$. Since G is an u.s.c. multivalued map with nonempty compact values, G is closed. Hence, $v \in G(y)$. Clearly, $u_{\alpha_\lambda} = z_{\alpha_\lambda} - v_{\alpha_\lambda} \to z - v$. Since M is closed, $z - v \in M(y)$ and $z \in v + M(y) \subseteq M(y) + G(y)$. This shows that $\overline{Gr(G + M)} = Gr(G + M)$ and $(G + M) : Y \multimap U$ is closed. □

Theorem 3.3 *Suppose conditions (i) and (ii) of Theorem 3.1 are satisfied. For each $i \in I$, suppose that*

(iii) $H_i : X \times Y \multimap Z_i$ is a closed multivalued map with with nonempty values and for each $x \in X$, $y \multimap H_i(x, y)$ is affine [1];
(iv) G_i is a l.s.c. multivalued map such that for each $(x, v_i) \in X \times Y_i$, $y \multimap G_i(x, y, v_i)$ is affine and $G_i(x, y, v_i) - (Z_i \setminus H_i(x, y)) \neq Z_i$ for all $(x, y, v_i) \in X \times Y \times Y_i$;
(v) For each $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ is $\{0\}$- quasiconvex [1] and $G_i(x, y, y_i) \subseteq H_i(x, y)$.

Then there exists $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) \subseteq H_i(\bar{x}, \bar{y})$ for all $v_i \in T_i(\bar{x})$.

Proof Let $P_i : X \times Y \times Y_i \multimap Z_i$ be defined by $P_i(x, y, v_i) = G_i(x, y, v_i) - (Z_i \setminus H_i(x, y))$. By (iii) and (iv), P_i has open graph and $P_i(x, y, v_i) \neq Z_i$ for all $(x, y, v_i) \in X \times Y \times Y_i$. For each $x \in X$, let $Q_i(x) = \{y_i \in T_i(x) : 0 \notin P_i(x, y, v_i)$ for all $v_i \in T_i(x)$ and for $y = (y_i)_{i\in I} \in Y\}$. It is easy to see that $Q_i(x) = \{y_i \in T_i(x) : G_i(x, y, v_i) \subseteq H_i(x, y)$ for all $v_i \in T_i(x)$ and $y = (y_i)_{i\in I} \in Y\}$ and $Q_i(x)$ is a convex set for each $i \in I$. Indeed, if $y_i^1, y_i^2 \in Q_i(x)$ and $\lambda \in [0, 1]$. Let $y^1 = (y_i^1)_{i\in I}$, $y^2 = (y_i^2)_{i\in I}$. Then $y^1, y^2 \in Y$, $y_i^1, y_i^2 \in T_i(x)$ and $G_i(x, y^1, v_i) \subseteq H_i(x, y^1)$, $G_i(x, y^2, v_i) \subseteq H_i(x, y^2)$ for all $v_i \in T_i(x)$. By (iii) and (iv), $G_i(x, \lambda y^1 + (1 - \lambda)y^2, v_i) = \lambda G_i(x, y^1, v_i) + (1 - \lambda)G_i(x, y^2, v_i) \subseteq$

$\lambda H_i(x, y^1) + (1-\lambda)H_i(x, y^2) = H_i(x, \lambda y^1 + (1-\lambda)y^2)$. We also have $\lambda y^1 + (1-\lambda)y^2 \in Y$ and $\lambda y_i^1 + (1-\lambda)y_i^2 \in T_i(x)$. This shows that $\lambda y_i^1 + (1-\lambda)y_i^2 \in Q_i(x)$ and $Q_i(x)$ is convex. By (v), for each $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$, $v_i \multimap P_i(x, y, v_i)$ is $\{0\}$-quasiconvex and $0 \notin P_i(x, y, y_i)$. Then by Theorem 3.2, there exists $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $0 \notin P_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$. That is, $G_i(\bar{x}, \bar{y}, v_i) \subseteq H_i(\bar{x}, \bar{y})$ for all $v_i \in T_i(\bar{x})$. □

As a simple consequence of Theorem 3.3, we have the following existence theorems of systems of variational inclusions problems and systems of equilibrium problems.

Theorem 3.4 *Assume conditions (i) and (ii) of Theorem 3.1 are satisfied. For each $i \in I$, suppose that:*

(iii) $C_i : X \multimap Z_i$ *is a closed multivalued map such that* $C_i(x)$ *is a convex set and* $0 \in C_i(x)$ *for each* $x \in X$;
(iv) G_i *is a continuous multivalued map with nonempty closed values such that for each* $x \in X$, $(y, v_i) \multimap G_i(x, y, v_i)$ *is affine;*
(v) *For each* $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ *is* $C_i(x)$*-quasiconvex [1] and* $G_i(x, y, v_i) - [Z_i \setminus (G_i(x, y, y_i) + C_i(x))] \neq \emptyset$.

Then there exists $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ *such that for each* $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, *and* $G_i(\bar{x}, \bar{y}, v_i) \subseteq G_i(\bar{x}, \bar{y}, \bar{y}_i) + C_i(\bar{x})$ *for all* $v_i \in T_i(\bar{x})$.

Proof Let $H_i : X \times Y \multimap Z_i$ be defined by $H_i(x, y) = G_i(x, y, y_i) + C_i(x)$ for each $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$. By (iii), (iv) and Lemma 3.2 that H_i is a closed multivalued map with nonempty values and for each $x \in X$, $y \multimap H_i(x, y)$ is affine. Since $0 \in C_i(x)$ for all $x \in X$, $G_i(x, y, y_i) \subseteq G_i(x, y, y_i) + C_i(x)$ for each $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$. Then Theorem 3.4 follows from Theorem 3.3. □

Remark 3.2 In Theorem 3.4 we do not assume that $C_i(x)$ is a cone, but in Theorem 3.6 [11] and Theorem 3.6 [12], $C_i(x)$ is a constant closed convex cone. The proof of Theorem 3.4 is much simple than the proofs of Theorem 3.6 in [11,12]. Indeed, Theorem 3.4 cannot be obtained from Theorem 3.6 in [11,12].

If we let $H_i(x, y) = C_i(x)$ for all $(x, y) \in X \times Y$, we have the following existence theorem of systems of equilibrium problem.

Corollary 3.1 *In Theorem 3.4, for each $i \in I$, suppose that*

(iv) G_i *is a l.s.c multivalued map such that for each* $x \in X$, $(y, v_i) \multimap G_i(x, y, v_i)$ *is affine and* C_i *is a closed multivalued map with nonempty values and* $C_i(x)$ *is a convex set for each* $x \in X$;
(v) *For each* $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ *is* $C_i(x)$*-quasiconvex,* $G_i(x, y, y_i) \subseteq C_i(x)$ *and* $G_i(x, y, v_i) - [Z_i \setminus (C_i(x))] \neq Z_i$ *for all* $(x, y, v_i) \in X \times Y \times Y_i$.

Then there exists $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ *such that for each* $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, *and* $G_i(\bar{x}, \bar{y}, v_i) \subseteq C_i(\bar{x})$ *for all* $v_i \in T_i(\bar{x})$.

If we let $H_i(x, y) = Z_i \setminus (-int\ C_i(x))$ for all $(x, y) \in X \times Y$, we have the following existence theorem of systems of equilibrium problem.

Corollary 3.2 *Assume that conditions (i), (ii), and (iii) of Theorem 3.1 are satisfied. For each $i \in I$, suppose that* *(iv)* $W_i : X \multimap Z_i$ *is an u.s.c. multivalued map with nonempty values, where* $W_i(x) = Z_i \setminus (-int\ C_i(x))$ *and* $C_i : X \multimap Z_i$ *is a multivalued map such that* $int\ C_i(x) \neq \emptyset$ *for all* $x \in X$*;*

(v) *For each* $(x, y) \in X \times Y$, $y = (y_i)_{i \in I}$, $v_i \multimap G_i(x, y, v_i)$ *is* $C_i(x)$*-quasiconvex,* $G_i(x, y, v_i) \cap (-int\ C_i(x)) = \emptyset$ *and* $G_i(x, y, v_i) - [Z_i \setminus (-int\ C_i(x))] \neq Z_i$ *for all* $(x, y, v_i) \in X \times Y \times Y_i$.

Then there exists $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times Y$ *such that for each* $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$*, and* $G_i(\bar{x}, \bar{y}, v_i) \cap (-int\ C_i(\bar{x})) = \emptyset$ *for all* $v_i \in T_i(\bar{x})$.

Proof Let $H_i(x, y) = Z_i \setminus (-int\ C_i(x))$. Then Corollary 3.2 follows immediately from Theorem 3.3. □

Remark 3.3 In Corollaries 3.1 and 3.2, we do not assume that $C_i(x)$ is a cone for each $x \in X$. Therefore, Corollaries 3.1 and 3.2 are different from Theorems 3.1 and 3.6 in [13]. Our proof of Corollaries 3.1 and 3.2 are much simple than Theorems 3.1 and 3.6 in [13].

Theorem 3.5 *Suppose conditions (i) and (ii) of Theorem 3.1 are satisfied. For each* $i \in I$*, suppose that*

(iii) $H_i : X \times Y \multimap Z_i$ *is a l.s.c. multivalued map and for each* $x \in X$, $y \multimap H_i(x, y)$ *is affine;* $H_i(x, y) - [Z_i \setminus G_i(x, y, v_i)] \neq Z_i$ *for all* $(x, y, v_i) \in X \times Y \times Y_i$*;*

(iv) G_i *is a closed multivalued map with nonempty values and for each* $x \in X$, $y \multimap G_i(x, y, v_i)$ *is affine;*

(v) *For each* $(x, (y_i)_{i \in I}) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ *is* $\{0\}$*-quasiconvex-like [1] and* $H_i(x, y) \subseteq G_i(x, y, y_i)$.

Then there exists $(\bar{x}, \bar{y}) = (\bar{x}, (\bar{y}_i)_{i \in I}) \in X \times Y$ *such that for each* $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$*, and* $H_i(\bar{x}, \bar{y}) \subseteq G_i(\bar{x}, \bar{y}, v_i)$ *for all* $v_i \in T_i(\bar{x})$.

Proof Let $F_i : X \times Y \times Y_i \multimap Z_i$ be defined by $F_i(x, y, v_i) = H_i(x, y) - [Z_i \setminus G_i(x, y, v_i)]$ for all $(x, y, v_i) \in X \times Y \times Y_i$. By (iii), (iv) and Lemma 3.1 that F_i is a multivalued map with open graph. For each $x \in X$, let $Q_i(x) = \{y_i \in T_i(x) :\ 0 \notin F_i(x, y, v_i)$ for all $v_i \in T_i(x)$ and for $y = (y_i)_{i \in I} \in Y\}$. Then $Q_i(x) = \{y_i \in T_i(x) : H_i(x, y) \subseteq G_i(x, y, v_i)$ for all $v_i \in T_i(x)$ and for $y = (y_i)_{i \in I} \in Y\}$. By (iii) and (iv), $Q_i(x)$ is a convex for each $x \in X$. By (v), $0 \notin F_i(x, y, y_i)$ for each $(x, y) = (x, (y_i)_{i \in I}) \in X \times Y$. By (iii), $F_i(x, y, v_i) \neq Z_i$ for all $(x, y, v_i) \in X \times Y \times Y_i$, By (v), for each $(x, y) \in X \times Y$, $v_i \multimap F_i(x, y, v_i)$ is $\{0\}$-quasiconvex. Then Theorem 3.5 follows from Theorem 3.2. □

As a simple consequence of Theorem 3.5, we have the following existence theorem of systems of generalized lower quasi-variational inclusions problems.

Theorem 3.6 *Suppose conditions (i) and (ii) of Theorem 3.1 are satisfied. For each* $i \in I$*, suppose that*

(iii) $C_i : X \multimap Z_i$ *is a closed multivalued map such that* $C_i(x)$ *is a convex set and* $0 \in C_i(x)$ *for each* $x \in X$*;*

(iv) G_i *is a continuous multivalued map with nonempty closed values, and for each* $x \in X$, $(y, v_i) \multimap G_i(x, y, v_i)$ *is affine, and for each* $x \in X$, $y = (y_i)_{i \in I} \in Y$, $v_i \in Y_i$, $G_i(x, y, y_i) \subseteq G_i(x, y, v_i) - C_i(x)$*;*

(v) *For each* $x \in X$, $y = (y_i)_{i \in I} \in Y$, $v_i \multimap G_i(x, y, v_i)$ *is* $\{0\}$*-quasiconvex-like and* $G_i(x, y, y_i) - [Z_i \setminus (G_i(x, y, v_i) - C_i(x))] \neq Z_i$.

Then there exists $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ *such that for each* $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, *and* $G_i(\bar{x}, \bar{y}, \bar{y}_i) \subseteq G_i(\bar{x}, \bar{y}, v_i) - C_i(\bar{x})$ *for all* $v_i \in T_i(\bar{x})$.

Proof Apply Theorem 3.5 and follow the same argument as in Theorem 3.4, we can prove Theorem 3.6. □

Theorem 3.7 *Theorem 3.1, 3.2 and 3.5 are equivalent.*

Proof We see that Theorem 3.1 implies Theorem 3.2, and Theorem 3.2 implies Theorem 3.5. We want to show that Theorem 3.5 implies Theorem 3.1. Under the assumptions of Theorem 3.1. For each $i \in I$, let $H_i : X \times Y \multimap Z_i$ be defined by $H_i(x, y) = \{0\}$ for all $(x, y) \in X \times Y$. Since $G_i(x, y, v_i) \neq \emptyset$ for all $(x, y, v_i) \in X \times Y \times Y_i$, $H_i(x, y) - (Z_i \setminus G_i(x, y, v_i)) = -(Z_i \setminus G_i(x, y, v_i)) \neq Z_i$. Then Theorem 3.1 follows from Theorem 3.5. Therefore, Theorems 3.1, 3.2 and 3.5 are equivalent. □

Theorem 3.8 *In Theorem 3.4, if we assume further that* $C_i(x)$ *is a convex cone for each* $x \in X$ *and for each* $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$, $IMin(G_i(x, y, y_i)/C_i(x)) \neq \emptyset$. *Then there exits* $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ *such that for each* $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, *and* $G_i(\bar{x}, \bar{y}, \bar{y}_i) \cap IMin(G_i(x, y, y_i)/C_i(x)) \neq \emptyset$.

Proof Let $H_i(x, y) = G_i(x, y, y_i) + C_i(x)$. Since $IMin(G_i(x, y, y_i)/C_i(x)) \neq \emptyset$. It is easy to see that $IMin(H_i(x, y)/C_i(x)) \neq \emptyset$. Then Theorem 3.8 follows from Theorem 3.3. □

Theorem 3.9 *Suppose condition (i), (iii), (iv) and (v) of Theorem 3.4. for each* $i \in I$, *suppose that*

(ii) $S_i : X \multimap X_i$ *is an compact u.s.c. multivalued map with nonempty closed convex values;*
(vi) $F_i : X \times Y \multimap Z_i$ *is a l.s.c. multivalued map with nonempty closed values and for each* $x \in X$, *there exists* $y = (y_i)_{i\in I} \in Y$ *such that* $y_i \in T_i(x)$, $F_i(x, y) \subseteq C_i(x)$ *and* $y \multimap F_i(x, y)$ *is* $C_i(x)$*-quasiconvenx-like.*

Then there exists $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ *such that for each* $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$, $F_i(\bar{x}, \bar{y}) \subseteq C_i(\bar{x})$ *and* $G_i(\bar{x}, \bar{y}, \bar{y}_i) \subseteq G_i(\bar{x}, \bar{y}, v_i) + C_i(\bar{x})$ *for all* $v_i \in T_i(\bar{x})$.

Proof For each $i \in I$, let $L_i : X \multimap Z_i$ be defined by $L_i(x) = \{y_i \in T_i(x) : F_i(x, y) \subseteq C_i(x)$ for $y = (y_i)_{i\in I} \in Y\}$ for each $x \in X$. It is easy to see that L_i is a compact u.s.c. multivalued map with nonempty closed convex values. Then Theorem 3.9 follows from Theorem 3.4. □

Theorem 3.10 *Let X be a nonempty subset of a topological vector space E, I be any index set. For each* $i \in I$, *let* Y_i *be a nonempty convex subset of a t.v.s.* V_i, Z_i *be a real t.v.s.. For each* $i \in I$, *suppose that*

(i) $C_i : X \multimap Z_i$ *is a multivalued map such that for each* $x \in X$, $C_i(x)$ *is a nonempty closed convex cone;*
(ii) $G_i : X \times Y_i \times Y_i \multimap Z_i$ *is an u.s.c. multivalued map with nonempty compact values such that for each* $(x, v_i) \in X \times Y_i$, $y_i \multimap G_i(x, y_i, v_i)$ *is l.s.c. and for each* $(x, v_i) \in X \times Y_i$, $y_i \multimap G_i(x, y_i, v_i)$ *is* $C_i(x)$*-quasiconvex;*
(iii) $T_i : X \multimap Y_i$ *is a multivalued map with nonempty closed convex values;*
(iv) There exist a nonempty compact subset K of Y and a nonempty compact convex subset D_i *of* Y_i *for all* $i \in I$ *such that for each* $y = (y_i)_{i\in I} \in Y \setminus K$ *and each* $x \in X$, *there exist* $j \in I$ *and* $u_j \in T_j(x)$ *such that* $G_j(x, y_j, u_j) \not\subseteq G_j(x, y_j, y_j) + C_j(x)$.

Then for each $x \in X$, there exists $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that for each $i \in I$, $\bar{y}_i \in T_i(x)$, and $G_i(x, \bar{y}_i, v_i) \subseteq G_i(x, \bar{y}_i, \bar{y}_i) + C_i(x)$ for all $v_i \in T_i(x)$.

Proof For each $i \in I$ and $x \in X$, let $P_i(x) : \prod_{i\in I} T_i(x) \multimap T_i(x)$ be defined by $P_i(x)(y) = \{v_i \in T_i(x) : \; G_i(x, y_i, v_i) \nsubseteq G_i(x, y_i, y_i) + C_i(x)\}$ for each $y = (y_i)_{i\in I} \in Y$. Then for each $y = (y_i)_{i\in I}$, $y_i \notin P_i(x)(y)$. For each $i \in I$, $x \in X$ and $y \in \prod_{i\in I} T_i(x)$, $P_i(x)(y)$ is convex. Indeed, let $v_i^1, v_i^2 \in P_i(x)(y)$ and $\lambda \in [0, 1]$, then $v_i^1, v_i^2 \in T_i(x)$ and $G_i(x, y_i, v_i^1) \nsubseteq G_i(x, y_i, y_i) + C_i(x)$ and $G_i(x, y_i, v_i^2) \nsubseteq G_i(x, y_i, y_i) + C_i(x)$. Assume $v_i^\lambda = \lambda v_i^1 + (1 - \lambda)v_i^2$. Then $v_i^\lambda \in T_i(x)$. Suppose that there exists $\lambda_0 \in (0, 1)$ such that $v_i^{\lambda_0} \notin P_i(x)(y)$, then $G_i(x, y_i, v_i^{\lambda_0}) \subseteq G_i(x, y_i, y_i) + C_i(x)$. Since for each $(x, y_i) \in X \times Y_i$, $v_i \multimap G_i(x, y_i, v_i)$ is $C_i(x)$-quasiconvex,

$$\begin{aligned}&\text{either } G_i(x, y_i, v_i^1) \subseteq G_i(x, y_i, v_i^{\lambda_0}) + C_i(x)\\ &\qquad \subseteq G_i(x, y_i, y_i) + C_i(x) + C_i(x) \subseteq G_i(x, y_i, y_i) + C_i(x),\\ &\text{or } G_i(x, y_i, v_i^2) \subseteq G_i(x, y_i, v_i^{\lambda_0}) + C_i(x) \subseteq G_i(x, y_i, y_i) + C_i(x).\end{aligned}$$

This leads to a contradiction. Therefore, $v_i^\lambda \in P_i(x)(y)$ and $P_i(x)(y)$ is convex for each $y \in \prod_{i\in I} T_i(x)$.

$[\prod_{i\in I} T_i(x)] \setminus [P_i(x)]^{-1}(u_i)$ is a closed set in $\prod_{i\in I} T_i(x)$ for each $u_i \in T_i(x)$. Indeed, if $y \in \overline{[\prod_{i\in I} T_i(x)] \setminus [P_i(x)]^{-1}(u_i)}$, then there exists a net $\{y^\alpha\}_{\alpha\in\Lambda}$ in $[\prod_{i\in I} T_i(x)] \setminus [P_i(x)]^{-1}(u_i)$ such that $y^\alpha = (y_i^\alpha)_{i\in I}$ for all $\alpha \in \Lambda$ and $y^\alpha \to y$. One has $y_i^\alpha \in T_i(x)$ and $G_i(x, y_i^\alpha, u_i) \subseteq G_i(x, y_i^\alpha, y_i^\alpha) + C_i(x)$. Let $z_i \in G_i(x, y_i, u_i)$. By assumption, for each $(x, u_i) \in X \times Y_i$, $y_i \multimap G_i(x, y_i, u_i)$ is l.s.c., there exist a net $\{z_i^\alpha\}_{\alpha\in\Lambda}$ such that $z_i^\alpha \in G_i(x, y_i^\alpha, u_i)$ for all $\alpha \in \Lambda$ and $z_i^\alpha \to z_i$. We follow the same arguments as in Theorem 3.1, we show that $[\prod_{i\in I} T_i(x)] \setminus [P_i(x)]^{-1}(u_i)$ is closed in $\prod_{i\in I} T_i(x)$. Therefore, $[P_i(x)]^{-1}(u_i)$ is open in $\prod_{i\in I} T_i(x)$.

By (iv), for each $y \in \prod_{i\in I} T_i(x) \setminus K$ and for each $x \in X$ there exist $j \in I$, $u_j \in T_j(x)$ such that for each $x \in X$, $y \in [P_i(x)]^{-1}(u_i)$. Then by Theorem 2.1 that for each $x \in X$ there exists $\bar{y} = (\bar{y}_i)_{i\in I} \in \prod_{i\in I} T_i(x)$ such that $P_i(x)(\bar{y}) = \emptyset$. Then for each $i \in I$, $\bar{y} \in T_i(x)$ and $G_i(x, \bar{y}_i, v_i) \subseteq G_i(x, \bar{y}_i, \bar{y}_i) + C_i(x)$ for all $v_i \in T_i(x)$. □

Corollary 3.3 *Theorem 3.10 is true if condition (iv) of Theorem 3.10 is replaced by (iv′), where*

(iv′) $T_i : X \multimap Y_i$ is a multivalued map with nonempty compact convex values.

Proof Since $T_i(x)$ is a compact set for each $x \in X$, $\prod_{i\in I} T_i(x)$ is a compact set for each $x \in X$. Then condition (iv) of Theorem 2.1 is satisfied by taking $Y = \prod_{i\in I} T_i(x) = K$. □

4 Applications to bilevel problem

As a consequence of Theorems 3.4 and 3.10, we establish an existence theorem of mathematical program with system of variational inclusion constrains from which we establish that existence theorems of bilevel problem.

Theorem 4.1 *Let I, E_i, V_i, X_i, Y_i, X, Y, T_i and Z_i be the same as in Theorem 3.4. Let Z_0 be a real t.v.s. and C_0 be a proper closed convex cone in Z_0. For each $i \in I$, suppose that*

(i) $C_i : X \multimap Z_i$ is a closed multivalued map such that $C_i(x)$ is a convex set and $0 \in C_i(x)$ for each $x \in X$;

(ii) $F_i : X \times Y \multimap Z_i$ *is a l.s.c. multivalued map such that for each* $x \in X$, $y \multimap F_i(x, y)$ *is* $C_i(x)$*-quasiconvex-like;*

(iii) $S_i : X \multimap X_i$ *is a compact u.s.c. multivalued map with nonempty closed convex values;*

(iv) *For each* $x \in X$, *there exists* $w = (w_i)_{i\in I} \in Y$ *such that* $w_i \in S_i(x)$, $F_i(w, y) \subseteq C_i(x)$*;*

(v) $G_i : X \times Y \times Y_i \multimap Z_i$ *is a continuous multivalued map with nonempty compact values such that for each* $x \in X$, $(y, v_i) \multimap G_i(x, y, v_i)$ *is affine and for each* $(x, y) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ *is* $C_i(x)$*-quasiconvex; and* $G_i(x, y, v_i) - [Z_i \setminus (G_i(x, y, y_i)+C_i(x))] \neq Z_i$ *for all* $(x, y, v_i) \in X \times Y \times Y_i$*;*

(vi) $f : X \times Y \multimap Z_0$ *is an u.s.c. multivalued map with nonempty compact values.*

Then there exists a solution to the following problem:
$Min(f(M)/C_0) \neq \emptyset$, *where* $M = \{(x, y) : x = (x_i)_{i\in I} \in X, y = (y_i)_{i\in I}$ *such that for all* $i \in I$, $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x, y) \subseteq C_i(x)$, *and* $G_i(x, y, v_i) \subseteq G_i(x, y, y_i) + C_i(x)$ *for all* $v_i \in T_i(x)$ $\}$.

Proof For each $i \in I$, let $M_i = \{(x, y) : x = (x_i)_{i\in I} \in X, y = (y_i)_{i\in I} \in Y, x_i \in S_i(x), y_i \in T_i(x), F_i(x, y) \subseteq C_i(x)$, and $G_i(x, y, v_i) \subseteq G_i(x, y, y_i) + C_i(x)$ for all $v_i \in T_i(x)\}$. Then $M = \cap_{i\in I} M_i$. By Theorem 3.9, $M \neq \emptyset$.

For each $i \in I$, M_i is closed. Indeed, if $(x, y) \in \overline{M_i}$, then there exists a net $\{(x^\alpha, y^\alpha) : \alpha \in \Lambda\}$ in M_i such that $(x^\alpha, y^\alpha) \to (x, y)$. One has $x^\alpha = (x_i^\alpha)_{i\in I} \in X$, $y^\alpha = (y_i^\alpha)_{i\in I} \in Y$, $x_i^\alpha \in S_i(x^\alpha)$, $y_i^\alpha \in T_i(x^\alpha)$, $F_i(x^\alpha, y^\alpha) \subseteq C_i(x^\alpha)$ and $G_i(x^\alpha, y^\alpha, v_i) \subseteq G_i(x^\alpha, y^\alpha, y_i^\alpha) + C_i(x^\alpha)$ for all $v_i \in T_i(x^\alpha)$. Let $v_i \in T_i(x)$. Since T_i is l.s.c., there exists a net $\{v_i^\alpha\}_{\alpha\in\Lambda}$ such that $v_i^\alpha \in T_i(x^\alpha)$ for all $\alpha \in \Lambda$ and $v_i^\alpha \to v_i$. We have $G_i(x^\alpha, y^\alpha, v_i^\alpha) \subseteq G_i(x^\alpha, y^\alpha, y_i^\alpha)+C_i(x^\alpha)$. Let $u_i \in G_i(x, y, v_i)$. Since G_i is l.s.c., there exists a net $\{u_i^\alpha\}_{\alpha\in\Lambda}$ such that $u_i^\alpha \in G_i(x^\alpha, y^\alpha, v_i^\alpha)$ for all $\alpha \in \Lambda$ and $u_i^\alpha \to u_i$. We have $u_i^\alpha = w_i^\alpha + c_i^\alpha$ for some $c_i^\alpha \in C_i(x^\alpha)$, and $w_i^\alpha \in G_i(x^\alpha, y^\alpha, y_i^\alpha)$. Let $K = \{x^\alpha : \alpha \in \Lambda\} \cup \{x\}$, $L = \{y^\alpha : \alpha \in \Lambda\} \cup \{y\}$ and $L_i = \{y_i^\alpha : \alpha \in \Lambda\} \cup \{y_i\}$. Then K, L and L_i are compact sets. Since G_i is an u.s.c. multivalued map with compact values, $G_i(K \times L \times L_i)$ is a compact set (see [7]). Hence, $\{w_i^\alpha\}_{\alpha\in\Lambda}$ has a subnet $\{w_i^{\alpha_\lambda}\}_{\alpha_\lambda\in\Lambda}$ such that $w_i^{\alpha_\lambda} \to w_i \in G_i(K \times L \times L_i)$. But $c_i^{\alpha_\lambda} = u_i^{\alpha_\lambda} - w_i^{\alpha_\lambda} \in C_i(x^\alpha)$, $c_i^{\alpha_\lambda} \to u_i - w_i$, and C_i is closed, $u_i - w_i \in C_i(x)$. By assumption $(x, y) \multimap G_i(x, y, y_i)$ is closed, $w_i \in G_i(x, y, y_i)$ and $u_i \in w_i + C_i(x) \subseteq G_i(x, y, y_i) + C_i(x)$. This shows that $G_i(x, y, v_i) \subseteq G_i(x, y, y_i) + C_i(x)$ for all $v_i \in T_i(x)$.

By assumption, S_i and T_i are closed. Hence $x_i \in S_i(x)$ and $y_i \in T_i(x)$ and $y = (y_i)_{i\in I} \in Y$. Let $z_i \in F_i(x, y)$. Since F_i is l.s.c., there exists a net $\{z_i^\alpha\}_{\alpha\in\Lambda}$ such that $z_i^\alpha \in F_i(x^\alpha, y^\alpha)$ for all $\alpha \in \Lambda$ and $z_i^\alpha \to z_i$. We see $z_i^\alpha \in C_i(x^\alpha)$. Since C_i is closed, $z_i \in C_i(x)$. This shows that $F_i(x, y) \subseteq C_i(x)$. By assumption, X is a closed set, we have $x \in X$. Therefore $(x, y) \in M_i$ and M_i is a closed set for each $i \in I$. But $M_i \subseteq (\prod_{i\in I} \overline{S_i(X)}) \times (\prod_{i\in I} \overline{T_i(X)})$ and S_i and T_i are compact, we see M_i is a compact set for each $i \in I$, and $M = \cap_{i\in I} M_i$ is a nonempty compact set.

Since $f : X \times Y \multimap Z_0$ is an u.s.c. multivalued amp with nonempty compact values, $f(M)$ is a compact set [7]. $Min(f(M)/C_0) \neq \emptyset$ [7] and Theorem 4.1 follows. □

Remark 4.1 If I is a singleton, Theorem 4.1 is still different from Theorem 4.1 in [18].

Remark 4.2 In Theorem 4.1, if $f : X \times Y \to \mathbb{R}$ is a l.s.c. function, then there exists a solution of the problem:

$Min_{(x,y)} f(x, y)$ such that $x \in X$, $y \in Y$, for each $i \in I$, $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x, y) \subseteq C_i(x)$, and $G_i(x, y, v_i) \subseteq G_i(x, y, y_i) + C_i(x)$ for all $v_i \in T_i(x)$.

Theorem 4.2 *In Theorem 4.1, if we assume furthermore that for each* $i \in I$, $IMin(G_i(x, y, y_i)/C_i(x)) \neq \emptyset$ *for each* $(x, y) = (x, (y_i)_{i\in I}) \in X \times Y$. *Then there exists a solution to the following problem:*
$Min(f(K)/C_0) \neq \emptyset$, *where* $K = \{(x, y) : x = (x_i)_{i\in I} \in X, y = (y_i)_{i\in I}$ *such that for each* $i \in I$, $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x, y) \subseteq C_i(x)$, *and* $G_i(x, y, v_i) \cap IMin(G_i(x, y, T_i(x))/C_i(x)) \neq \emptyset\}$.

Proof By assumption, $IMin(G_i(x, y, y_i)/C_i(x)) \neq \emptyset$. Then

$$G_i(x, y, v_i) \subseteq G_i(x, y, y_i) + C_i(x) \text{ for all } v_i \in T_i(x)$$

if and only if $G_i(x, y, y_i) \cap IMinG_i(x, y, T_i(x)) \neq \emptyset$. Then Theorem 4.2 follows immediately from Theorem 4.1. □

If $Z_i = \mathbb{R}$ and $C_i(x) = [0, \infty)$ for all $x \in X$, the following Corollary follows immediately from Theorem 4.2.

Corollary 4.1 *In Theorem 4.1, if conditions (ii), (iv), and (v) are replaced by (ii′), (iv′), and (v′), where*

- *(ii′)* $F_i : X \times Y \to \mathbb{R}$ *is a continuous function such that for each* $y \in Y$, $x_i \to F_i(x, y)$ *is quasiconvex;*
- *(iv′) For each* $(x, y) \in X \times Y$, *there exists* $w = (w_i)_{i\in I} \in Y$ *such that* $F_i(w, y) \geq 0$ *and* $w_i \in S_i(x)$;
- *(v′)* $G_i : X \times Y \times Y_i \to \mathbb{R}$ *is a continuous function such that for each* $x \in X$, $(y, v_i) \to G_i(x, y, v_i)$ *is affine and for each* $(x, y) \in X \times Y$, $v_i \to G_i(x, y, v_i)$ *is quasiconvex.*

Then there exists a solution to the following problem:
$Min(f(x, y)/C_0) \neq \emptyset$, $x \in X$, $y \in Y$ *such that for each* $i \in I$, $x_i \in S_i(x)$, $y_i \in T_i(x)$, $F_i(x, y) \geq 0$, *and* y_i *is a solution of the problem* $Min_{v_i \in T_i(x)}(G_i(x, y, v_i))$.

Remark 4.3 Corollary 4.1 is different from Corollary 5.3 [13], Corollary 3.1 [16].

Theorem 4.3 *For each* $i \in I$, *suppose that* X_i *is compact and (i), (iv) of Theorem 3.10. Conditions (ii) and (iii) of Theorem 3.10 are replaced by (ii′) and (iii′), respectively, where*

- *(ii′)* $G_i : X \times Y \times Y_i \multimap Z_i$ *is a continuous multivalued map such that for each* $(x, y) \in X \times Y$, $v_i \multimap G_i(x, y, v_i)$ *is* $C_i(x)$*-quasiconvex; and*
- *(iii′)* $T_i : X \multimap Y_i$ *is a continuous multivalued map with nonempty closed convex values.*

Suppose further that Z_0 *is a real t.v.s,* C_0 *is a proper closed convex cone in* Z_0 *and* $h : X \times Y \multimap Z_0$ *is an u.s.c. multivalued map with nonempty compact values. Then there exists a solution to the following problem:*
$Min(h(x, y)/C_0) \neq \emptyset$, $x \in X$, $y \in Y$ *such that for each* $i \in I$, $y_i \in T_i(x)$ *and* $G_i(x, y, v_i) \subseteq G_i(x, y, y_i) + C_i(x)$ *for all* $v_i \in T_i(x)$.

Proof For each $i \in I$, let $M_i = \{(x, y)$: $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I}$, $y_i \in T_i(x)$, and $G_i(x, y, v_i) \subseteq G_i(x, y, y_i) + C_i(x)$ for all v_i in $T_i(x)\}$. By Theorem 3.10, $\cap_{i\in I} M_i \neq \emptyset$. Let $M = \cap_{i\in I} M_i$. By condition (iv) of Theorem 3.10, if $(x, y) \in M$, then $y \in K$. By assumption, X_i is compact for each $i \in I$. Therefore, $X = \prod_{i\in I} X_i$ is compact. Hence X is closed. It is easy to see M_i is closed for each $i \in I$. Therefore, $\cap_{i\in I} M_i$ is closed. Then M is a nonempty closed subset of $X \times K$ and $X \times K$ is compact, M is compact. Since h is an u.s.c. multivalued map with nonempty compact values, $h(M)$ is a nonempty compact set [7] and $Min(h(M)/C_0) \neq \emptyset$ [4]. Therefore there exists a solution to the problem: $Min_{(x,y)} h(x, y)$,

$x \in X$, $y \in Y$ such that for each $i \in I$, $y_i \in T_i(x)$ and $G_i(x, y, v_i) \subseteq G_i(x, y, y_i) + C_i(x)$ for all $v_i \in T_i(x)$. □

Following the same arguments as in Theorem 4.2, we have the following Theorem.

Theorem 4.4 *In Theorem 4.3, if we assume further that for each $x \in X$ and $y \in Y$, $IMin(G_i(x, y_i, y_i)/C_i(x)) \neq \emptyset$. Then there exists a solution to the problem: $Min_{(x,y)}h(x, y)$, $x \in X$, $y \in Y$, $y_i \in T_i(x)$ and $G_i(x, y_i, y_i) \cap IMin(G_i(x, y_i, u_i)/C_i(x)) \neq \emptyset$.*

Apply Theorem 4.3 and follow the same arguments as in Corollary 4.1, we have the following Corollary.

Corollary 4.2 *In Theorem 4.3, if condition (ii′) is replaced by (ii″) and condition (iv) of Theorem 3.10 is replaced by (iv′), where (ii″) (iv′)*

- *(ii″) $G_i : X \times Y_i \times Y_i \to \mathbb{R}$ is a continuous function and for each $(x, y_i) \in X \times Y_i$, $v_i \to G_i(x, y_i, v_i)$ is quasiconvex; and*
- *(iv′) There exist a nonempty compact subset K of Y and a nonempty compact conves subset D_i of Y_i for all $i \in I$ such that for each $y \in Y \setminus K$ and each $x \in X$, there exist $j \in I$ and $u_j \in T_j(x) \cap D_j$ such that $G_j(x, y_j, u_j) < G_j(x, y_j, y_j)$.*

Then there exists a solution to the following problem:
$Min_{(x,y)}h(x, y)$, $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I}$ such that for each $i \in I$, $y_i \in T_i(x)$, and $G_i(x, y_i, v_i) \geq G_i(x, y_i, y_i)$ for all $v_i \in T_i(x)$.

Lemma 4.1 *Let I be any index set. For each $i \in I$, let X be a nonempty convex subset of t.v.s. E, Y_i be a nonempty convex subset of t.v.s. V_i, Z_i be a real t.v.s.. For each $i \in I$, suppose that*

- *(i) $G_i : X \times Y_i \times Y_i \multimap Z_i$ is an affine multivalued map;*
- *(ii) $T_i : X \multimap Y_i$ is a convex and concave mutltivalued map; and*
- *(iii) $C_i : X \multimap Z_i$ is a concave multivalued map.*

Let $M_i = \{(x, y) \in X \times Y : y = (y_i)_{i\in I}, G_i(x, y_i, v_i) \subseteq G_i(x, y_i, y_i) + C_i(x)$ for all $v_i \in T_i(x)\}$. Then M_i is a convex set for all $i \in I$.

Proof Let $(x, y), (x', y') \in M_i$, and $\lambda \in [0, 1]$. Then $x, x' \in X$, $y = (y_i)_{i\in I} \in Y$, $y' = (y'_i)_{i\in I} \in Y$, $y_i \in T_i(x)$, $y'_i \in T_i(x')$, $G_i(x, y_i, v_i) \subseteq G_i(x, y_i, y_i) + C_i(x)$ for all $v_i \in T_i(x)$ and $G_i(x, y'_i, v_i) \subseteq G_i(x, y'_i, y'_i) + C_i(x)$ for all $v_i \in T_i(x)$.

We have $(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y') \in X \times Y$. Since T_i is concave, $\lambda y_i + (1-\lambda)y'_i \in T_i(\lambda x + (1-\lambda)x')$. Let $u_i \in T_i(\lambda x + (1-\lambda)x')$. Since T_i is convex, there exist $v_i \in T_i(x)$, $v'_i \in T_i(x')$ such that $u_i = \lambda v_i + (1-\lambda)v'_i$. By (i) and (iii),

$$\begin{aligned}
&G_i(\lambda x + (1-\lambda)x', \lambda y_i + (1-\lambda)y'_i, u_i)\\
&\quad = G_i(\lambda x + (1-\lambda)x', \lambda y_i + (1-\lambda)y'_i, \lambda y_i + (1-\lambda)y'_i)\\
&\quad = \lambda G_i(x, y_i, v_i) + (1-\lambda)G_i(x', y'_i, v'_i)\\
&\quad \subseteq \lambda G_i(x, y_i, y_i) + \lambda C_i(x) + (1-\lambda)G_i(x', y'_i, y'_i) + (1-\lambda)C_i(x')\\
&\quad \subseteq G_i(\lambda x + (1-\lambda)x', \lambda y_i + (1-\lambda)y'_i, \lambda y_i + (1-\lambda)y'_i) + C_i(\lambda x + (1-\lambda)x').
\end{aligned}$$

Therefore, $(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y') \in M_i$ and M_i is convex. □

Theorem 4.5 *Let X be a nonempty convex subset of a t.v.s. E, I be any index set. For each $i \in I$, let Y_i be a nonempty convex subset of a Housdorff t.v.s. V_i, Z_i be a real t.v.s. $Y = \prod_{i\in I} Y_i$. For each $i \in I$, suppose that*

(i) $C_i : X \multimap Z_i$ *is a concave multivalued amp such that for each* $x \in X$*,* $C_i(x)$ *is a nonempty closed convex cone;*
(ii) $G_i : X \times Y_i \times Y_i \multimap Z_i$ *is an affine u.s.c. multivalued map with nonempty compact values such that for each* $(x, v_i) \in X \times Y_i$*,* $y_i \multimap G_i(x, y_i, v_i)$ *is l.s.c. and for each* $(x, y_i) \in X \times Y_i$*,* $v_i \multimap G_i(x, y_i, v_i)$ *is* $C_i(x)$*-quasiconvex;*
(iii) $T_i : X \multimap Y_i$ *is a concave and convex multivalued map with nonempty closed convex values;*
(iv) $h : X \times Y \to \mathbb{R}$ *is a l.s.c. quasiconvex function; and*
(v) There exist a nonempty compact subset K of Y and a nonempty compact convex subset D_i *of* Y_i *for all* $i \in I$ *such that for each* $y = (y_i)_{i \in I} \in Y \setminus K$ *and each* $x \in X$*, there exist* $j \in I$*, and* $u_j \in T_j(x) \cap D_j$ *such that* $G_j(x, y, u_j) \not\subseteq G_j(x, y, y_j) + C_j(x)$*; and*
(vi) There exist a nonempty compact subset L of M and a nonempty compact convex subset D of M such that for each $(x, y) \in M \setminus L$*, there exists* $(u, v) \in D$ *such that* $h(u, v) < h(x, y)$*, where M is defend as in the proof of Theorem 5.1.*

Then there exists a solution to the problem:
$Min(h(x, y)/C_0) \neq \emptyset$*,* $x \in X$*,* $y = (y_i)_{i \in I}$ *such that for all* $i \in I$*,* $y_i \in T_i(x)$*,* $G_i(x, y_i, v_i) \subseteq G_i(x, y, y_i) + C_i(x)$ *for all* $v_i \in T_i(x)$*.*

Proof Let M_i and M be defined as in Lemma 4.1. By theorem 3.10 that there exist $x \in X$, $y = (y_i)_{i \in I} \in Y$ such that for all $i \in I$, $y_i \in T_i(x)$ and

$$G_i(x, y, v_i) \subseteq G_i(x, y, y_i) + C_i(x) \text{ for all } v_i \in T_i(x).$$

That is, $M = \cap_{i \in I} M_i \neq \emptyset$. By Lemma 4.1 that M_i is convex for all $i \in I$. Therefore, M is a nonempty convex set in $X \times Y$. Let $P : M \multimap M$ be defend by $P(x, y) = \{(u, v) \in M : h(u, v) < h(x, y)\}$. Then $(x, y) \notin P(x, y)$ for all $(x, y) \in D$.

By (iv), $P(x, y)$ is convex for each $(x, y) \in M$ and $P^{-1}(u, v)$ is open in M for each $(u, v) \in M$. By (vi), for each $(x, y) \in M \setminus L$, there exists $(u, v) \in D$ such that $(x, y) \in P^{-1}(u, v)$.

By Theorem 2.1, that there exists $(\bar{x}, \bar{y}) \in M$ such that $P(\bar{x}, \bar{y}) = \emptyset$. That is, $h(u, v) \geq h(\bar{x}, \bar{y})$ for all $(u, v) \in M$. This completes the proof. □

Remark 4.4 In Theorem 4.5, if we assume further that for each $x \in X$, $y_i \in Y_i$, $I\text{Min}(G_i(x, y, y_i)/C_i(x)) \neq \emptyset$. Then there exists a solution to the problem: $\text{Min}(h(x, y)/C_0) \neq \emptyset$, $x \in X$, $y = (y_i)_{i \in I}$ such that for each $i \in I$, $y_i \in T_i(x)$, $G_i(x, y, y_i) \cap IMin(G_i(x, y, T_i(x))/C_i(x)) \neq \emptyset$.

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