J Glob Optim (2007) 39:393–407
DOI 10.1007/s10898-007-9143-3

ORIGINAL PAPER

On quasivariational inclusion problems of type I and related problems

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Received: 21 April 2006 / Accepted: 20 January 2007 / Published online: 17 May 2007

Abstract The quasivariational inclusion problems are formulated and sufficient conditions on the existence of solutions are shown. As special cases, we obtain several results on the existence of solutions of a general vector ideal (proper, Pareto, weak) quasi-optimization problems, of quasivariational inequalities, and of vector quasi-equilibrium problems. Further, we prove theorems on the existence for solutions of the sum of these inclusions. As corollaries, we shall show several results on the existence of solutions to another problems in the vector optimization problems concerning multivalued mappings.

Keywords Upper quasivariational inclusions · Lower quasivariational inclusions · α Quasi-optimization problems · Vector optimization problem · Quasi-equilibrium problems · Upper and lower C-quasiconvex multivalued mappings · Upper and lower C-continuous multivalued mappings

1 Introduction

Let Y be a topological vector space with a cone C. For a given subset $A \subset Y$, one can define efficient points of A with respect to C by different senses as: Ideal, Pareto, proper, weak,...(see Definition 2.1 below). The set of these efficient points is denoted

This work was supported by the National Science Council of the Republic of China and the Academy of Sciences and Technologies of Vietnam.
The authors wish to express their gratitude to the referees for their valuable suggestions.

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by $\alpha\text{Min}(A/C)$ with $\alpha = \text{I}; \alpha = \text{P}\,; \alpha = \text{ Pr}\,; \alpha = \text{W}; \ldots$ for the case of ideal, Pareto, proper, weak efficient points, respectively. Let D be a subset of another topological vector space X. By 2^D we denote the family of all subsets in D. For a given multivalued mapping $F : D \to 2^Y$, we consider the problem of finding $\bar{x} \in D$ such that

$$(GVOP)_\alpha \qquad F(\bar{x}) \cap \alpha\text{Min}(F(D)/C) \neq \emptyset.$$

This is called a general vector α optimization problem corresponding to D, F and C. The set of such points $\bar{x}$ is said to be a solution set of $(GVOP)_\alpha$. The elements of $\alpha\text{Min}(F(D)/C)$ are called α optimal values of $(GVOP)_\alpha$.

Now, let X, Y and Z be topological vector spaces, let $D \subset X, K \subset Z$ be nonempty subsets and let $C \subset Y$ be a cone. Given the following multivalued mappings:

$$\begin{aligned} S &: D \times K \to 2^D, \\ T &: D \times K \to 2^K, \\ F &: K \times D \times D \to 2^Y, \end{aligned}$$

we are interested in the problem of finding $(\bar{x}, \bar{y}) \in D \times K$ such that

$$(GVQOP)_\alpha \qquad \begin{aligned} &\bar{x} \in S(\bar{x}, \bar{y}), \\ &\bar{y} \in T(\bar{y}, \bar{x}), \end{aligned}$$

and

$$F(\bar{y}, \bar{x}, \bar{x}) \cap \alpha\text{Min}(F(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C) \neq \emptyset.$$

This is called a general vector α quasi-optimization problem (α is one of the following qualifications: ideal, Pareto, proper, weak, respectively). Such a pair $(\bar{x}, \bar{y})$ is said to be a solution of $(GVQOP)_\alpha$. The above multivalued mappings S, T, and F are said to be a constraint, a potential, and an utility mapping, respectively. These problems play a central role in the vector optimization theory concerning multivalued mappings and have many relations to the following problems

(UIQEP), Upper Ideal quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned} \bar{x} &\in S(\bar{x}, \bar{y}), \\ \bar{y} &\in T(\bar{x}, \bar{y}), \\ F(\bar{y}, \bar{x}, x) \subset C, \quad \text{for all} \quad x &\in S(\bar{x}, \bar{y}). \end{aligned}$$

(LIQEP), Lower ideal quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned} \bar{x} &\in S(\bar{x}, \bar{y}), \\ \bar{y} &\in T(\bar{x}, \bar{y}), \\ F(\bar{y}, \bar{x}, x) \cap C \neq \emptyset, \quad \text{for all} \quad x &\in S(\bar{x}, \bar{y}). \end{aligned}$$

(UPQEP), Upper Pareto quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned} \bar{x} &\in S(\bar{x}, \bar{y}), \\ \bar{y} &\in T(\bar{x}, \bar{y}), \\ F(\bar{y}, \bar{x}, x) \not\subset -(C \backslash l(C)), \quad \text{for all} \quad x &\in S(\bar{x}, \bar{y}). \end{aligned}$$

(LPQEP), Lower Pareto quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned} \bar{x} &\in S(\bar{x}, \bar{y}), \\ \bar{y} &\in T(\bar{x}, \bar{y}), \\ F(\bar{y}, \bar{x}, x) \cap -(C \backslash l(C)) = \emptyset, \quad \text{for all} \quad x &\in S(\bar{x}, \bar{y}). \end{aligned}$$

(UWQEP), Upper weak quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned} \bar{x} &\in S(\bar{x}, \bar{y}), \\ \bar{y} &\in T(\bar{x}, \bar{y}), \\ F(\bar{y}, \bar{x}, x) \not\subset \text{-int}(C), \quad \text{for all} \quad x &\in S(\bar{x}, \bar{y}). \end{aligned}$$

(UWQEP), Lower weak quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned} \bar{x} &\in S(\bar{x}, \bar{y}), \\ \bar{y} &\in T(\bar{x}, \bar{y}), \\ F(\bar{y}, \bar{x}, x) \cap \text{-int}(C) = \emptyset, \quad \text{for all} \quad x &\in S(\bar{x}, \bar{y}). \end{aligned}$$

These problems generalize many well-known problems in the optimization theory as quasi-equilibrium problems, quasivariational inequalities, fixed point problems, complementarity problems, saddle point problems, minimax problems as well as different others which have been studied by many authors, for examples, Park (2000), Chan and Pang (1982), Parida and Sen (1987), Guraggio and Tan (2002), etc. for quasi-equilibrium problems and quasivariational inequalities; Blum and Oettli (1993), Lin et al.(2002), Tan (2004), Minh and Tan (2000), Fan (1972), etc. for equilibrium and variational inequality problems and by some others in the references therein.

The purpose of this paper is to prove some new results on the existence of solutions to problems as follows:

(UQVIP), Upper quasivariational inclusion problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned} \bar{x} &\in S(\bar{x}, \bar{y}), \\ \bar{y} &\in T(\bar{x}, \bar{y}), \\ F(\bar{y}, \bar{x}, x) \subset F(\bar{y}, \bar{x}, \bar{x}) + C, \quad \text{for all} \quad x &\in S(\bar{x}, \bar{y}). \end{aligned}$$

(LQVIP), Lower quasivariational inclusion problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that

$$\begin{aligned} \bar{x} &\in S(\bar{x}, \bar{y}), \\ \bar{y} &\in T(\bar{x}, \bar{y}), \\ F(\bar{y}, \bar{x}, \bar{x}) \subset F(\bar{y}, \bar{x}, x) - C, \quad \text{for all} \quad x &\in S(\bar{x}, \bar{y}). \end{aligned}$$

In Tan(2004) the first author gave some existence theorems on the above problems. But, there are many rather strong conditions. For example: the cone C is supposed that the polar cone C' of C have weakly compact basis in the weak*topology; the multivalued mapping F have nonempty convex closed values etc. In this paper, we shall give some weaker sufficient conditions to improve his results on the existence of solutions to these problems and show several relationship with other problems mentioned above.

2 Preliminaries and definitions

Throughout this paper, as in the introduction, by X, Y and Z we denote real Hausdorff locally convex topological vector spaces. The space of real numbers is denoted by R. Given a subset $D \subset X$, we consider a multivalued mapping $F : D \to 2^Y$. The definition domain and the graph of F are denoted by

$$\mathrm{dom}F = \{x \in D/F(x) \neq \emptyset\},$$
$$\mathrm{Gr}(F) = \{(x, y) \in D \times Y/y \in F(x)\},$$

respectively. We recall that F is said to be a closed mapping if the graph $Gr(F)$ of F is a closed subset in the product space $X \times Y$ and it is said to be a compact mapping if the closure $\overline{F(D)}$ of its range $F(D)$ is a compact set in Y. A multivalued mapping $F : D \to 2^Y$ is said to be upper semicontinuous (u.s.c) at $\bar{x} \in D$ if for each open set V containing $F(\bar{x})$, there exists an open set U of $\bar{x}$ such that $F(x) \subset V$ for each $x \in U$.F is said to be u.s.c on D if it is u.s.c at all $x \in D$. Further, let Y be a topological vector space with a cone C. We denote $l(C) = C \cap (-C)$. If $l(C) = \{0\}$, C is said to be pointed. We recall the following definitions (see Definition 2.1, Chapter 2 in Luc (1989)).

Definition 2.1 Let A be a nonempty subset of Y. We say that:

(i) $x \in A$ is an ideal efficient (or ideal minimal) point of A with respect to C if $y - x \in C$ for every $y \in A$.
The set of ideal minimal points of A is denoted by $\mathrm{IMin}(A/C)$.

(ii) $x \in A$ is an efficient (or Pareto-minimal, or nondominated) point of A w.r.t. C if there is no $y \in A$ with $x - y \in C \backslash l(C)$.
The set of efficient points of A is denoted by $\mathrm{PMin}(A/C)$.

(iii) $x \in A$ is a (global) proper efficient point of A w.r.t. C if there exists a convex cone $\tilde{C}$ which is not the whole space and contains $C \backslash l(C)$ in its interior so that $x \in \mathrm{PMin}\left(A/\tilde{C}\right)$.
The set of proper efficient points of A is denoted by $\mathrm{PrMin}(A/C)$.

(iv) Supposing that $\mathrm{int}C$ is nonempty, $x \in A$ is a weak efficient point of A w.r.t. C if $x \in \mathrm{PMin}\,(A/\{0\} \cup \mathrm{int}\, C)$.
The set of weak efficient points of A is denoted by $\mathrm{WMin}(A/C)$.

We write $\alpha\mathrm{Min}(A/C)$ to denote one of $\mathrm{IMin}(A/C), \mathrm{PMin}(A/C), \ldots$.
We have the following inclusions

$$\mathrm{PrMin}(A/C) \subseteq \mathrm{PMin}(A/C) \subseteq \mathrm{WMin}(A/C).$$

Now, we introduce new definitions of the C-continuities.

Definition 2.2 Let $F : D \to 2^Y$ be a multivalued mapping.

(i) F is said to be upper (lower) C-continuous at $\bar{x} \in \mathrm{dom}\, F$ if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that:

$$F(x) \subset F(\bar{x}) + V + C$$

$$(F(\bar{x}) \subset F(x) + V - C, \text{respectively})$$

holds for all $x \in U \cap \mathrm{dom}F$.

(ii) If F is upper C-continuous and lower C-continuous at $\bar{x}$ simultaneously, we say that it is C-continuous at $\bar{x}$.

(iii) If F is upper, lower,..., C-continuous at any point of dom$F$, we say that it is upper, lower,..., C-continuous on D.

(iv) In the case $C = \{0\}$, a trivial one in Y, we shall only say F is upper, lower continuous instead of upper, lower 0-continuous. And, F is continuous if it is upper and lower continuous simultaneously.

Definition 2.3 Let F be a multivalued mapping from D to 2^Y. We say that:

(i) F is upper C-quasiconvex on D if for any $x_1,\ x_2 \in D,\ t \in \big[0,1\big]$, either

$$\begin{aligned} &F(x_1) \subset F(tx_1 + (1-t)x_2) + C \\ \text{or,}\quad &F(x_2) \subset F(tx_1 + (1-t)x_2) + C, \end{aligned}$$

holds.

(ii) F is lower C-quasiconvex on D if for any $x_1, x_2 \in D, t \in \big[0,1\big]$, either

$$\begin{aligned} &F(tx_1 + (1-t)x_2) \subset F(x_1) - C \\ \text{or,}\quad &F(tx_1 + (1-t)x_2) \subset F(x_2) - C, \end{aligned}$$

holds.

Now, we give some necessary and sufficient conditions on the upper and the lower C-continuities which we shall need in the next section.

Proposition 2.4 *Let $F : D \to 2^Y$ and $C \subset Y$ be a convex closed cone.*

(1) *If F is upper C-continuous at $x_o \in domF$ with $F(x_o) + C$ closed, then for any net $\{x_B\}$ in $domF$, $x_\beta \to x_o, y_\beta \in F(x_\beta) + C, y_\beta \to y_o$ imply $y_o \in F(x_o) + C$.*

Conversely, if F is compact and for any net $\{x_B\}$ in $domF$, $x_\beta \to x_o, y_\beta \in F(x_\beta) + C, y_\beta \to y_o$ imply $y_o \in F(x_o) + C$, then F is upper C-continuous at x_o.

(2) *If F is compact and lower C-continuous at $x_o \in domF$, then any net $\{x_B\}$ in $domF$, $x_\beta \to x_o, y_o \in F(x_o) + C$, there is a net $\{y_\beta\}, y_\beta \in F(x_\beta)$, which has a convergent subnet $\{y_{\beta_\gamma}\}, y_{\beta_\gamma} - y_o \to c \in C$(i.e $y_{\beta_\gamma} \to y_o + c \in y_o + C$).*

Conversely, if $F(x_o)$ is compact and for any net $\{x_B\}$ in $domF$, $x_\beta \to x_o, y_o \in F(x_o)+C$, there is a net $\{y_\beta\}, y_\beta \in F(x_\beta)$, which has a convergent subnet $\{y_{\beta_\gamma}\}, y_{\beta_\gamma} - y_o \to c \in C$, then F is lower C-continuous at x_o.

Proof (1) Assume first that F is upper C-continuous at $x_o \in domF$ and $x_\beta \to x_o, y_\beta \in F(x_\beta) + C, y_\beta \to y_o$. We suppose to the contrary that $y_o \notin F(x_o) + C$. We can find a convex balanced closed neighborhood V_o of the origin in Y such that

$$(y_o + V_o) \cap (F(x_o) + C) = \emptyset,$$

or,

$$(y_o + V_o/2) \cap (F(x_o) + V_o/2 + C) = \emptyset.$$

Since $y_\beta \to y_o$, one can find $\beta_1 \geq 0$ such that $y_\beta - y_o \in V_0/2$ for all $\beta \geq \beta_1$. Therefore, $y_\beta \in y_o + V_0/2$ and F is upper C-continuous at x_o, it implies that one can find a neighborhood U of x_o such that

$$F(x) \subset F(x_o) + V_o/2 + C \quad \text{for all} \quad x \in U \cap domF.$$

Since $x_\beta \to x_o$, one can find $\beta_2 \geq 0$ such that $x_\beta \in U$ and

$$y_\beta \in F(x_\beta) + C \subset F(x_o) + V_0/2 + C \quad \text{for all} \quad \beta \geq \beta_2.$$

This implies that

$$y_\beta \in (y_o + V_0/2) \cap (F(x_o) + V_0/2 + C) = \emptyset \quad \text{for all} \quad \beta \geq max\{\beta_1, \beta_2\}$$

and we have a contradiction. Thus, we conclude $y_o \in F(x_o) + C$. Now, assume that F is compact and for any net $x_\beta \to x_o, y_\beta \in F(x_\beta) + C, y_\beta \to y_o$ imply $y_o \in F(x_o) + C$. On the contrary, we assume that F is not upper C-continuous at x_o. This implies that there is a neighborhood V of the origin in Y such that for any neighborhood U_β of x_o one can find $x_\beta \in U_\beta$ such that

$$F(x_\beta) \not\subset F(x_o) + V + C.$$

We can choose $y_\beta \in F(x_\beta)$ with $y_\beta \notin F(x_o) + V + C$. Since $\overline{F(D)}$ is compact, we can assume, without loss of generality, that $y_\beta \to y_o$, and hence $y_o \in F(x_o) + C$. On the other hand, since $y_\beta \to y_o$, there is $\beta_o \geq 0$ such that $y_\beta - y_o \in V$ for all $\beta \geq \beta_o$. Consequently,

$$y_\beta \in y_o + V \subset F(x_o) + V + C, \quad \text{for all} \quad \beta \geq \beta_o$$

and we have a contradiction.

(2) Assume that F is compact and lower C-continuous at $x_o \in domF$, and $x_\beta \to x_o, y_o \in F(x_o)$. For any neighborhood V of the origin in Y there is a neighborhood U of x_o such that

$$F(x_o) \subset F(x) + V - C, \quad \text{for all} \quad x \in U \cap domF.$$

Since $x_\beta \to x_o$, there is $\beta_o \geq 0$ such that $x_\beta \in U$ and then

$$F(x_o) \subset F(x_\beta) + V - C, \quad \text{for all} \quad \beta \geq \beta_o.$$

For $y_o \in F(x_o)$, we can write

$$y_o = y_\beta + v_\beta - c_\beta \quad \text{with} \quad y_\beta \in F(x_\beta) \subset \overline{F(D)}, v_\beta \in V, c_\beta \in C.$$

Since $\overline{F(D)}$ is compact, we can choose $y_{\beta_\gamma} \to y^*, v_{\beta_\gamma} \to 0$. Therefore, $c_{\beta_\gamma} = y_{\beta_\gamma} + v_{\beta_\gamma} - y_o \to y^* - y_o \in C$, or $y_{\beta_\gamma} \to y^* \in y_o + C$. Thus, for any $x_\beta \to x_o, y_o \in F(x_o)$, one can find $y_{\beta_\gamma} \in F(x_{\beta_\gamma})$ with $y_{\beta_\gamma} \to y^* \in y_o + C$.

Now, we assume that $\overline{F(x_o)}$ is compact and for any net $x_\beta \to x_o, y_o \in F(x_o) + C$, there is a net $\{y_\beta\}, y_\beta \in F(x_\beta)$ which has a convergent subnet $y_{\beta_\gamma} - y_o \to c \in C$. By contrary, we suppose that F is not lower C-continuous at x_o. This implies that there is a neighborhood V of the origin in Y such that for any neighborhood U_β of x_o one can find $x_\beta \in U_\beta$ such that

$$F(x_o) \not\subset F(x_\beta) + V - C.$$

We can choose $z_\beta \in F(x_o)$ with $z_\beta \notin (F(x_\beta) + V - C)$. Since $F(x_o)$ is compact, we can assume, without loss of generality, that $z_\beta \to z_o \in F(x_o)$, and hence $z_o \in F(x_o) + C$. We may assume that $x_\beta \to x_o$. Therefore, there is a net $\{y_\beta\}, y_\beta \in F(x_\beta)$ which has a convergent subnet $\{y_{\beta_\gamma}\}, y_{\beta_\gamma} - z_o \to c \in C$. Without loss of generality, we suppose $y_\beta \to y^* \in z_o + C$. This implies that there is $\beta_1 \geq 0$ such that $z_\beta \in z_o + V/2, y_\beta \in y^* + V/2$ and $z_o \in y_\beta + V/2 - C$ for all $\beta \geq \beta_1$. Consequently,

$$z_\beta \in y_\beta + V/2 + V/2 - C \subset F(x_\beta) + V - C, \quad \text{for all} \quad \beta \geq \beta_1$$

and we have a contradiction. □

In the proof of the main results of our paper we use the following theorem.

Theorem 2.5 Browder (1968) *Let D be a nonempty convex compact subset of topological vector space X and $F : D \to 2^D$ be a multivalued mapping satisfying the following conditions:*

(1) *For each $x \in D$, $F(x)$ is a nonempty convex subset of D;*
(2) *For all $y \in D, F^{-1}(y)$ is open in D.*

Then there exists $\bar{x} \in D$ such that $\bar{x} \in F(\bar{x})$

3 Main results

Throughout this section, unless otherwise specify, by X, Y and Z we denote Hausdorff locally convex topological vector spaces. Let $D \subset X, K \subset Z$ be nonempty subsets, $C \subset Y$ be a convex closed cone. Given multivalued mappings S, T and F as in Introduction, we first prove the following proposition.

Proposition 3.1 *Let $B \subset D$ be a nonempty convex compact subset, $G : B \to 2^Y$ be an upper C-quasiconvex and upper C-continuous multivalued mapping with nonempty closed values. Then there exists $\bar{z} \in B$ such that*

$$G(z) \subset G(\bar{z}) + C, \quad \textit{for all} \quad z \in B.$$

Proof We define the multivalued mapping $N : B \to 2^B$ by

$$N(z) = \{z' \in B| \quad G(z') \not\subset G(z) + C\}.$$

It is clear that $z \notin N(z)$ for all $z \in B$. If $z_1, z_2 \in N(z)$, then

$$G(z_1) \not\subset G(z) + C,$$
$$G(z_2) \not\subset G(z) + C.$$

Together with the upper C-quasiconvexity of G we conclude

$$G(tz_1 + (1-t)z_2) \not\subset G(z) + C.$$

This implies $tz_1 + (1-t)z_2 \in N(z)$ for all $t \in [0, 1]$ and hence $N(z)$ is a convex set for any $z \in B$.

Further, we have

$$N^{-1}(z') = \{z \in B| \quad G(z') \not\subset G(z) + C\}.$$

Take $z \in N^{-1}(z')$, we deduce $z' \in N(z)$ and so

$$G(z') \not\subset G(z) + C.$$

The upper C-continuity of G implies that for any neighborhood V of the origin in Y there is a neighborhood U_V of z such that

$$G(x) \subset G(z) + V + C, \quad \text{for all} \quad x \in U_V \cap B.$$

This implies that if for all V

$$G(z') \subset G(x) + C, \quad \text{for some} \quad x \in U_V \cap B,$$

then

$$G(z') \subset G(x) + C \subset G(z) + V + C$$

and so

$$G(z') \subset G(z) + V + C, \quad \text{for all} \quad V.$$

Since $G(z)$ and C are closed, the last inclusion shows $G(z') \subset G(z) + C$ and we have a contradiction. Therefore, there exists V_0 such that

$$G(z') \not\subset G(x) + C, \quad \text{for all} \quad x \in U_{V_0} \cap B.$$

This gives

$$U_{V_0} \cap B \subset N^{-1}(z')$$

and so $N^1(z')$ is an open set in B. As it has been shown: $z \notin N(z), N(z)$ is convex for any $z \in B$ and $N^{-1}(z')$ is open in B for any $z' \in B$. Consequently, applying Theorem 2.5 in Sect. 2, we conclude that there exists $\bar{z} \in B$ with $N(\bar{z}) = \emptyset$. This implies

$$G(z) \subset G(\bar{z}) + C, \quad \text{for all} \quad z \in B.$$

Thus, the proof is complete. □

Analogically, we can prove the following proposition

Proposition 3.2 *Let $B \subset D$ be a nonempty convex compact subset, $G : B \to 2^Y$ be a lower C-quasiconvex and lower (C)-continuous multivalued mapping with nonempty closed values. Then there exists $\bar{z} \in B$ such that*

$$G(\bar{z}) \subset G(z) - C, \quad \textit{for all} \quad z \in B.$$

Corollary 3.3 *Assume that all assumptions of Proposition 3.1 are satisfied and for any $z \in B, IMin(G(z)/C) \neq \emptyset$. Then there exists $\bar{z} \in B$ such that*

$$G(\bar{z}) \cap IMin(G(B)/C) \neq \emptyset.$$

(This means that the general vector ideal optimization problem concerning G, B, C has a solution).

Proof Proposition 3.1 implies that there exists $\bar{z} \in B$ such that

$$G(z) \subset G(\bar{z}) + C, \quad \text{for all} \quad z \in B. \tag{1}$$

Take $v^* \in IMin(G(\bar{z})/C)$, we have $G(\bar{z}) \subset v^* + C$. Then, (1) yields

$$G(z) \subset v^* + C, \quad \text{for all} \quad z \in B.$$

This shows that $v^* \in IMin(G(B)/C)$ and the proof is complete. □

Similarly, we have

Corollary 3.4 *Assume that all assumptions of Proposition 3.1 are satisfied and G has compact values. Then there exists $\bar{z} \in B$ such that*

$$G(\bar{z}) \cap PMin(G(B)/C) \neq \emptyset.$$

(This means that the general vector Pareto optimization problem concerning G, B, C has a solution).

Corollary 3.5 *If $B \subset D$ is a nonempty convex compact subset has the following property: For any $x_1, x_2 \in B, t \in [0,1]$ either $x_1 - (tx_1 + (1-t)x_2) \in C$ or, $x_2 - (tx_1 + (1-t)x_2) \in C$, Then there exist $x^*, x^{**} \in B$ such that*

$$x^{**} \succeq x \succeq x^* \quad \text{for all} \quad x \in B,$$

where $x \succeq y$ denotes $x - y \in C$.

Proof Apply Corollaries 3.3 and 3.4 with $G(z) = -z$ and then $G(z) = z$. □

Theorem 3.6 *Let D and K be nonempty convex compact subsets of Hausdorff locally convex topological vector space X and Z, respectively. Let $C \subset Y$ be a closed convex cone. Let $S : D \times K \to 2^D$ be a continuous multivalued mapping with nonempty convex closed values, $T : D \times K \to 2^K$ be an upper continuous multivalued mapping with nonempty convex closed values. Let $F : K \times D \times D \to 2^Y$ be a lower $(-C)$ and upper C-continuous mapping with nonempty closed values such that for any fixed $(x, y) \in D \times K$, the multivalued mapping $F(y, x, .) : D \to 2^Y$ is upper C-quasiconvex.*

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y},)$ and

$$F(\bar{y}, \bar{x}, x) \subset F(\bar{y}, \bar{x}, \bar{x}) + C, \quad \text{for all} \quad x \in S(\bar{x}, \bar{y}). \tag{2}$$

Proof We define the multivalued mapping $M : D \times K \to 2^D$ by

$$M(x, y) = \{x' \in S(x, y) | F(y, x, z) \subset F(y, x, x') + C, \text{for all} \quad z \in S(x, y)\}, (x, y) \in D \times K.$$

For any fixed $(x, y) \in D \times K$, we apply Proposition 3.1 with $B = S(x, y)$ and $G(z) = F(y, x, z)$ to show that there exists $\bar{z} \in B$ with

$$F(y, x, z) \subset F(y, x, \bar{z}) + C, \quad \text{for all} \quad z \in S(x, y).$$

This implies $\bar{z} \in M(x, y)$ and therefore $M(x, y)$ is nonempty. Now, we prove that $M(x, y)$ is convex. Indeed, for any $x_1, x_2 \in M(x, y)$ and $t \in [0, 1]$, we have from the convexity of $S(x, y)$, $tx_1 + (1-t)x_2 \in S(x, y)$ and

$$F(y, x, z) \subset F(y, x, x_1) + C, \quad \text{for all} \quad z \in S(x, y),$$
$$F(y, x, z) \subset F(y, x, x_2) + C, \quad \text{for all} \quad z \in S(x, y).$$

Since $F(y, x, .)$ is upper C-quasiconvex, we then conclude

$$F(y, x, z) \subset F(y, x, tx_1 + (1-t)x_2) + C, \quad \text{for all} \quad t \in [0, 1], z \in S(x, y).$$

This shows $tx_1 + (1-t)x_2 \in M(x, y)$ and $M(x, y)$ is a convex set.

Further,we claim that M is a closed multivalued mapping. Indeed, assume that $x_\beta \to x, y_\beta \to y, x'_\beta \in M(y_\beta, x_\beta), x'_\beta \to x'$. We have to show $x' \in M(x, y)$. Since $x'_\beta \in S(x_\beta, y_\beta)$, the upper continuity of S with closed values implies $z' \in S(x, y)$. For $z_\beta \in M(x_\beta, y_\beta)$, one can see

$$F(y_\beta, x_\beta, z) \subset F(y_\beta, x_\beta, x'_\beta) + C, \quad \text{for all} \quad z \in S(x_\beta, y_\beta). \tag{3}$$

The lower continuity of S and $x_\beta \to x, y_\beta \to y$ implies that for any $z \in S(x, y)$ there exist $z_\beta \in S(x_\beta, y_\beta), z_\beta \to z$ and (2) gives

$$F(y_\beta, x_\beta, z_\beta) \subset F(y_\beta, x_\beta, x'_\beta) + C, \quad \text{for all} \quad z_\beta \in S(x_\beta, y_\beta). \tag{4}$$

Since $(y_\beta, x_\beta, z_\beta) \to (y, x, z)$ and F is lower $(-C)$-continuous at (y, x, z), for any neighborhood V of the origin in Y, there is β_1 such that

$$F(y, x, z) \subset F(y_\beta, x_\beta, z_\beta) + V + C, \quad \text{for all} \quad \beta \geq \beta_1. \tag{5}$$

Since $(y_\beta, x_\beta, x'_\beta) \to (y, x, x)$ and F is upper (C)-continuous at (y, x, x'), there exists β_2 such that

$$F(y_\beta, x_\beta, x'_\beta) \subset F(y, x, x') + V + C, \quad \text{for all} \quad \beta \geq \beta_2. \tag{6}$$

Setting $\beta_0 = max\{\beta_1, \beta_2\}$, the combination of (3), (4) and (5) yields

$$F(y, x, z) \subset F(y, x, x') + 2V + C, \quad \text{for all} \quad z \in S(x, y).$$

The closeness of C and the closed values of F show

$$F(y, x, z) \subset F(y, x, x') + C, \quad \text{for all} \quad z \in S(x, y).$$

This means that $x' \in M(x, y)$ and then M is a closed multivalued mapping.

Lastly, we define the multivalued mapping $P : D \times K \to 2^{D\times K}$ by

$$P(x, y) = M(x, y) \times T(x, y), \quad (x, y) \in D \times K.$$

We can easily verify that $P(x, y) \neq \emptyset, P(x, y)$ is convex for all $(x, y) \in D \times K$ and P is a closed multivalued mapping with closed values. Moreover, since $P(D \times K) \subset M(D \times K) \times T(D \times K) \subset D \times K$, then it follows from Proposition 2.4 that P is also a compact upper continuous multivalued mapping with nonempty closed convex values. Applying the fixed point theorem of Kakutani type (see for example, in Kakutani (1941)), we conclude that there exists a point $(\bar{x}, \bar{y}) \in D \times K$ with $(\bar{x}, \bar{y}) \in M(\bar{x}, \bar{y}) \times T(\bar{x}, \bar{y})$. This implies $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ and

$$F(\bar{y}, \bar{x}, x) \subset F(\bar{y}, \bar{x}, \bar{x}) + C, \quad \text{for all} \quad x \in S(\bar{x}, \bar{y}),$$

and so the proof of the theorem is complete. □

Theorem 3.7 *Let D, K, S, T be the same as in Theorem 3.6. Let $F : K \times D \times D \to 2^Y$ be a lower C and upper $(-C)$-continuous mapping with nonempty closed values such that for any fixed $(x, y) \in D \times K$, the multivalued mapping $F(y, x, .) : D \to 2^Y$ is lower C-quasiconvex*

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ such that $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$ and

$$F(\bar{y}, \bar{x}, \bar{x}) \subset F(\bar{y}, \bar{x}, x) - C, \quad \textit{for all} \quad x \in S(\bar{x}, \bar{y}).$$

Proof We define the multivalued mapping $M : D \times K \to 2^D$ by

$$M(x, y) = \{x' \in S(x, y) | \quad F(y, x, x') \subset F(z, x, z) - C, \quad \text{for all} \quad z \in S(x, y)\}$$

and then proceed the proof exactly as the one of the above theorem. □

Theorem 3.8 *Let D and K be nonempty convex compact subsets of Hausdorff locally convex topological vector space X and Z, respectively. Let $C \subset Y$ be a closed convex cone. Let $S : D \times K \to 2^D$ be an upper continuous multivalued mapping with nonempty convex closed values, $T : D \times K \to 2^K$ be a continuous multivalued mapping with nonempty convex closed values. Let $F : D \times K \times K \to 2^Y$ be a lower $(-C)$ and upper C-continuous mapping with nonempty closed values such that for any fixed $(x, y) \in D \times K$, the multivalued mapping $F(x, y, .) : K \to 2^Y$ is upper C-quasiconvex.*

Then there exists $(\bar{x},\bar{y}) \in D \times K$ *such that* $\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y},)$ *and*

$$F(\bar{x},\bar{y},y) \subset F(\bar{x},\bar{y},\bar{y}) + C, \quad \textit{for all} \quad y \in T(\bar{x},\bar{y}).$$

Proof We prove this theorem by the same method as Theorem 3.6 with the multivalued mapping $M : D \times K \to K$ defined by

$$M(x,y) = \{y' \in T(x,y)| \quad F(x,y,z) \subset F(x,y,y') + C, \quad \text{for all} \quad z \in T(x,y)\}.$$

□

Theorem 3.9 *Let D and K be nonempty convex compact subsets of Hausdorff locally convex topological vector space X and Z, respectively. Let* $C \subset Y$ *be a closed convex cone. Let* $S : D \times K \to 2^D$ *be an upper continuous multivalued mapping with nonempty convex closed values,* $T : D \times K \to 2^K$ *be a continuous multivalued mapping with nonempty convex closed values. Let* $F : D \times K \times K \to 2^Y$ *be a lower C and upper* $(-C)$*-continuous mapping with nonempty closed values such that for any fixed* $(x,y) \in D \times K$*, the multivalued mapping* $F(x,y,.) : K \to 2^Y$ *is lower C-quasiconvex.*

Then there exists $(\bar{x},\bar{y}) \in D \times K$ *such that* $\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y},)$ *and*

$$F(\bar{x},\bar{y},\bar{y}) \subset F(\bar{x},\bar{y},y) - C, \quad \textit{for all} \quad y \in T(\bar{x},\bar{y}).$$

Proof We prove this theorem by the same manner as Theorem 3.6 with the multivalued mapping $M : D \times K \to K$ defined by

$$M(x,y) = \{y' \in T(x,y)| \quad F(x,y,y') \subset F(x,y,z) - C, \quad \text{for all} \quad z \in T(x,y)\}.$$

The following corollaries are special cases of Theorems 3.6, 3.7, 3.8 and 3.9. The proof of them imply immediately from the above theorems. But, they are used to get other results later, therefore, we also formulate them here. □

Corollary 3.10 *Let D and K be nonempty convex compact subsets of Hausdorff locally convex topological vector space X and Z, respectively. Let* $C \subset Y$ *be a closed convex cone. Let* $S : D \to 2^D$ *be a continuous multivalued mapping with nonempty convex closed values. Let* $F : D \times D \to 2^Y$ *be a lower* $(-C)$ *and upper C-continuous mapping with nonempty closed values such that for any fixed* $x \in D$*, the multivalued mapping* $F(x,.) : D \to 2^Y$ *is upper C-quasiconvex.*

Then there exists $\bar{x} \in D$ *such that* $\bar{x} \in S(\bar{x})$ *and*

$$F(\bar{x},x) \subset F(\bar{x},\bar{x}) + C, \quad \textit{for all} \quad x \in S(\bar{x}).$$

Corollary 3.11 *Let D and K be nonempty convex compact subsets of Hausdorff locally convex topological vector space X and Z, respectively. Let* $C \subset Y$ *be a closed convex cone. Let* $S : D \to 2^D$ *be a continuous multivalued mapping with nonempty convex closed values. Let* $F : D \times D \to 2^Y$ *be a upper* $(-C)$ *and lower C-continuous mapping with nonempty closed values such that for any fixed* $x \in D$*, the multivalued mapping* $F(x,.) : D \to 2^Y$ *is lower C-quasiconvex.*

Then there exists $\bar{x} \in D$ *such that* $\bar{x} \in S(\bar{x})$ *and*

$$F(\bar{x},x) \subset F(\bar{x},\bar{x}) - C, \quad \textit{for all} \quad x \in S(\bar{x}).$$

Theorem 3.12 *Let D and K be nonempty convex compact subsets of Hausdorff locally convex topological vector space X and Z, respectively. Let* $C \subset Y$ *be a closed convex cone. Let* $S : D \times K \to 2^D$ *and* $T : D \times K \to 2^K$ *be continuous multivalued mappings* *with nonempty convex closed values. Let* $F_1 : K \times D \times D \to 2^Y, F_2 : D \times K \times K \to 2^Y$ *be lower* $(-C)$ *and upper* C*-continuous mappings with nonempty closed values such that for any fixed* $(x, y) \in D \times K$*, the multivalued mapping* $G : D \times K \to 2^Y$ *defined by*

$$G(v) = F_1(y, x, x') + F_2(x, y, y'), \quad v = (x', y') \in D \times K$$

is upper C*-quasiconvex .*

Then there exists $(\bar{x}, \bar{y}) \in D \times K$ *such that* $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y},)$ *and*

$$F_1(\bar{y}, \bar{x}, x) + F_2(\bar{x}, \bar{y}, y) \subset F_1(\bar{y}, \bar{x}, \bar{x}) + F_2(\bar{x}, \bar{y}, \bar{y}) + C, \textit{ for all } \quad x \in S(\bar{x}, \bar{y}), y \in T(\bar{x}, \bar{y}).$$

Proof Setting $\tilde{D} = D \times K$, we define the multivalued mappings $\tilde{S} : \tilde{D} \to 2^{\tilde{D}}$ by $\tilde{S}(u) = S(x, y) \times T(x, y)$, and $\tilde{F} : \tilde{D} \times \tilde{D} \to 2^Y$ by

$$\tilde{F}(u, v) = F_1(x, y, x') + F_2(x, y, y'), \quad u = (x, y), v = (x', y') \in \tilde{D}.$$

It is easily to verify that $\tilde{F}, \tilde{D}, \tilde{K}, \tilde{S}$ satisfy all conditions of Corollary 3.10. Applying this corollary, we con conclude that there exists $\tilde{u} \in \tilde{D}$ such that $\tilde{u} \in \tilde{S}(\tilde{u})$ and

$$\tilde{F}(\tilde{u}, u) \subset \tilde{F}(\bar{u}, \tilde{u}) + C, \quad \text{for all} \quad u \in \tilde{S}(\tilde{u}).$$

Say $\tilde{u} = (\bar{x}, \bar{y})$, then it follows from the definitions of $\tilde{S}, \tilde{F}$ that $(\bar{y}, \bar{x}) \in D \times K$ such that $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{y}, \bar{x},)$ and

$$F_1(\bar{y}, \bar{x}, x) + F_2(\bar{x}, \bar{y}, y) \subset F_1(\bar{y}, \bar{x}, \bar{x}) + F_2(\bar{x}, \bar{y}, \bar{y}) + C, \text{ for all } \quad x \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{y}, \bar{x}).$$

This completes the proof of the theorem. □

Analogically, we have

Theorem 3.13 *Let* D *and* K *be nonempty convex compact subsets of Hausdorff locally convex topological vector space* X *and* Z*, respectively. Let* $C \subset Y$ *be a closed convex cone. Let* $S : D \times K \to 2^D$ *and* $T : D \times K \to 2^K$ *be continuous multivalued mappings with nonempty convex closed values. Let* $F_1 : K \times D \times D \to 2^Y, F_2 : D \times K \times K \to 2^Y$ *be upper* $(-C)$ *and lower* C*-continuous mappings with nonempty closed values such that for any fixed* $(x, y) \in D \times K$*, the multivalued mapping* $G : D \times K \to 2^Y$ *defined by*

$$G(v) = F_1(y, x, x') + F_2(x, y, y'), \quad v = (x', y') \in D \times K$$

is lower C*-quasiconvex .*

Then there exists $(\bar{y}, \bar{x}) \in D \times K$ *such that* $\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y},)$ *and*

$$F_1(\bar{y}, \bar{x}, \bar{x}) + F_2(\bar{x}, \bar{y}, \bar{y}) \subset F_1(\bar{y}, \bar{x}, x) + F_2(\bar{x}, \bar{y}, y) - C, \textit{ for all } \quad x \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y}).$$

Proof The proof is similar as the one of Theorem 3.13. □

Corollary 3.14 *Let* D, K, C, S, T *and* F *be as in Theorem 3.6. In addition, assume that* $F(y, x, x) \subset C$ *for all* $(x, y) \in D \times K$*. Then Problem* $(UIQEP)$ *has a solution.*

Proof It is obvious. □

Corollary 3.15 *Let* D, K, S, T *and* F *be as in Theorem 3.6 and* $IMin(F(y, x, x)/C) \neq \emptyset$ *for all* $(x, y) \in D \times K$*. Then* $(\bar{x}, \bar{y})$ *satisfies (2) if and only if it is a solution of* $(GVIOP)_I$*.*

Proof First we assume that $(\bar{x}, \bar{y})$ satisfies (2). Take $v^* \in \mathrm{IMin}(F(\bar{y}, \bar{x}, \bar{x})/C)$. It is clear that $F(\bar{y}, \bar{x}, \bar{x}) \subset v^* + C$. Together with (2) we have

$$F(\bar{x}, \bar{y}, x) \subset F(\bar{y}, \bar{x}, \bar{x}) + C \subset v^* + C \quad \text{for all} \quad x \in S(\bar{x}, \bar{y}).$$

This follows $v^* \in \mathrm{IMin}(F(\bar{x}, \bar{y}, S(\bar{x}, \bar{y})/C)$ and hence

$$F(\bar{y}, \bar{x}, \bar{x}) \cap \mathrm{IMin}(F(\bar{x}, \bar{y}, S(\bar{x}, \bar{y}))/C) \neq \emptyset. \tag{7}$$

This shows that $(\bar{x}, \bar{y})$ is a solution of $(GVQOP)_I$. Now, assume that (6) holds. Take $v^* \in F(\bar{y}, \bar{x}, \bar{x}) \cap \mathrm{IMin}(F(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C)$, we have

$$F(\bar{y}, \bar{x}, x) \subset v^* + C \subset F(\bar{y}, \bar{x}, \bar{x}) + C \quad \text{for all} \quad x \in S(\bar{x}, \bar{y}).$$

This means that $(\bar{x}, \bar{y})$ satisfies (2). □

Corollary 3.16 *Let D, K, C, S, T and F be as in Theorem 3.6. In addition, assume that there exists a convex cone $\tilde{C}$ which is not the whole space and contains $C\backslash\{0\}$ in its interior. Then the problem $(GVQOP)_{Pr}$ has a solution.*

Proof Since C has the fore mentioned property as above, then any compact set A in Y has $\mathrm{PrMin}(A/C) \neq \emptyset$ (by using the cone $C^* = \{0\} \cup \mathrm{int}\tilde{C}$ one can verify $\mathrm{PMin}(A/C^*) \neq \emptyset$, see, for example, Corollary 3.15, Chapter 2 in Ref. 10). We then apply Theorem 3.6 to obtain $(\bar{x}, \bar{y}) \in D \times K$ such that:

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$$

and

$$F(\bar{y}, \bar{x}, x) \subset F(\bar{y}, \bar{x}, \bar{x}) + C, \quad \text{for all} \quad x \in S(\bar{x}, \bar{y}). \tag{8}$$

The compactness of $F(\bar{y}, \bar{x}, \bar{x})$ implies

$$\mathrm{PrMin}(F(\bar{y}, \bar{x}, \bar{x})/C) \neq \emptyset.$$

Take $\bar{v} \in \mathrm{PrMin}(F(\bar{y}, \bar{x}, \bar{x})/C)$, we show that $\bar{v} \in \mathrm{PrMin}(F(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C)$. By contrary, we suppose that $\bar{v} \notin \mathrm{PrMin}(F(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C)$. Then, there is $v^* \in F(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))$ such that

$$\bar{v} - v^* \in C^* \backslash l(C^*). \tag{9}$$

Assume that $v^* \in F(\bar{y}, \bar{x}, x^*)$, for some $x^* \in S(\bar{x}, \bar{y})$. We can conclude from (7) that there exists $v^o \in F(\bar{y}, \bar{x}, \bar{x})$ such that $v^* - v^o = c \in C$. If $c = 0$, then $v^* = v^o$ and then $\bar{v} - v^o \in C^* \backslash l(C^*)$. If $c \neq 0$, using (8), we conclude

$$\bar{v} - v^o = \bar{v} - v^* + v^* - v^o \in C^* \backslash l(C^*) + C \backslash \{0\} \subset C^* \backslash l(C^*).$$

Therefore, we obtain $\bar{v} - v^o \in C^* \backslash l(C^*)$. Due to $\bar{v} \in \mathrm{PrMin}(F(\bar{y}, \bar{x}, \bar{x})/C)$ and $v^o \in F(\bar{y}, \bar{x}, \bar{x})$, we then have a contradiction. Consequently,

$$F(\bar{y}, \bar{x}, \bar{x}) \cap \mathrm{PrMin}(F(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C) \neq \emptyset$$

and $(\bar{x}, \bar{y})$ is a solution of the problem $(GVQOP)_{Pr}$. □

Corollary 3.17 *If D, K, C, S, T, F are as in Theorem 3.6 and $F(y, x, x)$ is compact for all $(x, y) \in D \times K$, then the problem $(GVQOP)_P$ has a solution.*

Proof By Theorem 3.6, there is $(\bar{x}, \bar{y}) \in D \times K$ such that:

$$\bar{x} \in S(\bar{x}, \bar{y}), \bar{y} \in T(\bar{x}, \bar{y})$$

and

$$F(\bar{y}, \bar{x}, x) \subset F(\bar{y}, \bar{x}, \bar{x}) + C, \quad \text{for all} \quad x \in S(\bar{x}, \bar{y}). \tag{10}$$

We claim that

$$F(\bar{y}, \bar{x}, \bar{x}) \cap \mathrm{PMin}(F(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C) \neq \emptyset.$$

The compactness of $F(\bar{y}, \bar{x}, \bar{x})$ implies

$$\mathrm{PMin}(F(\bar{y}, \bar{x}, \bar{x}))/C) \neq \emptyset.$$

Assume $\bar{v} \in \mathrm{PMin}(F(\bar{y}, \bar{x}, \bar{x}))/C)$ and $\bar{v} \notin \mathrm{PMin}(F(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C)$. It implies that there is $v \in F(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))$, say $v \in F(\bar{y}, \bar{x}, x)$ with some $x \in S(\bar{x}, \bar{y})$, such that

$$\bar{v} - v \in C \backslash l(C) \tag{11}$$

(9) implies that $v \in F(\bar{y}, \bar{x}, \bar{x}) + C$ and so

$$v = v^* + c, \quad \text{with some} \quad v^* \in F(\bar{y}, \bar{x}, \bar{x}), c \in C,$$

or

$$v - v^* \in C. \tag{12}$$

A combination of (10) and (11) gives

$$\bar{v} - v^* = \bar{v} - v + v - v^* \in C \backslash l(C) + C \subset C \backslash l(C).$$

This contradicts $\bar{v} \in \mathrm{PMin}(F(\bar{y}, \bar{x}, \bar{x})/C)$. Therefore, we obtain

$$F(\bar{y}, \bar{x}, \bar{x}) \cap \mathrm{PMin}(F(\bar{y}, \bar{x}, S(\bar{x}, \bar{y}))/C) \neq \emptyset.$$

This completes the proof of the corollary. □

Corollary 3.18 *Let D, K, C, S, T and F be as in Theorem 3.7. In addition, assume that $F(y, x, x) \cap C \neq \emptyset$ for all $(x, y) \in D \times K$. Then the problem $(LIQEP)$ has a solution.*

Proof It follows immediately from the fact that

$$F(\bar{y}, \bar{x}, \bar{x}) \subset F(\bar{y}, \bar{x}, x) - C, \quad \text{for all} \quad x \in S(\bar{x}, \bar{y})$$

and $F(\bar{y}, \bar{x}, \bar{x}) \cap C \neq \emptyset$,we show that $F(\bar{y}, \bar{x}, x) \cap C \neq \emptyset$ for all $x \in S(\bar{x}, \bar{y})$. □

Corollary 3.19 *Let D, K, C, S, T and F be as in Theorem 3.4. If $(\bar{x}, \bar{y})$ is a solution of the problem $(LIQEP)$, then it is also a solution of the following Pareto quasi-equilibrium problem: Find $(\bar{x}, \bar{y}) \in D \times K$ such that*

$$\bar{x} \in S(\bar{x}, \bar{y}),$$

$$(PQEP) \quad \bar{y} \in T(\bar{x}, \bar{y}),$$

$$F(\bar{y}, \bar{x}, x) \not\subset -(C \backslash l(C)), \quad \textit{for all} \quad x \in S(\bar{x}, \bar{y}).$$

Proof Indeed, on the contrary we assume that there is $x^* \in S(\bar{x}, \bar{y})$ such that $F(\bar{y}, \bar{x}, x^*) \subset -(C \backslash l(C))$. Since $F(\bar{y}, \bar{x}, x^*) \cap C \neq \emptyset$, we can take $v^* \in F(\bar{y}, \bar{x}, x^*) \cap C$. This yields $v^* \in C \cap (-(C \backslash l(C)) \subset -l(C), v^* \in F(\bar{y}, \bar{x}, x^*) \subset -(C \backslash l(C))$. It is impossible, because of $v^* \in -l(C)$. This completes the proof of the corollary. □

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