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The study of the system of generalized vector quasi-equilibrium problems

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Abstract In this paper, we study the system of generalized vector quasi-equilibrium problems, which includes as special cases the system of vector quasi-equilibrium problems and the system of generalized vector equilibrium problems, and establish the existence and essential components of the solution set under perturbations of its best-reply map. Moreover, we also derive a new existence theorem of Ky Fan's points for a set-valued map.

Keywords The system of generalized vector quasi-equilibrium problems · Best-reply map · Upper *C*-semicontinuous · *C*-quasiconvex-pseudo · Essential component

1 Introduction

The system of generalized vector quasi-equilibrium problems (briefly, SGVQEP) includes as special cases the system of vector quasi-equilibrium problems (briefly, SVQEP) and the system of generalized vector equilibrium problems (briefly, SGVEP). Recently, the study with respect to the SGVQEP has attracted much attention. For existence results of solutions in this direction, we refer to Wu and Shen (1996), Yu and Yuan (1998), Deguire et al. (1999), Ansari et al. (2002), Yu (2003), Wu and Yuan (2003) and reference therein.

Essential component plays a important role in the study of stability. In 1950, Fort introduced the notion of essential fixed points of a continuous map. In 1952, Kinoshita introduced the notion of essential components of the set of fixed points of single-valued map. In 1963, Jiang introduced the notion of essential components of the set of Nash equilibrium points for n-person noncooperative game and proved the existence of essential components of the set of Nash equilibrium points. In 1986, Kohlberg and

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Mertens studied the stability of Nash equilibrium points and suggested that a satisfactory solution for a noncooperative game should be set-wise, and they proved that such a solution is just an essential component of Nash equilibrium points. In 1990, Hillas proposed another version of stability of Nash equilibrium points and established the stability results of the set of Nash equilibrium points under perturbations of its best-reply map for game problems. For other results of essential components in this direction, we refer to Yu and Luo (1999), Yu and Xiang (1999), Yang and Yu (2002) and reference therein.

In this paper, we study the SGVQEP and establish the existence and essential components of the solution set under perturbations of its best-reply map. Moreover, we also derive a new existence theorem of Ky Fan's points for a set-valued map. Our results are new and differ from those results in the literatures.

2 Preliminaries

Let C be a cone of a topological vector space Y. C is convex if and only if $C+C=C$, and pointed if and only if $C\cap(-C)=\{\theta\}$, where θ denotes the zero element of Y. Denote by 2^Y the family of all nonempty subset of Y.

Definition 1 Let X and Y be two topological vector spaces and K a nonempty convex subset of X, and f: $K\to Y$ be a vector-valued function. f is called C-continuous at $x_0\in K$ if, for any open neighborhood V of the zero element θ in Y, there exists an open neighborhood U of x_0 in K such that, for all $x\in U$,

$$f(x)\in f(x_0)+V+C$$

and C-continuous on K if it is C-continuous at every point of K.

Definition 2 Let X and Y be two topological vector spaces and K a nonempty convex subset of X, and F: $K\to 2^Y$ be a set-valued map.

(1) F is called upper C-semicontinuous at $x_0\in K$ if, for any open neighborhood V of the zero element θ in Y, there exists an open neighborhood U of x_0 in K such that, for all $x\in U$,

$$F(x)\subset F(x_0)+V+C$$

and upper C-semicontinuous on K if it is upper C-semicontinuous at every point of K;

(2) F is called lower C-semicontinuous at $x_0\in K$ if, for any open neighborhood V of the zero element θ in Y, there exists an open neighborhood U of x_0 in K such that, for all $x\in U$,

$$F(x)\cap(F(x_0)+V+C)\neq\emptyset$$

and lower C-semicontinuous on K if it is lower C-semicontinuous at every point of K;

(3) F is called C-continuous at $x_0\in K$ if, it is upper C-semicontinuous and lower C-semicontinuous at $x_0\in K$; and C-continuous on K if it is C-continuous at every point of K.

When F is a vector-valued function, we use symbol $\in$ instead of symbol $\subset$. In this case, both of upper C-semicontinuous and lower C-semicontinuous coincide with C-continuous.

Let X be a topological vector space and K a nonempty, convex and compact subset of X. Denote by M the set of all upper semicontinuous maps from K to 2^K with convex compact values. For any $F, G \in M$, we define

$$\rho(F, G) = \sup_{x \in K} h(F(x), G(x)),$$

where h is the Hausdorff metric defined on X. It is easy to verify that (M, ρ) is a metric space.

For each $F \in M$, we denote by $S(F)$ the set of all fixed points of F. By Kakutani-Fan-Glicksberg's fixed points Theorem (see Aliprantis and Border 1999, pp. 550), $S(F)$ is a nonempty compact set.

Definition 3 For each $F \in M$, the component of a point $x \in S(F)$ is the union of all connected subsets of $S(F)$ containing x.

Note that the components are connected closed subsets of $S(F)$ (see Engelking 1989, pp. 356), thus they are connected and compact. Since the components of two distinct points of $S(F)$ either coincide or are disjoint, the components of $S(F)$ form a decomposition as

$$S(F) = \bigcup_{\alpha \in \Lambda} S_\alpha,$$

where Λ is an index set and for any $\alpha \in \Lambda, S_\alpha$ is a nonempty connected compact subset of $S(F)$ and, for any $\alpha, \beta \in \Lambda, \alpha \neq \beta, S_\alpha \cap S_\beta = \emptyset$.

Definition 4 For each $F \in M$, let A be a nonempty closed subset of $S(F)$. A is said to be an essential set of $S(F)$ with respect to M if, for each open set $O \supset A$, there exists an open neighborhood U of F in M such that $S(F') \cap O \neq \emptyset$ whenever $F' \in U$. If a component S_α of $S(F)$ is an essential set with respect to M, then S_α is said to be an *essential component* of $S(F)$ with respect to M.

The following result can be found in Jiang (1963).

Lemma 1 *For any $F \in M$, there is at least one essential component of $S(F)$ with respect to M.*

The following result is a basic fact and its proof can be found in Yang and Yu (2002).

Lemma 2 *Let Y be a Banach space with a closed, convex, and pointed cone C with $intC \neq \emptyset$, where $intC$ denotes the interior of C. Then we have $intC + C \subset intC$.*

The following result is a particular form of a maximal element theorem for a family of set-valued maps due to Deguire et al. (1999, Theorem 1).

Lemma 3 *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Suppose that $A\colon K \to 2^K \cup \{\emptyset\}$ is a set-valued map with following conditions:*

(1) *for each* $x \in K$, $A(x)$ *is convex;*
(2) *for each* $x \in K$, $x \notin A(x)$*;*
(3) *for each* $y \in K$, $A^{-1}(y) = \{x \in K : y \in A(x)\}$ *is open in* K.

Then there exists $\overline{x} \in K$ *such that* $A(\overline{x}) = \emptyset$.

Throughout this paper, unless otherwise specified, assume that the index I has at least two element. For each $i \in I$, let X_i and Y_i be two Banach spaces and K_i a nonempty convex compact subset of X_i. For each $i \in I$, let C_i be a closed, convex and pointed cone of Y_i with $intC_i \neq \emptyset$, where $intC_i$ denotes the interior of C_i. Denote by 2^{K_i} the family of all nonempty subsets of K_i.

Denote that $K_{\hat{\imath}} = \prod_{j \in I, j \neq i} K_j, K = \prod_{i \in I} K_i = K_i \times K_{\hat{\imath}},\ X = \prod_{i \in I} X_i$, where the product space X is a Tychonoff product space. For each $x \in K$, we can write $x = (x_i, x_{\hat{\imath}})$. For each $i \in I$, let G_i: $K_{\hat{\imath}} \to 2^{K_i}$ and F_i: $K_i \times K_{\hat{\imath}} \times K_i \to 2^{Y_i}$ be two set-valued maps. The system of generalized vector quasi-equilibrium problems is: find $\overline{x} = (\overline{x}_i, \overline{x}_{\hat{\imath}}) \in K$ such that for each $i \in I$,

$$\overline{x}_i \in G_i(\overline{x}_{\hat{\imath}}) \quad \text{and} \quad F_i(\overline{x}_i, \overline{x}_{\hat{\imath}}, y_i) \not\subset -intC_i \quad \text{for all } y_i \in G_i(\overline{x}_{\hat{\imath}}),$$

where $\overline{x} = (\overline{x}_i, \overline{x}_{\hat{\imath}})$ is said to be a solution of the SGVQEP. A SGVQEP is denoted by $\{K_i, G_i, F_i\}_{i \in I}$ (briefly, (G, F)).

If $F_i = \varphi_i$ is a vector-valued function for each $i \in I$, then the SGVQEP coincides with the SVQEP. A SVQEP is usually denoted by $\{K_i, G_i, \varphi_i\}_{i \in I}$ (briefly, (G, φ)).

If setting $G_i(x_{\hat{\imath}}) = K_i$ for each $i \in I$ and each $x_{\hat{\imath}} \in X_{\hat{\imath}}$, then the SGVQEP coincides with the SGVEP, which has been studied in Ansari et al. (2002). A SGVEP is usually denoted by $\{K_i, F_i\}_{i \in I}$ (briefly, F).

The SVQEP includes as a special case the following multiobjective generalized game problems:

For each $i \in I$, let f_i: $K \to Y_i$ be a vector-valued function and let G_i: $K_{\hat{\imath}} \to 2^{K_i}$ be a feasible strategy map. The multiobjective generalized game problem is: find $(\overline{x}_i, \overline{x}_{\hat{\imath}}) \in K$ such that for each $i \in I, \overline{x}_i \in G_i(\overline{x}_{\hat{\imath}})$,

$$f_i(y_i, \overline{x}_{\hat{\imath}}) - f_i(\overline{x}_i, \overline{x}_{\hat{\imath}}) \notin -intC_i \quad \text{for all } y_i \in G_i(\overline{x}_{\hat{\imath}}),$$

where $\overline{x}$ is said to be a weakly Pareto–Nash equilibrium point.

For each $i \in I$, setting

$$\varphi_i(x_i, x_{\hat{\imath}}, y_i) = f_i(y_i, x_{\hat{\imath}}) - f_i(x_i, x_{\hat{\imath}})$$

the SVQEP coincides with the multiobjective generalized game problem, which has been studied by Yu and Luo (1999) but for real function. A multiobjective generalized game problem is usually denoted by $\{K_i, G_i, f_i\}_{i \in I}$ (briefly, (G, f)).

For each $i \in I$, setting $G_i(x_{\hat{\imath}}) = K_i$, the multiobjective generalized game problem coincides with the multiobjective game problem, which has been studied in Yu and Xiang (1999) and Yang and Yu (2002). Note that the SGVEP includes as a special case multiobjective game problems.

3 Existence and essential components

We first establish the existence of solutions for the SGVQEP.

Definition 5 Let X and Y be two topological vector spaces and K a nonempty convex subset of X and C a closed, convex and pointed cone of Y with $intC \neq \emptyset$. Let F: $K \to 2^Y$ be a set-valued map.

(1) F is called C-convex if, for any $x_1, x_2 \in K$ and each $t \in [0,1]$,

$$F(tx_1 + (1-t)x_2) \subset [tF(x_1) + (1-t)F(x_2)] - C$$

and C-concave if $-F$ is C-convex;

(2) F is called C-quasiconvex-pseudo if, for any $x_1, x_2 \in K$ and each $t \in [0,1]$,

$$\text{either} \quad F(x_1) \subset F(tx_1 + (1-t)x_2) + C \quad \text{or} \quad F(x_2) \subset F(tx_1 + (1-t)x_2) + C$$

and C-quasiconcave-pseudo if $-F$ is C-quasiconvex-pseudo.

Remark 1 In particular, if $Y = R$ and $C = R_+ = [0, +\infty)$, then C-convexity and C-quasiconvexity-pseudo is equivalent to the convexity and the quasiconvexity, respectively.

Example 1 Let $N = \{1,2\}, X = [-2,-1], Y = R^2, C = R^2_+ = [0,+\infty) \times [0,+\infty)$.
If $f = (f_1, f_2) = (-x, x)$, it is easy to verify that f is R^2_+- convex, but not R^2_+- quasiconvex-pseudo.
If $g = (g_1, g_2) = (\frac{1}{x}, \frac{1}{x})$, it is easy to verify that g is R^2_+- quasiconvex-pseudo, but not R^2_+- convex.

Remark 2 Example 1 shows that C-convexity does not imply C-quasiconvexity-pseudo in the general case, even though convexity does imply quasiconvexity.

For the SGVQEP $\{K_i, G_i, F_i\}_{i \in I}$, we define its best-reply map H: $K \to 2^K \cup \{\emptyset\}$ by $H(x) = \prod_{i \in I} H_i(x_{\hat{i}})$, where

$$H_i(x_{\hat{i}}) = \{z_i \in G_i(x_{\hat{i}}) : F_i(z_i, x_{\hat{i}}, y_i) \not\subset -intC_i \text{ for all } y_i \in G_i(x_{\hat{i}})\}. \tag{1}$$

Clearly, x is a solution of the SGVQEP if and only if x is a fixed point of H, where H_i is defined by (1). Denote by $S(H)$ the set of all fixed points of H.

Theorem 1 *Consider a SGVQEP $\{K_i, G_i, F_i\}_{i \in I}$. For each $i \in I$, assume that*

(1) *G_i is continuous on $K_{\hat{i}}$ with convex compact values;*
(2) *$F_i(\cdot,\cdot,\cdot)$ is upper $-C_i$ semicontinuous on $K_i \times K_{\hat{i}} \times K_i$ with compact values;*
(3) *for each $(x_i, x_{\hat{i}}) \in K_i \times K_{\hat{i}}$, $F_i(x_i, x_{\hat{i}}, \cdot)$ is C_i- convex;*
(4) *for each $(x_{\hat{i}}, y_i) \in K_{\hat{i}} \times K_i$, $F_i(\cdot, x_{\hat{i}}, y_i)$ is $-C_i$ quasiconvex-pseudo;*
(5) *for each $(x_i, x_{\hat{i}}) \in K_i \times K_{\hat{i}}$, if $x_i \in G_i(x_{\hat{i}})$, then $F_i(x_i, x_{\hat{i}}, x_i) \not\subset -intC_i$.*

Then the SGVQEP has a solution.

Proof Define the best-reply map $H(x) = \prod_{i \in I} H_i(x_{\hat{i}})$, where H_i is defined by (1). For each $i \in I$,

(1) for each $x_{\hat{i}} \in K_{\hat{i}}$, define a set-valued map A_i: $G_i(x_{\hat{i}}) \to 2^{G_i(x_{\hat{i}})} \cup \{\emptyset\}$ by

$$A_i(x_i) = \{y_i \in G_i(x_{\hat{i}}) : F_i(x_i, x_{\hat{i}}, y_i) \subset -intC_i\} \text{ for each } x_i \in G_i(x_{\hat{i}}).$$

  (a) For any $x_i \in G_i(x_{\hat{i}})$, Lemma 2 and the condition (3) and the convexity of $G_i(x_{\hat{i}})$ imply that $A_i(x_i)$ is convex.
  (b) For any $x_i \in G_i(x_{\hat{i}})$, the condition (5) implies that $x_i \notin A_i(x_i)$.

(c) For any $y_i \in G_i(x_{\hat{i}})$, the condition (2) implies that the set $A_i^{-1}(y_i) = \{x_i \in G_i(x_{\hat{i}}): y_i \in A_i(x_i)\} = \{x_i \in G_i(x_{\hat{i}}): F_i(x_i, x_{\hat{i}}, y_i) \subset -intC_i\}$ is open in $G_i(x_{\hat{i}})$.

By Lemma 3, there exists a $\overline{x}_i \in G_i(x_{\hat{i}})$ such that $A_i(\overline{x}_i) = \emptyset$, i.e., $H_i(x_{\hat{i}}) \neq \emptyset$.

(2) For each $x_{\hat{i}} \in K_{\hat{i}}$, next we verify that $H_i(x_{\hat{i}})$ is convex.
For any $z_i^1, z_i^2 \in H_i(x_{\hat{i}})$ and any $t \in [0,1]$, the convexity of $G_i(x_{\hat{i}})$ imply that $tz_i^1 + (1-t)z_i^2 \in G_i(x_{\hat{i}})$. By condition (4), assume without loss of generality that $F_i(z_i^1, x_{\hat{i}}, y_i) \subset F_i(tz_i^1 + (1-t)z_i^2, x_{\hat{i}}, y_i) - C_i$. If $tz_i^1 + (1-t)z_i^2 \notin H_i(x_{\hat{i}})$, then there exists a $y_i^0 \in G_i(x_{\hat{i}})$ such that $F_i(tz_i^1 + (1-t)z_i^2, x_{\hat{i}}, y_i^0) \subset -intC_i$. We have $F_i(z_i^1, x_{\hat{i}}, y_i^0) \subset F_i(tz_i^1 + (1-t)z_i^2, x_{\hat{i}}, y_i^0) - C_i \subset -C_i - intC_i \subset -intC_i$, a contradiction. Thus $H_i(x_{\hat{i}})$ is convex.

(3) Now we verify that H_i is upper semicontinuous on $K_{\hat{i}}$ with compact values. By Theorem 7.16 in Klein and Thompson (1984, pp. 78), it suffices to show that the Graph(H_i) is closed in K, where

$$\text{Graph}(H_i) = \{(z_i, x_{\hat{i}}) \in K: z_i \in H_i(x_{\hat{i}})\}.$$

Let $(z_i^n, x_{\hat{i}}^n)$ be any sequence in Graph(H_i) with $(z_i^n, x_{\hat{i}}^n) \to (z_i^0, x_{\hat{i}}^0)$. The condition (1) implies that $z_i^0 \in G_i(x_{\hat{i}}^0)$. If $z_i^0 \notin H_i(x_{\hat{i}}^0)$, there exist $y_i^0 \in G_i(x_{\hat{i}}^0)$ such that $F_i(z_i^0, x_{\hat{i}}^0, y_i^0) \subset -intC_i$, which implies that there exists an open neighborhood V_i of the zero element θ_i such that

$$F_i(z_i^0, x_{\hat{i}}^0, y_i^0) + V_i \subset -intC_i.$$

By the condition (2), there exists an open neighborhood $U(z_i^0, x_{\hat{i}}^0, y_i^0)$ of $(z_i^0, x_{\hat{i}}^0, y_i^0)$ such that

$$F_i(z_i', x_{\hat{i}}', y_i') \subset F_i(z_i^0, x_{\hat{i}}^0, y_i^0) + V_i - C_i \subset -intC_i - C_i \subset -intC_i,$$

whenever $(z_i', x_{\hat{i}}', y_i') \in U(z_i^0, x_{\hat{i}}^0, y_i^0)$. By condition (1), there exist $y_i^n \in G_i(x_{\hat{i}}^n)$ with $y_i^n \to y_i^0$. Thus there exists a positive integer N such that $(z_i^n, x_{\hat{i}}^n, y_i^n) \in U(z_i^0, x_{\hat{i}}^0, y_i^0)$ whenever $n > N$, which implies that

$$F_i(z_i^n, x_{\hat{i}}^n, y_i^n) \subset -intC_i$$

whenever $n > N$, a contradiction.

Thus, by Tychonoff Product Theorem and Theorem 7.3.14 in Klein and Thompson 1984, pp. 88, the best-reply map H is upper semicontinuous with nonempty, convex and compact values, which imply the best-reply map H is closed with nonempty and convex values. By Kakutani-Fan-Glicksberg's fixed points Theorem, the result follows.

For the SVQEP, since φ_i is a vector-valued function, we have following result.

Theorem 2 *Consider a SVQEP* $\{K_i, G_i, \varphi_i\}_{i \in I}$. *For each* $i \in I$, *assume that*

(1) *G_i is continuous on $K_{\hat{i}}$ with convex compact values;*
(2) *$\varphi_i(\cdot,\cdot,\cdot)$ is $-C_i-$ continuous on $K_i \times K_{\hat{i}} \times K_i$;*
(3) *for each $(x_i, x_{\hat{i}}) \in K_i \times K_{\hat{i}}, \varphi_i(x_i, x_{\hat{i}}, \cdot)$ is C_i-convex or C_i-quasiconvex-pseudo;*
(4) *for each $(x_{\hat{i}}, y_i) \in K_{\hat{i}} \times K_i, \varphi_i(\cdot, x_{\hat{i}}, y_i)$ is $-C_i$-quasiconvex-pseudo;*
(5) *for each $(x_i, x_{\hat{i}}) \in K_i \times K_{\hat{i}}$, if $x_i \in G_i(x_{\hat{i}})$, then $\varphi_i(x_i, x_{\hat{i}}, x_i) \notin -intC_i$.*

Then the SVQEP has a solution, i.e., there exists a point $\overline{x} \in K$ such that $\overline{x}_i \in G_i(\overline{x}_{\hat{i}})$ and

$$\varphi_i(\overline{x}_i, \overline{x}_{\hat{i}}, y_i) \notin -intC_i, \quad \text{for all } y_i \in G_i(\overline{x}_{\hat{i}}).$$

The proof of Theorem 2 is completely analogous to that of Theorem 1 and is omitted.

Remark 3 The convexity of $G_i(x_{\hat{i}})$ and the condition (3) in Theorem 2 imply that $A_i(x_i) = \{y_i \in G_i(x_{\hat{i}}) : \varphi_i(x_i, x_{\hat{i}}, y_i) \in -intC_i\}$ is convex for each $x_{\hat{i}} \in K_{\hat{i}}$ and each $x_i \in G_i(x_{\hat{i}})$, but analogous statement is not true in Theorem 1. Thus, Theorem 1 does not contain Theorem 2 as a special case.

By Theorem 2, we have following result.

Corollary 1 *Consider a multiobjective generalized game problem* $\{K_i, G_i, f_i\}_{i\in I}$. *For each* $i \in I$, *assume that*

(1) *G_i is continuous on $K_{\hat{i}}$ with convex compact values;*
(2) *f_i is continuous on K;*
(3) *for each $x_{\hat{i}} \in K_{\hat{i}}, f_i(\cdot, x_{\hat{i}})$ is C_i- quasiconvex-pseudo.*

Then the multiobjective generalized game problems has a solution, i.e., there exists a point $\overline{x} \in K$ such that $\overline{x}_i \in G_i(\overline{x}_{\hat{i}})$ and

$$f_i(y_i, \overline{x}_{\hat{i}}) - f_i(\overline{x}_i, \overline{x}_{\hat{i}}) \notin -intC_i \quad \text{for all } y_i \in G_i(\overline{x}_{\hat{i}}).$$

Proof For each $i \in I$, setting

$$\varphi_i(x_i, x_{\hat{i}}, y_i) = f_i(y_i, x_{\hat{i}}) - f_i(x_i, x_{\hat{i}}),$$

it is easy to verify that the conditions of Theorem 2 hold. Hence the result follows.

Remark 4 Corollary 1 is a new existence theorem of weakly Pareto–Nash equilibrium points for the multiobjective generalized game problems.

For the SGVEP, since there has not the constraint map, by Theorem 1, we have following result.

Theorem 3 *Consider a SGVEP* $\{K_i, F_i\}_{i\in I}$. *For each* $i \in I$, *assume that*

(1) *for each $y_i \in K_i$, $F_i(\cdot, \cdot, y_i)$ is upper $-C_i$-semicontinuous on $K_i \times K_{\hat{i}}$ with compact values;*
(2) *for each $(x_i, x_{\hat{i}}) \in K_i \times K_{\hat{i}}, F_i(x_i, x_{\hat{i}}, \cdot)$ is C_i-convex;*
(3) *for each $(x_{\hat{i}}, y_i) \in K_{\hat{i}} \times K_i, F_i(\cdot, x_{\hat{i}}, y_i)$ is $-C_i$-quasiconvex-pseudo;*
(4) *for each $(x_i, x_{\hat{i}}) \in K_i \times K_{\hat{i}}$, $F_i(x_i, x_{\hat{i}}, x_i) \not\subset -intC_i$.*

Then the SGVEP has a solution, i.e., there exists a point $\overline{x} \in K$ such that

$$F_i(\overline{x}_i, x_{\hat{i}}, y_i) \not\subset -intC_i \quad \text{for all } y_i \in K_i.$$

The proof of Theorem 3 is completely analogous to that of Theorem 1 and is omitted.

Remark 5 Note that Theorem 1 does not contain Theorem 3 as a special case.

If I is a singleton, the method used in Theorem 1 is invalid. In the case, we obtain following existence theorem of Ky Fan's points for a set-valued map directly by Lemma 3.

Theorem 4 *Let F: $K \times K \to 2^Y$ be a set-valued map. Assume that*

(1) *for each $y \in K$, $F(\cdot, y)$ is upper $-C$-semicontinuous on K with compact values;*
(2) *for each $x \in K, F(x, \cdot)$ is C-convex;*
(3) *for each $x \in K$, $F(x, x) \not\subset -intC$.*

Then there exists a point $\overline{x} \in K$ *such that*

$$F(\overline{x}, y) \not\subset -intC \quad \text{for all } y \in K.$$

Proof Define the set-valued map A: $K \to 2^K \cup \{\emptyset\}$ by

$$A(x) = \{y \in K : F(x, y) \subset -intC\} \quad \text{for each } x \in K.$$

The condition (2) and Lemma 2 imply that for each $x \in K$, $A(x)$ is convex, and the condition (3) implies that for each $x \in K$, $x \notin A(x)$.

For each $y \in K$, $A^{-1}(y) = \{x \in K : y \in A(x)\} = \{x \in K : F(x, y) \subset -intC\}$, i.e., for each $x \in A^{-1}(y)$, we have $F(x, y) \subset -intC$, which implies that there is an open neighborhood V of the zero element θ of Y such that $F(x, y) + V \subset -intC$. The condition (1) implies that there exists an open neighborhood $O(x)$ of x such that $F(x', y) \subset F(x, y) + V - C \subset -intC - C \subset -intC$ whenever $x' \in O(x)$, i.e., the set $A^{-1}(y)$ is open in K.

Hence the result follows by Lemma 3.

Remark 6 Theorem 4 is a new existence theorem of Ky Fan's points for a set-valued map and it contains as a special case the existence theorem of Ky Fan's points of a vector-valued function in Yang and Yu (2002).

Next we establish the existence of essential components of the solution set for the SGVQEP.

Let Q be the collection of all SGVQEP satisfying the conditions of Theorem 1. For any $q \in Q$, Theorem 1 implies that q has at least one solution. We denote by $N(q)$ the solution set of q. Clearly, $N(q) = S(H)$, where H is the best-reply map of q.

Definition 6 Let $q \in Q$ and S_α a component of $N(q)$. S_α is said to be essential if it, as a component of $S(H)$, is an essential component of $S(H)$ with respect to M, where H is the best-reply map of q.

Theorem 5 *For any* $q \in Q$*, there is at least one essential component of* $N(q)$*.*

Proof For any $q \in Q$, by the proof of Theorem 1, we know $H \in M$, where H is the best-reply correspondence of q. Hence the result follows by Lemma 1.

Remark 7 Those results of essential components for multiobjective (generalized) game problems in Jiang (1963), Kohlberg and Mertens (1986), Yu and Xiang (1999), Yu and Luo (1999) and Yang and Yu (2002) are established under perturbations of the payoff function and feasible strategy correspondence, but Theorem 5 is established under perturbations of the best-reply map. Thus, Theorem 5 does not contain them as special cases even though the SGVQEP does contain the multiobjective (generalized) gave problem as a special case.

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