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Equilibrium Problems under Generalized Convexity and Generalized Monotonicity

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Abstract. Generalized convex functions preserve many valuable properties of mathematical programming problems with convex functions. Generalized monotone maps allow for an extension of existence results for variational inequality problems with monotone maps. Both models are special realizations of an abstract equilibrium problem with numerous applications, especially in equilibrium analysis (e.g., Blum and Oettli, 1994). We survey existence results for equilibrium problems obtained under generalized convexity and generalized monotonicity. We consider both the scalar and the vector case. Finally existence results for a system of vector equilibrium problems under generalized convexity are surveyed which have applications to a system of vector variational inequality problems. Throughout the survey we demonstrate that the results can be obtained without the rigid assumptions of convexity and monotonicity.

Key words: Equilibrium Problems, Generalized Convexity, Generalized Monotonicity

1. Introduction

Mathematical programming problems can be viewed as variational inequality problems. Furthermore both models are special realizations of an abstract equilibrium problem. A classical assumption in the theory and in the algorithms for mathematical programming problems is the convexity of the objective function. A corresponding assumption for variational inequality problems and equilibrium problems is monotonicity of the defining map and bifunction, respectively.

These classical assumptions are sufficient, but not necessary. In this survey we show how various generalizations of convexity and monotonicity allow extensions of earlier results to certain non-convex and non-monotone problems. Extending recent results for non-monotone variational inequalities, existence of a solution is established for equilibrium problems, vector equilibrium problems and systems of vector equilibrium problems.

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2. From Mathematical Programming Problems to Variational Inequality Problems

To provide the necessary background, we briefly contrast mathematical programming problems and variational inequality problems.

The *mathematical programming problem* is defined as follows:

$$\text{(MP)} \qquad \text{find } \widetilde{x} \in C \text{ such that } g(\widetilde{x}) = \min\{g(x) \mid x \in C\} \tag{1}$$

where $g : C \to R$ for some set $C \subseteq R^n$. As it is well known, numerous applications of MP arise in management, economics, engineering, natural sciences, applied mathematics and statistics, for example.

The *variational inequality problem* (in its simplest form) can be stated as follows:

$$\text{(VIP)} \qquad \text{find } \widetilde{x} \in C \text{ such that } \langle G(\widetilde{x}), y - \widetilde{x} \rangle \geqslant 0 \text{ for all } y \in C \tag{2}$$

where $G : C \to R^n$ for some set $C \subseteq R^n$. Many applications of VIP are found in the natural sciences, often in an infinite-dimensional setting, as well as in economics and management. In fact, a large number of equilibrium problems in economics, game theory, mechanics and traffic analysis can be cast into this format.

A differentiable MP can be reduced to a VIP where $G = \nabla g$ if G is (generalized) monotone and g possesses the corresponding (generalized) convexity property (Konnov, 2001a). A VIP is a considerably more general model since a differentiable map G is a gradient map if and only if the Jacobian of G is a symmetric matrix.

In the classical theory of MP and VIP it is assumed that, in addition to convexity and closedness of C, the function g is convex and the map G is monotone, respectively. Moving from a differentiable MP to the equivalent VIP, the assumption of monotonicity of G is not surprising since convexity of g corresponds to monotonicity of $G = \nabla g$.

Although these classical assumptions are often satisfied, in many cases they are not. At this point it is important to realize that they are sufficient, but not necessary conditions in the theory and algorithms for MP and VIP, respectively.

In case of MP, a theory of *generalized convex* functions has been developed and their usefulness has been demonstrated during the last several decades (e.g., Schaible and Ziemba, 1981; Avriel et al., 1988; Singh and Dass, 1989; Cambini et al., 1990; Komlosi et al., 1994; Crouzeix et al., 1998; Hadjisavvas et al., 2001).

On the other hand, a systematic study of *generalized monotone* maps has only begun during the last decade. Since Karamardian and Schaible (1990) appeared, a sharp increase in interest in the topic can be noticed. Well over two hundred publications have appeared since then on concepts of generalized monotonicity and their uses. While the theory of generalized convex MP is well developed, some good progress at best has been made in the study of generalized monotone VIP.

3. Basic concepts of Generalized Convex Functions and Generalized Monotone Maps

Given the connection between MP and VIP, it is only natural to define different kinds of generalized monotonicity in such a way that in the special case of a gradient map $G = \nabla g$ generalized monotonicity of some kind is equivalent to generalized convexity of some kind of the underlying function g.

In MP the following basic types of (generalized) convex functions are commonly used (Avriel et al., 1988):

- convex (cx), strictly convex (s.cx);
- quasiconvex (qcx), strictly quasiconvex (s.qcx), semistrictly quasiconvex (ss.qcx);
- pseudoconvex (pcx), strictly pseudoconvex (s.pcx).

These functions are related to each other as follows where all the inclusions are proper. (A ss.qcx function is qcx, if it is lower semicontinuous.)

$$\begin{array}{ccccc} & & & & \text{qcx} \\ & & & & \uparrow \\ \text{cx} & \rightarrow & \text{pcx} & \rightarrow & \text{ss.qcx} \\ \uparrow & & \uparrow & & \uparrow \\ \text{s.cx} & \rightarrow & \text{s.pcx} & \rightarrow & \text{s.qcx} \end{array}$$

We now present the definitions (Avriel et al.,1988). Consider $g : C \to R$ where $C \subseteq R^n$ is convex.

- g is *convex* if for all $x, y \in C$ and $t \in (0, 1)$,

$$g(tx + (1 - t)y) \leqslant tg(x) + (1 - t)g(y); \tag{3}$$

- g is *strictly convex* if (3) is a strict inequality for $x \neq y$;
- g is *quasiconvex* if for all $x, y \in C$ such that $g(x) \leqslant g(y)$, $t \in (0, 1)$,

$$g(tx + (1 - t)y) \leqslant g(y); \tag{4}$$

- g is *strictly quasiconvex* if (4) is a strict inequality for $x \neq y$;
- g is *semistrictly quasiconvex* if for all $x, y \in C$ such that $g(x) < g(y)$, the inequality in (4) is strict.

For the remaining two types of generalized convex functions one assumes differentiability of g on an open set $C \subseteq R^n$, although more general definitions are available (Avriel et al.,1988):

- g is *pseudoconvex* if for all $x, y \in C$

$$\langle \nabla g(x), y - x \rangle \geqslant 0 \Longrightarrow g(y) \geqslant g(x); \tag{5}$$

- g is *strictly pseudoconvex* if or all $x, y \in C$ the second inequality in (5) is strict.

Different kinds of generalized convexity preserve different properties of convex functions. E.g., the characteristic of a pseudoconvex function is that a stationary point is a global minimum. Furthermore, for a semistrictly quasiconvex function a local is a global minimum. For a quasiconvex function the lower level sets are convex. Finally, the qualifier 'strict' indicates that a global minimum is unique. In contrast to convex functions, inflection points are admissible for all types of generalized convex functions.

Turning now to (generalized) monotone maps, the following basic types are commonly used in VIP (Karamardian, 1976; Hassouni, 1983; Karamardian and Schaible, 1990; Hadjisavvas and Schaible, 1993):

- monotone (m), strictly monotone (s.m.);
- quasimonotone (qm), strictly quasimonotone (s.qm), semistricly quasimonotone (ss.qm);
- pseudomonotone (pm), strictly pseudomonotone (s.pm).

These maps are related to each other as follows where all the inclusions are proper:

$$\begin{array}{ccccc} & & & & \text{qm} \\ & & & & \uparrow \\ \text{m} & \rightarrow & \text{pm} & \rightarrow & \text{ss.qm} \\ \uparrow & & \uparrow & & \uparrow \\ \text{s.m} & \rightarrow & \text{s.pm} & \rightarrow & \text{s.qm} \end{array}$$

We now present the definitions (Hadjisavvas and Schaible, 1993; Hassouni, 1983; Karamardian, 1976; Karamardian and Schaible, 1990). Consider $G : C \rightarrow R^n$ where $C \subseteq R^n$.

- G is *monotone* on C if for all $x, y \in C$,

$$\langle G(y) - G(x), y - x \rangle \geqslant 0; \tag{6}$$

- G is *strictly monotone* on C if for all $x, y \in C$, $x \neq y$,

$$\langle G(y) - G(x), y - x \rangle > 0; \tag{7}$$

- G is *quasimonotone* on C if for all $x, y \in C$,

$$\langle G(x), y - x \rangle > 0 \Longrightarrow \langle G(y), y - x \rangle \geqslant 0; \tag{8}$$

- G is *strictly quasimonotone* on C if G is quasimonotone on C and for all $x, y \in C, x \neq y$, there exists $z = tx + (1-t)y, t \in (0, 1)$, such that

$$\langle G(z), y - x \rangle \neq 0; \tag{9}$$

- G is *semistrictly quasimonotone* on C if G is quasimonotone on C and for all $x, y \in C, x \neq y$,

$$\langle G(x), y - x \rangle > 0 \Longrightarrow \langle G(z), y - x \rangle > 0 \tag{10}$$

for some $z = tx + (1-t)y, t \in \left(0, \frac{1}{2}\right)$;

- G is *pseudomonotone* on C if for all $x, y \in C$,

$$\langle G(x), y - x \rangle \geqslant 0 \Longrightarrow \langle G(y), y - x \rangle \geqslant 0 \tag{11}$$

which is equivalent to

$$\langle G(x), y - x \rangle > 0 \Longrightarrow \langle G(y), y - x \rangle > 0;$$

- G is *strictly pseudomonotone* on C if for all $x, y \in C, x \neq y$,

$$\langle G(x), y - x \rangle \geqslant 0 \Longrightarrow \langle G(y), y - x \rangle > 0. \tag{12}$$

If G is continuous, quasimonotonicity does not have to be required explicitely for strictly/semistrictly quasimonotone maps since it is implied by (9), (10), respectively. In terms of original references for the concepts above, see (Karamardian, 1976) for pseudomonotone maps, (Hassouni, 1983) and independently (Karamardian and Schaible, 1990) for quasimonotone maps, (Karamardian and Schaible, 1990) for strictly pseudomonotone maps and (Hadjisavvas and Schaible, 1993) for strictly and semistrictly quasimonotone maps.

We mention that the definitions of semistrict quasiconvexity for functions and semistrict quasimonotonicity for maps do not quite conform to each other in the general case. While semistrictly quasiconvex functions need not be quasiconvex (unless they are lower semicontinuous), semistrictly quasimonotone maps are quasimonotone by definition. To overcome this incongruence in the terminology, the latter are often called explicitly quasimonotone since semistrictly quasiconvex functions which are quasiconvex have been called explicitly quasiconvex in the past (Konnov, 2001a).

In the special case of a gradient map $G = \nabla g$, where g is differentiable on the open convex set $C \subseteq R^n$ it can be shown (Karamardian, 1976; Hassouni, 1983; Karamardian and Schaible, 1990; Hadjisavvas and Schaible, 1993):

THEOREM 1. *The map $G = \nabla g$ is quasimonotone (respectively, strictly quasimonotone, semistrictly quasimonotone, pseudomonotone, strictly pseudomonotone) if and only if the function g is quasiconvex (respectively, strictly quasiconvex, semistrictly quasiconvex, pseudoconvex, strictly pseudoconvex).*

4. From Variational Inequality Problems to Equilibrium Problems

Both MP and VIP as well as several other classical problems can be viewed as special realizations of an abstract equilibrium problem (Blum and Oettli, 1994; Brezis et al., 1972; Zukhovitskii et al., 1969). Given a set $C \subseteq R^n$, consider a bifunction $F : C \times C \to R$ such that $F(x, x) \geqslant 0$ for all $x \in C$. The *equilibrium problem* is defined as follows:

$$\text{(EP)} \qquad \text{find } \widetilde{x} \in C \text{ such that } F(\widetilde{x}, y) \geqslant 0 \text{ for all } y \in C. \tag{13}$$

According to (Blum and Oettli, 1994), the following classical problems can be cast into this format:

- general MP: $F(x, y) = g(y) - g(x)$;
- Gateaux differentiable convex MP: $F(x, y) = \langle Dg(x), y - x\rangle$;
- saddle point problem: $F(x, y) = h(y_1, x_2) - h(x_1, y_2)$ where $x = (x_1, x_2)$, $y = (y_1, y_2)$;
- Nash equilibrium problem in a non-cooperative game:

  $$F(x, y) = \sum_{i \in I} (f_i(x^i, y_i) - f_i(x))$$

  where f_i is the loss function of player i and x^i is the vector obtained from x by deleting component i;
- fixed point problem: $F(x, y) = \langle x - G(x), y - x\rangle$ where G is the operator of the fixed point problem;
- VIP: $F(x, y) = \langle G(x), y - x\rangle$;
- generalized VIP: $F(x, y) = \max_{z \in G(x)} \langle z, y - x\rangle$ where G is a multivalued map.

Each of these classical problems has numerous applications, including but not limited to equilibrium problems in economics, game theory, traffic analysis and mechanics. Hence collectively the equilibrium problem EP covers a vast range of applications. This very general model was studied by Zukhovitskii et al. (1969); Brezis et al. (1972) and was revisited in the 1990s by Blum and Oettli (1994). In these studies the bifunction F is usually assumed to be monotone. We note that Brezis et al. also considered pseudomonotone maps (without using this term).

F is *monotone* (m) if for all $x, y \in C$

$$F(x, y) + F(y, x) \leqslant 0. \tag{14}$$

In case of VIP monotonicity of $F(x, y) = \langle G(x), y - x\rangle$ is equivalent to monotonicity of the map G.

As it is clear from the above realizations of the EP, the bifunction is often not monotone. Meanwhile generalized monotone bifunctions have been introduced for EP. The following basic types of generalized monotone bifunctions are considered below (Bianchi and Schaible, 1996):

- F is *quasimonotone* (qm) if for all $x, y \in C$

$$F(x, y) > 0 \Longrightarrow F(y, x) \leqslant 0; \tag{15}$$

- F is *strictly quasimonotone* (s.qm) if it is quasimonotone and for all $x, y \in C, x \neq y$, there exists z in the open interval between x and y such that

$$\text{either } F(z, x) \neq 0 \text{ or } F(z, y) \neq 0; \tag{16}$$

- F is *pseudomonotone* (pm) if for all $x, y \in C$

$$F(x, y) \geqslant 0 \Longrightarrow F(y, x) \leqslant 0; \tag{17}$$

- F is *strictly pseudomonotone* (s.pm) if for all $x, y \in C, x \neq y$,

$$F(x, y) \geqslant 0 \Longrightarrow F(y, x) < 0. \tag{18}$$

In (17) both inequalities can be replaced by strict inequalities.

The following inclusions hold which are proper:

$$\begin{array}{ccccc} \text{m} \rightarrow & \text{pm} & \rightarrow & \text{qm} \\ & \uparrow & & \uparrow \\ & \text{s.pm} & \rightarrow & \text{s.qm} \end{array}$$

The last inclusion is true if $F(\cdot, y)$ is hemicontinuous for all $y \in C$. In case of a bifunction $F(x, y) = \langle G(x), y - x\rangle$ generalized monotonicity of F is equivalent to the corresponding generalized monotonicity of the map G.

In our review of generalized convex MP and generalized monotone VIP and EP we adopted a finite-dimensional setting, partially since MP were our starting point. A fine example of employing a generalized monotone VIP in solving the problem of minimizing a generalized convex functional over a convex set of an infinite-dimensional space is given by Yao (1994). From now on we will review existence results for generalized monotone EP. Therefore we adopt an infinite-dimensional setting throughout the remainder of this survey.

5. Equilibrium Problems under Quasimonotonicity

Most of the literature on generalized monotone VIP and EP is concerned with the existence of a solution. Among the publications most deal with the case which is the easiest from an analysis point of view, namely with pseudomonotone VIP. A departure from this direction can be found in the work of Hadjisavvas and Schaible (1996), where the existence of a solution for quasimonotone VIP is studied as well. In a follow-up paper by Bianchi and Schaible (1996) the results in (Hadjisavvas and Schaible, 1996) were extended to generalized monotone EP. Both the quasimonotone and pseudomonotone case are treated again. In this section we summarize some of the major results in this work.

Since applications of EP sometimes involve an infinite-dimensional space, the results in (Bianchi and Schaible, 1996) are derived in an abstract setting. Let X be a real topological Hausdorff vector space. Given a nonempty closed convex set $C \subseteq X$ and a bifunction $F : C \times C \to R$ such that $F(x, x) \geqslant 0$ for all $x \in C$, we are interested in conditions for the non-monotone and non-convex case which guarantee the existence of a solution for EP. Following an approach similar to that for quasimonotone VIP in (Hadjisavvas and Schaible, 1996), we can show (Bianchi and Schaible, 1996):

THEOREM 2. *Suppose conditions* (i)-(vi) *hold*:

- **(i)** $F(\cdot, y)$ *is hemicontinuous for all* $y \in C$;
- **(ii)** $F(x, \cdot)$ *is lower semicontinuous and semistrictly quasiconvex for all* $x \in C$;
- **(iii)** $F(x, y)$ *is quasimonotone on* $C \times C$;
- **(iv)** *there exist a compact set* $B \subseteq X$ *and* $y_0 \in B \cap C$ *such that* $F(x, y_0) < 0$ *for all* $x \in C \backslash B$ (*coercivity*);
- **(v)** $F(x, y) = 0, F(x, \widetilde{y}) > 0$ *imply* $F(x, t\widetilde{y} + (1 - t)y) > 0$ *for all* $t \in (0, 1)$ *and* $x \in C$ (*which holds if* $F(x, \cdot)$ *is semistictly quasiconcave for all* $x \in C$ *or* $F(x, y) = \langle G(x), y - x \rangle$);
- **(vi)** *the algebraic interior of C is nonempty.*
  Then EP has a solution.

In the proof Ky Fan's Lemma (Fan, 1961) plays an important role.

In the special case where the bifunction F is even pseudomonotone, conditions (v) and (vi) are not needed. It can be shown that in this case the solution set is nonempty, compact and convex. If F is strictly pseudomonotone, then the solution is unique. In the quasimonotone case additional results can be obtained for the set of nontrivial solutions, i.e., solutions $\widetilde{x}$ such that $F(\widetilde{x}, y) > 0$ for some $y \in C$ (Bianchi and Schaible, 1996).

6. Vector Equilibrium Problems under Quasimonotonicity

The existence result in (Bianchi and Schaible, 1996) can further be extended to the case of equilibrium problems with vector-valued bifunctions as shown in the paper by Bianchi et al. (1997). In addition to the assumptions of the previous section we consider a real locally convex vector space Y and a bifunction $F : C \times C \to Y$. With help of a pointed closed convex cone K with nonempty interior $int\ K$ we introduce the partial order

$$x \leqslant y \text{ if and only if } y - x \in K.$$

Moreover

$$x \not< y \text{ if and only if } y - x \notin int\ K.$$

Following the scalar case we assume $F(x, x) \geqslant 0$ for all $x \in C$.

The *vector equilibrium problem* is defined as follows (Bianchi et al., 1997):

$$\text{(VEP)} \qquad \text{find } \widetilde{x} \in C \text{ such that } F(\widetilde{x}, y) \not< 0 \text{ for all } y \in C. \tag{19}$$

Special cases of the VEP are the *vector variational inequality problem* and the *vector optimization problem* (Bianchi et al., 1997).

The following extensions of generalized monotonicity concepts for bifunctions to the vector case have proved useful (Bianchi et al., 1997):

- F is *monotone* (m) if for all $x, y \in C$
$$F(x, y) + F(y, x) \leqslant 0; \tag{20}$$
- F is *quasimonotone* (qm) if for all $x, y \in C$
$$F(x, y) > 0 \Longrightarrow F(y, x) \leqslant 0; \tag{21}$$
- F is *pseudomonotone* (pm) if for all $x, y \in C$
$$F(x, y) \not< 0 \Longrightarrow F(y, x) \not> 0; \tag{22}$$
- F is *strictly pseudomonotone* (s.pm) if for all $x, y \in C$, $x \neq y$,
$$F(x, y) \not< 0 \Longrightarrow F(y, x) < 0. \tag{23}$$

The following inclusions hold which are proper:

$$\begin{array}{ccccc} m & \to & pm & \to & qm \\ & & \uparrow & & \\ & & s.pm & & \end{array}$$

Before presenting the main existence results in (Bianchi et al., 1997), we point out that for vector-valued functions and bifunctions an extension of hemicontinuity, lower semicontinuity, quasiconvexity, semistrict quasiconvexity can be defined with help of the partial order in Y (Bianchi et al., 1997). An additional concept is needed which was introduced earlier in (Jeyakumar et al., 1993). A function $f : C \to Y$ is called *-semistrictly quasiconvex (quasiconcave) if $\varphi \circ f : C \to R$ is semistrictly quasiconvex (quasiconcave) for all nontrivial φ in the dual cone of K. One can show that a lower semicontinuous *-semistrictly quasiconvex function is quasiconvex and semistrictly quasiconvex.

We can then prove (Bianchi et al., 1997):

THEOREM 3. *Suppose conditions* (i)–(vi) *hold*:

- **(i)** $F(\cdot, y)$ *is hemicontinuous for all* $y \in C$;
- **(ii)** $F(x, \cdot)$ *is lower semicontinuous and *-semistrictly quasiconvex for all* $x \in C$;
- **(iii)** $F(x, y)$ *is quasimonotone on* $C \times C$;
- **(iv)** *there exist a compact set* $B \subseteq X$ *and* $y_0 \in B \cap C$ *such that* $F(x, y_0) < 0$ *for all* $x \in C \backslash B$ $(coercivity)$;
- **(v)** $F(x, \cdot)$ *is *-semistrictly quasiconcave for all* $x \in C$;
- **(vi)** *the algebraic interior of C is nonempty.*
  Then VEP has a solution.

Like in the scalar case, the proof makes use of Ky Fan's Lemma (Fan, 1961).

In the special case of a pseudomonotone bifunction F assumptions (v) and (vi) are not needed and in (ii) *-semistrict quasiconvexity can be relaxed to semistrict quasiconvexity. Under these conditions the solution set of VEP is nonempty and compact. But it is not convex in general as in the scalar case. This is not surprising since the VEP includes vector optimization problems as a special case where the solution set is known to be non-convex in general. Finally, for a strictly pseudomonotone bifunction in a VEP the solution is unique.

In the follow-up study (Hadjisavvas and Schaible, 1998), inspired by the work in (Oettli, 1997), Hadjisavvas and Schaible arrived at stronger existence results for the VEP. It could be shown that the rather strong assumption $F(x, x) \geqslant 0$ in the vector case can be relaxed to $F(x, x) \not< 0$ for all $x \in C$. Furthermore other conditions in Theorem 3 can be weakened. For details as well as related existence results for vector variational inequality problems we refer to (Hadjisavvas and Schaible, 1998).

7. A System of Vector Equilibrium Problems

In a very recent study by Ansari et al. (2000) a system of vector equilibrium problems has been introduced; i.e., a family of vector equilibrium problems defined on a product set. A special case, a system of (scalar) variational inequality problems was considered earlier by Pang (1985). He showed that traffic equilibrium problems, spatial equilibrium problems and general equilibrium programming problems give rise to a system of variational inequalities rather than to a single variational inequality. Later this model was studied also by Cohen and Chaplais (1988) and by Bianchi (1993). Existence results were derived in (Bianchi, 1993) assuming pseudomonotonicity extended to product sets.

Let I be an index set. For each $i \in I$, X_i denotes a real topological Hausdorff vector space and $C_i \subseteq X_i$ a nonempty convex set. Let $X = \prod_{i \in I} X_i$ and $C = \prod_{i \in I} C_i$. Consider further a real topological Hausdorff vector space Y. Given a pointed closed convex cone $K \subseteq Y$ with nonempty interior $int\ K$, we introduce a partial order in Y in the usual way. Consider a set of bifunctions $F_i : C \times C_i \to Y$. The *system of vector equilibrium problems* is defined as follows:

$$\text{(SVEP)} \qquad \text{find } \widetilde{x} \in C : \forall\, i \in I, F_i(\widetilde{x}, y_i) \notin -int\ K\ \forall y_i \in C_i. \tag{24}$$

The following theorem, the main result in (Ansari et al., 2000), involves the concept of a K-quasiconvex vector-valued function $f : M \to Y$ where M is a nonempty convex subset of a topological vector space (Luc, 1989): f is called K-quasiconvex if for all $\alpha \in Y$ the set $\{x \in M : f(x) - \alpha \in -K\}$ is convex. In the result below, x^i denotes the vector derived from $x \in X$ by omitting component x_i.

Using a recent fixed point theorem by Ansari and Yao (1999), we can show (Ansari et al., 2000):

THEOREM 4. *For each $i \in I$, let $C_i \subseteq X_i$ be nonempty convex and compact, and $F_i : C \times C_i \to Y$ such that $F_i(x, x_i) = 0$ for all $x = (x^i, x_i) \in C$. Suppose that the following conditions hold*:

(i) *$y_i \to F_i(x, y_i)$ is $K-$quasiconvex for all $i \in I$, $x \in C$.*
(ii) *F_i is continuous on $C \times C_i$ for all $i \in I$.*

Then the solution set of the SVEP is nonempty and compact.

If C_i is not compact, the existence of a solution can still be established if instead of condition (ii) continuity of F_i on all compact convex subsets of $C \times C_i$ is assumed and a coercivity assumption is added (Ansari et al., 2000).

The general existence result in Theorem 4 was specialized in (Ansari et al., 2000) to a system of vector variational inequality problems. For each $i \in I$, let $G_i : C \to L(X_i, Y)$ be a given map, where $L(X_i, Y)$ denotes the space of all continuous linear operators from X_i to Y. The *system of vector variational inequality problems* is defined as follows:

$$\text{(SVVIP)} \qquad \text{find } \tilde{x} \in C : \forall\, i \in I, \langle G_i(\tilde{x}), y_i - \tilde{x}_i \rangle \notin -int\, K \;\forall y_i \in C_i \qquad (25)$$

where $\langle s, x_i \rangle$ denotes the evaluation of $s \in L(X_i, Y)$ at $x_i \in X_i$.If $Y = R$ and $K = \{x \in R : x \geqslant 0\}$, then the SVVIP becomes the system of variational inequalities studied in (Ansari and Yao, 1999; Bianchi, 1993; Cohen and Chaplais, 1988; Pang, 1985). Specializing Theorem 4, it follows (Ansari et al., 2000):

COROLLARY 7.1. *For each $i \in I$, let $C_i \subseteq X_i$ be nonempty convex and compact, let $L(X_i, Y)$ be equipped with the uniform convergence topology and G_i be continuous on C. Then there exists a solution $\tilde{x}$ of the SVVIP.*

Based on Corollary 5, an existence result for a system of vector optimization problems was derived in (Ansari et al., 2000). This in turn can be used to establish the existence of a solution of the associated Nash equilibrium problem involving vector-valued functions.

Results more general than Theorem 4 and Corollary 5 have been obtained in a follow-up study by Ansari et al. (2002) allowing for multivalued bifunctions.

A *system of generalized vector equilibrium problems* is defined as follows:

$$\text{(SGVEP)} \qquad \text{find } \tilde{x} \in C : \forall\, i \in I, F_i(\tilde{x}, y_i) \subsetneq -int\, K_i(\tilde{x}) \;\forall y_i \in C_i. \qquad (26)$$

Compared with SVEP above, three extensions have been introduced in this model: F_i is multi-valued, the ordering cone $K_i(x)$ depends on i and it is moving with $x \in C$. (For multi-valued generalized monotone maps see (Hadjisavvas and Schaible (2001)). The existence result obtained in (Ansari et al., 2002) is based on a maximal element theorem by Deguire et al. (1999). Like for SVEP, a certain generalized convexity assumption is used to prove the existence of a solution for SGVEP.

8. Conclusion

In this survey we have seen how certain concepts of generalized convexity and generalized monotonicity prove to be sufficient to derive the existence of a solution for a series of increasingly more complex models. In all these results the rigid assumptions of convexity and monotonicity can be relaxed to various kinds of generalized convexity and generalized monotonicity.

Generalized convexity of functions was developed for MP to relax the restrictive assumption of convexity in applications (Avriel et al., 1988). Moving from MP to the more general VIP (Section 2), necessitates the introduction of generalized monotone maps (Section 3). Moving from there to the considerably more general EP, leads to concepts of generalized monotone bifunctions (Section 4). As Sections 5 and 6 demonstrate, existence results for EP and VEP make use of both generalized convexity and generalized monotonicity concepts. Finally existence results for the even more general models SVEP and SGVEP in Section 7 are derived under certain generalized convexity assumptions.

Throughout this survey it has become clear that neither the restrictive assumption of convexity of a function nor the rigid assumption of monotonicity of a map or bifunction are necessary to guarantee valuable properties of the increasingly more complex models MP, VIP, EP, VEP, SVEP and SGVEP. There is a growing literature which proves this point. For this we refer the reader to the references in the publications selected for the present survey.

As an example of a very recent development we point out the effort to specialize generalized monotonicity concepts for scalar and vector variational inequalities over product sets; (e.g., see Allevi et al., 2001; Konnov, 2001b). This promising approach allows to derive new existence and uniqueness results.

Most studies on generalized monotone VIP and EP are concerned with existence and uniqueness questions. By contrast, Konnov in his very recent monograph (Konnov, 2001a) emphasizes solution methods for generalized monotone problems. Clearly there is a need for more work in this important direction.

References

Allevi, E., Gnudi, A. and Konnov, I.V. (2001), Generalized vector variational inequalities over product sets, *Nonlinear Analysis* 47, 573–582.

Ansari, Q.H. and Yao, J.-C. (1999), A fixed point theorem and its applications to the system of variational inequalities, *Bullettin of the Australian Mathematical Society* 59, 433–442.

Ansari, Q.H., Schaible, S. and Yao, J.-C. (2000), System of vector equilibrium problems and its applications, *Journal of Optimization Theory and Applications* 107, 547–557.

Ansari, Q.H., Schaible, S. and Yao, J.-C. (2002), The system of generalized vector equilibrium problems with applications, *Journal of Global Optimization*, 22, 3–16.

Avriel, M., Diewert, W.E., Schaible, S. and Zang, I. (1988), *Generalized Concavity*, Plenum Publishing Corporation, New York.

Bianchi, M. (1993), Pseudo P-monotone operators and variational inequalities, report N°.6, Istituto di Econometria e Matematica per le Decisioni Economiche, Università Cattolica del Sacro Cuore, Milan, Italy.

Bianchi, M. and Schaible, S. (1996), Generalized monotone bifunctions and equilibrium problems, *Journal of Optimization Theory and Applications* 90, 31–43.

Bianchi, M., Hadjisavvas, N. and Schaible, S. (1997), Vector equilibrium problems with generalized monotone bifunctions, *Journal of Optimization Theory and Applications* 92, 527–542.

Blum, E. and Oettli, W. (1994) From optimization and variational inequalities to equilibrium problems, *The Mathematics Student* 63, 123–145.

Brezis, H., Niremberg, L. and Stampacchia, G. (1972), A remark on Fan's minimax principle, *Bollettino della Unione Matematica Italiana* 6, 293–300.

Cambini, A., Castagnoli, E., Martein, L., Mazzoleni, P. and Schaible, S. (eds.) (1990), *Generalized Convexity and Fractional Programming with Economic Applications,* Volume 345 of *Lecture Notes in Economics and Mathematical Systems.* Springer, Berlin, Heidelberg, New York.

Cohen, G. and Chaplais, F. (1988), Nested monotonicity for variational inequalities over a product of spaces and convergence of iterative algorithms, *Journal of Optimization Theory and Applications* 59, 360–390.

Crouzeix, J-P., Martinez-Legaz, J.-E. and Volle, M. (eds.) (1998) *Generalized Convexity, Generalized Monotonicity.* Kluwer Academic Publishers, Dordrecht, Boston, London.

Deguire, P., Tan, K.K. and Yuan, G.X.-Z. (1999), The study of maximal elements, fixed points for L_s-majorized mappings and their applications to minimax and variational inequalities in the product topological spaces, *Nonlinear Analysis: Theory, Methods and Applications* 37, 933–951.

Fan, K. (1961), A generalization of Tychonoff's fixed point theorem, *Mathematische Annalen* 142, 305–310.

Hadjisavvas, N. and Schaible, S. (1993), On strong pseudomonotonicity and (semi) strict quasimonotonicity, *Journal of Optimization Theory and Applications* 79, 139–155.

Hadjisavvas, N. and Schaible, S. (1996), Quasimonotone variational inequalities in Banach spaces, *Journal of Optimization Theory and Applications* 90, 95–111.

Hadjisavvas, N. and Schaible, S. (1998), From scalar to vector equilibrium problems in the quasimonotone case, *Journal of Optimization Theory and Applications* 96, 297–309.

Hadjisavvas, N. and Schaible, S. (2001), Generalized monotone multi-valued maps, In: Floudas, C.A. and Pardalos, P.M. (eds), *Encyclopedia of Optimization,* Volume II (E-Integer), pp. 224–229.

Hadjisavvas, N., Matinez-Legaz, J.-E. and Penot, J.-P. (eds.) (2001), *Generalized Convexity and Generalized Monotonicity.* Volume 502 of *Lecture Notes in Economics and Mathematical Systems.* Springer, Berlin, Heidelberg, New York.

Hassouni, A. (1983). *Sous-differentiels des fonctions quasiconvexes,* These de troisieme cycle, Universite Paul Sabatier, Toulouse.

Jeyakumar, V., Oettli, W. and Natividad, M. (1993), A solvability theorem for a class of quasiconvex mappings with applications to optimization, *Journal of Mathematical Analysis and Applications* 179, 537–546.

Karamardian, S. (1976), Complementarity over cones with monotone and pseudomonotone maps, *Journal of Optimization Theory and Applications* 18, 445–454.

Karamardian, S. and Schaible, S. (1990), Seven kinds of monotone maps, *Journal of Optimization Theory and Applications* 66, 37–46.

Komlosi, S., Rapcsak, T. and Schaible, S. (eds.) (1994), *Generalized Convexity,* Volume *405 of Lecture Notes in Economics and Mathematical Systems.* Springer, Berlin, Heidelberg, New York.

Konnov, I.V. (2001a), *Combined Relaxation Methods for Variational Inequalities,* Springer, Berlin, Heidelberg, New York.

Konnov, I.V. (2001b), Relatively monotone variational inequalities over product sets, *Operations Research Letters* 28, 21–26.

Luc, D.T. (1989), *Theory of vector optimization,* Volume *319 of Lecture Notes in Economics and Mathematical Systems.* Springer, Berlin, Heidelberg, New York.

Oettli, W. (1997), A remark on vector-valued equilibria and generalized monotonicity, *Acta Matematica Vietnamica* 22, 213–221.

Pang, J.S. (1985), Asymmetric variational inequality problems over product sets, applications and iterative methods, *Mathematical Programming* 31, 206–219.

Schaible, S. and Ziemba, W.T. (eds.) (1981), *Generalized Concavity in Optimization and Economics.* Academic Press, New York.

Singh, C. and Dass, B.K. (eds) (1989), *Continuous-Time, Fractional and Multiobjective Programming.* Analytic Publishing Company, New Delhi.

Yao, J.C. (1994), Variational inequalities with generalized monotone operators, *Mathematics of Operations Research,* 19, 391–403.

Zukhovitskii, S.I., Polyak, R.A. and Primak, M.E. (1969), Two methods of search for equilibrium points of n-person concave games, *Soviet Mathematics Doklady* 10, 279–282.