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Vector Variational Inequalities with Semi-monotone Operators★

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Abstract. In this paper, the concept of a vector semi-monotone operator is introduced. This concept is applied to establish several existence results of vector variational inequalities. These extend some results of others.

Key words: semi-monotone operator, vector valued, vector variational inequality

1. Introduction

In scalar optimization, a variational inequality problem has shown to be a very useful model both for unifying the mathematical analysis and for overcoming the difficulty of defining the objective function. The same advantages are expected in the vector cases. About two decades ago, a vector variational inequality problem was introduced in finite dimensional spaces by Giannessi (1980). Since then, this problem has become one of the most active fields in mathematics. A vector variational inequality problem has shown to be a powerful tool in the mathematical investigation of optimization topics. It has many applications in the study of vector equilibrium problems, and vector extremal problems. Chen and many others have intensively studied vector variational inequalities problem in abstract spaces, where the dimension of the space is not necessarily finite, and made a lot of progresses.

It is well known that the compactness and monotonicity of operators are two important concepts in nonlinear functional analysis and its applications. They play important roles in the study of both ordinary and partial differential equations, variational inequality problems, fixed point theory, etc. It was Browder (1968) who first combined the compactness and accretion of operators, and posed the concept of a semi-accretive operator. More recently, motivated by this idea, Chen (1999) posed the concept of

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a semi-monotone operator, which combines the compactness and monotonicity of an operator, and applied it to the study of scalar value variational inequalities.

In this paper, we pose the concept of a vector semi-monotone operator, investigate a vector variational inequality problem and obtain some existence results. These extend the results of Chen (1999).

2. Preliminaries

In order to study a vector variational inequality problem, we need some background about the partial orderings in a Banach space. Let's first give a brief recall.

Let X be a real Banach space. A nonempty subset P of X is called a convex cone if $\lambda P \subset P$ for all $\lambda \geqslant 0$ and $P + P = P$. A cone P is called a pointed cone if $P \cap (-P) = 0$, where the 0 denotes the zero vector of X. Also a cone is called proper if it is properly contained in X. The partial order $\leqslant_P$ in X, induced by the pointed cone P, is defined by declaring $x \leqslant_P y$ if and only if $y - x \in P$ for all x, y in X, and P is called a positive cone of X. An ordered Banach space is a pair (X, P), where X is a real Banach space and P a pointed convex cone with the partial order induced by P. The weak order $\nleqslant_{\text{int}P}$ in an ordered Banach space (X, P) with $\text{int}P \neq \emptyset$ is defined as $x \nleqslant_{\text{int}P} y$ if and only if $y - x \notin \text{int}P$ for x, y in X, where int denotes the interior of a subset.

In this paper, X, Y are always real Banach spaces, $L(X, Y)$ denotes the set of all the bounded linear operators from X to Y. $T \in L(X, Y), x \in X$, $\langle T, x \rangle$ denotes the value of T at x.

Now, we give the definition of a vector semi-monotone operator. In order to do so, the following definition of a vector monotone operator is needed, which was posed by Chen (1992).

DEFINITION 2.1. (Chen (1992)). Let $T: K \to L(X, Y)$ be a mapping, $K \subset X$ be a nonempty, closed, and convex subset in X. Let $\{C(x) : x \in K\}$ be a family of closed, pointed, and convex cones of Y such that $\text{int}C(x) \neq \emptyset$ for each $x \in K$. Suppose $C_- = \cap_{x \in K} C(x) \neq \emptyset$. T is said to be C_--monotone on K if and only if it satisfies the following condition:

$$\langle T(y) - T(x), y - x \rangle \geqslant_{C_-} 0,$$

for all $x, y \in K$.

We now give the concept of a vector semi-monotone operator.

DEFINITION 2.2. Let $\{C(x) : x \in K\}$ be a family of closed, pointed and convex cones of Y satisfying $\text{int}C(x) \neq \emptyset$ for all $x \in K$. Suppose $C_- =$

$\cap_{x\in K} C(x) \neq \emptyset$. We say $A: K \times K \to L(X, Y)$ is a C_- semi-monotone operator, if and only if the following two conditions are satisfied:

(1) for every $u \in K$, $A(u, \cdot)$ is a C_--monotone operator;
(2) for every $v \in K, A(\cdot, v)$ is completely continuous, that is, when $u_n \to^w u, A(u_n, v) \to A(u, v)$ (by the norm of operators), where $\to^w$ denotes the weak convergence.

Now we will give an example of a vector semi-monotone operator.

EXAMPLE 2.1. Let $X = Y = R^2$, $K = [0, 1] \times [0, 1]$, where R denotes the set of all real numbers. A vector of R^2 will be denoted by $x = (x_1, x_2)$, and let $C: K \to 2^Y$ be defined by

$$C(x) = \{(y_1, y_2) \in Y \,|\, y_1 \geqslant 0, y_2 \geqslant 0\}, \quad x \in K.$$

So $C_- = \{(y_1, y_2) \in Y \,|\, y_1 \geqslant 0, y_2 \geqslant 0\}$. Let $A: K \times K \longrightarrow L(X, Y)$ be defined by

$$A(x, y) = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.$$

The norm of A is defined as follows

$$\|A\| = |x_1| + |x_2| + |y_1| + |y_2|.$$

Next we show that A is a vector semi-monotone operator. Indeed, for each $u = (u_1, u_2), v = (v_1, v_2) \in K$,

$$\begin{aligned}\langle A(y, u) - A(y, v), u - v\rangle &= \begin{pmatrix} 0 & 0 \\ u_1 - v_1 & u_2 - v_2 \end{pmatrix} \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ (u_1 - v_1)^2 + (u_2 - v_2)^2 \end{pmatrix} \geqslant_{C_-} 0.\end{aligned}$$

On the other hand, for fixed $u \in K$, if $y_n \in K, y \in K, y_n \longrightarrow^w y$, it is easy to see that $\|A(u, y_n) - A(u, y)\| \longrightarrow 0$. Hence the operator defined as above is a vector semi-monotone operator.

REMARK 2.1. When $Y = R$, this is just the definition of a semi-monotone operator introduced by Chen (1999).

3. Lemmas and Main Results

Let $T: X \to L(X, Y)$ be a mapping, $K \subset X$ be a nonempty, closed, and convex subset in X. Let $\{C(x): x \in K\}$ be a family of closed, pointed, and convex cones of Y such that $\mathrm{int}C(x) \neq \emptyset$ for each $x \in K$. Suppose $C_- = \cap_{x\in K} C(x) \neq \emptyset$. In Chen (1992), G.Y. Chen studied the following vector variational inequality problem of finding an $x_0 \in K$ such that

$$\langle T(x_0), x - x_0\rangle \not\leqslant_{\mathrm{int}C(x_0)} 0, \quad \forall x \in K.$$

If $Y = R$, $L(X, Y)$ is the dual space X^* of X. In Chen (1999), Y.Q. Chen investigated the following variational inequality problem of finding a $w_0 \in K$ such that

$$(f(w_0, w_0), u - w_0) \geqslant 0, \quad \forall u \in K.$$

where $f: X \times X \longrightarrow X^*$, (f, x) is the value of the functional $f \in X^\star$ at $x \in X$. He obtained some existence results and discussed their applications in partial differential equations of divergence form.

We are now in a position to pose the main problem of our study in this paper. We will study the following vector variational inequality problem of finding a $w_0 \in K$ such that

$$\langle A(w_0, w_0), u - w_0\rangle \not\leqslant_{\mathrm{int}C(w_0)} 0, \quad \forall u \in K,$$

where $A: K \times K \longrightarrow L(X, Y)$.

The following famous fixed point theorem will play a key role in the proof of the main theorem.

LEMMA 3.1 (Zeidle (1998)) (Fan, Glicksberg). *The set-valued mapping* F: $M \rightrightarrows M$ *has a fixed point if the following conditions are satisfied:*

(1) M *is a compact, convex, and nonempty set in a locally convex space*;
(2) $F(x)$ *is convex, closed, and nonempty for every* $x \in M$; *and*
(3) F *is upper semi-continuous on* M.

The next lemma is an existence result about a vector variational inequality problem related to a C_--monotone operator, which is needed in our proof of the main theorem.

LEMMA 3.2 (Chen (1992)). *Let X be a reflexive Banach space*, *Y a Banach space. Let* $K \subset X$ *be a nonempty, bounded, closed, and convex subset in* X. *Let* $C: K \to 2^Y$ *be a set-valued mapping such that for all* $x \in K, C(x)$ *is a closed, pointed, and convex cone of Y with* $\mathrm{int}C(x) \neq \emptyset$ *and* $C_- = \cap_{x\in K} C(x)$ *with* $\mathrm{int}C_- \neq \emptyset$. *Suppose the set-valued mapping* $W(x) = Y \setminus -\mathrm{int}C(x)$ *is upper semi-continuous on* K, *and* $T: K \longrightarrow L(X, Y)$ *is* C_-*-monotone and hemicontinuous mapping on* K. *Then there exists an* $u_0 \in K$ *such that*

$$\langle T(u_0), u - u_0\rangle \not\leqslant_{\mathrm{int}C(u_0)} 0, \quad \forall u \in K.$$

LEMMA 3.3 (Chen (1992)). *Let $K \subset X$ be a nonempty, closed, and convex subset in X. Let $\{C(x) : x \in K\}$ be a family of closed, pointed, and convex cones of Y such that* $\text{int}C(x) \neq \emptyset$ *for each $x \in K$. Suppose $C_- = \cap_{x\in K} C(x) \neq \emptyset$. Let $T : K \to L(X, Y)$ be a C_--monotone and hemicontinuous mapping on K. Then the following problems* (*i*) and (*ii*) *are equivalent:*

(i) $\quad x \in K, \quad \langle T(x), y - x\rangle \not\leq_{\text{int}C(x)} 0, \quad \forall y \in K,$

(ii) $\quad x \in K, \quad \langle T(y), y - x\rangle \not\leq_{\text{int}C(x)} 0, \quad \forall y \in K.$

THEOREM 3.1. *Let X be a reflexive Banach space, Y a Banach space. Let $K \subset X$ be a nonempty, bounded, closed, and convex subset in X. Let $C : K \to 2^Y$ be a setvalued mapping such that for all $x \in K, C(x)$ is a closed, pointed, and convex cone with* $\text{int}C(x) \neq \emptyset$ *and $C_- = \cap_{x\in K} C(x)$ with* $\text{int}C_- \neq \emptyset$. *Suppose the set-valued mapping $W(x) = Y \setminus -\text{int}C(x)$ is weakly upper semi-continuous $(w - w)$ on K, and satisfies*

$$\lambda W(x) + (1 - \lambda)W(y) \subseteq W(\lambda x + (1 - \lambda)y),$$

for all $0 < \lambda < 1$. If *$A : X \times X \to L(X, Y)$ satisfies the following conditions*:

(a) *A is a C_--semi-monotone operator*;
(b) *for all $v \in K$, $A(v, \cdot)$ is continuous on each finite dimensional subspace of X.*

Then there exists an $u_0 \in K$ such that

$$\langle A(u_0, u_0), u - u_0\rangle \not\leq_{\text{int}\,C(u_0)} 0, \quad \forall u \in K. \tag{1}$$

Proof. Let F be a finite dimensional subspace of X and $K_F \equiv K \cap F \neq \emptyset$. For each $v \in K$, the operator $A(v, \cdot)$ satisfies the conditions of Lemma 3.2, so there exists an $u_0 \in K_F$ such that

$$\langle A(v, u_0), u - u_0\rangle \not\leq_{\text{int}\,C(u_0)} 0, \quad \forall u \in K_F.$$

Define a set-valued mapping $F : K_F \rightrightarrows K_F$ as follows:

$$F(v) = \{w \in K_F : \langle A(v, m), u - w\rangle \not\leq_{\text{int}\,C(w)} 0, \forall u \in K_F\}.$$

By the linearization lemma, we have

$$\begin{aligned} F(v) &= \{w \in K_F : \langle A(v, w), u - w\rangle \not\leq_{\text{int}\,C(w)} 0, \forall u \in K_F\} \\ &= \{w \in K_F : \langle A(v, u), u - w\rangle \not\leq_{\text{int}\,C(w)} 0, \forall u \in K_F\}. \end{aligned}$$

Now we shall use the fixed point theorem to verify the existence of the solution of the variational inequality problem in a finite dimensional subspace. Obviously, K_F is compact, since F is of finite dimension.

First, we know that $\forall v \in K_F$, $F(v)$ is a nonempty bounded convex subset. Indeed, if $w_1, w_2 \in F(v)$, then $\langle A(v,u), u - w_i\rangle \in W(w_i)$, $i=1,2$. Thus for all $0 < \lambda < 1$, we have

$$\begin{aligned}\langle A(v,u), u-(\lambda w_1+(1-\lambda)w_2)\rangle &\in \lambda W(w_1)+(1-\lambda)W(w_2)\\ &\subseteq W(\lambda w_1+(1-\lambda)w_2).\end{aligned}$$

This means that $\lambda w_1 + (1-\lambda)w_2 \in F(v)$, i.e., $F(v)$ is convex.
We say that $F(v)$ is closed. In fact, let $w_j \to w$, $w_j \in F(v)$, then $\langle A(v,u), u - w_j\rangle \in W(w_j)$, and $A(v,u) \in L(X,Y)$, $\langle A(v,u), u-w_j\rangle \to \langle A(v,u), u-w\rangle$. Since W is upper semi-continuous,

$$\langle A(v,u), u-w\rangle \in W(w).$$

This means $w \in F(v)$, hence $F(v)$ is closed.
We say that F is upper semi-continuous. Let $v_j \to v$, $w_j \in F(v_j)$, $w_j \longrightarrow w$,

$$\langle A(v_j,u), u-w_j\rangle \in W(w_j).$$

From the complete continuity of $A(\cdot,u)$,

$$\langle A(v_j,u), u-w_j\rangle \longrightarrow \langle A(v,u), u-w\rangle,$$

and the upper semi-continuity of W, we have

$$\langle A(v,u), u-w\rangle \in W(w).$$

This means $w \in F(v)$, thus F is upper semi-continuous.
By applying the fixed point theorem, Lemma 3.1, there exists a $v_0 \in F(v_0)$, i.e., there exists a $v_0 \in K_F$ such that

$$\langle A(v_0,v_0), u-v_0\rangle \nleq_{\mathrm{int}C(v_0)} 0, \quad u \in K_F.$$

We now generalize this result to the whole space. Let

$$\begin{aligned}&\Lambda \equiv \{F \subset E : \dim F < \infty, F\cap K \neq \emptyset\},\\ &W_F \equiv \{w \in K : \langle A(w,u), u-w\rangle \nleq_{\mathrm{int}C(w)} 0, \forall u \in K_F\}.\end{aligned}$$

From above we know for $\forall F \in \Lambda$, $W_F \neq \emptyset$.
Let $\overline{W_F}^{w}$ denote the weak closure of W_F . Obviously we have

$$W_{\cup_{i=1}^{n} F_i} \subseteq \cap_{i=1}^{n} W_{F_i} \subseteq \cap_{i=1}^{n} \overline{W_{F_i}}^{w}.$$

So $\cap_{i=1}^{n} \overline{W_{F_i}}^{w} \neq \emptyset$. Since K is weakly compact, from the finite intersection property, we have $\cap_{F\in\Lambda} \overline{W_F^w} \neq \emptyset$. Take a $w_0 \in \cap_{F\in\Lambda} \overline{W_F^w}$, we see that

$$\langle A(w_o, w_0), u - w_0\rangle \not\leq_{\mathrm{int}C(w_o)} 0.$$

Indeed, for $\forall u \in K$, take an F from Λ such that $u \in K_F$, $w_0 \in K_F$. From $w_0 \in \overline{W_F}^{w}$, there exists $w_j \in W_F$, i.e. $\langle A(w_j, u), u - w_j\rangle \notin -\mathrm{int}C(w_j)$, which also means that $\langle A(w_j, u), u - w_j\rangle \in W(w_j)$, such that $w_j \to^{w} w_0$. From the property of $A(\cdot, u)$, we have

$$\langle A(w_j, u), u - w_j\rangle \to^{w} \langle A(w, u), u - w\rangle.$$

It follows from the weak upper semi-continuity of W that

$$\langle A(w_0, u), u - w_0\rangle \in W(w_0),$$

i.e.,

$$\langle A(w_0, u), u - w_0\rangle \not\leq_{\mathrm{int}C(w_0)} 0.$$

From the linearization lemma

$$\langle A(w_0, w_0), u - w_0\rangle \not\leq_{\mathrm{int}C(w_0)} 0.$$

This completes the proof. □

Note that the subset K in the last theorem is bounded, when it is unbounded we have the following theorem.

THEOREM 3.2. *Let X be a reflexive Banach space, Y a Banach space. Let $K \subset X$ be a nonempty, unbounded, closed, and convex subset in X and $0 \in K$. Let $C : K \to 2^Y$ be a set-valued mapping such that for all $x \in K$, $C(x)$ is a closed, pointed, and convex cone with* $\mathrm{int}C(x) \neq \emptyset$ *and $C_- = \cap_{x\in K} C(x)$ with* $\mathrm{int}C_- \neq \emptyset$. *Let the set-valued mapping $W(x) = Y \setminus -\mathrm{int}C(x)$ be weakly upper semi-continuous on K, and satisfy*

$$\lambda W(x) + (1-\lambda) W(y) \subseteq W(\lambda x + (1-\lambda) y),$$

for all $0 < \lambda < 1$. If $A : X \times X \to L(X, Y)$ satisfies the following conditions:

(a) *A is a C_--semi-monotone operator*;
(b) *for all $v \in K, A(v, \cdot)$ is continuous on each finite dimensional subspace of X*;

(c) $\lim_{\|u\|\to\infty}\langle A(u,u),u\rangle \in C_+$, where $C_+ = \text{int}C_-$.

Then the vector variational inequality (1) has a solution.

Proof. For each $r>0$, let $B[0,r]$ denote the closed ball in the Banach space X with center 0 and radius r. From Theorem 3.1, for each $n \in N$, there exists a $w_n \in B[0,n]\cap K$ such that

$$\langle A(w_n, w_n), u-w_n\rangle \not\leq_{\text{int}C(w_n)} 0, \quad \forall u \in B[0,n]\cap K.$$

Note $0 \in K$, thus

$$\langle A(w_n, w_n), w_n\rangle \not\leq_{-\text{int}C(w_n)} 0.$$

We say that the family $\{wn\}_{n\in N}$ is bounded. For otherwise, without loss of generality, let's suppose $\|w_n\| \to \infty$, when $n\to\infty$. From (3),

$$\lim_{n\to\infty}\langle A(w_n, w_n), w_n\rangle \in C_+.$$

From this, we know that when n is sufficiently large,

$$\langle A(w_n, w_n), w_n\rangle \in C_+ \subseteq \text{int}C(w_n),$$

i.e.,

$$\langle A(w_n, w_n), w_n\rangle \in \text{int}C(w_n).$$

This is a contradiction. Since X is reflexive, we can assume $w_n \to^w w$. From the complete continuity of A and the weak upper semi-continuity of W, we have

$$\langle A(w,u), u-w\rangle \not\leq_{\text{int}C(w)} 0.$$

Utilizing the linearization lemma again, we obtain

$$\langle A(w,w), u-w\rangle \not\leq_{\text{int}C(w)} 0.$$

This completes the proof. □

At the end of this section, we give three corollaries which appeared in Chen (1999) as main results.

COROLLARY 3.1 (Chen (1999)). *Let E be a reflexive Banach space, and $K\subset E$ a bounded, closed, and convex subset. Let $A: K\times K\to E^*$ be a mapping satisfying*

(a) A is semi-monotone;
(b) *For each* $u \in K$, $A(u, \cdot): K \to E^\star$ *is continuous on each finite dimensional subspace of* E.

Then the following variational inequality problem

$$(A(w, w), u - w) \geqslant 0, \quad \forall u \in K,$$

has a solution $w \in K$.

COROLLARY 3.2 (Chen (1999)). *Let* E *be a reflexive Banach space, and* $K \subset E$ *an unbounded, closed, and convex subset with* $0 \in K$. *Let* $A: K \times K \to E^\star$ *be a mapping satisfying*

(a) *A is semi-monotone;*
(b) *For each* $u \in K$, $A(u, \cdot): K \to E^\star$ *is continuous on each finite dimensional subspace of* E;
(c) $\liminf_{\|u\| \to \infty} (A(u, u), u) > 0$.

Then the following variational inequality

$$(A(w, w), u - w) \geqslant 0, \quad u \in K,$$

has a solution $w \in K$.

COROLLARY 3.3 (Chen (1992)). *Let* E *be a reflexive Banach space, and* $K \subset E$ *a nonempty, unbounded, closed, and convex subset of* X. *Let* $C: K \to 2^Y$ *be a set-valued mapping such that for all* $x \in K$, $C(x)$ *is a closed, pointed, and convex cone of* Y *with* $\mathrm{int} C(x) \neq \emptyset$ *and* $C_- = \cap_{x \in K} C(x)$ *with* $\mathrm{int} C_- \neq \emptyset$. *Suppose the set-valued mapping* $W(x) = Y \setminus -\mathrm{int} C(x)$ *is upper semi-continuous on* K, *and* $T: K \to L(X, Y)$ *is* C_-*-monotone coercive and hemicontinuous on* K. *Then there exists an* $u_0 \in K$ *such that*

$$\langle T(u_0), u - u_0 \rangle \nleq_{\mathrm{int} C(u_0)} 0, \quad \forall u \in K.$$

Proof. Define $A: K \times K \to L(X, Y), A(x, y) = T(y)$. Since T is a C_--monotone operator, $A(u, \cdot)$ is a C_--monotone operator for every $u \in K$. Obviously, for fixed $v \in K$, $A(\cdot, v)$ is completely continuous on K. It is easy to verify that the conditions of Theorem 3.2 are satisfied, so by Theorem 3.2, there exists an $u_0 \in K$ such that

$$\langle A(u_0, u_0), u - u_0 \rangle \nleq_{\mathrm{int} C(u_0)} 0, \quad \forall u \in K,$$

i.e.,

$$\langle T(u_0), u - u_0 \rangle \nleq_{\mathrm{int} C(u_o)} 0, \quad \forall u \in K. \qquad \square$$

4. Concluding Remarks

Several extensions of the preceding results are conceivable. For instance, it is interesting to pose the concept of a set-valued semi-monotone operators, and then study a vector variational inequality problem with this kind of operator, which promise nice results in applications. The extension of the theory of duality for this kind of vector variational inequalities is also interesting both from a theoretical point of view and for finding solutions.

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