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AN EXAMPLE OF AN ACCURATE SOLUTION OF A PROBLEM ON STRATIFIED FLOWS CAUSED BY SPATIAL INHOMOGENEITIES OF TRANSFER COEFFICIENTS

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In the recent literature, attention has been drawn to the previously uninvestigated mechanism of the occurrence of flows in a stratified fluid in a gravity field. Such flows can occur in the absence of pulse and buoyancy sources due to horizontal inhomogeneity of transfer coefficients. For the first time, this investigation provides an example of an exact analytical solution of such a problem free from assumptions of amplitude smallness.

Keywords: *stratified media, inhomogeneous transfer coefficients, gravity flows, accurate analytical solution.*

Introduction. In recent investigations [1, 2], attention has been drawn to the previously uninvestigated mechanism of the occurrence of flows in a stratified fluid in a gravity field. It is obvious that in a steadily stratified medium, there is heat (buoyancy) diffusion directed from top to bottom. If the thermal conductivity coefficient is spatially inhomogeneous (depends on horizontal coordinates), this results in the occurrence of horizontal inhomogeneities in distributions of buoyancy and hydrostatic pressure (weight of the medium column) and hence in the occurrence of horizontal flows. Spatial inhomogeneities of effective transfer coefficients are especially characteristic of turbulent exchange [1–3]. Therefore, such density currents must exist, for example, in the atmosphere [1].

Due to the complexity of mathematical problems with spatially inhomogeneous transfer coefficients, accurate analytical solutions of such problems have been almost absent so far in the literature. This paper provides an example of such a solution for the simplest two-layer model.

Problem Formulation. Let an unbounded medium be stratified steadily; the constant vertical temperature gradient $\gamma > 0$, and the thermal diffusivity K_u = const. In a stationary mode, in this medium, there is a homogeneous descending heat flux proportional to γK_u. Suppose that at a certain instant of time, in the region below certain inclined interface $n = 0$ (Fig. 1), the thermal diffusivity decreased to the constant value $K_d < K_u$. On the inclined interface, there occurs heat flux discontinuity: above the boundary, it was proportional to γK_u and below, it would be equal for a moment to a smaller γK_d value. Hence, in the vicinity of the inclined interface, heat accumulation will start. Due to buoyancy deviation, an ascending motion must occur along this boundary. This motion brings from below colder fluid volumes, which results in the compensation of excessive heat accumulated due to the difference in the transfer coefficients. We may assume that, as a result, a stationary flow occurs along the inclined interface, providing heat balance (the geometry of the problem is schematically shown in Fig. 1.

Stationary Solution. We seek an appropriate stationary solution for a two-layer one-dimensional problem in a system of coordinates (s, n). For simplicity, we limit ourselves to the case when the values of heat transfer and momentum transfer coefficients coincide (the value of the Prandtl number is everywhere equal to unity, which is a widely accepted hypothesis in describing turbulent exchange). The system of equations for the dynamics and transfer of heat in a Boussinesq approximation above the inclined interface is similar to the model of Prandtl downslope flows and has the form [4–9]:

$$0 = K_u \frac{d^2 u}{dn^2} + \alpha g \theta \sin\varphi , \quad \gamma u \sin\varphi = K_u \frac{d^2\theta}{dn^2} . \tag{1}$$

Here, θ is temperature deviation (for air in the atmosphere, potential temperature [4]) from the background.

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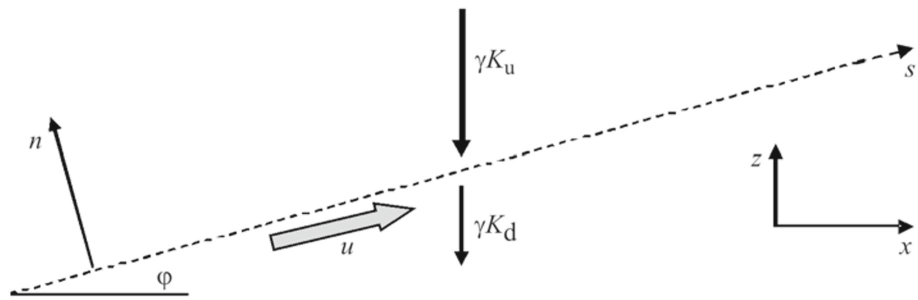

Fig. 1. Downward vertical arrows indicate schematically the diffusion heat fluxes above and below the interface (at different values of thermal diffusivity).

Note that the system of hydrodynamics and heat transfer equations was reduced to a linear system (1) only due to the symmetry of the problem without any assumptions about the smallness of perturbation amplitudes. Excluding one of the variables, we come to the equation

$$\frac{d^4u}{dn^4} + \frac{u}{h_\mathrm{u}^4} = 0 , \tag{2}$$

where the spatial scale

$$h_\mathrm{u} = \left(\frac{2K_\mathrm{u}}{N \sin \varphi}\right)^{1/2} ,$$

and the buoyancy frequency $N = (\alpha g \gamma)^{1/2}$. The general solution of Eq. (2) represents a linear combination of four exponents with complex indices. Considering the disturbance decay conditions at $n \to \infty$, we have

$$u_u = \left(C_1 \sin \frac{n}{h_\mathrm{u}} + C_2 \cos \frac{n}{h_\mathrm{u}}\right) \exp\left(-\frac{n}{h_\mathrm{u}}\right) . \tag{3}$$

Below the interface, the solution appears similar but with the change of sign of the exponent and the replacement of the subscript "u" with "d":

$$u_\mathrm{d} = \left(C_3 \sin \frac{n}{h_\mathrm{d}} + C_4 \cos \frac{n}{h_\mathrm{d}}\right) \exp\left(\frac{n}{h_\mathrm{d}}\right) , \quad h_\mathrm{d} = \left(\frac{2K_\mathrm{d}}{N \sin \varphi}\right)^{1/2} . \tag{4}$$

Using (1), it is also not difficult to obtain expressions for temperature deviations. The solutions on both sides of the interface join at $n = 0$ where the following conditions are fulfilled:

$$u_\mathrm{d} = u_\mathrm{u} , \quad \theta_\mathrm{u} = \theta_\mathrm{d} , \quad K_\mathrm{d} \frac{du_\mathrm{d}}{dn} = K_\mathrm{u} \frac{du_\mathrm{u}}{dn} , \quad K_\mathrm{d} \left(\frac{d\theta_\mathrm{d}}{dn} + \gamma \cos \varphi\right) = K_\mathrm{u} \left(\frac{d\theta_\mathrm{u}}{dn} + \gamma \cos \varphi\right) .$$

The latter condition is an equality of diffusion heat fluxes on both sides of the interface. From the above-mentioned conditions, it is not difficult to obtain expressions for integration constants:

$$C_1 = C_2 = C_4 = -C_3 = U \equiv \left(\frac{N}{2 \sin \varphi}\right)^{1/2} (K_\mathrm{u}^{1/2} - K_\mathrm{d}^{1/2}) \cos \varphi .$$

Thus, the solution will have the form

$$u = U \left(\pm \sin \frac{n}{h_\mathrm{u,d}} + \cos \frac{n}{h_\mathrm{u,d}}\right) \exp\left(\mp \frac{n}{h_\mathrm{u,d}}\right) , \quad \theta = \frac{NU}{\alpha g} \left(\mp \sin \frac{n}{h_\mathrm{u,d}} + \cos \frac{n}{h_\mathrm{u,d}}\right) \exp\left(\mp \frac{n}{h_\mathrm{u,d}}\right) ,$$

where the top sign and the subscript "u" correspond to the region above the interface.

Solution Analysis. We can see that if the intensity of exchange in the lower region is weaker than in the upper one ($K_d < K_u$), then $u > 0$; near the interface $n = 0$, as was to be expected, there is a positive temperature deviation and an ascending motion along this boundary. It is interesting to note that with decrease in the angle of inclination φ this effect of the inhomogeneity of transfer coefficients increases: the layers′ thickness $h_{u,d}$ increases; flatly inclined flows transfer little heat, and a rather high velocity is required to sustain the heat balance near the interface. But this refers to the stationary mode under investigation whose setting time must increase with decrease in the inclination angle since the spatial scales $h_{u,d}$ grow.

We assume the parameter values characteristic of turbulent exchange in the atmosphere: $N = 10^{-2}$ s^{-1}, $K_u = 5$ m^2/s, $K_d = 1$ m^2/s. Then at $\varphi = 0.05$, the velocity of the occurring flow is ~0.4 m/s, and the temperature deviation is about 0.1 K.

Conclusions. In a quite substantive body of literature on the theory of downslope flows (see, for example, [5–9] and the references in these works), it is commonly believed that these flows are due to thermal disturbances on the lower boundary. A qualitatively new result follows from the above-said: at a horizontal imhomogeneity of transfer coefficients, when the density gradient does not coincide in direction with the force of gravity, the static state in nonequilibrium stratified media is impossible. Significant flows may occur in the absence of sources of buoyancy and/or momentum in the medium.

This also means the existence of the effect of occurrence of ordered flows in a turbulized density-stratified medium near hard inclined boundaries. Indeed, near hard boundaries, turbulent exchange is weakened, hence, there is spatial inhomogeneity of effective transfer coefficients, which, as shown above, must result in the occurrence of regular flows.

NOTATION

C_j, integration constants, m/s; g, free-fall acceleration, m/s^2; h, spatial scale, m; K, transfer coefficient, m/s^2; N, buoyancy frequency (Brunt–Väisälä frequency), s^{-1}; n, coordinate in the direction normal to the slope, m; s, slope coordinate, m; U, velocity scale, m/s; u, velocity component in the direction of the s axis, m/s; x and z, horizontal and vertical coordinates, m; α, coefficient of thermal expansion of a medium, K^{-1}; γ, background vertical gradient of temperature (of potential temperature), K/m; θ, temperature deviation, K; φ, lower boundary angle of inclination to the horizon, rad. Subscripts: d and u refer to the lower and upper layers, respectively; j, integration constant number.

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