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HEAT CONDUCTION AND HEAT TRANSFER IN TECHNOLOGICAL PROCESSES

UNIQUENESS AND STABILITY OF SOLVING THE INVERSE PROBLEM OF THERMOELASTICITY. PART 2. REGULARIZATION

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Based on the analysis of direct variational methods used in the Hilbert space — the regularization method and the iterative regularization method — an iterative variational method was developed for regularization of the mathematically incorrect solution of nonlinear inverse thermoelasticity problems described by partial differential equations. Using the quadratic functional of the regularization method, an integral equation of the first kind is obtained, which connects the norms of increments of the direct and inverse thermoelasticity problems. The solution of the inverse problem is linearized by calculating the norms in the Hilbert space of square-integrable functions. The integral equation is regularized by reducing it to the Euler equation. The discretization of the boundary-value problem, described by the Euler equation, is performed, and the resulting system of linear algebraic equations is solved. A computational experiment was carried out for the simultaneous identification of two nonlinear temperature functions that confirms the efficiency of the method and shows that in the iterative selection of a quasi-solution for simultaneous determination of several functions, one experimental mode can be used.

Keywords: *thermoelasticity, inverse problems, regularization, variational method, finite-difference method.*

Introduction. The article considers the problem of determining the function w on the right-hand side of the operator equation [1]:

$$Au = f(w) , \quad f \in F , \quad u \in U , \tag{1}$$

where F, $U \subset R$ are the metric spaces, and the right side of the equation f includes the boundary and initial conditions, If this part is given exactly, then, if the solution of the equation exists and is unique, it is usually stable, i.e., depends continuously on f.

The inverse problem for the desired function w is formulated similarly:

$$Bw = u , \quad B = A^{-1} f(w) , \quad f \in F , \quad u \in U . \tag{2}$$

In inverse problems, the element w is determined by experimental (identification) or specially set (control) function u_δ, which has a dispersion or is known approximately. Further, we will talk about Eq. (1), implying the same for Eq. (2).

The solution of Eq. (1) is correct according to Tikhonov if it is known that for the exact value $f = f_0$ there is a unique solution to the equation $Au_0 = f_0$, which belongs to the given compact D_A. In this case, the operator A^{-1} is continuous (outside these restrictions A^{-1} may not be continuous) on the set $R_A = AD_A$, and if instead of the element f_0 the element f_δ is known such that $\rho_F(f_0, f_\delta) \le \delta$ and $f_\delta \in R_A$, then as an approximate solution of Eq. (1) with the right side $f = f_\delta$ we can take the element $u_\delta = A^{-1} f_\delta$. As $\delta \to 0$, u_δ tends to u_0. The set $U_1 \subset \mathrm{U}$, on which the solution of Eq. (1) is correct, is called the correctness class. So, if the operator A is continuous and performs a one-to-one mapping, then the compact D_A, to which u_0 belongs, is the correctness class for Eq. (1) [2].

In practical problems, due to the errors in the initial data, instead of the exact value of the right side f_0, its approximate value f_δ is known, which may belong to the set $R_A = AD_A$, which is not compact. In these cases, it is impossible to construct an approximate solution of Eq. (1) using the formula $u_\delta = A^{-1} f_\delta$, since the symbol $A^{-1} f_\delta$ may not make sense. For this reason,

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the concept of a quasi-solution $u_\delta \in U$ is introduced and the selection method under the condition of the compactness of the set U allows one to find approximation to the quasi-solution [2]:

$$\rho_F(Au_\delta, f) = \inf_{u \in U} \rho_F(Au, f) . \tag{3}$$

If U is a compact, then the quasi-solution exists for any $f \in F$, and if, moreover, $f \in AU$, then $u_\delta = u_0$. There may be more than one quasi-solution. In this case, a quasi-solution is understood as any element from a set of quasi-solutions. The quasi-solution is not correct according to Tikhonov.

Solution (3) is similar to the solution of Eq. (1) by the direct variational method of least squares with the only difference that it was obtained without taking into account the correctness of the problem formulation, determined by the linearity of the operator A, as well as by the uniqueness and stability of the solution [1].

The computational practice makes a wide use of the quasi-solution subset method, in which for the elements u of some predetermined subset of possible solutions $M \subset U$ the operator Au is calculated, i.e., the direct problem is solved. As an approximate solution, such element u_δ is taken from the set M, on which the residual minimum (3) is reached. Selection of a quasi-solution can be carried out iteratively [2]:

$$w^{(l+1)} = w^{(l)} + \Delta w^{(l)} , \quad l = 0 , 1 , \dots . \tag{4}$$

In the case of functional (pointwise) identification of the operator equation (1), the desired function w is discretized on the time grid τ as a vector $\mathbf{w}$:

$$\boldsymbol{\tau} = (\tau_0, \tau_2, \dots , \tau_M)^{\mathrm{T}} , \quad \mathbf{w} = (w_0, w_2, \dots , w_M)^{\mathrm{T}} .$$

Then expression (4) is written for each coordinate of the vector $\mathbf{w}$:

$$w_m^{(l+1)} = w_m^{(l)} + \Delta w_m^{(l)} , \quad l = 0 , 1 , \dots ; \quad m = 1 , \dots , M .$$

In the case of parametric identification, the function w is represented as a series, which is built using the basis $\varphi_k(x, \tau)$, $k = 1, 2, \dots$:

$$w(x, \tau) = \sum_{k=1}^{\infty} a_k \varphi_k ,$$

with x conditionally denoting the required number of spatial coordinates x_1, x_2, ... In numerical implementation of algorithms, they are limited to the partial sum of the series that includes K first terms of the series. Then the following vector becomes the desired one:

$$\mathbf{a} = (a_1, a_2, \dots , a_K)^{\mathrm{T}} ,$$

and the increment is calculated for each coordinate of this vector:

$$a_k^{(l+1)} = a_k^{(l)} + \Delta a_k^{(l)} , \quad l = 0 , 1 , \dots ; \quad k = 1 , 2 , \dots , K .$$

Variational Method for Regularizing an Unstable Solution of the Equation $Au = f_\delta$ with an Approximately Given Right-Hand Side. Equation (1) with an approximately given right side f_δ that belongs to the range of the values of R_A of operator A, which is not compact, describes an essentially ill-posed problem whose quasi-solution is conditionally correct according to Tikhonov. Significantly ill-posed problems exclude the last of the four conditions for the applicability of the least squares method — the condition for the uniqueness of the solution, but leave the possibility of minimizing the norm of the difference between the left and right sides of the equation

$$Au = f_\delta , \tag{5}$$

and, consequently, a quadratic functional of the form of (3), which ensures the convergence of the elements f of the space F. For simultaneous convergence of elements u of space U we will add a stabilizing term (stabilizer) [2]:

$$Mu = \|Au - f_\delta\|^2 + \gamma\Omega(u) = \rho_F^2(Au, f_\delta) + \gamma\Omega(u) . \tag{6}$$

Here $\gamma > 0$ is the regularization parameter determined uniquely by the known discrepancy δ:

$$||Au(\gamma) - f_\delta||^2 = \delta.$$

The pseudo-solution of the system of equations (5) is the element u, which minimizes the norm $||Au - f||$ over the entire space F. The system of equations (5) may have more than one pseudo-solution. Let D_A be a set of all pseudo-solutions of system (5) and u_δ be a certain fixed element from R^N, which is usually determined by the statement of the problem. The normal solution for the element u_δ (normal solution) of system (5) is the pseudo-solution u_0 with the minimum norm $||u - u_\delta||$ [2]:

$$\|u_0 - u_\delta\| = \inf_{u \in D_A} \|u - u_\delta\| .$$

For any system of the form (1) a normal solution exists, and it is unique. Since the normal solution is a pseudo-solution, the sequence of elements converging to the normal solution in the space U ($D_A \subset U$) converges simultaneously also in the space F (as, for example, in the method of least squares for a linear operator A):

$$u \to u_0, \; Au \to Au_0 = f_0 .$$

Therefore, it is logical to choose the stabilizer Ω in the form of the square of the distance between the elements u and $u_0 \in D_A \subset U$:

$$\Omega(u) = \tilde{\rho}_U^2(u, u_0) = \|u - u_0\|^2 \tag{7}$$

Then the quadratic functional (6) can be written in the form

$$\begin{gathered} Mu = \tilde{\rho}_F^2(Au, f_0) + \gamma \tilde{\rho}_U^2(u, u_0) = \|Au - f_0\|^2 + \gamma \|u - u_0\|^2 \\ = (Au - f_0, Au - f_0) + \gamma(u - u_0, u - u_0) \\ = (Au, Au) - 2(Au, f_0) + (f_0, f_0) + \gamma(u, u) - 2\gamma(u, u_0) + \gamma(u_0, u_0) . \end{gathered} \tag{8}$$

The regularization method is applicable for both functional and parametric identification of the function w included in the right side of the operator equation of the first kind $Au = f(w)$. As shown above, such problems are inverse in nature and often have an unstable solution for the function w, since the function u_δ, necessary to calculate the right side of $f_\delta = Au_\delta$, is determined experimentally or is set approximately, which leads to large fluctuations of the differential operator Au for small fluctuations of the function u. Therefore, it is expedient to lead the solution of inverse problems to the sequence of solutions of direct problems, avoiding direct differentiation of the experimental function and using it as a normal solution of the direct problem u_0. Then the approximate solution of the direct problem minimizes the functional (8):

$$\begin{gathered} \frac{dMu}{du} = \frac{d(Au, Au)}{du} - 2\frac{d(Au, f_0)}{du} + \gamma \frac{d(u, u)}{du} - 2\gamma \frac{d(u, u_0)}{du} = 0; \\ \frac{d}{du}\left[(Au, Au) - 2(Au, f_0) + \gamma(u, u) - 2\gamma(u, u_0)\right] = 0; \\ Mu - (f_0, f_0) - \gamma(u_0, u_0) = \text{const}; \\ Mu_0 = 0, \Rightarrow \text{const} = -(f_0, f_0) - \gamma(u_0, u_0); \\ (Au, Au) - 2(Au, f_0) + \gamma(u, u) - 2\gamma(u, u_0) = \text{const}; \\ 2\gamma(u, u_0) - \gamma(u, u) = (Au, Au) - 2(Au, f_0) - \text{const}; \\ \gamma(u, 2u_0 - u) = (Au, Au - 2f_0) - \text{const} = (f, f - 2f_0) - \text{const}; \\ \gamma(u, 2u_0 - u) = (f, f - 2f_0) + (f_0, f_0) + \gamma(u_0, u_0); \end{gathered} \tag{9}$$

$$-\gamma(u,\, 2u_0 - u) = -(f,\, f - 2f_0) - (f_0,\, f_0) - \gamma(u_0,\, u_0);$$

$$\gamma(u_0,\, u_0) - \gamma(u,\, 2u_0 - u) = (f,\, 2f_0 - f) - (f_0,\, f_0);$$

$$\gamma\left[(u_0,\, u_0) - 2(u,\, u_0) + (u,\, u)\right] = -(f,\, f) + 2(f,\, f_0) - (f_0,\, f_0);$$

$$\gamma\left[(u_0,\, u_0) - 2(u,\, u_0) + (u,\, u)\right] = -\left[(f,\, f) - 2(f,\, f_0) + (f_0,\, f_0)\right];$$

$$\gamma(u - u_0,\, u - u_0) = -(f - f_0,\, f - f_0);$$

$$\gamma \left\|u - u_0\right\|^2 = -\left\|f - f_0\right\|^2;$$

$$\rho_F^2(f,\, f_0) = -\gamma\rho_U^2(u,\, u_0)\,.$$

To avoid complex values, we choose the stabilizer (7) with the opposite sign:

$$\Omega(u) = -\rho_U^2(u,\, u_0) = -\left\|u - u_0\right\|^2 \Rightarrow Mu = \left\|Au - f_0\right\|^2 - \gamma\left\|u - u_0\right\|^2\,.$$

The same result is obtained if we consider the metric space U imaginary:

$$\Omega(u) = \rho_U^2(u,\, u_0) = -\left\|u - u_0\right\|^2 \Rightarrow \rho_U(u,\, u_0) = i\left\|u - u_0\right\|,$$

where i is an imaginary unit. Then

$$\rho_F^2(f,\, f_0) = \gamma\rho_U^2(u,\, u_0)\,.$$

For an iterative process, assuming at each iteration that $f^{(l+1)} = f_0$, we have

$$\left(f^{(l+1)} - f^{(l)},\, f^{(l+1)} - f^{(l)}\right) = \gamma\left(u^{(l)} - u_0,\, u^{(l)} - u_0\right);$$

$$\left\|\Delta f^{(l)}\right\|^2 = \left\|f^{(l+1)} - f^{(l)}\right\|^2 = \gamma\left\|u^{(l)} - u_0\right\|^2; \qquad l = 0,\, 1,\, 2,\, \ldots,$$

$$\rho_F^2\left(f^{(l+1)},\, f^{(l)}\right) = \gamma\rho_U^2\left(u^{(l)},\, u_0\right),$$

i.e., the square of the increment norm of the right side of the equation $Au = f(w)$ (in the metric space F) is equal to the square of the norm of the difference of the solution of this equation (in the metric space U) for an approximate value of $u^{(l)}$ known at the current iteration $l = 0, 1, 2, \ldots$, and the normal solution (found experimentally) multiplied by the regularization parameter $\gamma > 0$. Then we have the equation

$$f(w)^{(l+1)} = f^{(l)}(w) + \Delta f^{(l)}(w);\ l = 0,\, 1,\, 2,\, \ldots\,.$$

The regularization parameter has the meaning of a dimensional factor that matches the units of the left and right sides of this equation and speeds up or slows down the iterative process depending on the convergence of the solution, actually being in the interval $\gamma > 0$.

If the metric spaces U and F are linear, then by virtue of the existence of their norm they are Banach. Since the norm is introduced on the basis of the scalar product, these Banach spaces are pre-Hilbert spaces and, if they have the completeness property, then they are also Hilbert spaces. Calculating the norm in the Hilbert space, we linearize the solution, subjecting its convergence to the regularities of the complete linear (Hilbert) space.

We write Eq. (9) in the Hilbert space $L_2 = U = F$ on the time interval $0 \le \tau \le \tau_M$ in the spatial area G:

$$\rho_{L_2}^2\left(f^{(l+1)},\, f^{(l)}\right) = \gamma\rho_{L_2}^2\left(u^{(l)},\, u_0\right);\ \left\|\Delta f^{(l)}\right\|_{L_2}^2 = \gamma\left\|u^{(l)} - u_0\right\|_{L_2}^2;$$

$$\int_0^{\tau_M} d\tau \int_G \left(\Delta f^{(l)}(x,\, \tau)\right)^2 dx = \gamma\int_0^{\tau_M} d\tau \int_G \left(u^{(l)}(x,\, \tau) - u_0(x,\, \tau)\right)^2 dx,\ Q = G \times (0,\, \tau_M), \tag{10}$$

where x denotes, as usual, the required number of spatial coordinates. Relation (10) is an integral equation of the first kind with an unknown integrand Δf on the left side. The right-hand side of the equation is known approximately, since the value of the function u on iteration l is found from the solution of the problem $Au = f(w)$ using the approximately calculated function w.

The function of one independent variable, which is conveniently chosen to be the time τ, is unknown in integral equations. For simplicity, we restrict ourselves to the one-dimensional region $G = (0, l)$, introduce a uniform grid

$$x_n = n\Delta x\,, \quad n = 0, 1, 2, \ldots, N\,, \quad \Delta x = \frac{l}{N}\,,$$

and write the difference analogues of the space integrals on the left- and right-hand sides of the indicated equations:

$$\sum_{n=1}^{N} \int_0^{\tau_M} \left(\Delta f^{(l)}(x_n, \tau)\right)^2 d\tau\Delta x = \gamma \sum_{n=1}^{N} \int_0^{\tau_M} \left(u^{(l)}(x_n, \tau) - u_0(x_n, \tau)\right)^2 d\tau\Delta x\,,$$

$$\sum_{n=1}^{N} \int_0^{\tau_M} \left(\Delta f^{(l)}(x_n, \tau)\right)^2 d\tau = \gamma \sum_{n=1}^{N} \int_0^{\tau_M} \left(u^{(l)}(x_n, \tau) - u_0(x_n, \tau)\right)^2 d\tau\,.$$

Equating the terms with the same values of n on the left and right sides of these equations, we obtain a system of linear integral equations for each coordinate x_n:

$$\int_0^{\tau_M} \left(\Delta f^{(l)}(x_n, \tau)\right)^2 d\tau = \gamma \int_0^{\tau_M} \left(u^{(l)}(x_n, \tau) - u_0(x_n, \tau)\right)^2 d\tau\,.$$

The square of the increment of the desired function is determined by the relation

$$z_n(\tau) = z(x_n, \tau) = \left(\Delta f^{(l)}(x_n, \tau)\right)^2\,, \tag{11}$$

with account for which we obtain the equation

$$\int_0^{\tau_M} z_n(\tau) d\tau = \gamma \int_0^{\tau_M} \left(u^{(l)}(x_n, \tau) - u_0(x_n, \tau)\right)^2 d\tau\,.$$

and an integral equation of the first kind

$$Bz_n = \int_0^{\tau_M} K(x_n, \tau) z_n(\tau) d\tau = y_\delta(x_n)\,, \quad x \in [0, l]\,, \tag{12}$$

with the kernel $K(x_n, \tau) = 1$.

There may exist three main, not equivalent to each other, approaches to the discretization of the problem of finding approximate (regularized) solutions of Eq. (12) at $Bz = y_\delta$ [2].

1. The initial equation (12) is discretized at $Bz = y_\delta$ by replacing the integral by the integral sum using some quadrature formula. As a result, a degenerate or ill-conditioned system of linear algebraic equations is obtained, an approximate solution of which, stable to small changes in the right-hand side of the equations, must be found. This can be done by the regularization method. If we use the variational principle, then, taking the discrete analogue of the stabilizer $\Omega[z]$, it is possible to form a discrete analogue of the smoothing functional. Then a transition is made to the Euler equation that describes it, which is a regularized system of linear algebraic equations. The solutions of this system with the corresponding values of the regularization parameter ξ will be approximate solutions of Eq. (12) [2].

2. The smoothing functional is discretized, and the problem of minimization of the function of many variables with subsequent determination of the regularization parameter ξ is solved [2].

3. Discretization of the boundary-value problem for the Euler equation (12) is carried out, and then the resulting system of linear algebraic equations is solved.

Let us consider in more detail the third approach. We will carry out discretization on a uniform time grid, assuming that the kernel $K(x, \tau)$ in Eq. (12) is a real function continuous in the domain $\{0 \le \tau \le \tau_M, 0 \le x \le l\}$ at $Bz = y_\delta$. Let us take a stabilizing functional of the form [2]

$$\Omega[z] = \int_0^{\tau_M} \left\{ qz^2 + p(z')^2 \right\} d\tau ,$$

where $q, p > 0$ are positive numbers.

Let the exact solution $z_{ex}(\tau)$ belong to the correctness class F_1 and satisfy one of the boundary conditions $z_{ex}(0) = 0$, $z_{ex}(\tau_M) = 0$, $z'_{ex}(0) = 0$, and $z'_{ex}(\tau_M) = 0$. Physically, the choice of such conditions is explained by the fact that both the initial and final states of the nonstationary regime are quasistationary. Therefore, during a certain period of time at the beginning and end of the regime, while the values of the derivatives are set, the desired functions cannot be determined or will be singular. In the singularity region, the identifiable functions are assumed to be constant and equal to the first (for the beginning of the interval) or the last (for the end of the interval) non-singular value. Then, as regularized solutions $z(\tau)$ of Eq. (12) at $Bz = y_\delta$ one can take functions that are solutions of the following boundary-value problem for the Euler equation [1]:

$$\int_0^{\tau_M} \bar{K}(\tau, t) z(t) dt + \xi \left\{ qz(\tau) - pz''(\tau) \right\} = g(\tau) , \quad \bar{K}(\tau, t) = \int_0^{\Delta x} K(x, \tau) K(x, t) dx = \Delta x ;$$
$$g(\tau) = \int_0^{\Delta x} K(x, \tau) y_{\ddot{a}}(x) dx = \gamma \int_0^{\Delta x} dx \int_0^{\tau_M} \left(u^{(l)}(x_n, \tau) - u_0(x_n, \tau) \right)^2 d\tau = \gamma \Delta x \int_0^{\tau_M} \left(u^{(l)}(x_n, \tau) - u_0(x_n, \tau) \right)^2 d\tau \qquad (13)$$

with boundary conditions $z'(a) = 0$ and $z'(b) = 0$.

Let us write the difference analogue of Eq. (13) on a uniform grid with a step h. Let us split the segment $[0, \tau_M]$ into M equal parts and take as nodal points the ends of the resulting segments [2] $\tau_i = ih$, $i = 1, 2, \ldots, M$, and $h = \tau_M / M$. Replacing the integral on the left side of Eq. (13) with the integral sum corresponding to it, for example, according to the formula of rectangles, and $z''(s)$ with the corresponding difference relation, we obtain [2]

$$\Delta x \sum_{j=1}^{M} hz_j + q\xi z_i + p\xi \frac{2z_i - z_{i-1} - z_{i+1}}{h^2} = g_i , \quad i = 1, 2, \ldots, M ,$$
$$g(\tau_i) = \gamma \Delta x \sum_{m=1}^{M} \left(u^{(l)}(x_n, \tau_m) - u_0(x_n, \tau_m) \right)^2 (\tau_m - \tau_{m-1}) . \qquad (14)$$

The values of $\bar{K}(\tau_i, t_j)$ and g_i are either calculated analytically, or using the corresponding quadrature formulas. In this case, the numbers of the grid points along the coordinates x and τ are not interrelated. At $i = 1$ and $i = M$, the system of linear algebraic equations (14) relative to the vector $\mathbf{z} = (z_1, z_2, ..., z_M)$ includes the undefined values of z_0 and z_{M+1}. To satisfy the boundary conditions, we set that $z_0 = z_1$ and $z_{M+1} = z_M$ taking into account the remark on the singularity of values at the beginning and end of the interval.

Thus, the problem of finding approximate (regularized) solutions to Eq. (12) at $Bz = y_\delta$ is reduced to solving a system of linear algebraic equations in the vector $\mathbf{z} = (z_1, z_2, \ldots, z_M)$ for each point x_n ($n = 1, 2, \ldots, N$). The considered problem can easily be generalized to the multidimensional domain G, as well as to a system of ordinary differential equations (in this case, each coordinate x_n corresponds to a point of the system with lumped parameters.) Since the right-hand side f of the original equation formally includes the boundary and initial conditions, then the presented method is also applicable to determination of boundary (diagnostics) and initial (retrospective problem) conditions.

Calculation of Functions on the Right Side of the Nonstationary Wave Equation. During identification of the functions $E(T) = E(x)$ and $\beta(T) = \beta(x)$ or the function $P = P(x, \tau)$ we have a nonstationary wave equation

$$Au = f,\ u = u(x, \tau),\ A = \frac{\partial^2}{\partial \tau^2},\ f = \frac{1}{\rho}\left[\frac{\partial}{\partial x}\left[E(x)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial x}\left[\beta(x)T\right] + P(x, \tau)\right].$$

Assuming that the functions $E(x)$ and $\beta(x)$ depend weakly on the coordinate for a small calculated volume, in the absence of their time dependence, we will obtain the constant functions E and β. However, since the right side of this equation is a function of time, we will assume that $E = E(\tau)$ and $\beta = \beta(\tau)$:

$$f = \frac{1}{\rho}\left[E(\tau)\frac{\partial^2 u}{\partial x^2} + \beta(\tau)\frac{dT}{dx} + P(x, \tau)\right]. \tag{15}$$

The increment of the right side of this equation depends on the increments of the desired and wave functions:

$$\Delta f_1(x_n, \tau) = f\left[\delta E(x_n, \tau), \delta u_1(x_n, \tau)\right],$$

$$\Delta f_2(x_n, \tau) = f\left[\delta\beta(x_n, \tau), \delta u_2(x_n, \tau)\right],$$

$$\Delta f_3(x_n, \tau) = f\left[\delta P(x_n, \tau), \delta u_3(x_n, \tau)\right],$$

where, according to (11), $\Delta f^{(l)}(x_n, \tau) = \pm\sqrt{z(x_n, \tau)}$.

According to the theorem of optimal planning a thermophysical experiment [5, 6], for each of the desired functions such a regime is necessary that all modes taken together could be linearly independent and their Gram determinant be different from zero. Differentiating Eq. (15) with respect to temperature and multiplying by the temperature differential, we obtain expressions for the increments of the desired functions:

$$\Delta f_1 = \frac{1}{\rho}\left[\frac{\partial^2 \delta u}{\partial x^2}E + \frac{\partial^2 u}{\partial x^2}\delta E\right],\ \Delta f_2 = \frac{1}{\rho}\left[E\frac{\partial^2 \delta u}{\partial x^2} + \frac{dT}{dx}\delta\beta\right],\ \Delta f_3 = \frac{1}{\rho}\left[E\frac{\partial^2 \delta u}{\partial x^2} + \delta P(x, \tau)\right],$$

where $\delta u = u^{(0)} - u$. Then

$$\Delta f_1 = \frac{1}{\rho}\left[-\frac{\partial^2 u}{\partial x^2}E + \frac{\partial^2 u}{\partial x^2}\delta E\right],\ \Delta f_2 = \frac{1}{\rho}\left[-E\frac{\partial^2 u}{\partial x^2} + \frac{dT}{dx}\delta\beta\right],\ \Delta f_3 = \frac{1}{\rho}\left[-E\frac{\partial^2 u}{\partial x^2} + \delta P(x, \tau)\right], \tag{16}$$

or

$$\delta E = \frac{\rho\Delta f_1 + \dfrac{\partial^2 u}{\partial x^2}E}{\dfrac{\partial^2 u}{\partial x^2}},\quad \delta\beta = \frac{\rho\Delta f_2 + \dfrac{\partial^2 u}{\partial x^2}E}{\dfrac{dT}{dx}},\quad \delta P = \rho\Delta f_3 + \frac{\partial^2 u}{\partial x^2}E,$$

i. e.,

$$\delta E(x_n, \tau_m) = \frac{\rho\Delta f_1(x_n, \tau_m)}{\dfrac{\partial^2 u(x_n, \tau_m)}{\partial x^2}} + E(x_n, \tau_m),\quad \delta\beta(x_n, \tau_m) = \frac{\rho\Delta f_2(x_n, \tau_m) + \dfrac{\partial^2 u(x_n, \tau_m)}{\partial x^2}E(x_n, \tau_m)}{\dfrac{dT}{dx}},$$

$$\delta P(x_n, \tau_m) = \rho\Delta f_3(x_n, \tau_m) + \frac{\partial^2 u(x_n, \tau_m)}{\partial x^2}E(x_n, \tau_m),\ n = 1, 2, \ldots, N-1;\ m = 0, 1, 2, \ldots, M.$$

Writing the temperature derivatives in finite differences, we obtain the relations

$$\delta E_{n,m} = \frac{\rho\Delta f_{1,n,m}}{\dfrac{\partial^2 u_{n,m}}{\partial x^2}} + E_{n,m}, \qquad \delta\beta_{n,m} = \frac{\rho\Delta f_{2,n,m} + \dfrac{\partial^2 u_{n,m}}{\partial x^2} E_{n,m}}{\dfrac{dT_n}{dx}},$$

$$\delta P_{n,m} = \rho\Delta f_{3,n,m} + \frac{\partial^2 u_{n,m}}{\partial x^2} E_{n,m}; \qquad \frac{\partial^2 u_{n,m}}{\partial x^2} = \frac{u_{n-1,m} - 2u_{n,m} + u_{n+1,m}}{2(x_n - x_{n-1})}; \tag{17}$$

$$\frac{dT_n}{dx} = \frac{T_n - T_{n-1}}{x_n - x_{n-1}},$$

$$n = 1, 2, ..., N-1; \; m = 0, 1, 2, ..., M.$$

Then

$$E_{n,m}^{(l+1)} = E_{n,m}^{(l)} + t_E \delta E_{n,m}^{(l)}, \; \beta_{n,m}^{(l+1)} = \beta_{n,m}^{(l)} + t_\beta \delta\beta_{n,m}^{(l)},$$

$$P_{n,m}^{(l+1)} = P_{n,m}^{(l)} + t_P \delta P_{n,m}^{(l)}, \; n = 1, 2, ..., N-1; \; m = 0, 1, 2, ..., M. \tag{18}$$

For constant functions E = const and β = const calculations can be performed for one of the values $n = 1, 2, \ldots, N-1$.

In expressions (16) for the increment of the right side of the equations Δf, only one desired function varies each time. When calculating the increment (17) of this function, the rest of the desired functions are taken from the previous iteration and are conditionally considered known. Therefore, in the iterative selection of the quasi-solution according to the proposed algorithm to determine several functions, one mode may be sufficient: $\Delta f_1 = \Delta f_2 = \Delta f_3 = \Delta f$.

Computational Experiment for the Wave Equation with Regularization. The results of identification of the known functions $E(T)$ and $\beta(T)$ for the temperature T_M obtained as a result of cooling the sample with liquid nitrogen, according to the presented algorithm of the iterative-variational method are presented in Figs. 1–4. The wave equation is solved with the boundary conditions

$$u\,|_{\tau=0} = \varphi_1(x) = 0, \quad \left(\frac{\partial u}{\partial \tau}\right)_{\tau=0} = \varphi_2(x) = 0, \quad u\,|_{x=0} = f_1(\tau) = 0, \quad u\,|_{x=l} = f_2(\tau) = 0. \tag{19}$$

To calculate the increment Δf the parameters of the regularization of the iterative process $\gamma = 10^{10}$ and $\xi = 1$ are used, as well as the coefficients in the general functions that control the shape of the profile of the increment Δf at each iteration, $q = 10^{10}$ and $p = 1$.

The regularization parameters t_E and t_β in expressions (18) together with the values of the functional depending on the iteration number are presented in Table 1. Usually, these regularizing parameters are equal to unity at the beginning of the iteration process and decrease as the desidered solution is approached. When automating the iterative process by software, it is possible, for simplicity, not to optimize the values of these parameters, but to set their minimum values at once, which will only lead to an increase in the number of iterations and will practically not affect the results of calculations.

As a result of identification, distributions of the desired functions in time are obtained. Since for the given temperature value the functions $E(T)$ and $\beta(T)$ are constant, at each iteration the average value of the obtained distribution of the function is chosen. The computational experiment showed that to identify the functions $E(T)$ and $\beta(T)$ by the iterative-variational method, one mode is sufficient, which determines the increment of the right side of the wave equation Δf at the current iteration. Then, using this increment, the increments δE and $\delta\beta$ are calculated by formulas (1), (3), and (7) which use the distributions of the elasticity modulus and the function β from the previous iteration.

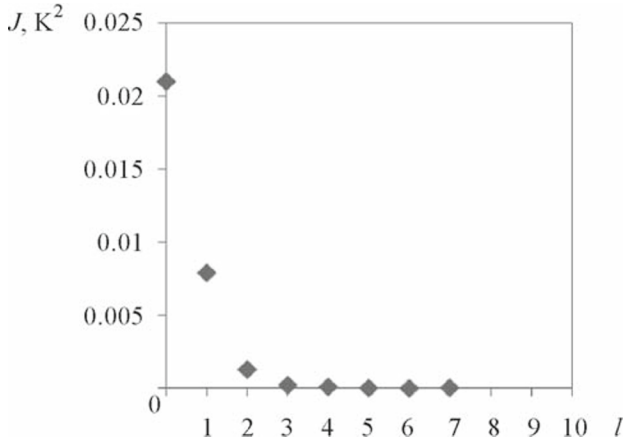

Fig. 1. Distribution of the functional J depending on the number of iteration l during identification of the functions E and β.

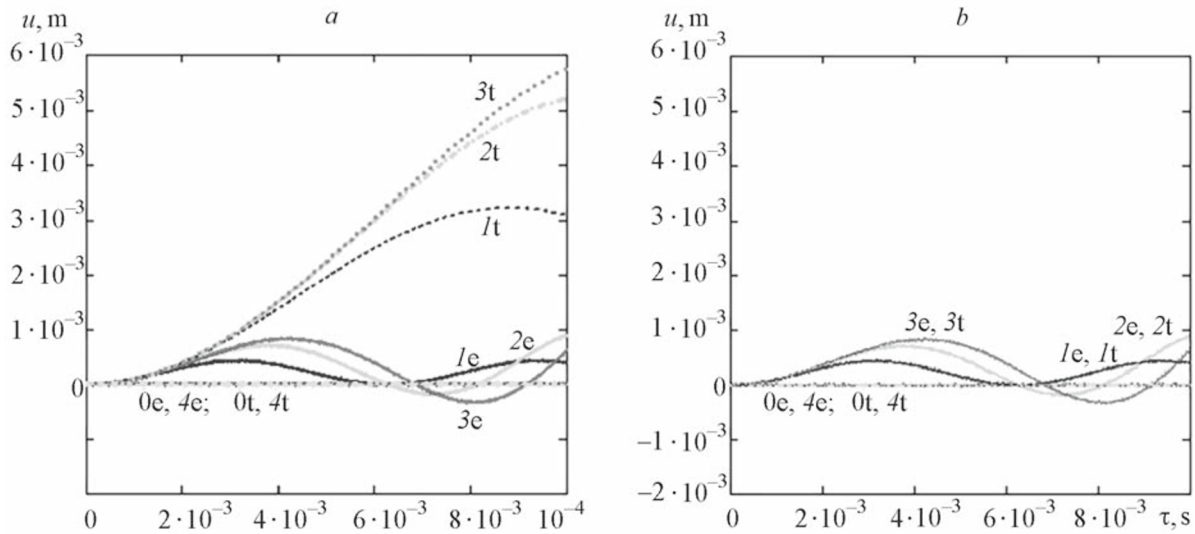

Fig. 2. Solution of the wave equation with boundary-value conditions (19) for the mode with a constant pressure $P(x, \tau) = 10^9$ Pa/m of duration 10^{-2} s for the initial approximations $E^{(0)} = 10^4$ Pa and $\beta^{(0)} = 10^{-3}$ Pa/m (a) and at the target iteration $l = 6$ (b): 0) $x = 0$ m; 1) $0.4 \cdot 10^{-2}$; 2) $0.8 \cdot 10^{-2}$; 3) $1.2 \cdot 10^{-2}$; 4) $1.6 \cdot 10^{-2}$; e) solution for known functions $E^{(\infty)} = 7.866 \cdot 10^4$ Pa and $\beta^{(\infty)} = 13 \cdot 10^{-3}$ Pa/m with dispersion 10^{-5} K; t) solutions obtained at the iteration l.

TABLE 1. Values of the Functional in the Iterative Process when Identifying the Modulus of Elasticity Modulus E and Function β

Iteration	J	t_E	t_β
0	0.021000	1.00	1.00
1	0.007907	1.00	1.00
2	0.001286	1.00	0.10
3	0.000225	0.50	0.10
4	0.000098	0.10	0.01
5	0.000021	0.10	0.01
6	0.000001	0.10	0.01
7	0.000039	0.10	0.01

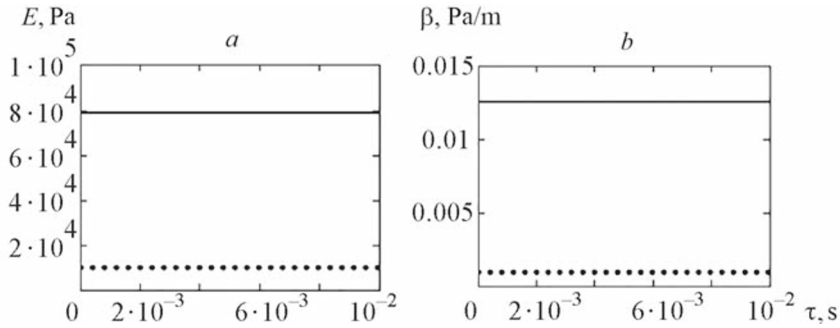

Fig. 3. Time dependences of functions $E^{(0)} = 10^4$ Pa (a) and $\beta^{(0)} = 10^{-3}$ Pa/m (b): dashed lines, initial approximations; solid lines, set values.

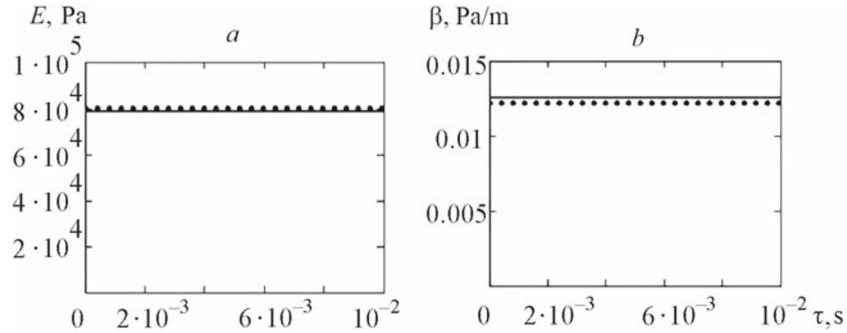

Fig. 4. Time dependences of functions $E^{(\infty)}$ (a) and $\beta^{(\infty)}$ (b): dashed lines, identified values; solid lines, set values.

CONCLUSIONS

1. The regularization method [2] and the method of iterative regularization [3] have smoothing properties with respect to inverse problems with partial differential equations.

2. In the method of iterative regularization, the solution of the inverse problem of determining the functions of the right-hand side of equation $Au = f$ (which includes the boundary and initial conditions) is carried out by the variational method of minimization of the quadratic functional $||u - u_0||^2$. At every iteration l the boundary-value problem is solved for determining $u^{(l)}$ on the right side $f^{(l)}$, which is found by iterative-gradient methods. The norm in the expression of the functional is calculated in the Hilbert space U, which ensures the linearization of the solution.

3. In the regularization method, the quadratic functional has two terms. The first term has the form $||Au - f_0||^2$ and the second term is $\gamma||u - u_0||^2$, where $\gamma > 0$ is the regularization parameter formally required to match the units of the first and second terms, but actually calculated from the condition $||Au - f_0|| = \delta$. The second term is a stabilizer and can take different forms. Minimization of the smoothing functional ensures simultaneous convergence in the spaces of the parameters $U(u \in U)$ and $F(f \in F)$. Thus, the functional of the regularization method is constructed similarly to the functional of the Courant method, in which the functional of the least squares method (the first term responsible for the convergence in the space F) is supplemented with a second term, which is responsible for convergence in the space U. In this case, in the regularization method the conditions for the correct formulation of the problem are eliminated (except for the existence of a solution), which are characteristic of direct variational methods — the linearity of the operator A, the uniqueness of the solution and its stability.

4. In the iterative-variational regularization method, the smoothing functional is written in the form $||f - f_0||^2 + \gamma||u - u_0||^2$ and is used to connect the increment of the solution of the boundary-value problem $\Delta u^{(l)} = u^{(l)} - u_0$ and the increment of the solution of the inverse problem $\Delta f^{(l)}$. The norm is also computed in the Hilbert space, which provides linearization of the solution. For partial differential equations, it is convenient to choose L_2 as such a space. Then

the minimization of the smoothing functional leads to an integral equation of the first kind with respect to Δf, whose solution is well studied and constructed in the regularization method. As a result, at each iteration l it is necessary to solve an integral equation.

5. According to the optimal planning theorem [4, 5], the number of linearly independent modes $T(x, \tau)$ of a thermophysical experiment conducted to identify a mathematical model should be equal to the number of unknown parameters or functions, if they are determined by direct calculations using these modes.

6. For simultaneous identification of several functions of the wave equation (modulus of elasticity, function β) by the iterative method of selecting a quasi-solution, one mode is sufficient, since for a specific desired function at the current iteration, the remaining unknown functions are taken from the previous iteration and are conditionally considered known. In particular, in this case the second function is always considered to be known at the current iteration.

7. For simultaneous identification of the elasticity modulus and function β by one wave mode several iterations by the iterative-variational method are sufficient if the order of the initial approximation coincides with the order of the desired function. Since the temperature regime is quasi-stationary with respect to the process of propagation of elastic waves, the functions $E = E(T)$ and $\beta = \beta(T)$ for a particular temperature value are constant. To obtain the temperature dependence of these functions, it is necessary to identify them several times for different temperatures.

8. The number of regularization parameters in the iterative-variational method doubles: the parameter of the smoothing functional for the increments of solutions of the direct and inverse problems γ is chosen from the condition of ensuring the required order of the identified function, and the parameter of the smoothing functional for regularizing the solution of the integral equation ξ is usually equal to one. The functions p and q (in a particular case, constant), allowing one to adjust the profile of the solution of the integral equation.

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