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HYDROGASDYNAMICS IN TECHNOLOGICAL PROCESSES

ON STRATIFIED FLOWS CAUSED BY SPATIAL INHOMOGENEITIES OF TRANSFER COEFFICIENTS

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The previously unstudied mechanism of the emergence of stratified flows caused by spatial inhomogeneities of exchange coefficients is investigated analytically. In the calculations, a linear approximation is used with specified horizontally harmonic variations of the thermal conductivity coefficient of a relatively small amplitude. Explicit analytical expressions are obtained for temperature perturbations of the environment and velocity of flows in it. The possibility of intensifying such perturbations for some values of the medium parameters is shown.

Keywords: *stratified medium, spatially inhomogeneous exchange coefficients, buoyancy, diffusion, density flows.*

Introduction. In a recent work [1], attention was drawn to the previously unstudied mechanism of the occurrence of convection in a stratified liquid/gaseous medium in a gravity field. In a stably stratified medium, a downward diffusion of heat (buoyancy) obviously takes place. If the thermal conductivity coefficient is spatially inhomogeneous (depends on horizontal coordinates), this leads to the appearance of horizontal inhomogeneities in the distributions of buoyancy and hydrostatic pressure (the weight of the medium column) and, consequently, to the occurrence of horizontal flows. Spatial inhomogeneities of effective transfer coefficients are especially typical of turbulent exchange, which is reflected, for example, in [1–3]. Therefore, such density flows should exist, for example, in the atmosphere [1].

As an example of possible applications, we can cite one of the familiar methods for combating frost in the ground layer of the atmosphere [4, 5]. The ground layer of air is artificially turbulized above the protected area of soil with the help of special fans. It is assumed that this makes it possible to mix the cooled air layer near the soil surface with a warmer layer, which under certain conditions can be located higher. But with such an artificial intensification of vertical heat transfer in a limited horizontal area, heat exchange with a cold surface is enhanced and ordered flows arise that influence heat transfer. Thus, there are meaningful problems at hand, which have been little studied so far.

The corresponding mathematical problems with variable transfer coefficients are, generally speaking, quite complex. Work [1] considers only the simplest case of flows over an infinite inclined surface. In the present work, a more general case is considered that allows an analytical study and, consequently, the identification of fairly general trends. If the amplitude of spatial variations of the thermal conductivity coefficient of a medium is relatively small, the amplitudes of the flows arising in it are also small, which gives grounds to consider the problem linearized in perturbations, which is the subject of this work.

Formulation of the Problem. The paper considers a semibounded stably temperature-stratified (in the atmosphere, in terms of potential temperature [3]) medium bounded from below by a horizontal surface. For simplicity, the consideration is limited to a two-dimensional problem with exchange coefficients dependent on the horizontal coordinate x and vertical coordinate z (the z axis is directed upwards). The thermal diffusivity coefficient is assigned as $K = K_0 + K_1(x, z)$, where $K_0 = \text{const}$, with the second term being much smaller than the first in absolute value. If one neglects the second term, there is a static solution with a constant vertical temperature gradient (with a potential temperature): $\gamma > 0$ (stable background stratification). The presence of the second term leads to the appearance of horizontal thermal inhomogeneities and to perturbations of this background state. The relative smallness of $|K_1(x, z)|$ gives grounds to consider linear perturbations.

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For simplicity, let us assume that the exchange coefficients coincide for all substances: Pr = 1. This hypothesis is widely used in describing turbulent exchange. Generalization to the case Pr ≠ 1 presents no fundamental difficulty. The corresponding linearized system of equations for the two-dimensional stationary problem of hydrothermodynamics in the Boussinesq approximation has the form

$$0 = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial x} + K_0\Delta_2 u\,, \quad 0 = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial z} + K_0\Delta_2 w + g\alpha\theta\,, \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\,, \quad \gamma w = K_0\Delta_2\theta + \gamma\,\frac{\partial K_1(x, z)}{\partial z}\,. \tag{2}$$

On the lower horizontal boundary (on the surface $z = 0$) the impermeability and no-slip conditions are assumed, as well as the constancy of temperature (absence of temperature disturbances):

$$u = w = 0\,, \quad \theta = 0\,. \tag{3}$$

It is assumed that the variants of the exchange coefficient are nonzero in the region of the finite thickness near the lower boundary. Accordingly, at $z \to \infty$ damping of perturbations is assumed.

From the heat transfer equation [the last equation (2)] it can be seen that the variability of the thermal diffusivity coefficient leads to the appearance of an effective horizontally inhomogeneous heat source/sink $\gamma(\partial K_1(x, z)/\partial z)$ in the problem.

Solution of the Problem. Excluding all unknowns, except for one, from the system of equations (1), (2), we obtain

$$\Delta_2^3 w + \frac{N^2}{K_0^2}\frac{\partial^2 w}{\partial x^2} = \frac{N^2}{K_0^2}\frac{\partial^3 K_1}{\partial x^2\partial z}\,, \tag{4}$$

where $N = (\alpha g\gamma)^{1/2}$. It is convenient to analyze the model with a harmonic dependence of the exchange coefficients on the horizontal coordinate:

$$K_1(x, z) = \kappa(z)\cos kx\,. \tag{5}$$

In this case, the solution is also sought in the form of a horizontal harmonic:

$$u(x, z) = U(z)\sin kx\,, \quad w(x, z) = W(z)\cos kx\,,$$

$$\theta(x, z) = \Theta(z)\cos kx\,, \quad p(x, z)/\bar{\rho} = P(z)\cos kx\,.$$

As a result, the following equation is obtained:

$$\left(\frac{d^2}{dZ^2} - 1\right)^3 W - RW = -R\,\frac{d\kappa}{dz}\,, \quad R \equiv \frac{N^2}{K_0^2 k^4}\,. \tag{6}$$

Here, the dimensionless variables $Z = kz$ are introduced, as well as the dimensionless parameter R, which is some analogue of the Rayleigh number [3, 6].

The solution of the last equation is sought in the standard way as the sum of the general solution of the homogeneous equation and of the particular solution of the inhomogeneous equation. The general solution can be represented as a linear combination of exponents of the type exp $(\sigma_j kz)$, where σ_j are the roots of the characteristic equation

$$(\sigma^2 - 1)^3 - R = 0\,. \tag{7}$$

With regard to the damping of perturbations at $z \to \infty$, of the six roots σ_j three with negative real parts are selected (it is assumed here that these roots are different):

$$W_h(z) = \sum_{j=1}^{3} C_j \exp(k\sigma_j z) , \tag{8}$$

where C_j are integration constants.

It will be assumed that variations of the exchange coefficient decreases with height according to the exponential law:

$$\kappa = \kappa_0 \exp(-z/h) , \tag{9}$$

where $\kappa_0 > 0$. In this case, it is easy to find a particular solution of the inhomogeneous equation (6):

$$W_i = W_0 \exp(-z/h) , \quad W_0 = \frac{\kappa_0}{h[-1 + (1-\delta^2)^3/(\delta^6 R)]} , \quad \delta = hk . \tag{10}$$

Subject to Eq. (1) and the continuity equation, the solution of the problem can be presented as

$$\begin{aligned} w &= \left[\sum_{j=1}^{3} C_j \exp(k\sigma_j z) + W_0 \exp(-z/h)\right] \cos kx , \\ u &= \left[-\sum_{j=1}^{3} C_j \sigma_j \exp(k\sigma_j z) + (W_0/hk) \exp(-z/h)\right] \sin kx , \\ \theta &= \frac{K_0 k^2}{\alpha g} \left[\sum_{j=1}^{3} C_j (\sigma_j^2 - 1)^2 \exp(k\sigma_j z) + \frac{W_0}{\delta^4} (1-\delta^2)^2 \exp(-z/h)\right] \cos kx . \end{aligned} \tag{11}$$

Using boundary-value conditions (3), a system of equations is obtained for determining the integration constants C_j:

$$\sum_{j=1}^{3} C_j = -W_0 , \quad \sum_{j=1}^{3} \sigma_j C_j = W_0/\delta , \quad \sum_{j=1}^{3} (\sigma_j^2 - 1)^2 C_j = -\frac{W_0}{\delta^4} (1-\delta^2)^2 . \tag{12}$$

As can be seen from Eq. (7), the quantity $\sigma_j^2 - 1$ can take the values $R^{1/3}$ and $R^{1/3} \exp(\pm 2\pi i/3)$. Expressions for the roots σ_j in the general case are somewhat cumbersome. It makes sense to dwell on the limiting case of large values of the parameter R. For example, if in the surface layer of the atmosphere $N = 10^{-2}$ s and $K_0 = 3$ m^2/s (quite characteristic values), then at $k = 10^{-2}$ m^{-1} (half-wave length about 300 m) $R = 10^3$. In the indicated limit $|\sigma_j| >> 1$, and the roots with negative real parts are equal to

$$\begin{aligned} \sigma_1 &\approx -R^{1/6} , \quad \sigma_2 \approx -R^{1/6}\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = R^{1/6} \exp(-2\pi i/3) , \\ \sigma_3 &\approx -R^{1/6}\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = R^{1/6} \exp(2\pi i/3) . \end{aligned} \tag{13}$$

The approximate solution of the system of equations (12) has the form

$$\begin{aligned} C_1 &\approx \frac{1}{2}(W_1 - W_2) , \\ C_2 &\approx \frac{1}{\sqrt{3}}\left[W_0 \exp\left(-\frac{5}{6}\pi i\right) + \frac{1}{2} W_1 \exp\left(\frac{5}{6}\pi i\right) + \frac{\sqrt{3}}{2} W_2 \exp\left(\frac{\pi i}{3}\right)\right] , \\ C_3 &\approx \frac{1}{\sqrt{3}}\left[W_0 \exp\left(\frac{5}{6}\pi i\right) + \frac{1}{2} W_1 \exp\left(-\frac{5}{6}\pi i\right) + \frac{\sqrt{3}}{2} W_2 \exp\left(-\frac{\pi i}{3}\right)\right] , \end{aligned} \tag{14}$$

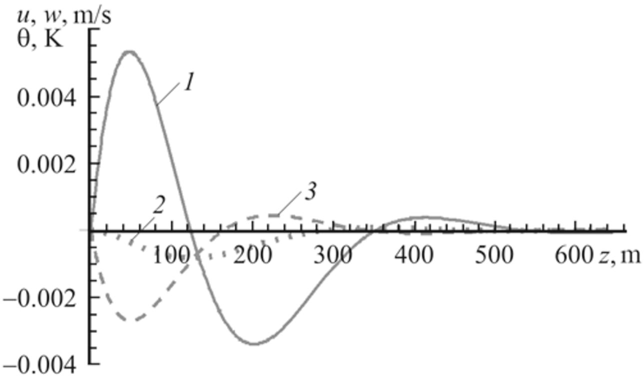

Fig. 1. Profiles of the velocity components u (1) and w (2) on the verticals $kx = \pi/2$ and $x = 0$, respectively, and of temperature deviation θ on the vertical $x = 0$ (normalized to $K_0k_2R^{2/3}/2\alpha g$) (3) at $N = 10^{-2}$ s, $K_0 = 3$ m^2/s, $\kappa_0 = 0.3$ m^2/s, $h = 50$ m, and $k = 2\cdot10^{-3}$ m^{-1}.

where $W_1 = -\dfrac{W_0}{b^4}(1-\delta^2)^2$ and $W_2 = W_0/b$ are the scales of the velocity and $b = \delta R^{1/6}$.

It is also convenient to use the height scale $H = 1/kR^{1/6}$. Subject to (14), the solution of the problem has the form

$$w \approx \left\{\frac{1}{2}(W_1 - W_2)\exp\left(-\frac{z}{H}\right) + \frac{2}{\sqrt{3}}\exp\left(-\frac{1}{2}\frac{z}{H}\right)\left[W_0\cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} + \frac{5}{6}\pi\right)\right.\right.$$

$$\left.\left. + \frac{1}{2}W_1\cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} - \frac{5}{6}\pi\right) + \frac{\sqrt{3}}{2}W_2\cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} - \frac{\pi}{3}\right)\right] + W_0\exp\left(-\frac{z}{h}\right)\right\}\cos kx\,,$$

$$u \approx R^{1/6}\left\{\frac{1}{2}(W_1 - W_2)\exp\left(-\frac{z}{H}\right) - \frac{2}{\sqrt{3}}\exp\left(-\frac{1}{2}\frac{z}{H}\right)\left[W_0\sin\left(\frac{\sqrt{3}}{2}\frac{z}{H}\right)\right.\right.$$

$$\left.\left. + \frac{1}{2}W_1\cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}W_2\cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} + \frac{\pi}{3}\right)\right] + \frac{W_0}{b}\exp\left(-\frac{z}{h}\right)\right\}\sin kx\,,$$

$$\theta \approx \frac{K_0k^2}{\alpha g}R^{2/3}\left\{\frac{1}{2}(W_1 - W_2)\exp\left(-\frac{z}{H}\right) + \frac{2}{\sqrt{3}}\exp\left(-\frac{1}{2}\frac{z}{H}\right)\left[W_0\sin\left(\frac{\sqrt{3}}{2}\frac{z}{H}\right)\right.\right.$$

$$\left.\left. + \frac{1}{2}W_1\cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}W_2\cos\left(\frac{\sqrt{3}}{2}\frac{z}{H} + \frac{\pi}{3}\right)\right] - W_1\exp\left(-\frac{z}{h}\right)\right\}\cos kx\,.$$

Solution Analysis. For definiteness, let us dwell on the solution near the vertical $x = 0$. In this region, $K_1 > 0$ (exchange is intensified), and the effective "heat source" $\gamma(\partial K_1(x, z)/\partial z)$ in the last equation (2) is negative. This is explained as follows. Since in the background state the medium is cooled form below, the intensification of exchange in this region leads to additional cooling of the medium. Consequently, in this region negative temperature deviations, downward motions, and horizontal spreading of the medium can be observed, which is demonstrated by graphs in Fig. 1 plotted according to the solution obtained.

Conclusions. An example of an analytical solution of a previously unstudied type of stratified flows is given. It has been established that horizontal inhomogeneities of transfer coefficients lead to inhomogeneities in the distributions of buoyancy and pressure and, consequently, to the appearance of flows. In the given specific numerical example, the perturbation amplitude is very small: the order of the vertical velocity amplitude is 10^{-3} m/s, while in the atmosphere, several times greater vertical velocities are considered to be noticeable, which are determined depending on the horizontal scale of movements. However, it should be kept in mind that the linear approximation used in the present calculations does not allow us to correctly consider the effects of a larger amplitude. Therefore, rather weak variations of the transfer coefficients are taken as the source of perturbations. In the surface layer of the atmosphere, these variations may well be much larger, which corresponds to a much stronger response. Moreover, even with the considered small variations of the exchange coefficients, the perturbations, in principle, can be much more intense. The fact is that the denominator in expression (10) can vanish at a certain ratio of parameters, which corresponds to a strong intensification of perturbations. This happens when $\delta = (1 + R^{1/3})^{-1/2}$. Calculations have shown that such a ratio of parameters can actually be achieved even on a relatively small change in the accepted value of the parameter h (the thickness of the layer in which transfer coefficients vary), and the amplitude of perturbations can increase many times over.

NOTATION

g, free fall acceleration, m/s^2; h, vertical scale of exchange coefficient variations, m; i, imaginary unit; K, exchange coefficient, m^2/s; k, wave number, m^{-1}; N, buoyancy frequency (Brunt–Väisälä frequency), s^{-1}; p, pressure disturbance, Pa; P, amplitude of pressure perturbation normalized to the average density of the medium, m^2/s^2; Pr, Prandtl number; u and w, velocity components in the direction of the axes x and z, m/s; U and W, amplitudes of horizontal and vertical velocities, m/s; W_0, vertical speed scale, m/s; x, z, horizontal and vertical coordinates, m; α, thermal coefficient of expansion of the medium, K^{-1}; γ, background vertical temperature gradient (of potential temperature), K/m; Δ_2, symbol of the two-dimensional Laplacian, m^{-2}; θ, temperature perturbation, K; Θ, temperature perturbation amplitude, K; κ, amplitude of exchange coefficient variations, m^2/s; κ_0, value of κ at the lower boundary, m^2/s; $\overline{\rho}$, average (reference) density of the medium, kg/m^3.

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