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THE EFFECT OF DIFFUSION AND MICROCONCENTRATION ON PLANE WAVES IN A GENERALIZED THERMOELASTIC MATERIAL

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The governing equations for a linear, isotropic, homogeneous, thermoelastic material with diffusion and microconcentration in a plane have been stated in accordance with the Lord and Shulman theory of generalized thermoelasticity. The plane harmonic solutions of these equations have been obtained. It has been shown that there exist four dispersive coupled longitudinal waves and two uncoupled transverse waves. A half-space with thermally insulated surface has been taken for exploring the reflection of these plane waves. For an incident plane wave, the coefficients of reflection and energy shares of the reflected waves have been presented graphically. The numerical results have made possible to observe the effects of the diffusion and microconcentration parameters on the speeds, reflection coefficients, and the energy ratios.

Keywords*: generalized thermoelasticity, diffusion, microconcentration, plane waves, reflection coefficients, energy ratios.*

Introduction. Due to wide applications of thermoelastic problems in daily life, various thermoelastic theories with additional parameters have been developed. Starting from the Biot coupled thermoelastic theory [1], the thermoelastic governing equations have been extended by many researchers. Lord and Shulman [2] have formulated the generalized thermoelasticity theory with one relaxation time to eliminate the shortcoming of the classical thermoelasticity. Green and Lindsay [3] have developed the theory of generalized thermoelasticity with two relaxation times. Green and Naghdi [4] have proposed another generalized thermoelasticity theory with energy dissipation. The problem of the plane wave propagation in a thermoelastic solid has been studied by Puri [5]. A. N. Sinha and S. B. Sinha [6] have studied the reflection of plane waves at a solid half-space with a thermal relaxation time. Agarwal [7] investigated time-dependent thermoelastic plane waves in the context of the Green and Lindsay theory. Various other wave propagation and reflection problems in the context of these theories have been studied in [8–13].

The phenomenon of themrodiffusion has many applications in integrated resistors, semiconductors, computer circuit fabrication, electronics, and geophysical sciences. Nowacki [14–16] has presented the classical thermoelastic diffusion theory, formulating the relationship between deformation and heat and mass diffusion. Sherief et al. [17] have derived the theory of a generalized thermoelastic diffusive medium with one relaxation time which allows a finite speed of waves. Aouadi [18] used equations of generalized thermoelastic diffusion in anisotropic media and derived the uniqueness and reciprocity theorem. Ezzat and Fayik [19] used the methodology of fractional calculus and proposed a new theory of thermodiffusion. Aouadi [20] has discussed the classical and generalized thermoelastic theories with the effect of diffusion. El-Karamany and Ezzat [21] have derived a new theory of thermodiffusion, using a kernel function. Singh [22] has studied the reflection of plane waves in a thermoelastic diffusive material. Sharma [23] has investigated the propagation of plane diffusive waves in a heat conducting solid. Various other wave propagation problems in thermoelastic solids with diffusive and other parameters have been considered by many researchers, including Singh [24], Bijarnia and Singh [25], Kumar and Gupta [26], and Singh and Singla [27].

The study of continuum theories with microstructure and various other physical fields is crucial in several actual applications, namely, in aviation, nanomaterials, biology, and chemical industry. Grot [28] has studied thermodynamics of a continuum with microstructure, adding the first-order moment of the energy equation to the balance laws of a continuum

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with microtemperature. Riha [29] has also developed a theory of thermoelastic material with microtemperature. Eringen [30] has introduced the concept of microdeformation in his theory of micromorphic continua. Various thermoelastic theories [31–37] with microstructure, microtemperature, and microconcentration have been developed, where microelements were assumed to have different temperatures or mass concentrations.

Aouadi et al. [38] have derived a consistent theory of a thermoelastic diffusive material with microtemperature and microconcentration. Bazarra et al. [39] have revisited the Aouadi theory and numerically analyzed the dynamic thermoelastic problem with microtemperatures and microconcentrations. Deswal et al. [40] and Gunghas et al. [41] have studied the reflection of plane waves in a thermodiffusive material with microtemperature or microconcentration in the context of the coupled theory of thermoelasticity. Motivated by the works of Aouadi et al. [38] and Bazarra et al. [39], we consider the plane wave propagation in a generalized thermoelastic material with diffusion and microconcentration. In the present paper, the governing equations of a thermoelastic medium with diffusion and microconcentration are reduced with the help of the Lord–Shulman theory of generalized thermoelasticity; the plane harmonic solutions of these equations are obtained and discussed for some special cases; the problem on the reflection of plane waves at a thermally insulated surface of a half-space is solved in terms of the reflection coefficients and energy ratios; a quantitative example is taken for numerical simulations. The effects of the diffusion and microconcentration parameters on the wave characteristics are analyzed, and the conclusions are summarized.

Two-Dimensional Formulation. According to [2, 38], the field equations for an isotropic and homogeneous thermoelastic diffusive material with microconcentration in the absence of the body forces and heat sources are the following:

$$\mu \frac{\partial^2 u_i}{\partial x_j^2} + (\mu + \lambda_0) \frac{\partial^2 u_j}{\partial x_i \partial x_j} - \gamma_1 \frac{\partial T}{\partial x_i} - \gamma_2 \frac{\partial P}{\partial x_i} = \rho \frac{\partial^2 u_i}{\partial t^2} , \tag{1}$$

$$T_0 \left(1 + \tau_1 \frac{\partial}{\partial t}\right)\left(\gamma_1 \frac{\partial e_{kk}}{\partial t} + C_e \frac{\partial T}{\partial t} + K \frac{\partial P}{\partial t}\right) = \kappa \frac{\partial^2 T}{\partial x_j^2} , \tag{2}$$

$$\left(1 + \tau_2 \frac{\partial}{\partial t}\right)\left(\gamma_2 \frac{\partial e_{kk}}{\partial t} + K \frac{\partial T}{\partial t} + m \frac{\partial P}{\partial t}\right) = g \frac{\partial^2 P}{\partial x_j^2} + g_1 \frac{\partial C_i}{\partial x_i} , \tag{3}$$

$$\left(1 - \tau_3 \frac{\partial}{\partial t}\right)\left(g_6 \frac{\partial^2 C_i}{\partial x_j^2} + (g_4 + g_5) \frac{\partial^2 C_j}{\partial x_i \partial x_j}\right) + \tau_2 \frac{\partial}{\partial t}\left(g \frac{\partial P}{\partial x_i} + g_1 C_i\right) - g_2 C_i - g_3 \frac{\partial P}{\partial x_i} = m_1 \frac{\partial C_i}{\partial t} , \tag{4}$$

where

$$\gamma_1 = \beta_1 + \frac{a\beta_2}{b} , \quad \gamma_2 = \frac{\beta_2}{b} , \quad \lambda_0 = \lambda - \frac{\beta_2^2}{b} , \quad C_e = \frac{\rho c_E}{T_0} + \frac{a^2}{b} , \quad K = \frac{a}{b} , \quad m = \frac{1}{b} ,$$

$$\beta_1 = (3\lambda + 2\mu)\alpha_t , \quad \beta_2 = (3\lambda + 2\mu)\alpha_c .$$

We consider a thermoelastic half-space with diffusion and microconcentration in rectangular Cartesian coordinate system (x_1, x_2, x_3) with the surface bounding the half-space as the plane $x_3 = 0$ (see Fig. 1). The present analysis is restricted to x_1x_3 plane. We take the displacement and microconcentration vectors as $\mathbf{u} = (u_1, 0, u_3)$ and $\mathbf{C} = (C_1, 0, C_3)$, respectively. Using the Helmholtz decomposition theorem on vectors, the components of these vectors are written in terms of the scalar potential components ϕ_1, ϕ_3, ψ_1, and ψ_3 as

$$u_1 = \frac{\partial \phi_1}{\partial x_1} - \frac{\partial \phi_3}{\partial x_3} , \quad u_3 = \frac{\partial \phi_1}{\partial x_3} + \frac{\partial \phi_3}{\partial x_1} , \tag{5}$$

$$C_1 = \frac{\partial \psi_1}{\partial x_1} - \frac{\partial \psi_3}{\partial x_3} , \quad C_3 = \frac{\partial \psi_1}{\partial x_3} + \frac{\partial \psi_3}{\partial x_1} . \tag{6}$$

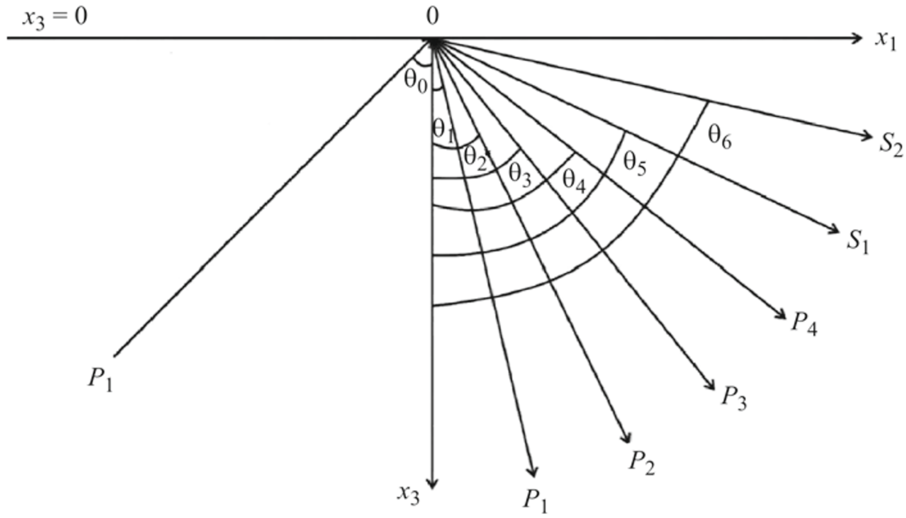

Fig. 1. Geometry of the incident and reflected waves.

Substituting Eqs. (5) and (6) into Eqs. (1)–(4), we obtain

$$(\lambda_0 + 2\mu)\nabla^2\phi_1 - \gamma_1 T - \gamma_2 P = \rho \frac{\partial^2 \phi_1}{\partial t^2}, \tag{7}$$

$$T_0\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\left(\gamma_1 \frac{\partial}{\partial t}\nabla^2\phi_1 + C_e \frac{\partial T}{\partial t} + K\frac{\partial P}{\partial t}\right) = \kappa\nabla^2 T, \tag{8}$$

$$\left(1 + \tau_2 \frac{\partial}{\partial t}\right)\left(\gamma_2 \frac{\partial}{\partial t}\nabla^2\phi_1 + K \frac{\partial T}{\partial t} + m\frac{\partial P}{\partial t}\right) = g\nabla^2 P + g_1\nabla^2\psi_1, \tag{9}$$

$$\left(1 - \tau_3 \frac{\partial}{\partial t}\right)(g_4 + g_5 + g_6)\nabla^2\psi_1 + \tau_2\left(g\frac{\partial P}{\partial t} + g_1\frac{\partial \psi_1}{\partial t}\right) - g_2\psi_1 - g_3 P = m_1\frac{\partial \psi_1}{\partial t}, \tag{10}$$

$$\mu\nabla^2\phi_3 = \rho\frac{\partial^2\phi_3}{\partial t^2}, \tag{11}$$

$$g_6\left(1 - \tau_3\frac{\partial}{\partial t}\right)\nabla^2\psi_3 + \tau_2 g_1\frac{\partial\psi_3}{\partial t} - g_2\psi_3 = m_1\frac{\partial\psi_3}{\partial t}, \tag{12}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_3^2}$. Equations (11) and (12) are uncoupled, whereas Eqs. (7)–(10) are coupled in ϕ_1, T, P, and ψ_1.

Plane Harmonic Solutions. We consider the following plane wave solutions:

$$\{\phi_1, \phi_3, T, P, \psi_1, \psi_3\} = \{\bar{\phi}_1, \bar{\phi}_3, \bar{T}, \bar{P}, \bar{\psi}_1, \bar{\psi}_3\}\exp\{ik(x_1\sin\theta + x_3\cos\theta - vt)\}, \tag{13}$$

where $\bar{\phi}_1$, $\bar{\phi}_3$, $\bar{T}$, $\bar{P}$, $\bar{\psi}_1$, and $\bar{\psi}_3$ are arbitrary constants, k is the wave number, v is the complex wave speed, and $\sin\theta$ and $\cos\theta$ relate to the normal wave projections on the x_1x_3 plane. Inserting Eq. (13) into Eqs. (7)–(12), we get

$$(c_1^2 - v^2)k^2\bar{\phi}_1 + \bar{\gamma}_1\bar{T} + \bar{\gamma}_2\bar{P} = 0, \tag{14}$$

$$K_1 k^2 v^2 \overline{\phi}_1 + (K_2 - v^2)\overline{T} - K_3 v^2 \overline{P} = 0 , \tag{15}$$

$$d_1 k^2 v^2 \overline{\phi}_1 - d_2 v^2 \overline{T} + (d_3 - v^2)\overline{P} + d_4 \overline{\psi}_1 = 0 , \tag{16}$$

$$H_1 v^2 \overline{P} + [(H_2 + H_3) - H_4 v^2]\overline{\psi}_1 = 0 , \tag{17}$$

$$(c_2^2 - v^2)\overline{\phi}_3 = 0 , \tag{18}$$

$$(H_2 - H_4 v^2)\overline{\psi}_3 = 0 , \tag{19}$$

where

$$c_1^2 = \frac{\lambda_0 + 2\mu}{\rho} , \quad \overline{\gamma}_1 = \frac{\gamma_1}{\rho} , \quad \overline{\gamma}_2 = \frac{\gamma_2}{\rho} , \quad c_2^2 = \frac{\mu}{\rho} , \quad d_1 = \frac{\gamma_2}{m} , \quad d_2 = \frac{K}{m\tau_2^*} , \quad d_3 = \frac{g}{m\tau_2^*} , \quad d_4 = \frac{g_1}{m\tau_2^*} ,$$

$$K_1 = \frac{\gamma_1}{C_e} , \quad K_2 = \frac{\kappa}{C_e T_0 \tau_1^*} , \quad K_3 = \frac{K}{C_e} , \quad H_1 = g\left(\tau_2 - \frac{ig_3}{\omega g}\right) , \quad H_2 = g_6 \tau_3^* \omega^2 , \quad H_3 = (g_4 + g_5)\tau_3^* \omega^2 ,$$

$$H_4 = -g_1\left(\tau_2 + \frac{ig_2}{\omega g_1}\right) + m_1 , \quad \tau_1^* = \tau_1 + \frac{i}{\omega} , \quad \tau_2^* = \tau_2 + \frac{i}{\omega} , \quad \tau_3^* = \tau_3 - \frac{i}{\omega} .$$

Equations (14)–(17) admit a nontrivial solution if

$$B_0 (v^2)^4 + B_1 (v^2)^3 + B_2 (v^2)^2 + B_3 v^2 + B_4 = 0 , \tag{20}$$

where B_i ($i = 0, 1, \dots , 4$) are given in Appendix. We present v_j^{-1} as $v_j^{-1} = V_j^{-1} + i\omega^{-1} Q_j$, where the phase velocity and attenuation coefficient of the coupled longitudinal waves P_1, P_2, P_3, and P_4 are written as

$$V_j = \frac{1}{\mathrm{Re}\,(v_j^{-1})} , \quad Q_j = \omega\,\mathrm{Im}\,(v_j^{-1}) , \quad j = 1 , \dots , 4 . \tag{21}$$

It follows from the numerical results that $V_1 > V_2 > V_3 > V_4$. The solutions of Eqs. (18) and (19) show that two transverse waves S_1 and S_2 propagate with the velocities $V_5 = \sqrt{\mu/\rho}$ and $V_6 = \sqrt{H_2/H_4}$, respectively.

Case of limited frequencies. We discuss the behavior of the speeds of different waves when frequency approaches zero. The effect of a limited frequency is significant only for coupled waves. As $\omega \to 0$ (i.e., at very low frequency), Eq. (20) reduces to

$$(d_2 K_3 - 1)v^2 + [c_1^2(1 - d_2 K_3) - \overline{\gamma}_1(-K_1 + d_1 K_3) + \overline{\gamma}_2(-K_1 d_2 + d_1)] = 0 , \tag{22}$$

which shows that three roots of Eq. (20) reduce to zero and hence the corresponding waves will not exist. Therefore, in the absence of the thermal and diffusion parameters, the speed in Eq. (22) reduces to the classical longitudinal wave speed. Hence, when frequency approaches zero, there exist only two waves, one of which is the classical longitudinal and the second is the classical shear wave.

Absence of microconcentration. In the absence of microconcentration, Eq. (20) reduces to the following cubic equation in v^2:

$$\begin{aligned}(d_2 K_3 - 1)(v^2)^3 &+ (c_1^2 + d_3 + K_2 + d_1\overline{\gamma}_2 + \overline{\gamma}_1 K_1 - c_1^2 d_2 K_3 - d_1 \overline{\gamma}_1 K_3 - d_2 \overline{\gamma}_2 K_1)(v^2)^2 \\ &+ (-c_1^2 d_3 - c_1^2 K_2 - d_3 K_2 - d_1 \overline{\gamma}_2 K_2 - d_3 \overline{\gamma}_1 K_1) v^2 + c_1^2 d_3 K_2 = 0 .\end{aligned} \tag{23}$$

Therefore, the slowest wave P_4 with the speed V_4 will disappear. This fact has been verified by the numerical results.

Absence of microconcentration and diffusion. In the absence of microconcentration and diffusion, Eq. (20) reduces to the following quadratic equation in v^2:

$$(v^2)^2 - (K_2 + c_1^2 + K_1\bar{\gamma}_1)v^2 + c_1^2 K_2 = 0 \,. \tag{24}$$

Therefore, two slower coupled longitudinal waves, i.e., P_3 and P_4, will not exist. This fact has been checked by the numerical results.

Reflection Coefficients and Energy Ratios. The relevant boundary conditions at the free surface $x_3 = 0$ are taken as the following vanishing quantities: normal and shear components of the stress force, normal components of the heat flux and diffusing mass vectors, and normal and shear components of the flux moment tensor of mass diffusion, i.e.,

$$\tau_{33} = 0\,, \quad \tau_{31} = 0\,, \quad q_3 = 0\,, \quad \eta_3 = 0\,, \quad \eta_{33} = 0\,, \quad \eta_{13} = 0\,, \tag{25}$$

where

$$\tau_{33} = \lambda_0 \nabla^2 \phi_1 + 2\mu \left(\frac{\partial^2 \phi_1}{\partial x_3^2} + \frac{\partial^2 \phi_3}{\partial x_1 \partial x_3} \right) - \gamma_1 T - \gamma_2 P \,,$$

$$\tau_{31} = \frac{\partial^2 \phi_1}{\partial x_3 \partial x_1} + \frac{\partial^2 \phi_3}{\partial x_1^2} - \frac{\partial^2 \phi_3}{\partial x_3^2} \,,$$

$$q_3 = \kappa \frac{\partial T}{\partial x_3} \,, \quad \eta_3 = g \frac{\partial P}{\partial x_3} + g_1 \left(\frac{\partial \psi_1}{\partial x_3} + \frac{\partial \psi_3}{\partial x_1} \right),$$

$$\eta_{33} = g_4 \nabla^2 \psi_1 + (g_5 + g_6) \left(\frac{\partial^2 \psi_1}{\partial x_3^2} + \frac{\partial^2 \psi_3}{\partial x_1 \partial x_3} \right),$$

$$\eta_{13} = (g_5 + g_6) \frac{\partial^2 \psi_1}{\partial x_1 \partial x_3} + \left(g_6 \frac{\partial \psi_3}{\partial x_3^2} - g_5 \frac{\partial^2 \psi_3}{\partial x_1 \partial x_3} \right).$$

The appropriate potentials for the incident and reflected waves in a half-space are

$$\phi_1 = A_0 \exp\{ik_1(x_1 \sin\theta_0 - x_3 \cos\theta_0 - V_1 t)\} + \sum_{i=1}^{4} A_i \exp\{ik_i(x_1 \sin\theta_i + x_3 \cos\theta_i - V_i t)\}\,, \tag{26}$$

$$T = \xi_1 A_0 \exp\{ik_1(x_1 \sin\theta_0 - x_3 \cos\theta_0 - V_1 t)\} + \sum_{i=1}^{4} \xi_i A_i \exp\{ik_i(x_1 \sin\theta_i + x_3 \cos\theta_i - V_i t)\}\,, \tag{27}$$

$$P = \zeta_1 A_0 \exp\{ik_1(x_1 \sin\theta_0 - x_3 \cos\theta_0 - V_1 t)\} + \sum_{i=1}^{4} \zeta_i A_i \exp\{ik_i(x_1 \sin\theta_i + x_3 \cos\theta_i - V_i t)\}\,, \tag{28}$$

$$\psi_1 = \chi_1 A_0 \exp\{ik_1(x_1 \sin\theta_0 - x_3 \cos\theta_0 - V_1 t)\} + \sum_{i=1}^{4} \chi_i A_i \exp\{ik_i(x_1 \sin\theta_i + x_3 \cos\theta_i - V_i t)\}\,, \tag{29}$$

$$\phi_3 = B_1 \exp\{ik_5(x_1 \sin\theta_5 + x_3 \cos\theta_5 - V_5 t)\}\,, \tag{30}$$

$$\psi_3 = D_1 \exp\{ik_6(x_1 \sin\theta_6 + x_3 \cos\theta_6 - V_6 t)\} . \tag{31}$$

The expressions for the coupling coefficients ξ_p/k_p^2, ζ_p/k_p^2, χ_p/k_p^2 $(p = 1, \dots, 4)$ are given in Appendix. The amplitude ratios, namely, A_i/A_0 $(i = 1, \dots, 4)$, B_1/A_0, and D_1/A_0 are the reflection coefficients for the reflected P_i, S_1, and S_2 waves, respectively. The potentials in Eqs. (26)–(31) satisfy boundary conditions (25) if the following Snell law holds:

$$k_0 \sin\theta_0 = k_i \sin\theta_i , \quad i = 1, 2, \dots, 6 . \tag{32}$$

We obtain the following nonhomogeneous system of six equations:

$$\sum_{j=1}^{6} a_{ij} Z_j = b_i , \tag{33}$$

where

$$a_{1p} = \left(\lambda_0 + 2\mu\cos^2\theta_p + \gamma_1 \frac{\xi_p}{k_p^2} + \gamma_2 \frac{\zeta_p}{k_p^2}\right)\left(\frac{k_p}{k_1}\right)^2 , \quad a_{15} = 0 , \quad a_{16} = \mu \sin 2\theta_6 \left(\frac{k_6}{k_1}\right)^2 ,$$

$$a_{2p} = \sin 2\theta_p \left(\frac{k_p}{k_1}\right)^2 , \quad a_{25} = 0 , \quad a_{26} = -\cos 2\theta_6 \left(\frac{k_6}{k_1}\right)^2 ,$$

$$a_{3p} = \kappa \cos\theta_p \frac{\xi_p}{k_p^2}\left(\frac{k_p}{k_1}\right)^3 , \quad a_{35} = a_{36} = 0 ,$$

$$a_{4p} = \cos\theta_p \left(g \frac{\zeta_p}{k_p^2} + g_1 \frac{\chi_p}{k_p^2}\right)\left(\frac{k_p}{k_1}\right)^3 , \quad a_{45} = g_1 \sin\theta_5 \frac{k_5}{k_1^3} , \quad a_{46} = 0 ,$$

$$a_{5p} = \left\{g_4 + [(g_5 + g_6)\cos^2\theta_p] \frac{\chi_p}{k_p^2}\right\}\left(\frac{k_p}{k_1}\right)^4 , \quad a_{55} = \sin\theta_5 \cos\theta_5 (g_5 + g_6) \frac{k_5^2}{k_1^4} ,$$

$$a_{56} = 0 , \quad a_{6p} = \sin\theta_p \cos\theta_p (g_5 + g_6) \frac{\chi_p}{k_p^2}\left(\frac{k_p}{k_1}\right)^4 ,$$

$$a_{65} = (g_6 \sin^2\theta_5 - g_5 \sin\theta_5 \cos\theta_5) \frac{k_5^2}{k_1^4} , \quad a_{66} = 0 ,$$

$$b_1 = -a_{11} , \quad b_2 = -a_{21} , \quad b_3 = a_{31} , \quad b_4 = a_{41} , \quad b_5 = -a_{51} , \quad b_6 = a_{61} ,$$

$$Z_p = \frac{A_p}{A_0} \quad (p = 1, \dots, 4) , \quad Z_5 = \frac{B_1}{A_0} , \quad Z_6 = \frac{D_1}{A_0} .$$

According to [42], the rate of energy transmission per unit surface area is given as

$$P^* = \tau_{33} \frac{\partial u_3}{\partial t} + \tau_{31} \frac{\partial u_1}{\partial t} + \eta_{33} \frac{\partial C_3}{\partial t} + \eta_{31} \frac{\partial C_1}{\partial t} . \tag{34}$$

Substituting Eqs. (5) and (6) into Eq. (34) and then using Eqs. (26)–(31), we obtain the following expression for the energy ratios of various reflected waves:

$$|E_p| = \frac{\langle P^*_{\mathrm{ref},P_p} \rangle}{\langle P^*_{\mathrm{inc},P_1} \rangle} \quad (p = 1, \dots, 4) , \quad |E_5| = \frac{\langle P^*_{\mathrm{ref},S_1} \rangle}{\langle P^*_{\mathrm{inc},P_1} \rangle} , \quad |E_6| = \frac{\langle P^*_{\mathrm{ref},S_2} \rangle}{\langle P^*_{\mathrm{inc},P_1} \rangle} , \tag{35}$$

where

$$\langle P^*_{\mathrm{inc},P_1} \rangle = V_1 \cos\theta_1 \left\{ \lambda_0 + 2\mu \cos^2\theta_1 + \gamma_1 \frac{\xi_1}{k_1^2} + \gamma_2 \frac{\zeta_1}{k_1^2} + 2\mu \sin^2\theta_1 + \left[g_4 + (g_5 + g_6) \frac{\chi_1}{k_1^2} \right] \right\} k_1^4 A_0^2 ,$$

$$\langle P^*_{\mathrm{ref},P_p} \rangle = V_p \cos\theta_p \left\{ \lambda_0 + 2\mu \cos^2\theta_p + \gamma_1 \frac{\xi_p}{k_p^2} + \gamma_2 \frac{\zeta_p}{k_p^2} + 2\mu \sin^2\theta_p + \left[g_4 + (g_5 + g_6) \frac{\chi_p}{k_p^2} \right] \right\} k_p^4 A_p^2 ,$$

$$\langle P^*_{\mathrm{ref},S_1} \rangle = - V_5 \sin\theta_5 \,(\mu \sin 2\theta_5 + \cos 2\theta_5) B_1^2 k_5^4 ,$$

$$\langle P^*_{\mathrm{ref},S_2} \rangle = V_6 \cos\theta_6 \,[-(g_5 + g_6) \sin^2\theta_6 + \cos\theta_6 \sin\theta_6 \,(g_6 \sin\theta_6 - g_5)] k_6^4 D_1^2 .$$

Numerical Results and Discussion. According to [24], a quantitative example for a medium was considered to compute the phase speeds, reflection coefficients, and the energy ratios. The microconcentration coefficients following from [38] have been taken as

$$\lambda = 3.17 \cdot 10^{10}\ \mathrm{N \cdot m^{-2}} , \quad \mu = 1.639 \cdot 10^{10}\ \mathrm{N \cdot m^{-2}} , \quad T_0 = 300\ \mathrm{K} , \quad \alpha_t = 0.05\ \mathrm{K^{-1}} ,$$

$$\alpha_c = 0.05\ \mathrm{m^3 \cdot kg^{-1}} , \quad \rho = 1740\ \mathrm{kg \cdot m^{-3}} , \quad c_E = 2361\ \mathrm{J \cdot kg^{-1} \cdot K^{-1}} ,$$

$$m_1 = 0.04\ \mathrm{kg \cdot m^{-1}} , \quad a = 0.05\ \mathrm{m^2 \cdot s^{-2} \cdot K^{-1}} , \quad b = 0.05\ \mathrm{m^5 \cdot s^{-2} \cdot kg^{-1}} ,$$

$$\tau_1 = 0.05\ \mathrm{s} , \quad \tau_2 = 0.05\ \mathrm{s} , \quad \tau_3 = 0.05\ \mathrm{s} , \quad g_1 = 0.0028\ \mathrm{kg \cdot m^{-1} \cdot s^{-1}} ,$$

$$g_2 = 0.0038\ \mathrm{kg \cdot m^{-1} \cdot s^{-1}} , \quad g_3 = 0.0048\ \mathrm{kg \cdot s \cdot m^{-3}} , \quad g_4 = 0.0058\ \mathrm{kg \cdot m \cdot s^{-1}} ,$$

$$g_5 = 0.0068\ \mathrm{kg \cdot m \cdot s^{-1}} , \quad g_6 = 0.0038\ \mathrm{kg \cdot m \cdot s^{-1}} , \quad m = 0.45 \cdot 10^{-9}\ \mathrm{kg \cdot m^{-1}} .$$

Figure 2 shows the variations of the phase speeds of the coupled longitudinal waves P_1, P_2, P_3, and P_4 against the frequency (0.01 Hz $\le \omega \le$ 50 Hz). The speed V_1 of the P_1 wave is $1.9883 \cdot 10^5$ m·s^{-1} at ω = 0.01 Hz. It decreases very slowly with increase in ω to $1.9875 \cdot 10^5$ m·s^{-1} at ω = 50 Hz. The speeds V_2, V_3, and V_4 of other coupled longitudinal waves increase quadratically with frequency. It is seen that the speeds V_1 and V_2 become lower in the absence of microconcentration and the speed V_3 increases for all frequencies. The effects of microconcentration on V_2 and V_3 increase with frequency. The wave P_4 with the speed V_4 does not appear in the absence of microconcentration. The effect of microconcentration on the speed V_1 remains almost the same at all frequencies in the range considered.

Figure 3 shows the variations in the phase speeds of the coupled longitudinal waves P_1, P_2, P_3, and P_4 against the measure constant of thermodiffusion a for three different frequencies. The speeds V_1 and V_4 decrease with increase in a, whereas V_2 increases with a. However, the speed V_3 changes with a only slightly. It is seen that increasing the frequency enhances the effect of the constant a on all speeds.

The speeds of the coupled longitudinal waves P_1, P_2, P_3, and P_4 are plotted against τ_1 (thermal relaxation time), τ_2 (diffusion relaxation time), and τ_3 (microconcentration relaxation time) in Figs. 4, 5, and 6, respectively. Figure 4 shows that the speeds of almost all coupled longitudinal waves depend on the thermal relaxation time. It is seen from Fig. 5 that the speeds of almost all coupled longitudinal waves depend on the diffusion relaxation time too. The speeds V_2, V_3, and V_4 decrease with increase in τ_2, whereas V_1 increases with τ_2. Figure 6 shows that the speed V_3 depends on the microconcentration relaxation time τ_3 more significantly as compared to the speeds of other waves.

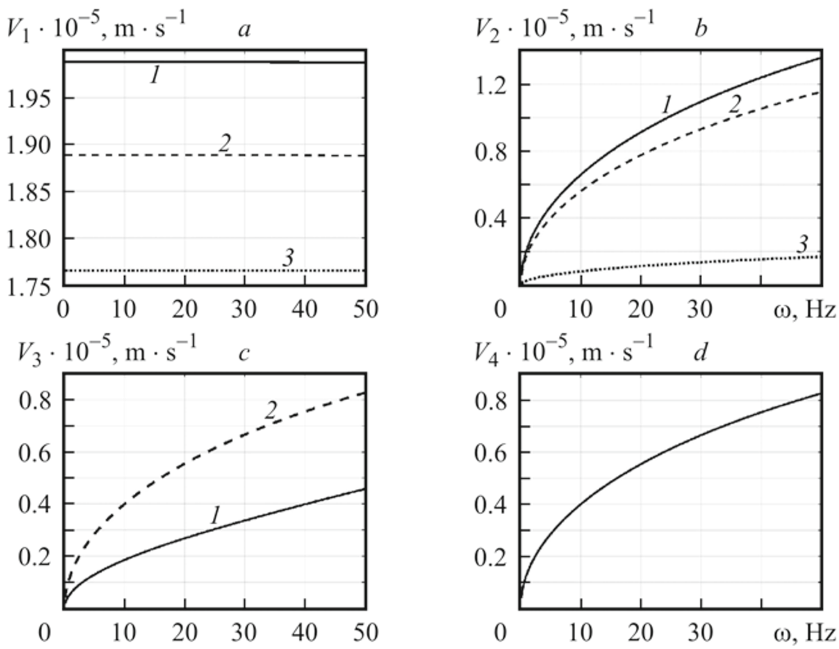

Fig. 2. Phase speeds V_1 (a), V_2 (b), V_3 (c), and V_4 (d) of the plane waves P_1, P_2, P_3, and P_4, respectively, against the frequency: with microconcentration and diffusion (1); without microconcentration (2); without microconcentration and diffusion (3).

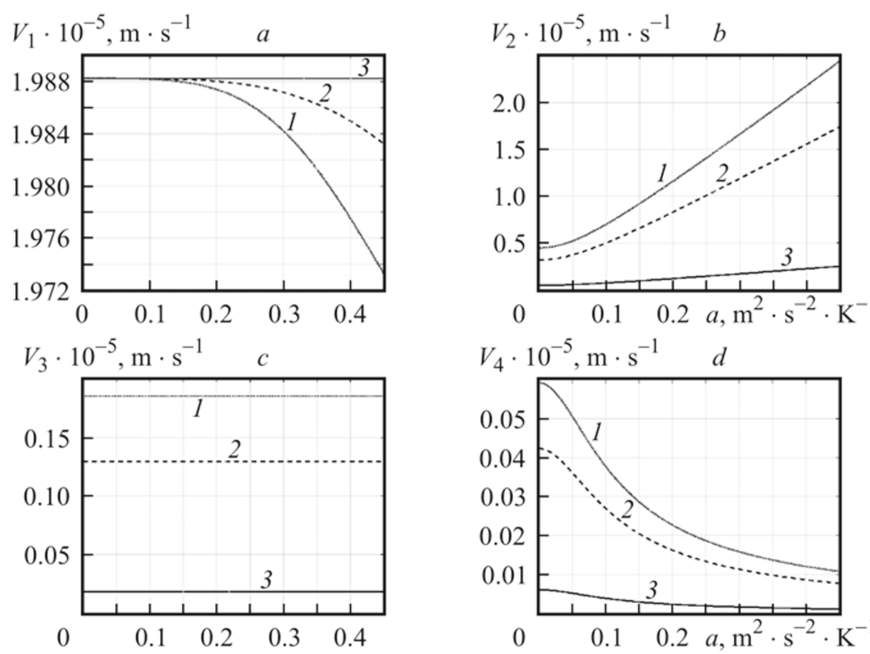

Fig. 3. Phase speeds V_1 (a), V_2 (b), V_3 (c), and V_4 (d) of the plane waves P_1, P_2, P_3, and P_4 against the measure of thermodiffusion at different frequencies: ω = 10 (1); 5 (2); 0.5 (3).

The reflection coefficients of all reflected waves for the incident wave P_1 against the angle of incidence are shown in Fig. 7. In the general case with diffusion and microconcentration, the reflection coefficient Z_1 at $\theta_0 = 0^{\mathrm{o}}$ is unity and then it decreases monotonically to a minimum value of 0.7651 at $\theta = 60^{\mathrm{o}}$. Further it increasesmonotonically to unity at $\theta = 90^{\mathrm{o}}$. The variations in the reflection coefficients Z_2, Z_3, and Z_4 are similar. They decrease from their maxima at $\theta = 0^{\mathrm{o}}$ to minima at

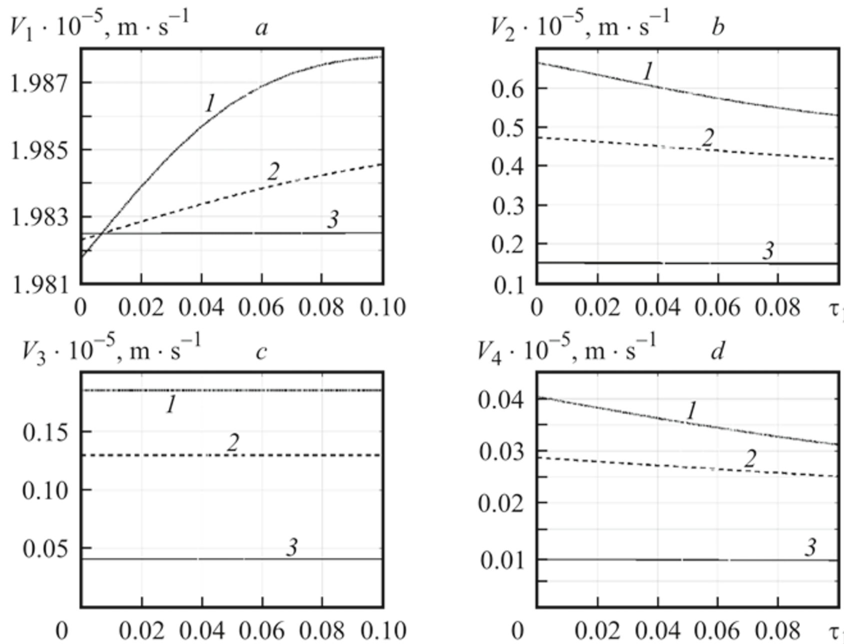

Fig. 4. Phase speeds V_1 (a), V_2 (b), V_3 (c), and V_4 (d) of the plane waves P_1, P_2, P_3, and P_4 against the thermal relaxation time τ_1 at different frequencies: $\omega = 10$ (1); 5 (2); 0.5 (3).

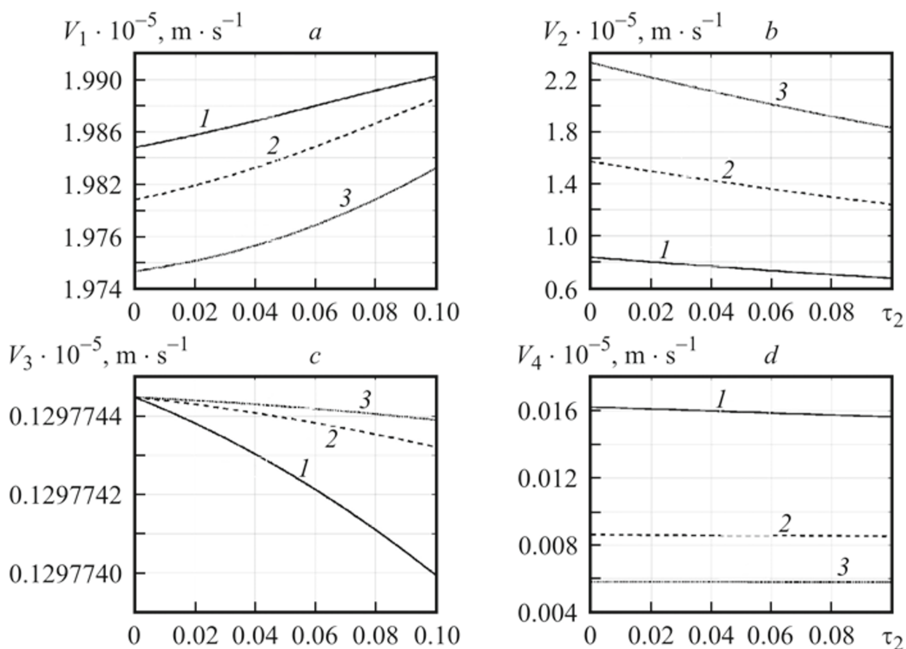

Fig. 5. Phase speeds V_1 (a), V_2 (b), V_3 (c), and V_4 (d) of the plane waves P_1, P_2, P_3, and P_4 against the diffusion relaxation time τ_2 at different measures of thermodiffusion: $a = 0.2$ (1); 0.4 (2); 0.6 (3).

$\theta = 90^{\circ}$. The reflection coefficient Z_5 at $\theta = 0^{\circ}$ is zero, and it increases monotonically to a maximum value of 0.1618 at $\theta = 42^{\circ}$. Then it decreases monotonically to zero at $\theta = 90^{\circ}$. The reflection coefficient Z_6 at $\theta = 0^{\circ}$ is 0.0119, and it decreases first very sharply and then slowly to zero at $\theta = 90^{\circ}$. Comparison of different curves in Fig. 7 shows the effects of diffusion and microconcentration on the reflection coefficients. It is seen that the reflection coefficients Z_4 and Z_6 take place only due

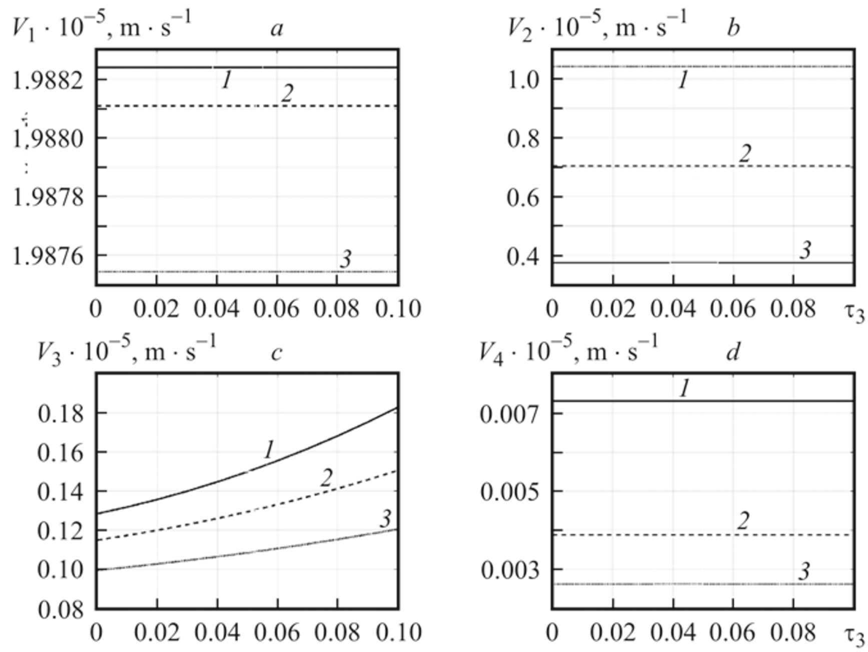

Fig. 6. Phase speeds V_1 (a), V_2 (b), V_3 (c), and V_4 (d) of the plane waves P_1, P_2, P_3, and P_4 against the microconcentration relaxation time τ_2 at different measures of thermodiffusion: a = 0.2 (1); 0.4 (2); 0.6 (3).

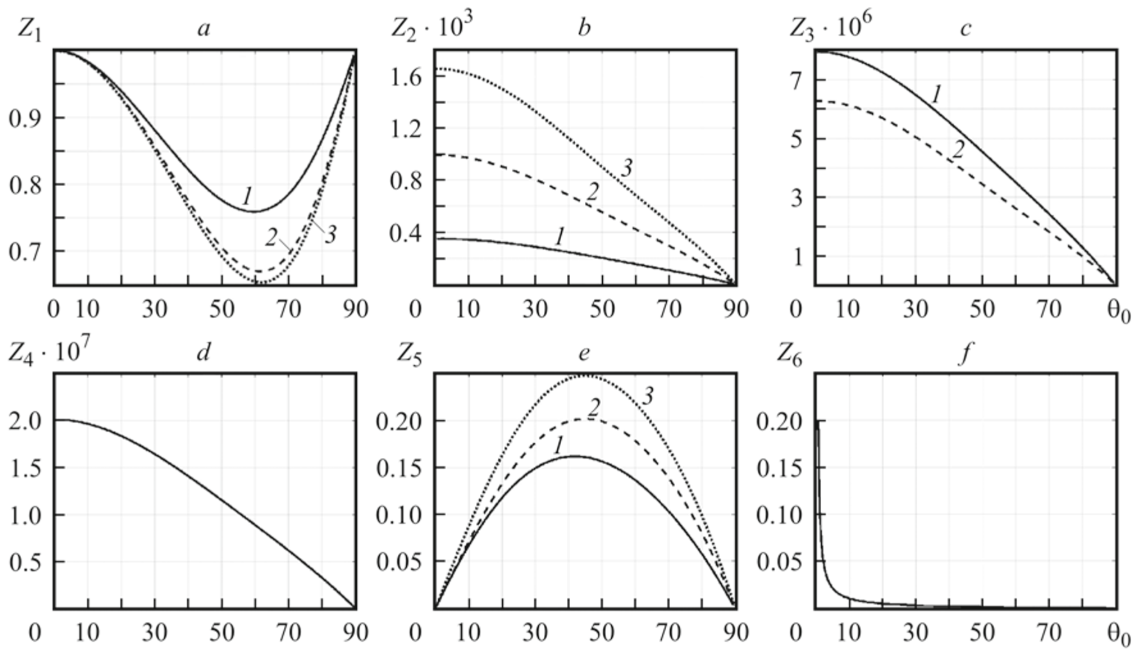

Fig. 7. Reflection coefficients of the reflected waves P_1 (a), P_2 (b), P_3 (c), P_4 (d), S_1 (e) and S_2 (f) against the angle of incidence: with microconcentration and diffusion (1); without microconcentration (2); without microconcentration and diffusion (3).

to the presence of microconcentration. The maximum effect of microconcentration on the reflection coefficients Z_1 and Z_5 is observed at the angles near 60° and 42°, respectively, and for the coefficients Z_2 and Z_3, at normal incidence.

To verify the reflection coefficients, the energy ratios $|E_i|$ (i = 1, 2, ... , 6) of the reflected waves P_1, P_2, P_3, P_4, S_1, and S_2 against the angle of incidence are given in Fig. 8. The sum of the energy ratios of all reflected waves is seen to

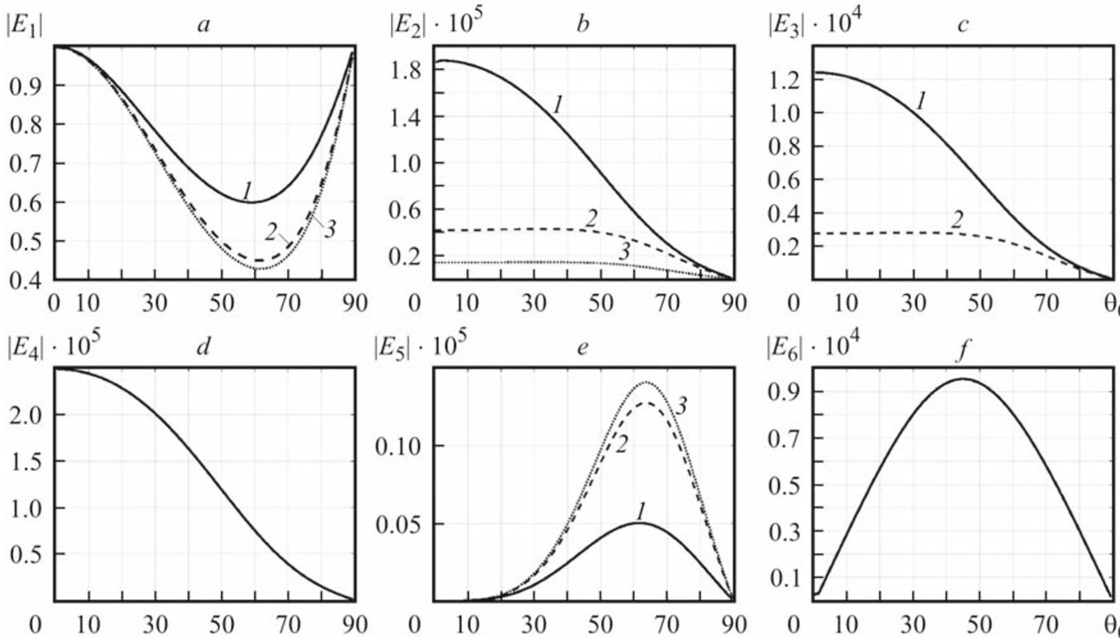

Fig. 8. Energy ratios of the reflected waves P_1 (a), P_2 (b), P_3 (c), P_4 (d), S_1 (e) and S_2 (f) against the angle of incidence: with microconcentration and diffusion (1); without microconcentration (2); without microconcentration and diffusion (3).

be less than or equal to unity at each angle of incidence. Comparison of the curves in this figure shows that the effects of microconcentration and diffusion on the energy ratios are similar to those for the reflection coefficients.

Conclusions. Plane harmonic solutions of the two-dimensional governing equations for a linear, isotropic, and homogeneous generalized thermoelastic medium with diffusion and microconcentration suggest the possibility of four dispersive coupled longitudinal waves and two transverse waves. The expressions for the reflection coefficients and energy ratios have been derived at a thermally insulated surface of a half-space of the medium. The speeds of the plane waves, reflection coefficients, and the energy ratios of the reflected waves have been computed for a quantitative example of the medium. The graphical illustration of the numerical results shows that the speeds, reflection coefficients, and the energy ratios depend on the microconcentration and diffusion effects, thermal relaxation time, diffusion relaxation time, microconcentration relaxation time, and the frequency. The following conclusions can be made from these numerical results:

1. The variations in the speeds of various coupled longitudinal waves with frequency in the absence of microconcentration are different: the speeds V_1 and V_2 become lower V_3 rises. The effect of microconcentration on V_2 and V_3 increases with frequency.

2. The waves with the speeds V_4 and V_6 do not appear in the absence of microconcentration.

3. The waves with the speeds V_3, V_4, and V_6 do not exist in the absence of both microconcentration and diffusion. If the thermal effect is neglected, the wave with the speed V_2 also disappears.

4. An increase in the diffusion relaxation time τ_2 decreases the speeds V_2, V_3, and V_4 and increases V_1.

5. The effect of the microconcentration relaxation time τ_3 on the speed V_3 is more significant as compared to those for the speeds of other coupled longitudinal waves.

6. The effect of microconcentration on the reflection coefficients and energy ratios depends on the angle of incidence. It becomes maximum or minimum at different angles for each reflected wave.

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NOTATION

a, measure of the thermodiffusion effect, $m^2 \cdot s^{-2} \cdot K^{-1}$; *b*, measure of the diffusive effect, $m^5 \cdot s^{-2} \cdot kg^{-1}$; c_E, specific heat at constant strain, $J \cdot kg^{-1} \cdot K^{-1}$; C_1, C_3, components of the microconcentration vector; *E*, energy ratio; g_1, g_2, constitutive coefficients of microconcentration, kg·m^{-1}·s^{-1}; g_3, constitutive coefficient of microconcentration, kg·s·m^{-3}; g_4, g_5, g_6, constitutive coefficients of microconcentration, kg·m·s^{-1}; m, constitutive coefficient of microconcentration, kg·m^{-1}; m_1, measure of microdiffusion conduction, kg·m^{-1}; T_0, temperature of the medium in natural state, K; u_1, u_3, components of the displacement vector, m; V, phase speed, m/s; Z, reflection coefficient; α_c, coefficient of linear diffusion expansion, m^3·kg^{-1}; α_t, coefficient of linear thermal expansion, K^{-1}; θ_0, angle of incidence, deg; λ, μ, Lamé constants, N·m^{-2}; ρ, density of the material, kg·m^{-3}.

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APPENDIX

The expressions for B_i ($i = 0, \dots, 4$) in Eq. (20) take the form:

$$B_0 = H_4 - d_2 H_4 K_3 ,$$

$$B_1 = d_2 H_2 K_3 - H_3 - c_1^2 H_4 - d_3 H_4 - H_4 K_2 - d_1 \overline{\gamma}_2 H_4 - H_2 + d_2 H_3 K_3$$

$$- \overline{\gamma}_1 H_4 K_1 + c_1^2 d_2 H_4 K_3 + d_1 \overline{\gamma}_1 H_4 K_3 + d_2 \overline{\gamma}_2 H_4 K_1 ,$$

$$B_2 = c_1^2 H_2 + c_1^2 H_3 + d_3 H_2 + d_3 H_3 + H_2 K_2 + H_3 K_2 + c_1^2 d_3 H_4 + d_1 \overline{\gamma}_2 H_2 + d_1 \overline{\gamma}_2 H_3 + c_1^2 H_4 K_2$$

$$+ d_3 H_4 K_2 + \overline{\gamma}_1 H_2 K_1 + \overline{\gamma}_1 H_3 K_1 - c_1^2 d_2 H_2 K_3 - c_1^2 d_2 H_3 K_3 - d_1 \overline{\gamma}_1 H_2 K_3$$

$$- d_2 \overline{\gamma}_2 H_2 K_1 - d_1 \overline{\gamma}_1 H_3 K_3 - d_2 \overline{\gamma}_2 H_3 K_1 + d_1 \overline{\gamma}_2 H_4 K_2 + d_3 \overline{\gamma}_1 H_4 K_1 ,$$

$$B_3 = -c_1^2 d_3 H_2 - c_1^2 d_3 H_3 - c_1^2 H_2 K_2 - c_1^2 H_3 K_2 - d_3 H_2 K_2 - d_3 H_3 K_2 - c_1^2 d_3 H_4 K_2$$
$$- d_1 \overline{\gamma}_2 H_2 K_2 - d_3 \overline{\gamma}_1 H_2 K_1 - d_1 \overline{\gamma}_2 H_3 K_2 - d_3 \overline{\gamma}_1 H_3 K_1 \ ,$$
$$B_4 = c_1^2 d_3 K_2 (H_2 + H_3) \ .$$

The expressions for the coupling coefficients ξ_p / k_p^2, ζ_p / k_p^2, χ_p / k_p^2 ($p = 1, \ldots, 4$) obtained by substituting Eqs. (26)–(29) into Eqs. (7)–(9) take the form:

$$\frac{\xi_p}{k_p^2} = \frac{K_3 V_p^2 (c_1^2 - V_p^2) + \overline{\gamma}_2 K_1 V_p^2}{-K_3 V_p^2 \overline{\gamma}_1 - \overline{\gamma}_2 (K_2 - v_p^2)} \ , \quad \frac{\zeta_p}{k_p^2} = \frac{-(c_1^2 - v_p^2) - \overline{\gamma}_1 \dfrac{\xi_p}{k_p^2}}{\overline{\gamma}_2} \ ,$$

$$\frac{\chi_p}{k_p^2} = \frac{-d_1 v_p^2 + d_2 v_p^2 \dfrac{\xi_p}{k_p^2} - (d_3 - v_p^2) \dfrac{\zeta_p}{k_p^2}}{d_4} \ .$$