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THE METHOD OF INTEGRAL TRANSFORMATIONS FOR SOLVING BOUNDARY-VALUE PROBLEMS FOR THE HEAT CONDUCTION EQUATION IN LIMITED AREAS CONTAINING A MOVING BOUNDARY

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A method of integral transformations for solving the boundary-value problems for the equation of heat conduction in limited regions containing a moving boundary of phase transition has been developed. New integral representations of the solutions of boundary-value problems for the heat conduction equation under different boundary conditions assigned on the outer fixed boundaries of a limited region are obtained. The analytical expressions obtained by the proposed method for solving the indicated boundary-value problems are convenient for calculating and studying the temperature fields, as well as the velocity of motion of the interface at large Fourier numbers.

Keywords: *integral transformation, phase transition, heat conduction, interface, limited region.*

Introduction. Mathematical models of the processes of heat conduction and diffusion in regions containing moving boundaries play an important role in the study and prediction of technological processes in which use is made of the processes of heat conduction (heat treatment, crystallization, melting) and diffusion (processes with phase transition in a solid state in multicomponent systems, solidification of solutions in the melt, processes of saturation of the material with a technologically important components) [1–5].

The indicated mathematical models are boundary-value problems for the equations of heat conduction or diffusion in regions separated by moving boundaries and are among the most difficult problems of mathematical physics. This is due to the fact that for technologically important cases, the laws governing the movement of mobile boundaries must be self-consistent with temperature or/and diffusion fields that satisfy the equations of heat conduction or/and diffusion from the conditions of heat or/and mass balance on them. Usually, this leads to the necessity of solving the corresponding nonlinear integro-differential equations numerically or analytically, if analytical representations of the problems are found, or, from the outset, of applying numerical methods to the solution of all boundary-value problems of the model.

Compared to numerical methods, analytical methods have the advantage that they allow one, especially in the case of a large number of parameters determining the kinetics of heat and mass transfer and phase transition processes, to readily study the influence of these parameters on the laws governing the development of the corresponding technological processes.

Due to the above-noted nonlinearity of the equations that determine the motion of interfaces, at the present time there is no standard analytical method for solving boundary-value problems for the equations of heat conduction or diffusion in regions separated by moving boundaries. In each individual case, a specific analytical method is developed for their solution [3–11]. Effective analytical methods for solving boundary-value problems of heat conduction and diffusion, which model the kinetics of phase transitions, are the Green′s function method [6, 12] and the method of integral transformations [6, 8, 13].

In [13], using the Fourier transform, integral representations of solutions of boundary-value problems of heat conduction and diffusion in a two-phase region with a moving interface were obtained with the assumption that the initial phase, as in most problems modeling the kinetics of phase transitions, occupies the entire space, i.e., not taking into account the limitedness of the region containing the moving boundary. This assumption limits the applicability of the majority of the existing models of the kinetics of phase transitions to the initial stages of the process.

If the inner boundary of the heat and mass transfer region is immovable (two-layer region), then in this case the analytical solution of the problem can be obtained by the method of integral transformations for limited regions, which is

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described in detail in [6]. However, in the presence of the moving boundary, it is impossible to find the eigenfunctions of the corresponding Sturm–Liouville problems and, consequently, to fully implement the advantages of the method of integral transformations. In this regard, work [14] proposes a method of extending the boundaries in problems with phase transitions allowing one to represent their solutions in the form of Fourier series in an orthogonal system of functions in a properly expanded area of constructing a solution.

Taking into account the above-described state of the art in the development of analytical methods of solving the boundary-value problems of heat conduction, which model the kinetics of phase transitions, in this paper we developed an analytical method, which, for solving the boundary-value problems for the equations of heat conduction in a limited region containing a moving interface, uses the method of integral transformations for limited regions which is similar to the method proposed in [13] for an unlimited region. The method makes it possible also to consider technologically important cases that were not considered in [14], when the entire material at the initial moment is only in the state of the original phase (for example, in the form of a melt).

Analogous boundary-value problems for the diffusion equation are used also to model the processes of phase transition in solutions (solid and liquid), so that the approach outlined below can also be applied to integrally formulate the solution of boundary-value problems for the equation of diffusion in a limited region containing a moving boundary.

Formulation of the Problem. Let us consider the following boundary-value problem for the equation of heat conduction in a limited region G containing an interface $S(t)$ moving in it and separating two nonoverlapping regions $G_s(t)$ and $G_{liq}(t)$ at the time t. The regions $G_s(t)$ and $G_{liq}(t)$ are limited also by immovable surfaces S_s and S_{liq}, respectively:

$$\partial\Theta_s(M,t)/\partial t = a_s\Delta\Theta_s(M,t)\,,\quad M\in G_s(t)\,,\quad t>0\,; \tag{1}$$

$$\Theta_s(M,0) = \Phi_s(M)\ 0,\quad M\in G_s(0)\,; \tag{2}$$

$$\Theta_s(M,t) = \Theta_{liq}(M,t) = \varphi(v_n(M))\,,\quad M\in S(t)\,,\quad t>0\,; \tag{3}$$

$$\beta_{11}(M)\partial\Theta_s(M,t)/\partial n - \beta_{12}(M)\Theta_s(M,t) = -\varphi_s(M,t)\,,\quad M\in S_s\,,\quad t>0\,; \tag{4}$$

$$\partial\Theta_{liq}(M,t)/\partial t = a_{liq}\Delta\Theta_{liq}(M,t)\,,\quad M\in G_{liq}(t)\,,\quad t>0\,; \tag{5}$$

$$\Theta_{liq}(M,0) = \Phi_{liq}(M)\,,\quad M\in G_{liq}(0)\,; \tag{6}$$

$$\beta_{21}(M)\partial\Theta_{liq}(M,t)/\partial n - \beta_{22}(M)\Theta_l(M,t) = -\varphi_{liq}(M,t)\,,\quad M\in S_{liq}\,,\quad t>0\,; \tag{7}$$

$$\rho q v_n = \lambda_s\partial\Theta_s(M,t)/\partial n - \lambda_{liq}\partial\Theta_{liq}(M,t)/\partial n\,,\quad M\in S(t)\,,\quad t>0\,; \tag{8}$$

$$\beta_{11}^2+\beta_{12}^2\neq 0\,,\quad \beta_{21}^2+\beta_{22}^2\neq 0\,.$$

Boundary condition (3) determines the temperature at the interface depending on the mechanism of the growth of a new phase. Boundary conditions (3), (4), (7), and (8) together with initial conditions (2) and (6) are sufficient for unambiguous determination of the corresponding temperature fields, if the velocity of the interface is given. If the velocity of the interface (growth rate) is not assigned, it is necessary to set an additional condition for its determination. This is the condition of heat balance (8) on the interface, which significantly complicates the possibility of obtaining an analytical solution of the formulated boundary-value problem, since it is necessary to self-consistently find the intercorrelated temperature field and the function $S(t)$ [1]. Approximate analytical methods for solving self-consistent boundary-value problems for the heat conduction and diffusion equations are presented in [3–6].

Solution Method. We seek the solution of boundary-value problems (1)–(8) by the method of integral transformations, and in order to improve the convergence of the series obtained below, we seek the solution of these problems in the form

$$\Theta_s(M, t) = W_s(M, t) + U_s(M, t)\,, \tag{9}$$

$$\Theta_{liq}(M, t) = W_{liq}(M, t) + U_{liq}(M, t)\,, \tag{10}$$

where the functions $W_s(M, t)$ and $W_{liq}(M, t)$ are the solutions of the following boundary-value problems:

$$\partial W_s(M, t)/\partial t = a_s \Delta W_s(M, t) - \partial U_s(M, t)/\partial t\,, \quad M \in G_s(t)\,, \quad t > 0\,; \tag{11}$$

$$W_s(M, 0) = \Phi_s(M) - U_s(M, 0)\,, \quad M \in G_s(0)\,; \tag{12}$$

$$W_s(M, t) = \varphi(v_n(M)) - U_s(M, t)\,, \quad M \in S(t)\,, \quad t > 0\,; \tag{13}$$

$$\beta_{11}(M)\partial W_s(M, t)/\partial n - \beta_{12}(M) W_s(M, t) = 0\,, \quad M \in S_s\,, \quad t > 0\,; \tag{14}$$

$$\partial W_{liq}(M, t)/\partial t = a_{liq} \Delta W_{liq}(M, t) - \partial U_{liq}(M, t)/\partial t\,, \quad M \in G_{liq}(t)\,, \quad t > 0\,; \tag{15}$$

$$W_{liq}(M, 0) = \Phi_{liq}(M) - U_{liq}(M, 0)\,, \quad M \in G_{liq}(0)\,; \tag{16}$$

$$W_{liq}(M, t) = \varphi(v_n(M)) - U_{liq}(M, t)\,, \quad M \in S(t)\,, \quad t > 0\,; \tag{17}$$

$$\beta_{21}(M)\partial W_{liq}(M, t)/\partial n - \beta_{22}(M) W_{liq}(M, t) = 0\,, \quad M \in S_{liq}\,, \quad t > 0\,. \tag{18}$$

The functions $U_s(M, t)$ and $U_{liq}(M, t)$ are the solution of the following quasi-stationary problems:

$$\Delta U_s(M, t) = 0\,, \quad M \in G\,; \tag{19}$$

$$\beta_{11}(M)\partial U_s(M, t)/\partial n - \beta_{12}(M) U_s(M, t) = -\varphi_s(M, t)\,, \quad M \in S_s\,; \tag{20}$$

$$\beta_{21}(M)\partial U_s(M, t)/\partial n - \beta_{22}(M) U_s(M, t) = 0\,, \quad M \in S_{liq}\,; \tag{21}$$

$$\Delta U_{liq}(M, t) = 0\,, \quad M \in G\,; \tag{22}$$

$$\beta_{11}(M)\partial U_{liq}(M, t)/\partial n - \beta_{12}(M) U_{liq}(M, t) = 0\,, \quad M \in S_s\,; \tag{23}$$

$$\beta_{21}(M)\partial U_{liq}(M, t)/\partial n - \beta_{22}(M) U_{liq}(M, t) = -\varphi_{liq}(M, t)\,, \quad M \in S_{liq}\,. \tag{24}$$

In accordance with the method of integral transformations for limited regions, we seek for the functions $W_s(M, t)$ and $W_{liq}(M, t)$ in the form of Fourier series, assuming that these functions outside their domains of definition $G_s(t)$ and $G_{liq}(t)$ in their boundary-value problems are equal to zero:

$$W_s(M, t) = \sum_{k=1}^{\infty} \frac{\overline{W}_s(k, t)}{\|\Psi_k\|^2}\, \Psi_k(M)\,, \quad M \in G_s(t)\,, \quad t > 0\,; \tag{25}$$

$$W_{liq}(M, t) = \sum_{n=1}^{\infty} \frac{\overline{W}_{liq}(n, t)}{\|\Psi_n\|^2}\, \Psi_n(M)\,, \quad M \in G_{liq}(t)\,, \quad t > 0\,. \tag{26}$$

Here, $\Psi_k(M)$ is the system of functions orthogonal in the region G, i.e., $\iiint\limits_G \Psi_i(P)\Psi_k(P)dv = \delta_{ik}\|\Psi_k\|^2$, $\|\Psi_k\|^2 = \iiint\limits_G \Psi_k^2(P)dv$ is the square of the norm of the function $\Psi_k(M)$.

According to the theory of Fourier series,

$$\bar{W}_{\mathrm{s}}(k, t) = \iiint\limits_G W_{\mathrm{s}}(P, t)\Psi_k(P)dV = \iiint\limits_{G_{\mathrm{s}}(t)} W_{\mathrm{s}}(P, t)\Psi_k(P)dV, \tag{27}$$

$$\bar{W}_{\mathrm{liq}}(k, t) = \iiint\limits_G W_{\mathrm{liq}}(P, t)\Psi_k(P)dV = \iiint\limits_{G_{\mathrm{liq}}(t)} W_{\mathrm{liq}}(P, t)\Psi_k(P)dV. \tag{28}$$

To find the Fourier coefficients in expansions (25) and (26), we multiply both parts of Eqs. (11) and (15) by $W_{\mathrm{s}}(P, t)\Psi_k(P)dV$ and $W_{\mathrm{liq}}(P, t)\Psi_k(P)dV$, respectively, and integrate over the regions $G_{\mathrm{s}}(t)$ and $G_{\mathrm{liq}}(t)$. When integrating, we take into account that

$$\frac{d}{dt}\iiint\limits_{V(t)} f(M, t)dV = \iiint\limits_{V(t)} \frac{\partial}{\partial t} f(M, t)dV + \iiint\limits_{V(t)} \mathrm{div}\,(\mathbf{v}f(M, t))dV = \iiint\limits_{V(t)} \frac{\partial}{\partial t} f(M, t)dV + \iint\limits_{S(t)} v_n f(M, t)dS,$$

where $\mathbf{v} = d\mathbf{r}(M, t)/dt$, $M \in S(t)$, and $S(t) = \partial V(t)$ is the boundary of the region $V(t)$, and $\mathbf{r}(M, t)$, where $M \in S(t)$ is the radius vector of the point M that belongs to the surface $S(t)$; v_n is the normal velocity of motion of the boundary $S(t)$ of the region $V(t)$, and n is the outer normal to $S(t)$. As a result, we obtain the equations

$$d\bar{W}_{\mathrm{s}}(k, t)/dt = \iint\limits_{S(t)} v_n(P, t)W_{\mathrm{s}}(P, t)\Psi_k(P)dS + \iiint\limits_{G_s(t)} \Psi_k(P)(a_{\mathrm{s}}\Delta W_{\mathrm{s}}(P, t) - \partial U_{\mathrm{s}}(P, t)/\partial t)dV, \tag{29}$$

$$d\bar{W}_{\mathrm{liq}}(k, t)/dt = -\iint\limits_{S(t)} v_n(P, t)W_{\mathrm{liq}}(P, t)\Psi_k(P)dS_P + \iiint\limits_{G_{\mathrm{liq}}(t)} \Psi_k(P)(a_{\mathrm{liq}}\Delta W_{\mathrm{liq}}(P, t) - \partial U_{\mathrm{liq}}(P, t)/\partial t)dV. \tag{30}$$

Here, v_n is the normal velocity of motion of the boundary $S(t)$ of the region $G_{\mathrm{s}}(t)$ in the direction $\mathbf{n}$, which is the external normal to $S(t)$ relative to the region $G_{\mathrm{s}}(t)$. We transform the triple integrals in (29) and (30) by the second Green's formula. We have

$$\begin{aligned}\iiint\limits_{G_{\mathrm{s}}(t)} \Psi_k(P)\Delta W_{\mathrm{s}}(P, t)\Psi_k(P)dV &= \iiint\limits_{G_{\mathrm{s}}(t)} W_{\mathrm{s}}(P, t)\Delta\Psi_k(P)dV \\ &+ \iint\limits_{S(t)} (\Psi_k(P)\partial W_{\mathrm{s}}(P, t)/\partial n - W_{\mathrm{s}}(P, t)\partial\Psi_k(P)/\partial n)dS \\ &+ \iint\limits_{S_{\mathrm{s}}} (\Psi_k(P)\partial W_{\mathrm{s}}(P, t)/\partial n - W_{\mathrm{s}}(P, t)\partial\Psi_k(P)/\partial n)dS,\end{aligned} \tag{31}$$

$$\begin{aligned}\iiint\limits_{G_{\mathrm{liq}}(t)} \Psi_k(P)\Delta W_{\mathrm{liq}}(P, t)dV &= \iiint\limits_{G_{\mathrm{liq}}(t)} W_{\mathrm{liq}}(P, t)\Delta\Psi_k(P)dV \\ &- \iint\limits_{S(t)} (\Psi_k(P)\partial W_{\mathrm{liq}}(P, t)/\partial n - W_{\mathrm{liq}}(P, t)\partial\Psi_k(P)/\partial n)dS \\ &+ \iint\limits_{S_{\mathrm{liq}}} (\Psi_k(P)\partial W_{\mathrm{liq}}(P, t)/\partial n - W_{\mathrm{liq}}(P, t)\partial\Psi_k(P)/\partial n)dS.\end{aligned} \tag{32}$$

To obtain the equations for the Fourier coefficients $\bar{W}_{\mathrm{s}}(k, t)$ and $\bar{W}_{\mathrm{liq}}(k, t)$, we require that the functions $\Psi_k(P)$, $P \in G$ satisfy the equation

$$\Delta\Psi(P) + \gamma^2\Psi(P) = 0\,, \quad P \in G \tag{33}$$

and boundary conditions

$$\beta_{11}(P)\partial\Psi(P, t)/\partial n - \beta_{12}(P)\Psi(P, t) = 0\,, \quad P \in S_{\text{s}}\,, \tag{34}$$

$$\beta_{21}(P)\partial\Psi(P, t)/\partial n - \beta_{22}(P)\Psi(P, t) = 0\,, \quad P \in S_{\text{liq}}\,. \tag{35}$$

Thus, we have the Sturm–Liouville problem (33)–(35). As is known, the eigenfunctions $\Psi_i(P)$ and $\Psi_k(P)$ of this problem, corresponding to different eigenvalues γ_i^2, $i = 1, 2, \ldots$, γ_k^2, $k = 1, 2, \ldots$ ($\gamma_i^2 \neq \gamma_k^2$), are orthogonal in the region G, i.e., $\iiint\limits_G \Psi_i(P)\Psi_k(P)dV = \delta_{ik}\|\Psi_k\|^2$, and form a complete system, proving that expansions (25) and (26) are well founded.

Taking into account Eqs. (31)–(33) in the right-hand side of Eqs. (29) and (30), we obtain

$$\begin{aligned} d\overline{W}_{\text{s}}(k, t)/dt + a_{\text{s}}\gamma_k^2\overline{W}_{\text{s}}(k, t) &= \iint\limits_{S(t)} v_n(P, t)W_{\text{s}}(P, t)\Psi_k(P)dS \\ &+ a_{\text{s}}\iint\limits_{S(t)} (\Psi_k(P)\partial W_{\text{s}}(P, t)/\partial n - W_{\text{s}}(P, t)\partial\Psi_k(P)/\partial n)dS - \iiint\limits_{G_{\text{s}}(t)} \Psi_k(P)\partial U_{\text{s}}(P, t)/\partial t dV \\ &+ \iint\limits_{S_{\text{s}}} (\Psi_k(P)\partial W_{\text{s}}(P, t)/\partial n - W_{\text{s}}(P, t)\partial\Psi_k(P)/\partial n)dS\,, \end{aligned} \tag{36}$$

$$\begin{aligned} d\overline{W}_{\text{liq}}(k, t)/dt + a_{\text{liq}}\gamma_k^2\overline{W}_{\text{liq}}(k, t) &= -\iint\limits_{S(t)} v_n(P, t)W_{\text{liq}}(P, t)\Psi_k(P)dS \\ &- a_{\text{liq}}\iint\limits_{S(t)} (\Psi_k(P)\partial W_{\text{liq}}(P, t)/\partial n - W_{\text{liq}}(P, t)\partial\Psi_k(P)/\partial n)dS \\ &- \iiint\limits_{G_{\text{liq}}(t)} \Psi_k(P)\partial U_{\text{liq}}(P, t)/\partial t dV + \iint\limits_{S_{\text{liq}}} (\Psi_k(P)\partial W_{\text{liq}}(P, t)/\partial n - W_{\text{liq}}(P, t)\partial\Psi_k(P)/\partial n)dS\,. \end{aligned} \tag{37}$$

Let us further simplify the right-hand sides of Eqs. (36) and (37) using boundary conditions (14), (18). For this purpose, we consider the case where $\beta_{11} \neq 0$, $\beta_{21} \neq 0$. Analysis of other combinations of the values of coefficients β_{11}, β_{12}, β_{21}, and β_{22} is carried out in a similar way. Then

$$\begin{aligned} &\Psi_k(P)\partial W_{\text{s}}(P, t)/\partial n - W_{\text{s}}(P, t)\partial\Psi_k(P)/\partial n \\ &= \frac{\Psi_k(P)}{\beta_{11}}(\beta_{11}\partial W_{\text{s}}(P, t)/\partial n - \beta_{12}W_{\text{s}}(P, t)) - \frac{W_{\text{s}}(P, t)}{\beta_{11}}(\beta_{11}\partial\Psi_k(P)/\partial n - \beta_{12}\Psi_k(P)) = 0\,, \quad P \in S_{\text{s}}\,. \end{aligned} \tag{38}$$

Analogously,

$$\begin{aligned} \Psi_k(P)\partial W_{\text{liq}}(P, t)/\partial n - W_{\text{liq}}(P, t)\partial\Psi_k(P)/\partial n &= \frac{\Psi_k(P)}{\beta_{21}}(\beta_{21}\partial W_{\text{liq}}(P, t)/\partial n - \beta_{22}W_{\text{liq}}(P, t)) \\ &- \frac{W_{\text{liq}}(P, t)}{\beta_{21}}(\beta_{11}\partial\Psi_k(P)/\partial n - \beta_{22}\Psi_k(P)) = 0\,, \quad P \in S_{\text{liq}}\,. \end{aligned} \tag{39}$$

Taking into account Eqs. (38) and (39) in Eqs. (36) and (37), respectively, we obtain the following equations for the Fourier coefficients $\overline{W}_{\text{s}}(k, t)$ and $\overline{W}_{\text{liq}}(k, t)$:

$$d\overline{W}_{\text{s}}(k, t)/dt + a_{\text{s}}\gamma_k^2\overline{W}_{\text{s}}(k, t) = g_{\text{s},k}(t)\,, \tag{40}$$

where

$$g_{s,k}(t) = \iint\limits_{S(t)} v_n(P, t)W_s(P, t)\Psi_k(P)dS$$

$$+ a_s \iint\limits_{S(t)} (\Psi_k(P)\partial W_s(P, t)/\partial n - W_s(P, t)\partial\Psi_k(P)/\partial n)dS - \iiint\limits_{G_s(t)} \Psi_k(P)\partial U_s(P, t)/\partial t dV \ . \tag{41}$$

Analogously,

$$d\bar{W}_{liq}(k, t)/dt + a_{liq}\gamma_k^2\bar{W}_{liq}(k, t) = g_{liq,k}(t) \ , \tag{42}$$

where

$$g_{liq,k}(t) = -\iint\limits_{S(t)} v_n(P, t)\Theta_{liq}(P, t)\Psi_k(P)dS$$

$$- a_{liq} \iint\limits_{S(t)} (\Psi_k(P)\partial W_{liq}(P, t)/\partial n - W_{liq}(P, t)\partial\Psi_k(P)/\partial n)dS - \iiint\limits_{G_{liq}(t)} \Psi_k(P)\partial U_{liq}(P, t)/\partial t dV \ . \tag{43}$$

In accordance with Eqs. (12) and (16), the initial conditions for Eqs. (40) and (42) have the following form:

$$\bar{W}_s(k, 0) = \iiint\limits_{G_s(0)} \Psi_k(P)(\Phi_s(P) - U_s(P, 0))dV \ , \tag{44}$$

$$\bar{W}_{liq}(k, 0) = \iiint\limits_{G_{liq}(0)} \Psi_k(P)(\Phi_{liq}(P) - U_{liq}(P, 0))dV \ . \tag{45}$$

Then the solutions of Eqs. (40) and (42), with account for conditions (44) and (45), can be represented as

$$\bar{W}_s(k, t) = \exp(-a_s\gamma_k^2 t)\left(\bar{W}_s(k, 0) + \int\limits_0^t \exp(a_s\gamma_k^2\eta)g_{s,k}(\eta)d\eta\right) , \tag{46}$$

$$\bar{W}_{liq}(k, t) = \exp(-a_{liq}\gamma_k^2 t)\left(\bar{W}_{liq}(k, 0) + \int\limits_0^t \exp(a_l\gamma_k^2\eta)g_{l,k}(\eta)d\eta\right) . \tag{47}$$

The functions $U_s(M, t)$ and $U_{liq}(M, t)$, which are the solution of boundary-value problems (19)–(21) and (22)–(24), respectively, will be found using the same integral transformation that was used to find the functions $W_s(M, t)$ and $W_{liq}(M, t)$. We have

$$U_i(M, t) = \sum_{k=1}^{\infty} \frac{\bar{U}_i(k, t)}{\|\Psi_k\|^2}\Psi_k(M) \ , \quad i = \text{s, liq} \ , \quad \bar{U}_i(k, t) = \iiint\limits_G U_i(P, t)\Psi_k(P)dV \ . \tag{48}$$

The solution of problems of the type of (19)–(21) and (22)–(24) was considered in detail in [6]. Following [6] and taking into account boundary conditions (20), (21), and (38), (39), we obtain

$$\bar{U}_s(k, t) = \frac{1}{\gamma_k^2}\iint\limits_{S_{liq}} (\Psi_k(P)\partial U_s(P, t)/\partial n - U_s(P, t)\partial\Psi_k(P)/\partial n)dS$$

$$+ \frac{1}{\gamma_k^2}\iint\limits_{S_s} (\Psi_k(P)\partial U_s(P, t)/\partial n - U_s(P, t)\partial\Psi_k(P)/\partial n)dS = -\frac{1}{\beta_{11}\gamma_k^2}\iint\limits_{S_s} \Psi_k(P)\varphi_s(P, t)dS \ .$$

Analogously, taking into account Eqs. (23), (24) and (38), (39), we have

$$\bar{U}_{\text{liq}}(k,t) = -\frac{1}{\beta_{21}\gamma_k^2}\iint\limits_{S_{\text{liq}}} \Psi_k(P)\varphi_{\text{liq}}(P,t)dS\,.$$

Substituting the obtained results into formulas (9) and (10), we write down the analytical representations of the solutions of the posed problems:

$$\Theta_{\text{s}}(M,t) = -\frac{1}{\beta_{11}}\sum_{k=1}^{\infty}\frac{\Psi_k(M)}{\|\Psi_k\|^2\gamma_k^2}\iint\limits_{S_{\text{s}}}\Psi_k(P)\varphi_{\text{s}}(P,t)dS_P$$

$$+\sum_{k=1}^{\infty}\frac{\Psi_k(M)}{\|\Psi_k\|^2}\exp(-a_{\text{s}}\gamma_k^2 t)\left(\bar{W}_{\text{s}}(k,0)+\int_0^t \exp(a_{\text{s}}\gamma_k^2\eta)g_{\text{s},k}(\eta)d\eta\right),\quad M\in G_{\text{s}}(t)\,,\quad t>0\,; \tag{49}$$

$$\Theta_{\text{liq}}(M,t) = -\frac{1}{\beta_{21}}\sum_{k=1}^{\infty}\frac{\Psi_k(M)}{\|\Psi_k\|^2\gamma_k^2}\iint\limits_{S_{\text{liq}}}\Psi_k(P)\varphi_{\text{liq}}(P,t)dS$$

$$+\sum_{k=1}^{\infty}\frac{\Psi_k(M)}{\|\Psi_k\|^2}\exp(-a_{\text{liq}}\gamma_k^2 t)\left(\bar{W}_{\text{liq}}(k,0)+\int_0^t \exp(a_{\text{liq}}\gamma_k^2\eta)g_{\text{liq},k}(\eta)d\eta\right),\quad M\in G_{\text{liq}}(t)\,,\quad t>0\,, \tag{50}$$

where the functions $g_{\text{s},k}(\eta)$, $\bar{W}_{\text{s}}(k,0)$, $g_{\text{liq},k}(\eta)$, and $\bar{W}_{\text{liq}}(k,0)$ are defined by expressions (41), (44) and (43), (45), respectively.

The temperature fields $\Theta_{\text{s}}(M,t)$ and $\Theta_{\text{liq}}(M,t)$ can be expressed in terms of the Green′s functions of the boundary-value problem for the heat conduction equations (1) and (5), respectively, with boundary conditions (34), (35):

$$G_{\text{s}}(M,P,t-\eta) = \sum_{k=1}^{\infty}\frac{\Psi_k(M)\Psi_k(P)}{\|\Psi_k\|^2}\exp(-a_{\text{s}}\gamma_k^2(t-\eta))\,,\quad M\,,\quad P\in G\,,\quad t>\eta\,, \tag{51}$$

$$G_{\text{liq}}(M,P,t-\eta) = \sum_{k=1}^{\infty}\frac{\Psi_k(M)\Psi_k(P)}{\|\Psi_k\|^2}\exp(-a_l\gamma_k^2(t-\eta))\,,\quad M\,,\quad P\in G\,,\quad t>\eta \tag{52}$$

and in terms of the Green′s function of the stationary heat conduction equation with the same boundary conditions:

$$G(M,P) = \sum_{k=1}^{\infty}\frac{\Psi_k(M)\Psi_k(P)}{\|\Psi_k\|^2\gamma_k^2}\,,\quad M\,,\quad P\in G\,. \tag{53}$$

Then

$$\Theta_{\text{s}}(M,t) = -\frac{1}{\beta_{11}}\iint\limits_{S_{\text{s}}}G(M,P)\varphi_{\text{s}}(P,t)dS_P + \int_0^t d\eta\iint\limits_{S(\eta)}v_n(P,\eta)W_{\text{s}}(P,\eta)G_{\text{s}}(M,P,t-\eta)dS_P$$

$$+\,a_s\int_0^t d\eta\iint\limits_{S(\eta)}(G_{\text{s}}(M,P,t-\eta)\partial W_{\text{s}}(P,\eta)/\partial n_P - W_{\text{s}}(P,\eta)\partial G_{\text{s}}(M,P,t-\eta)/\partial n_P)dS_P \tag{54}$$

$$-\int_0^t d\eta\iiint\limits_{G_{\text{s}}(t)}G_{\text{s}}(M,P,t-\eta)\partial U_{\text{s}}(P,\eta)/\partial\eta\, dV_P + \iiint\limits_{G_{\text{s}}(0)}G_{\text{s}}(M,P,t)(\Phi_{\text{s}}(P)-U_s(P,0))dV_P\,,$$

$$\Theta_{\text{liq}}(M,t) = -\frac{1}{\beta_{21}}\iint\limits_{S_{\text{liq}}} G(M,P)\varphi_{\text{liq}}(P,t)dS_P - \int\limits_0^t d\eta \iint\limits_{S(\eta)} v_n(P,\eta)W_{\text{liq}}(P,\eta)G_{\text{liq}}(M,P,t-\eta)dS_P$$

$$- a_{\text{liq}}\int\limits_0^t d\eta \iint\limits_{S(\eta)} (G_{\text{liq}}(M,P,t-\eta)\partial W_{\text{liq}}(P,\eta)/\partial n_P - W_{\text{liq}}(P,\eta)\partial G_{\text{liq}}(M,P,t-\eta)/\partial n_P)dS_P \quad (55)$$

$$- \int\limits_0^t d\eta \iiint\limits_{G_{\text{liq}}(t)} G_{\text{liq}}(M,P,t-\eta)\partial U_{\text{liq}}(P,\eta)/\partial\eta dV_P + \iiint\limits_{G_{\text{liq}}(0)} G_{\text{liq}}(M,P,t)(\Phi_{\text{liq}}(P) - U_{\text{liq}}(P,0))dV_P \ ,$$

where the subscript P indicates integration or differentiation with respect to the variable P.

Analysis of Results. To get the final expression for $\Theta_s(M, t)$ and $\Theta_{\text{liq}}(M, t)$ it is necessary to find the values of $\partial W_s(P, t)/\partial n$ and $\partial W_{\text{liq}}(P, t)/\partial n$, where $P \in S(t)$, that enter into the quantities $g_{s,k}(t)$ and $g_{\text{liq},k}(t)$, respectively. For this purpose, it is necessary to take into account that on the interface $S(t)$ the series representing $W_s(M, t)$ converges in accordance with the theorem of convergence of the Fourier series at a point [according to Eq. (13)] to the quantity $(\varphi(v_n(M)) - U_s(M, t))/2$, and the series representing $W_{\text{liq}}(M, t)$ converges [according to Eq. (17)] to the quantity $(\varphi(v_n(M)) - U_{\text{liq}}(M, t))/2$. As a result, we obtain the following integral equations at $M \in S(t)$, $t > 0$, which are necessary to determine $\partial W_s(P, t)/\partial n$ and $\partial W_{\text{liq}}(P, t)/\partial n$:

$$\int\limits_0^t d\eta \iint\limits_{S(\eta)} v_n(P,\eta)W_s(P,\eta)G_s(M,P,t-\eta)dS_P$$

$$+ a_s\int\limits_0^t d\eta \iint\limits_{S(\eta)} (G_s(M,P,t-\eta)\partial W_s(P,\eta)/\partial n_P - W_s(P,\eta)\partial G_s(M,P,t-\eta)/\partial n_P)dS_P \quad (56)$$

$$- \int\limits_0^t d\eta \iiint\limits_{G_s(t)} G_s(M,P,t-\eta)\partial U_s(P,\eta)/\partial\eta dV_P$$

$$+ \iiint\limits_{G_s(0)} G_s(M,P,t)(\Phi_s(P) - U_s(P,0))dV_P = (\varphi(v_n(M)) - U_s(M,t))/2 \ ,$$

$$-\int\limits_0^t d\eta \iint\limits_{S(\eta)} v_n(P,\eta)W_{\text{liq}}(P,\eta)G_{\text{liq}}(M,P,t-\eta)dS_P$$

$$- a_{\text{liq}}\int\limits_0^t d\eta \iint\limits_{S(\eta)} (G_{\text{liq}}(M,P,t-\eta)\partial W_{\text{liq}}(P,\eta)/\partial n_P - W_{\text{liq}}(P,\eta)\partial G_{\text{liq}}(M,P,t-\eta)/\partial n_P)dS_P$$

$$-\int\limits_0^t d\eta \iiint\limits_{G_{\text{liq}}(t)} G_{\text{liq}}(M,P,t-\eta)\partial U_{\text{liq}}(P,\eta)/\partial\eta dV_P \quad (57)$$

$$+ \iiint\limits_{G_{\text{liq}}(0)} G_{\text{liq}}(M,P,t)(\Phi_{\text{liq}}(P) - U_{\text{liq}}(P,0))dV_P = (\varphi(v_n(M)) - U_{\text{liq}}(M,t))/2 \ .$$

Solving these equations, we find the quantities $\partial W_s(P, t)/\partial n$ and $\partial W_{\text{liq}}(P, t)/\partial n$, the substitution of which into the heat balance equation (8), with Eqs. (9) and (10) taken into account, we obtain an equation for determining the normal velocity of the surface $S(t)$:

$$\rho q v_n = \lambda_s(\partial W_s(M,t)/\partial n + \partial U_s(M,t)/\partial n)$$
$$- \lambda_{\text{liq}}(\partial W_{\text{liq}}(M,t)/\partial n + \partial U_{\text{liq}}(M,t)/\partial n) \ , \quad M \in S(t) \ , \quad t > 0 \ . \quad (58)$$

The representation of the solution of the posed problems in the form of Fourier series (49) and (50) is convenient for calculating the kinetics of phase transitions at sufficiently large values of t determined by conditions $a_s\gamma_1^2 t$, $a_{liq}\gamma_1^2 t >> 1$. In this case, in the expressions of $W_s(M, t)$ and $W_{liq}(M, t)$ it is possible to be restricted to the first terms in the series representing them.

Conclusions. The applicability of the method of integral transformations for limited regions containing a moving boundary is shown. The obtained expressions for the temperature fields are convenient for their calculation in the case of sufficiently large values of time and allow one to find the asymptotics of the velocity of the moving boundary.

The described approach can successfully be applied to solving stochastic differential equations of the mass, momentum, and energy conservation laws [15], to calculating the thermally stressed state in materials in nonuniform stationary temperature fields [16–18] when modeling heat transfer in a stationary nanofluid flow for calculating the corresponding temperature profiles [19], in stochastic models of heat conduction [20, 21].

NOTATION

a_s and a_{liq}, thermal diffusivity of the solid and liquid phases, respectively, m^2/s; q, specific heat of melting, J/kg; $S(t)$, interface; v_n, normal velocity of the interface, m/s; $W_s(M, t)$ and $U_s(M, t)$, terms of the temperature field $\Theta_s(M, t)$, K; $W_{liq}(M, t)$ and $U_{liq}(M, t)$, terms of the temperature field $\Theta_{liq}(M, t)$, K; β_{11} and β_{12}, parameters characterizing the heat exchange of the region transferring with the environment through the external fixed surface S_s with dimensions W/(m·K) and $W/(m^2 \cdot K)$, respectively; β_{21} and β_{22}, parameters characterizing heat exchange of the region transferring with the environment through the external fixed surface S_{liq} with dimensions W/(m·K) and $W/(m^2 \cdot K)$, respectively; δ_{nk}, Kronecker symbol; $\Theta_s(M, t)$, temperature at the point $M \in G_s(t)$ in the region $G_s(t)$ occupied by the solid phase at the moment t, K; $\Theta_{liq}(M, t)$, temperature at the point $M \in G_{liq}(t)$ in the region $G_{liq}(t)$, occupied by the liquid phase (melt) at the moment t, K; λ_s and λ_{liq}, thermal conductivity of the solid and liquid phases, respectively, W/(m·K); ρ, solid phase density, kg/m^3; $\Phi_s(M)$ and $\Phi_{liq}(M)$, initial temperatures of the solid and liquid phases, K; $\varphi(v_n(M))$, temperature on the interface $S(t)$, which generally is a function of the normal growth rate of the new phase and of the interface curvature, K; $\varphi_s(M, t)$ and $\varphi_{liq}(M, t)$, given functions of the control of heat exchange in the heat conduction region G with the environment through the external surface S_s and S_{liq}, respectively, W/m^2. Indices: liq, liquid; s, solid.

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