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REYNOLDS ANALOGY BASED ON THE THEORY OF STOCHASTIC EQUATIONS AND EQUIVALENCE OF MEASURES*

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A new dependence has been obtained to calculate the Reynolds analogy in a nonisothermal turbulent flow in a circular tube. The formula for the Reynolds analogy was obtained from stochastic turbulence theory, which is based on stochastic differential equations of the laws of conservation of mass, momentum, and energy, and also on the regularities of equivalence of measures between deterministic and random motions. A comparison has been made of the calculation results for the classical formula and for the new formula for various Prandtl numbers.

Keywords: *stochastic equations, equivalence of measures, Reynolds analogy.*

Introduction. Issues associated with determining friction and heat-transfer coefficients are known to assume three approaches: experimental, theoretical, and numerical. Although the results of numerical investigations make it possible to determine these coefficients using various procedures, such as RANS, LES, and DNS, they call for subsequent verification by experiment. At the same time, in experimental investigations of actual industrial machines, researchers often manage to obtain measurements of thermal characteristics, whereas friction has to be calculated from indirect dependences. One of these relations is the Reynolds analogy between momentum and heat transfer. The existing formulas to calculate the Reynolds analogy were obtained when criterion procedures were used. The use of these relations caused numerous questions since the dependences involved just the Prandtl number. In this connection, obtaining a dependence to calculate the Reynolds analogy from the new theories developed in recent years is of undoubted interest. In accordance with the results of analyzing the basic principles of turbulence theory [1–15], a physical regularity of equivalence of measures was determined in [16–29], which made it possible to create systems of stochastic equations to determine the onset of turbulence in isothermal and nonisothermal flows and to determine the basic characteristics of turbulent flows. On the basis of stochastic equations for continuous laws and equivalence of measures, analytical dependences have been obtained to calculate the critical point of the onset of turbulence and critical Reynolds and Taylor numbers for classical flows of a continuous medium. Relations have been derived that determine the classical distributions of velocity and temperature fields and of second-order correlations; dependences of the coefficients of friction and heat transfer on a plate and in tubes on the energy of perturbation and its scale and on the turbulent Reynolds number have been determined [21–31]. In [32–36], dependences have been determined and correlation dimensions of an attractor (number of the degrees of freedom) of the boundary layer have been calculated on a plane plate and in a circular tube and in the boundary layer of the earth′s atmosphere. This is a very important point, since the existing methods and the obtained relations to determine the dimensions of an attractor require, in essence, repeated multiple conduct of numerous thorough experimental and theoretical investigations of hydrodynamic turbulence that have been performed in the past 100 years. This new method of constructing a space-time portrait of the correlation dimension of an attractor for conditions in a tube, in the boundary layer on a plane plate, and in the boundary layer of the earth′s atmosphere enables us to check results of numerical calculations and direct numerical simulation DNS. Analytical dependences for the spectral function are presented in [37, 38].

System of Equations. The equations obtained in [16–21] are of the form:

the continuity equation

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$$\frac{d\rho_{\text{col,st}}}{d\tau} = -\frac{\rho_{\text{st}}}{\tau_{\text{cor}}} - \frac{d\rho_{\text{st}}}{d\tau}\ ; \tag{1}$$

the equation of motion

$$\frac{d(\rho\mathbf{U})_{\text{col,st}}}{d\tau} = \text{div}\,(\tau_{i,j})_{\text{col,st}} + \text{div}\,(\tau_{i,j})_{\text{st}} - \frac{(\rho\mathbf{U})_{\text{st}}}{\tau_{\text{cor}}} - \frac{d(\rho\mathbf{U})_{\text{st}}}{d\tau} + F_{\text{col,st},l} + F_{\text{st},l}\ , \tag{2}$$

and the energy equation

$$\frac{dE_{\text{col,st}}}{d\tau} = \text{div}\left(\lambda\,\frac{\partial T}{\partial x_j} + u_i\tau_{i,j}\right)_{\text{col,st}} + \text{div}\left(\lambda\,\frac{\partial T}{\partial x_j} + u_i\tau_{i,j}\right)_{\text{st}} - \left(\frac{E_{\text{st}}}{\tau_{\text{cor}}}\right) - \left(\frac{dE_{\text{st}}}{d\tau}\right) + (u_iF)_{\text{col,st},l} + (u_iF)_{\text{st},l}\ . \tag{3}$$

Here τ, ρ, $\mathbf{U}$, E, and T are the time, density, velocity vector, energy, and temperature. The stress tensor $\tau_{i,j}$ is determined from the following formula:

$$\tau_{i,j} = P + \sigma_{i,j}\ ,\quad \sigma_{i,j} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{i,j}\left(\xi - \frac{2}{3}\,\mu\right)\frac{\partial u_l}{\partial x_l}\ ,\quad i,\ j,\ l = 1,\ 2,\ 3\ ,$$

where ν, μ, and ξ are the kinematic, dynamic, and second viscosity. The quantities u_i, u_j, u_l, x_i, x_j, and x_l are the velocities and coordinate corresponding to i, j, and l; $\delta_{i,j} = l$ at $i = j$ and $\delta_{i,j} = 0$ at $i \neq j$; P is the pressure of the liquid or the gas; λ is the thermal conductivity; c_p and c_v are the specific heats at constant pressure and volume; F is the external force. The correlation time $\tau_{\text{cor}} = \tau^0_{\text{cor}}$ [21–25] determines the lifetime of a perturbation existing in the flow at the beginning of interaction with the deterministic motion

$$(\tau^0_{\text{cor}})_{1U,P} = \frac{L}{((E_{\text{st}})_{U,P}/\rho)^{1/2}}\ ,\quad (\tau^0_{\text{cor}})_{1T} = \frac{L_T}{((E_{\text{st}})_{u_j}/\rho)^{1/2}}\ .$$

In these expressions, L is the linear measure of perturbation (turbulence scale) and $L_T = \dfrac{L}{\text{Pr}}$, $\text{Pr} = \dfrac{\rho\nu c_p}{\lambda}$ is the Prandtl number. In what follows, $L = L_{U,P} = L_U$ is the turbulence scale. The subscripts U, P, and U refer to the velocity field, and the subscript T, to the temperature field. Then for the medium′s nonisothermal motion, using the definition of measures of equivalence between the deterministic and random process at the critical point, we determine the sets of stochastic energy, momentum, and mass equations for the following space-time domains: 1) onset of generation (subscript 1,0 or 1); 2) generation (subscript 1,1); 3) diffusion (1,1,1), and 4) dissipation of turbulent fields. These results make it possible to introduce the notion of a correlator, which has been determined for potential physical quantities and combinations (N, M). This correlator will structurally determine the possible range of motion in space depending on various combinations (M, N) and relevant values which determine the interval of the space–time correlation.

Analytical Solutions. For the domain of the onset of generation $r_{c0}(x_c + \Delta x_0, \tau_c + \Delta\tau_0) - r_c$ of the pair $(N, M) = (1, 0)$, as defined in [16–31], we have a system of mass, motion, and energy equations. The system of mass, momentum, and energy equations (1)–(3) for domain 1, which refers to the $(N, M) = (1, 0)$ pair, is equal to

$$\left(\frac{d\rho_{\text{col,st}}}{d\tau}\right)_{1,0} = -\frac{\rho_{\text{st}}}{\tau_{\text{cor}}}\ ,$$

$$\begin{cases}\left(\dfrac{d(\rho\mathbf{U})_{\text{col,st}}}{d\tau}\right)_{1,0} = -\left(\dfrac{(\rho\mathbf{U})_{\text{st}}}{\tau_{\text{cor}}}\right), \\ \text{div}\,(\tau_{i,j})_{\text{col,st1}} = \dfrac{(\rho\mathbf{U})_{\text{st}}}{\tau_{\text{cor}}}\ ,\end{cases} \qquad \begin{cases}\left(\dfrac{dE_{\text{col,st}}}{d\tau}\right)_{1,0} = -\left(\dfrac{E_{\text{st}}}{\tau_{\text{cor}}}\right)_{1,0}\ , \\ \text{div}\left(\lambda\,\dfrac{\partial T}{\partial x_j} + u_i\tau_{i,j}\right)_{\text{col,st1}} = \left(\dfrac{E_{\text{st}}}{\tau_{\text{cor}}}\right)_{1,0}\ .\end{cases} \tag{4}$$

According to [39–43], deterministic motion in a tube is determined by the quadratic Poiseuille equation of the velocity profile under the assumption of constancy of the medium′s thermophysical properties $u_1 = U_0\left(1 - \left(\dfrac{r}{R}\right)^2\right)$; U_0 and u_1 are the velocities on the axis and along x_1; x_1 and x_2 are the longitudinal and transverse coordinates; R and r are the tube radius and the running radius. In the case of substantial nonisothermicity, the character of distribution of the velocity and temperature profiles has been determined in [44–50]. As for the isothermal process, to find expressions of the critical point $(x_2)_{\text{cr}}$, we use the relation of equivalence of measures $d(E_{\text{col,st}})_{1,0} = -E_{\text{st}}$; then for $\left(\frac{r}{R}\right)_{\text{cr}} = 1 - \left(\frac{x_2}{R}\right)_{\text{cr}}$ we obtain

$$\left(\frac{r}{R}\right)_{\text{cr}} = k\left[2\left(\frac{\sqrt{E_{\text{st}}/\rho}}{U_0}\right)^2\left(\frac{R}{L}\right)\right]^{1/3}, \quad \text{where } k \sim 2.13\,. \tag{5}$$

The expression is approximately 20% smaller than the expression written previously in boundary-layer approximations as far as the numerical coefficient is concerned [16–18]. Then the critical Reynolds number will be determined analogously [16–22]: $\text{Re}_{\text{d.cp}} = 2\left(\frac{U_0}{\sqrt{E_{\text{st}}/\rho}}\right)^3\left(\frac{L}{R}\right)\left(3\left(\frac{r}{R}\right)^2 - 1\right)$ or, with account of formula (5), we write, according to [21]

$$\left(\text{Re}_{\text{d}}\right)_{\text{cr}} = 25.8\left(\frac{U_0}{\sqrt{E_{\text{st}}/\rho}}\right)^{5/3}\left(\frac{L}{R}\right)^{1/3}. \tag{6}$$

Next, $\left(\frac{x_2}{R}\right)_{\text{cr}} = \frac{x_2}{R}$. The system of mass, momentum, and energy equations (1)–(3) for domain 2, which refers to the $(N, M) = (1, 1)$ pair, is written in the form

$$\left(\frac{d\rho_{\text{col,st}}}{d\tau}\right)_{1,1} = -\left(\frac{d\rho_{\text{st}}}{d\tau_{\text{cor}}}\right), \quad \begin{cases}\left(\dfrac{d(\rho\mathbf{U})_{\text{col,st}}}{d\tau}\right)_{1,1} = -\left(\dfrac{d(\rho\mathbf{U})_{\text{st}}}{d\tau}\right), \\ \text{div}\,(\tau_{i,j})_{\text{col,st2}} = \dfrac{d(\rho\mathbf{U})_{\text{st}}}{d\tau},\end{cases} \quad \begin{cases}\left(\dfrac{dE_{\text{col,st}}}{d\tau}\right)_{1,1} = -\left(\dfrac{dE_{\text{st}}}{d\tau}\right)_{1,1}, \\ \text{div}\left(\lambda\dfrac{\partial T}{\partial x_j} + u_i\tau_{i,j}\right)_{\text{col,st2}} = \left(\dfrac{dE_{\text{st}}}{d\tau}\right)_{1,1}.\end{cases} \tag{7}$$

Using systems (6) and (7), we have obtained formulas for the velocity and temperature field [21]. Here, account should be taken of the fact that substantially nonisothermal fields are characterized by the non-self-similarity of distributions [39–41].

It is common knowledge [44–52] that in the region of developed turbulence, the profiles of the averaged characteristics of velocity and temperature have affine similarity

$$\left[\left(\frac{x_2}{R}\right)^{1/n} = \frac{u_1}{U_0}\right], \quad \left[\frac{T - T_{\text{w}}}{T_0 - T_{\text{w}}} = \left(\frac{x_2}{R}\right)^{1/n_T}\right], \quad T = T_{\text{w}} + (T_0 - T_{\text{w}})\left(\frac{x_2}{R}\right)^{1/n_T}. \tag{8}$$

Note that in laminar flow in the tube, $T = T_{\text{w}} + (T_0 - T_{\text{w}})\left(1 - \left(\frac{r}{R}\right)^4\right) = T = T_{\text{w}} + (T_0 - T_{\text{w}})\left(\frac{r}{R}\right)^4$. We can determine the exponents of the profiles n and n_T in (8) according to [19–28] on the basis of relations for the domain of generation of turbulence at the initial $\tau^0_{\text{cor1}U,P}$ and final $\tau^1_{\text{cor1}U,P}$ instants of time of generation of a turbulent field (domain 2), taking account of the characteristic times of the laminar $\tau_{\text{L}} = \left(\frac{\partial u}{\partial x_2}\right)_{\text{L}}$ and turbulent $\tau_{\text{turb}} = \left(\frac{\partial u}{\partial x_2}\right)_{\text{turb}}$ motion. Then the ratio of the turbulent energy to the deterministic (laminar) one produced over the same period $\Delta\tau$ is equal to

$$\frac{\displaystyle\int^{\Delta\tau}\left[\left(\text{div}\,(u_i\tau_{i,j})_{\text{col,st2}}\right)_{\tau^1_{\text{cor1}U,P}} + \left(\text{div}\left(\lambda\frac{\partial T}{\partial x_j}\right)_{\text{col,st2}}\right)_{\tau^1_{\text{cor1}T}}\right]d\tau}{\displaystyle\int^{\Delta\tau}\left[\left(\text{div}\,(u_i\tau_{i,j})_{\text{col,st1}}\right)_{\tau^0_{\text{cor1}U,P}} + \left(\text{div}\left(\lambda\frac{\partial T}{\partial x_j}\right)_{\text{col,st1}}\right)_{\tau^0_{\text{cor1}T}}\right]d\tau} = \frac{(E_{\text{st}})_{\tau^1_{\text{cor}}}}{(E_{\text{st}})_{\tau^0_{\text{cor}}}}. \tag{9}$$

Applying the mean-value theorem, we obtain

$$\frac{\left[\left(\operatorname{div}\,(u_i\tau_{i,j})_{\mathrm{col,st2}}\right)_{\tau^1_{\mathrm{cor1}U,P}} + \left(\operatorname{div}\left(\lambda\,\frac{\partial T}{\partial x_j}\right)_{\mathrm{col,st2}}\right)_{\tau^1_{\mathrm{cor1}T}}\right]_{\mathrm{mean}}(\Delta\tau)}{\left[\left(\operatorname{div}\,(u_i\tau_{i,j})_{\mathrm{col,st1}}\right)_{\tau^0_{\mathrm{cor1}U,P}} + \left(\operatorname{div}\left(\lambda\,\frac{\partial T}{\partial x_j}\right)_{\mathrm{col,st1}}\right)_{\tau^0_{\mathrm{cor1}T}}\right]_{\mathrm{mean}}(\tau^1_{\mathrm{cor1}U,P}/\tau_{\mathrm{L}})(\Delta\tau)} = \frac{(E_{\mathrm{st}})_{\tau^1_{\mathrm{cor}}}}{(E_{\mathrm{st}})_{\tau^0_{\mathrm{cor}}}}\,. \tag{10}$$

Following [19–27] and taking account of $(\tau_L/\tau^1_{\mathrm{cor1}U,P}) \approx \frac{1}{n}\left(\frac{1}{2}\,\frac{(n+1)(2n+1)}{2n^2}\,\frac{(\mathrm{Re_d})_{\mathrm{turb}}}{(\mathrm{Re_d})_{\mathrm{cr1}}}\right)^{-1}\left(\frac{x_2}{R}\right)^{1/n-2.5} \approx \left(\frac{x_2}{R}\right)^{1/n-2}$ and $(U_0)_{\mathrm{turb}}/(U_0)_L \approx \left(\frac{1}{2}\,\frac{(n+1)(2n+1)}{2n^2}\,\frac{(\mathrm{Re_d})_{\mathrm{turb}}}{(\mathrm{Re_d})_{\mathrm{cr1}}}\right)$, where $(U_0)_{\mathrm{L}}$ is the velocity on the tube axis at the first critical number $(\mathrm{Re_d})_{\mathrm{cr1}}$ and $(U_0)_{\mathrm{turb}}$ is the velocity on the tube axis at $(\mathrm{Re_d})_{\mathrm{turb}} \geq (\mathrm{Re_d})_{\mathrm{cr1}}$, we determine that at $8 \geq n \geq 2$ and $(\mathrm{Re_d})_{\mathrm{cr2}} \geq (\mathrm{Re_d})_{\mathrm{turb}} \geq (\mathrm{Re_d})_{\mathrm{cr1}}$, $(U_0)_{\mathrm{turb}}/(U_0)_{\mathrm{L}} \approx 1$–$1.1$ and $(U_0)_{\mathrm{turb}} \approx (U_0)_{\mathrm{L}} \approx U_0$. Then we obtain

$$\left|\frac{\frac{1}{n}\,\frac{2-n}{n}\,\mu\left(\frac{U_0}{R}\right)^2\left(\frac{x_2}{R}\right)^{(2/n)-2} + \frac{1-n_T}{n_T^2}\,\lambda\left(\frac{T_0-T_{\mathrm{w}}}{R^2}\right)\left(\frac{x_2}{R}\right)^{(1/n_T)-2}}{2\mu\left(\frac{U_0}{R}\right)^2\left(3\left(\frac{r}{R}\right)^2-1\right)+12\lambda\left(\frac{T_0-T_{\mathrm{w}}}{R}\right)^2\left(\frac{r}{R}\right)^2}\right|\left(\frac{x_2}{R}\right)^{(1/n)-2}$$

$$= \frac{(E_{\mathrm{st}})_{\tau^1_{\mathrm{cor}}}}{(E_{\mathrm{st}})_{\tau^0_{\mathrm{cor}}}} = \left(\mathrm{Re_{st}} - \frac{1}{\mathrm{Re_{st}}}\right), \quad \frac{(E_{\mathrm{st}})_{\tau^1_{\mathrm{cor}}}}{(E_{\mathrm{st}})_{\tau^0_{\mathrm{cor}}}} = \frac{\left\{|(\mathrm{Re_{st}})_U| + 2\,\frac{T_T}{\mathrm{EcTu}^2}\,|F(\mathrm{Re_{st}})_U|\right\}}{1+2\,\frac{T_T}{\mathrm{EcTu}^2}}\,. \tag{11}$$

Here $(\mathrm{Re_d})_{\mathrm{cr2}}$ is the second critical Reynolds number. Equation (11) includes the exponent of the profiles n_T and n; $\mathrm{Ec} = \frac{U_0^2}{c_p(T_0-T_{\mathrm{w}})}$ is the Eckert number, $T_T = |T_{\mathrm{st}}|/(T_0 - T_{\mathrm{w}})$, and $T_u = \sqrt{\frac{\sum (u_i^2)_{\mathrm{st}}}{U_0^2}}$ is the degree of turbulence. Following [20–28] and using the equations of motion of the stochastic system (7), we determine n:

$$\frac{\int^{\Delta\tau}\left[\left(\operatorname{div}\,(\tau_{i,j})_{\mathrm{col,st2}}\right)_{\tau^1_{\mathrm{cor1}U,P}}\right]d\tau}{\int^{\Delta\tau}\left[\left(\operatorname{div}\,(\tau_{i,j})_{\mathrm{col,st1}}\right)_{\tau^0_{\mathrm{cor1}U,P}}\right]d\tau} = \frac{((\rho U)_{\mathrm{st}})_{\tau^1_{\mathrm{cor}}}}{((\rho U)_{\mathrm{st}})_{\tau^0_{\mathrm{cor}}}} \quad \text{or} \quad \frac{\left[\left(\operatorname{div}\,(\tau_{i,j})_{\mathrm{col,st2}}\right)_{\tau^1_{\mathrm{cor1}U,P}}\right]_{\mathrm{mean}}(\Delta\tau)}{\left[\left(\operatorname{div}\,(\tau_{i,j})_{\mathrm{col,st1}}\right)_{\tau^0_{\mathrm{cor1}U,P}}\right]_{\mathrm{mean}}(\tau_{\mathrm{turb}}/\tau_{\mathrm{L}})(\Delta\tau)} = \frac{((\rho U)_{\mathrm{st}})_{\tau^1_{\mathrm{cor}}}}{((\rho U)_{\mathrm{st}})_{\tau^0_{\mathrm{cor}}}}\,.$$

Taking account of relations (10) and (11) and of formulas in [19–23], we have

$$\left|\frac{\frac{1-n}{n^2}\,\mu\left(\frac{U_0}{R^2}\right)\left(\frac{x_2}{R}\right)^{(1/n)-2}}{2\mu\left(\frac{U_0}{R^2}\right)}\right| = \frac{((\rho U)_{\mathrm{st}})_{\tau^1_{\mathrm{cor}}}}{((\rho U)_{\mathrm{st}})_{\tau^0_{\mathrm{cor}}}} = \sqrt{\left|\mathrm{Re_{st}} - \frac{1}{\mathrm{Re_{st}}}\right|}\,. \tag{12}$$

For the nonisothermal motion [20–23], the critical point will be defined as

$$\left(\frac{x_2}{R}\right) = 0.19\left[\frac{(E_{\mathrm{st}})_{U,P}}{U_0^2}\left(\frac{R}{L_U}\right)\right]^{1/3}\left[\left|\frac{\mathrm{EcPr}}{1+\mathrm{EcPr}}\right|\left(1+\frac{2T_T}{\mathrm{Tu}^2\mathrm{Ec}}\right)\right]^{1/3}, \tag{13}$$

and the critical Reynolds number (6) for the nonisothermal flow will be

$$\mathrm{Re}_{\mathrm{d}(T,U)} = \left(25.8\left(\frac{U_0}{\sqrt{E_{\mathrm{st}}/\rho}}\right)^{5/3}\left(\frac{L_U}{R}\right)^{1/3}\right)F_{\mathrm{turb}}, \tag{14}$$

where

$$F_{\mathrm{turb}} = \left(\frac{\left[1+\dfrac{2T_T}{|\mathrm{Ec}|\mathrm{Tu}^2}\right]^{2/3}}{\left[1+\left(\dfrac{\mathrm{Pr}((u_j^2)_{\mathrm{st}})^{1/2}}{((u_i^2)_{\mathrm{st}})^{1/2}}\right)\dfrac{2T_T}{|\mathrm{Ec}|\mathrm{Tu}^2}\right]}\left(\frac{\left(1+2\dfrac{1}{\mathrm{Pr}|\mathrm{Ec}|}\right)}{\left(1+\dfrac{1}{|\mathrm{PrEc}|}\right)^{2/3}}\right)\right)\frac{((u_j^2)_{\mathrm{st}})^{1/2}}{((u_i^2)_{\mathrm{st}})^{1/2}} \approx K_u = 0.3\text{–}0.5\,.$$

Then, according to [26], the friction factor is defined as

$$\frac{\lambda}{8}\rho\left[\frac{2n^2}{(n+1)(2n+1)}\right]^2 U_0^2 = \frac{1}{n}\mu\left(\frac{U_0}{R}\right)\left(\frac{x_2}{R}\right)^{(1/n)-1}. \tag{15}$$

In accordance with [27], the Nusselt number may be written as

$$\mathrm{Nu}_{\mathrm{d}} = \frac{\alpha 2R}{\lambda} = 2\left(\frac{x_2}{R}\right)^{(1/n_T)-1}\left[1+|\mathrm{PrEc}|\left(\frac{x_2}{R}\right)^{(2/n)-(1/n_T)}\right]. \tag{16}$$

In this case, the Reynolds analogy will be written as

$$\frac{\mathrm{Nu}_{\mathrm{d}}}{\mathrm{PrRe}_{\mathrm{d}}}\frac{2}{\lambda} = \frac{\mathrm{Pr}^{(1/n_T)}\,2\left(\dfrac{x_2}{R}\right)^{(1/n_T)-1}\left[1+|\mathrm{PrEc}|\left(\dfrac{x_2}{R}\right)^{(2/n)-(1/n_T)}\right]}{\mathrm{PrRe}_{\mathrm{d}}\dfrac{1}{\mathrm{Re}_{\mathrm{d}}}\dfrac{8(n+1)(2n+1)}{(n^2)n}\left(\dfrac{x_2}{R}\right)^{(1/n)-1}}$$

$$= \frac{(n^2)n}{4(n+1)(2n+1)}\mathrm{Pr}^{(1/n_T-1)}\left(\frac{x_2}{R}\right)^{(1/n)-(1/n_T)}\left[1+|\mathrm{PrEc}|\left(\frac{x_2}{R}\right)^{(2/n)-(1/n_T)}\right].$$

Taking account of the estimates [21–29] of the exponents of the profiles $n_T = 8$ and $n = 7$, we have

$$\frac{\mathrm{Nu}_{\mathrm{d}}}{\mathrm{PrRe}_{\mathrm{d}}}\frac{2}{\lambda} = 0.71\mathrm{Pr}^{-(0.856-0.875)}\left(\frac{x_2}{R}\right)_{\mathrm{cr}}^{(1/56)}\left[1+|\mathrm{PrEc}|\left(\frac{x_2}{R}\right)^{9/56}\right]. \tag{17}$$

Finally, we write

$$\frac{\mathrm{Nu}_{\mathrm{d}}}{\lambda/2} = 0.8\text{–}1.1\ \mathrm{Pr}^{-(0.856-0.875)}. \tag{18}$$

Results. On the basis of the solution of stochastic equations, we have obtained dependence (18) to calculate the Reynolds analogy in a circular tube. A comparison of this dependence with the classical dependence is presented in Table 1, which also gives a calculation of the Reynolds analogy for a plane plate according to [51].

TABLE 1. Comparison of the Results of Calculations of the Reynolds Analogy

Pr	Calculation from the classical empirical dependence $\frac{St}{\lambda/2} \approx Pr^{-(0.6-0.7)}$	Calculation for the circular tube from dependence (18) based on stochastic equations $\frac{Nu_d}{\lambda/2} = 0.8\text{–}1.1\ Pr^{-(0.856-0.875)}$	Calculation for the plane plate from the dependence based on stochastic equations [51] $\frac{St}{C_f/2} \approx 0.85Pr^{-0.75}$
0.05	6.03–8.14	10–13	8.03
0.70	1.24–1.28	1.0–1.5	1.11
3	0.52–0.46	0.31–0.45	0.373
7	0.311–0.25	0.146–0.21	0.197

Conclusions. Thus, the results show a satisfactory agreement between the classical experimental dependence and analytical formulas (18) and [51] obtained according to the stochastic theory of turbulence, which is based on stochastic differentials equations of the laws of conservation of mass, momentum, and energy, and also the regularity of the equivalence of measures between deterministic and random motions. It should be noted that the stochastic procedure finds increasing use in scientific investigations not only in motion of continuous media [7–10, 18–38, 53–55] but also in heat-conduction problems [56].

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