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INVERSE PROBLEM OF PIPELINE TRANSPORT OF WEAKLY-COMPRESSIBLE FLUIDS

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Nonstationary one-dimensional flow of a weakly-compressible fluid in a pipeline is considered. The flow is described by a nonlinear system of two partial differential equations for the fluid flow rate and pressure in the pipeline. An inverse problem on determination of the fluid pressure and flow rate at the beginning of the pipeline needed for the passage of the assigned quantity of fluid in the pipeline at a certain pressure at the pipeline end was posed and solved. To solve the above problem, a method of nonlocal perturbation of boundary conditions has been developed, according to which the initial problem is split at each discrete moment into two successively solvable problems: a boundary-value inverse problem for a differential-difference equation of second order for the fluid flow rate and a direct differential-difference problem for pressure. A computational algorithm was suggested for solving a system of difference equations, and a formula was obtained for approximate determination of the fluid flow rate at the beginning of the pipeline. Based on this algorithm, numerical experiments for model problems were carried out.

Keywords: *pipeline transport, weakly-compressible fluid, nonstationary flow, boundary-value inverse problem, differential-difference problem.*

Introduction. At the present time, for transporting various fluids (water, oil, oil products), pipelines of various dimensions are used, beginning from the smallest ones, used in laboratories and control-measuring apparatuses, up to main ones. Usually, in designing a pipeline the fluid flow rate in it is assigned; it determines the pipeline efficiency and the positions of its beginning and end. One of the main tasks here is the determination of the pressure drop along the pipeline length needed for the passage of a given quantity of fluid through it. In practice, in solving this problem, use is made of the assumption according to which the fluid motion in the pipeline is stationary and, proceeding from this assumption, the Darcy–Weisbach formula is used in calculations [1–3]:

$$\Delta P = \lambda \frac{\rho u^2}{2d} l \, . \tag{1}$$

It should be noted that it was possible to justify this formula and to obtain an explicit expression for the coefficient of hydraulic resistance of the pipeline only for a homogeneous incompressible stationary laminar fluid flow obeying the corresponding rheological laws. However, as the practice of the fluid transport through pipelines shows, the start-up or stopping of a pipeline, switching-in or switching-off of the pumping-over station, the beginning or stopping of the fluid takeoffs, and other technological operations lead to the appearance of nonstationary fluid flow in the pipeline.

In this connection, for the pipeline conveyance of fluids, of great importance is the investigation of nonstationary flow of compressible fluid in the pipeline with the aim of determining the hydrodynamic conditions at the beginning of the pipeline indispensable for achieving the transmission of the assigned quantity of fluid through the pipeline.

Problem Formulation. Nonstationary flow of a weakly-compressible viscous fluid in a horizontally positioned pipeline of length l is considered. The flow is described by a system of partial-differential equations in the fluid flow rate and pressure variables [1–3]:

$$\frac{\partial Q(x, t)}{\partial t} + \frac{Q(x, t)}{S} \frac{\partial Q(x, t)}{\partial x} = - \frac{S}{\rho_0} \frac{\partial P(x, t)}{\partial x} - \frac{\lambda \left| Q(x, t) \right|}{2dS} Q(x, t) \, , \tag{2}$$

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$$\frac{\partial P(x,t)}{\partial t} + \frac{\rho_0 c^2}{S}\frac{\partial Q(x,t)}{\partial x} = 0\,, \tag{3}$$

$$0 < x < l\,, \quad 0 < t \le T\,.$$

Let the state of the fluid flow in the pipeline at the initial moment $t = 0$ be known, i.e., for the system of equations (2) and (3) the following initial conditions are known:

$$Q(x, 0) = \varphi(x)\,, \tag{4}$$

$$P(x, 0) = \psi(x)\,. \tag{5}$$

It is assumed that the fluid is supplied to the beginning of the pipeline ($x = 0$) and that the fluid flow rate and pressure at the end of the pipeline ($x = l$) are given in conformity with its designed purpose. Then, for the system of equations (2), (3) we will have the following boundary-value conditions:

$$Q(l, t) = q(t)\,, \tag{6}$$

$$P(l, t) = \theta(t)\,, \tag{7}$$

where $q(t)$ and $\theta(t)$ are the functions describing the time history of the fluid flow rate and pressure in the outlet section of the pipeline. It is necessary to find the laws of variation of the pressure $P(0, t)$ and fluid flow rate $Q(0, t)$ over time at the beginning of the pipeline, which provide the assigned fluid flow rate $q(t)$ in the pipeline at a certain pressure $\theta(t)$ at its end. The problem (2)–(7) posed relates to a class of boundary-value inverse problems [4–6].

Method of Solution. The boundary-value inverse problem (2)–(7) was solved by the method of nonlocal perturbation of boundary conditions [5]. We will replace the boundary condition (6), to which problem (2)–(7) owes its incorrectness by the nonlocal boundary condition

$$Q(l, t) + \alpha Q(0, t) = q(t)\,, \tag{8}$$

where α is the parameter of nonlocal perturbation, which acts here as a regularization parameter: $\alpha > 0$.

We discretize problem (2)–(5), (7), (8) in time. For this purpose we introduce a uniform difference time grid in the region $[0 \le t \le T]$:

$$\bar{\omega}_t = \left\{ t_j = j\Delta t\,, \quad j = \overline{0,\, m} \right\},$$

with the step $\Delta t = \dfrac{T}{m}$. In the case of using the variable t_j, $j = \overline{1,\, m}$, the derivatives $\dfrac{\partial Q(x,t)}{\partial t}$ and $\dfrac{\partial P(x,t)}{\partial t}$ in Eqs. (2) and (3) are discretized by "backward" difference:

$$\left.\frac{\partial Q(x,t)}{\partial t}\right|_{t=t_j} \approx \frac{Q(x,t_j) - Q(x,t_{j-1})}{\Delta t}\,, \quad \left.\frac{\partial P(x,t)}{\partial t}\right|_{t=t_j} \approx \frac{P(x,t_j) - P(x,t_{j-1})}{\Delta t}\,.$$

Introducing the notations $Q^j(x) \approx Q(x, t_j)$ and $P^j(x) \approx P(x, t_j)$, we will write problem (2)–(5), (7), (8) in the form

$$\frac{Q^j(x) - Q^{j-1}(x)}{\Delta t} + \frac{Q^{j-1}(x)}{S}\frac{dQ^j(x)}{dx} = -\frac{S}{\rho_0}\frac{dP^j(x)}{dx} - \frac{\lambda |Q^{j-1}(x)|}{2dS} Q^j(x)\,, \tag{9}$$

$$\frac{P^j(x) - P^{j-1}(x)}{\Delta t} + \frac{\rho_0 c^2}{S}\frac{dQ^j(x)}{dx} = 0\,, \tag{10}$$

$$Q^j(l) + \alpha Q^j(0) = q^j\,, \tag{11}$$

$$P^j(l) = \theta^j , \quad j = 1, 2, \dots , m , \tag{12}$$

$$Q^0(x) = \varphi(x) , \quad P^0(x) = \psi(x) , \tag{13}$$

where $\theta^j = \theta(t_j)$ and $q^j = q(t_j)$. It is evident that the resulting differential-difference problem (9)–(13) at each fixed value $j = 1, 2, \dots , m$ can be split into two independent problems. We will find $P^j(x)$ from Eq. (10):

$$P^j(x) = P^{j-1}(x) - \frac{\upsilon_0 c^2 \Delta t}{S} \frac{dQ^j(x)}{dx} , \quad 0 < x < l , \tag{14}$$

and we substitute the obtained expression into Eq. (9). As a result, we will have a linear differential-difference equation of the second order:

$$\frac{Q^j(x) - Q^{j-1}(x)}{\Delta t} + \frac{Q^{j-1}(x)}{S} \frac{dQ^j(x)}{dx} = c^2 \Delta t \frac{d^2 Q^j(x)}{dx^2} - \frac{\lambda |Q^{j-1}(x)|}{2dS} Q^j(x) - \frac{S}{\rho_0} \frac{dP^{j-1}(x)}{dx} , \tag{15}$$

$$0 < x < l ,$$

$$Q^j(l) + \alpha Q^j(0) = q^j . \tag{16}$$

The boundary-value condition (16) is insufficient for singe-valued solution of Eq. (15). To obtain an additional condition we will write Eq. (14) for the case $x = l$:

$$P^j(l) = P^{j-1}(l) - \frac{\rho_0 c^2 \Delta t}{S} \frac{dQ^j(l)}{dx} .$$

Hence, with account for the boundary-value condition (12), we will obtain the additional condition for Eq. (15):

$$\frac{dQ^j(l)}{dx} = - \frac{S(\theta^j - \theta^{j-1})}{\rho_0 c^2 \Delta t} . \tag{17}$$

It is evident that on solving the problem (15)–(17) and determining the function $Q^j(x)$, $0 \le x \le l$, we can find the function $P^j(x)$ from the solution of the problem

$$P^j(x) = P^{j-1}(x) - \frac{\rho_0 c^2 \Delta t}{S} \frac{dQ^j(x)}{dx} , \quad 0 \le x < l , \quad P^j(l) = \theta^j . \tag{18}$$

Now, for numerical solution of the boundary inverse problem for the differential-difference equation (15) we discretize this problem in the variable x. Let us introduce a uniform difference spatial grid in the region $[0 \le x \le l]$:

$$\bar{\omega}_x = \{x_i = i\Delta x, \quad i = \overline{0, n} , \quad \Delta x = l/n\} ,$$

and represent the discrete analog of the problem (15)–(17) on the grid $\bar{\omega}$:

$$\frac{Q_i^j - Q_i^{j-1}}{\Delta t} + \frac{Q_i^{j-1}}{S} \frac{Q_i^j - Q_{i-1}^j}{\Delta x} = c^2 \Delta t \frac{Q_{i+1}^j - 2Q_i^j + Q_{i-1}^j}{\Delta x^2} - \frac{\lambda |Q_i^{j-1}|}{2dS} Q_i^j - \frac{S}{\rho_0} \frac{P_{i+1}^{j-1} - P_{i-1}^{j-1}}{2\Delta x} ,$$

$$i = 1, 2, \dots , n-1 , \quad \frac{Q_n^j - Q_{n-1}^j}{\Delta x} = - \frac{S(\theta^j - \theta^{j-1})}{\rho_0 c^2 \Delta t} , \quad Q_n^j + \alpha Q_0^j = q^j ,$$

where $Q_i^j \approx Q^j(x_i)$ and $P_i^j \approx P^j(x_i)$. We transform the obtained system of difference equations to the form

$$a_i Q_{i-1}^j - d_i Q_i^j + b_i Q_{i+1}^j = -f_i^{j-1} , \quad i = 1, 2, \dots , n-1 , \tag{19}$$

$$Q_n^j = Q_{n-1}^j - \frac{\Delta x S(\theta^j - \theta^{j-1})}{\rho_0 c^2 \Delta t}\ , \qquad (20)$$

$$Q_n^j + \alpha Q_0^j = q^j\ , \qquad (21)$$

$$a_i = \Delta t^2 c^2 S + Q_i^{j-1}\Delta t\Delta x\ , \quad b_i = \Delta t^2 c^2 S\ , \quad d_i = a_i + b_i + \Delta x^2 S + \frac{\lambda |Q_i^{j-1}|}{2d}\Delta x^2 \Delta t\ ,$$

$$f_i^{j-1} = \Delta x^2 Q_i^{j-1} S - \Delta t \Delta x S^2 (P_{i+1}^{j-1} - P_{i-1}^{j-1})/2\rho_0\ .$$

In the difference problem described by a system of linear algebraic equations (19)–(21), the approximate values of the sought function $Q^j(x)$ are considered to be unknown at the internal nodes of the difference grid $\overline{\omega}_x$, i.e., the function Q_i^j, $i = 0, 1, 2, \dots, n$, is considered. In order to divide this problem into mutually independent subproblems, each of which may be solved independently, we will present its solution at each fixed value $j = 1, 2, \dots, n$ in the following form [7, 8]:

$$Q_i^j = \xi_i^j + \eta_i^j Q_0^j\ , \quad i = 0, 1, \dots, n\ , \qquad (22)$$

where ξ_i^j and η_i^j are the unknown variables. Substitution of (22) into (19) and (20) yields the relations

$$\left[a_i \xi_{i-1}^j - d_i \xi_i^j + b_i \xi_{i+1}^j + f_i^{j-1}\right] + Q_0^j \left[a_i \xi_{i-1}^j - d_i \xi_i^j + b_i \xi_{i+1}^j\right] = 0\ ,$$

$$\left[\xi_n^j - \xi_{n-1}^j + \frac{\Delta x S(\theta^j - \theta^{j-1})}{\rho_0 c^2 \Delta t}\right] + Q_0^j \left[\eta_n^j - \eta_{n-1}^j\right] = 0\ ,$$

from which, with account for the relation $Q_0^j = \xi_0^j + \eta_0^j Q_0^j$, we obtain difference problems for determining the additional variables ξ_i^j and η_i^j:

$$a_i \xi_{i-1}^j - d_i \xi_i^j + b_i \xi_{i+1}^j = -f_i^{j-1}\ , \qquad (23)$$

$$\xi_0^j = 0\ , \qquad (24)$$

$$\xi_n^j = \xi_{n-1}^j - \frac{\Delta x S(\theta^j - \theta^{j-1})}{\rho_0 c^2 \Delta t}\ . \qquad (25)$$

$$a_i \eta_{i-1}^j - d_i \eta_i^j + b_i \eta_{i+1}^j = 0\ , \qquad (26)$$

$$\eta_0^j = 1\ , \qquad (27)$$

$$\eta_n^j = \eta_{n-1}^j\ . \qquad (28)$$

We can find the solution of the difference problems described by systems of linear algebraic equations with a three-diagonal matrix (23)–(25) and (26)–(28) at each fixed value of $j = 1, 2, \dots, m$ by the Thomas method [5]. Substitution of (22) into (21) yields the expression

$$\xi_n^j + \eta_n^j Q_0^j + \alpha Q_0^j = q^j\ ,$$

from which we obtain the following formula for determining the fluid flow rate at the beginning of the pipeline:

$$Q_0^j = \frac{q^j - \xi_n^j}{\eta_n^j + \alpha}\ . \qquad (29)$$

After determining Q_0^j by formula (29), we can successively find Q_1^j, Q_2^j, ... , Q_n^j by the recurrent formula (22). Finding the fluid flow rate distribution along the pipeline length, we can go over to the determination of pressure distribution. After constructing the discrete analog of problem (18) on the grid $\overline{\omega}_x$, we obtain the following computational formula for calculating the pressure:

$$P_i^j = P_i^{j-1} - \frac{\rho_0 c^2 \Delta t}{S} \frac{Q_{i+1}^j - Q_i^j}{\Delta x} , \quad i = 0, 1, 2, \dots , n-1 . \tag{30}$$

In particular, the pressure of the beginning of the pipeline is determined as

$$P_0^j = P_0^{j-1} - \frac{\rho_0 c^2 \Delta t}{S} \frac{Q_1^j - Q_0^j}{\Delta x} .$$

Thus, the computational algorithm of the solution of inverse problem (2)–(7) on recovering the pressure and the fluid flow rate at each discrete value of the time variable t_j, $j = 1, 2, \dots, m$ includes the solution of two linear difference problem of second order (23)–(25) and (26)–(28) for the additional variables ξ_i^j and η_i^j, $i = \overline{1,\ n}$, the determination of Q_0^j from (29) and computation of Q_i^j, $i = \overline{0,\ n}$, and P_i^j, $i = \overline{0,\ n-1}$ with the use of (22) and (30).

Results of Numerical Calculations. To elucidate the efficiency of practical application of the proposed computational algorithm, numerical experiments were carried out for model problems by the following scheme:

1) solution of the system of equations (2) and (3) at the initial conditions

$$Q(x, 0) = \varphi(x) , \quad P(x, 0) = \psi(x)$$

and boundary conditions

$$Q(0, t) = q_0(t) , \quad P(l, t) = \theta(t) ,$$

2) the found dependences $q(t) = Q(l, t)$ and $\theta(t)$ are taken to be exact data for numerical solution of the inverse problem on the recovery of $Q(0, t)$ and $P(0, t)$.

The first series of calculations was carried out with the use of nonperturbed data. The second series of calculations was carried out with superposition of a certain function that models the error of experimental data on $q(t)$:

$$\tilde{q}(t) = q(t) + \delta\sigma(t)q(t) .$$

In this expression δ is the error level and $\sigma(t)$ is a random quantity modeled with the aid of the random-number generator. The value of the nonlocal perturbation parameter is determined in accordance with the misclosure principle [5], i.e., the values of α_0 and γ are assigned and the succession $\alpha_{k+1} = \alpha_{k\gamma}$ is built. Computation is carried out until the following condition is fulfilled:

$$\left[\sum_{j=1}^{m} (\tilde{q}(t_j) - Q_n^j)^2 \Delta t \right]^{1/2} \le \varepsilon ,$$

where ε is the given error.

Numerical calculations were carried out in a spatial-time difference grid with steps $\Delta x = 5$ m and $\Delta t = 5$ s. The results of the numerical experiment carried out for $d = 1.2$ m, $\rho_0 = 900$ kg/m^3, $s = 1300$ m/s, $\theta(t) = 0.2$ MPa, $\psi(x) = 0$, $\varphi(x) = 0$, $q_0(t) = 2 + 0.5 \sin 2t$ m^3/s, $l = 1000$ m, $\lambda = 0.2$, $\varepsilon = 0.015$, $\alpha_0 = 2$, $\gamma = 0.1$, and $\alpha = 0.0002$ with the use of nonperturbed and perturbed inlet data are presented in Table 1. For perturbation of the input data, the error $\delta = 0.05$ was used. The results of numerical experiment show that in using nonperturbed input data the sought functions $Q(0, t)$ and $P(0, t)$ are recovered with high accuracy (the second and third, fifth and sixth columns in Table 1). In this case, the relative errors of the recovery of functions do not exceed 0.02%. In using perturbed input data the error of determination of which is fluctuating on character a weak dependence of the recovery of the indicated functions on this error is manifested. Thus, for an error of determining the input data of 4.23% the sought functions are determined with an error of 4.32%. On decrease in the error level, the solution is recovered more accurately. An analysis of the results of numerical experiment indicates that in the suggested algorithm the stability of the solution against the input data error is provided.

TABLE 1. Results of Numerical Calculations

t_j, s	$Q(0, t)$, m^3/s			$P(0, t)$, MPa		
	Q_0^t	$\bar{Q}_0$	$\tilde{Q}_0$	P_0^t	$\bar{P}_0$	$\tilde{P}_0$
60	2.290	2.290	2.345	0.4046	0.4045	0.4186
120	2.473	2.472	2.510	0.4523	0.4521	0.4609
180	2.479	2.479	2.600	0.4923	0.4858	0.5143
240	2.308	2.308	2.382	0.5112	0.5090	0.5262
300	2.022	2.022	2.051	0.4468	0.4467	0.4524
360	1.728	1.728	1.813	0.3633	0.3633	0.3830
420	1.535	1.535	1.584	0.2985	0.2984	0.3117
480	1.515	1.514	1.552	0.2834	0.2834	0.2906
540	1.675	1.675	1.705	0.3288	0.3288	0.3334
600	1.956	1.955	1.975	0.4141	0.4140	0.4194
660	2.253	2.253	2.288	0.5002	0.5000	0.5080
720	2.456	2.456	2.469	0.5539	0.5537	0.5568
780	2.490	2.490	2.523	0.5619	0.5618	0.5697
840	2.342	2.341	2.347	0.5268	0.5267	0.5280
900	2.066	2.066	2.112	0.4585	0.4584	0.4691
960	1.766	1.766	1.846	0.3749	0.3749	0.3953
1020	1.553	1.553	1.567	0.3053	0.3052	0.3080
1080	1.506	1.506	1.538	0.2818	0.2817	0.2864
1140	1.643	1.642	1.680	0.3191	0.3191	0.3272
1200	1.912	1.912	1.997	0.4009	0.4009	0.4227

Conclusions. Within the framework of the one-dimensional model of nonstationary flow of weakly-compressible fluid in the pipeline the boundary-value inverse problem of determining pressure and fluid flow rate is considered at the beginning of the pipeline, which provide the required regime of fluid flow in its outlet section. The proposed computational algorithm makes it possible to successively determine the distributions of the fluid flow rate and pressure along the pipeline length in each time layer only on the basis of the information on flow parameters in its outlet section.

NOTATION

c, velocity of sound in the fluid, m/s; d, pipeline diameter, m; l, pipeline length, m; P_0^t, exact value of pressure, MPa; $\bar{P}_0$ and $\tilde{P}_0$, values of pressure calculated at nonperturbed and perturbed input data, MPa; Q_0^t, exact value of the volumetric flow rate of fluid, m^3/s; $\bar{Q}_0$ and $\tilde{Q}_0$, values of the volumetric flow rate of fluid calculated at nonperturbed and perturbed input data, m^3/s; $S = \pi d^2/4$, cross-sectional area of pipeline, m^2; t, time, s; u, mean velocity of fluid flow in pipeline section, m/s; x, Cartesian coordinate, m; λ, coefficient of hydraulic resistance; ρ_0, fluid density, kg/m^3.

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