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RIEMANN AND SHOCK WAVES IN A POROUS LIQUID-SATURATED GEOMETRICALLY NONLINEAR MEDIUM

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Within the framework of the classical Biot theory, the propagation of plane longitudinal waves in a porous liquid-saturated medium is considered with account for the nonlinear connection between deformations and displacements of solid phase. It is shown that the mathematical model accounting for the geometrical nonlinearity of the medium skeleton can be reduced to a system of evolution equations for the displacements of the skeleton of medium and of the liquid in pores. The system of evolution equations, in turn, depending on the presence of viscosity, is reduced to the equation of a simple wave or to the equation externally resembling the Burgers equation. The solution of the Riemann equation is obtained for a bell-shaped initial profile; the characteristic wave breaking is shown. In the second case, the solution is found in the form of a stationary shock wave having the profile of a nonsymmetric kink. The relationship between the amplitude and width of the shock wave front has been established. It is noted that the behavior of nonlinear waves in such media differs from the standard one typical of dissipative nondispersing media, in which the propagation of waves is described by the classical Burgers equation.

Keywords: *porous medium (Biot medium), geometrical nonlinearity, evolution equation, Riemann wave, generalized Burgers equation, stationary shock wave.*

The mathematical models of deformed porous materials, both presented in the classical works of M. A. Biot [1, 2], Ya. I. Frenkel [3], L. Ya. Kosachevskii [4] and subsequently in modified forms in [5–30], are widely used in studying the processes proceeding in the geophysics and mechanics of natural and artificial composite materials. The authors of the majority of works in their investigations restrict themselves to the linear theory of pore-toughness despite the fact that, as shown experimentally in [31], the nonlinear effects in liquid-saturated porous media are substantial and are also of definite interest.

Moreover, in nature, technique, and technologies porous liquid-saturated materials are frequently encountered that contain cavities filled with a liquid and distributed chaotically. Under certain conditions, the cavities oscillate under the action of an elastic wave and exert a substantial influence on the laws governing the propagation of waves. It is shown in works [32–34] that along with the geometric nonlinearity (nonlinear connection between deformations and displacements) and physical nonlinearity (nonlinear connection between stresses and deformations) it is important to account for the cavity nonlinearity.

Works [35–37] consider the propagation of plane longitudinal waves in a porous liquid-saturated medium with cavities. The behavior of linear and nonlinear waves in cavity-porous media is studied. It is shown that in such media three longitudinal waves propagate: two waves as in the Biot medium and one wave due to the presence of cavities in the medium. The influence of the size of spherical cavities on the basic parameters of stationary waves, on the amplitude and width of the solitons are investigated.

The equations that describe the deformation of the Biot medium are presented in [2]. In this system, the relationship between deformations and displacements is linear. The equations that describe the motion of a porous liquid-saturated medium in a one-dimensional case with account for the geometrical nonlinearity take the form

$$\begin{cases} \rho_{11}\dfrac{\partial^2 U}{\partial t^2} + \rho_{12}\dfrac{\partial^2 V}{\partial t^2} + b\left(\dfrac{\partial U}{\partial t} - \dfrac{\partial V}{\partial t}\right) = \dfrac{\partial \sigma_{11}}{\partial x}\,, \\ \rho_{12}\dfrac{\partial^2 U}{\partial t^2} + \rho_{22}\dfrac{\partial^2 V}{\partial t^2} + b\left(\dfrac{\partial V}{\partial t} - \dfrac{\partial U}{\partial t}\right) = \dfrac{\partial s}{\partial x}\,, \end{cases} \tag{1}$$

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where $U = U(x, t)$ and $V = V(x, t)$ are the displacements of the skeleton and liquid in the pores along the coordinate x:

$$\sigma_{11} = Ae_U + 2Ne_{11} + Qe_V \ , \quad s = Qe_U + Re_V \ , \quad e_U = \frac{\partial U}{\partial x} \ , \quad e_V = \frac{\partial V}{\partial x} \ , \quad e_{11} = \frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial U}{\partial x}\right)^2 \ ,$$

where the viscosity coefficient is $b = (\eta/K_{per})\Phi^2$.

We seek the solution of the system of equations (1) in the form of asymptotic expansions in the small parameter $U = U_{(0)} + \varepsilon U_{(1)} + \ldots$, $V = V_{(0)} + \varepsilon V_{(1)} + \ldots$, where $\varepsilon << 1$. In this case, we introduce new variables $\xi = x - ct$ and $\tau = \varepsilon t$. Such selection of the variables is explained by the fact that the perturbation, while propagating with velocity c along the x axis, evolves slowly in time because of the nonlinearity, dispersion, and dissipation. We consider that the nonlinearity in system (1) has the first order of smallness in ε and the viscosity is a small quantity.

In the zero approximation we obtain the following system of equations:

$$(\rho_{11}c^2 - A - 2N)\frac{\partial^2 U}{\partial \xi^2} + (\rho_{12}c^2 - Q)\frac{\partial^2 V}{\partial \xi^2} = 0 \ , \quad (\rho_{12}c^2 - Q)\frac{\partial^2 U}{\partial \xi^2} + (\rho_{22}c^2 - R)\frac{\partial^2 V}{\partial \xi^2} = 0 \ .$$

The condition of the existence of the nonzero solution of this system yields the following biquadratic equation for determining the velocity:

$$m_1 c^4 + m_2 c^2 + m_3 = 0 \ ,$$

where $m_1 = \rho_{11}\rho_{22} - \rho_{12}^2$, $m_2 = 2\rho_{12}Q - \rho_{11}R - \rho_{22}(A + 2N)$, $m_3 = (A + 2N) - Q^2$. For the existence of two real and positive roots of the equation it is necessary that the inequalities of one of the two systems could be satisfied:

$$\begin{cases} m_1 > 0 \\ m_2 < 0 \\ m_3 > 0 \end{cases} \quad \text{or} \quad \begin{cases} m_1 < 0 \\ m_2 > 0 \\ m_3 < 0 \end{cases} .$$

The connections of the attached density of the mass ρ_{12} with the real densities of phases are given, for example, in [2] and have the form $\rho_{11} = \rho_1 - \rho_{12}$, $\rho_{22} = \rho_2 - \rho_{12}$, $\rho_{12} < 0$; ρ_1 and ρ_2 are the densities of the solid and liquid phases, respectively. It is evident that $m_1 > 0$.

The first approximation in ε leads to the system of evolution equations for the displacements of the medium skeleton U and of the liquid in pores V:

$$\begin{cases} cb\left(\frac{\partial U}{\partial \xi} - \frac{\partial V}{\partial \xi}\right) + 2\varepsilon c\left(\rho_{11}\frac{\partial^2 U}{\partial \xi \partial \tau} + \rho_{12}\frac{\partial^2 V}{\partial \xi \partial \tau}\right) + 2N\frac{\partial U}{\partial \xi}\frac{\partial^2 U}{\partial \xi^2} = 0 \ , \\ cb\left(\frac{\partial U}{\partial \xi} - \frac{\partial V}{\partial \xi}\right) - 2\varepsilon c\left(\rho_{12}\frac{\partial^2 U}{\partial \xi \partial \tau} + \rho_{22}\frac{\partial^2 V}{\partial \xi \partial \tau}\right) = 0 \ . \end{cases} \quad (2)$$

We will consider the case where the viscosity is absent in the medium. Then the system of equations (2) is reduced to one equation for $W = \dfrac{\partial U}{\partial \xi}$:

$$\frac{\partial W}{\partial \tau} + \frac{\rho_{22}N}{\varepsilon c(\rho_{11}\rho_{22} - \rho_{12}^2)} W \frac{\partial W}{\partial \xi} = 0 \ . \quad (3)$$

Equation (3) is the equation of a simple wave or the Riemann equation [38, 39] and can be solved by the method of characteristics as a first-order partial differential equation.

If we select a bell-shaped profile as the initial profile of the wave at the zero instant of time

$$W = \exp(-\xi^2) \ , \quad \tau = 0 \ , \quad -\infty < \xi < +\infty \ ,$$

then the equations for the characteristics of the wave and for its profile at an arbitrary moment will have the form

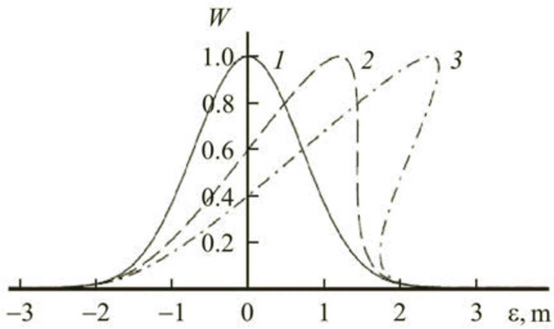


Fig. 1. Riemann wave profiles at different instants of time: 1) $\tau_0 = 0$; 2) τ_1; 3) τ_2 (dash-dotted curves); $\tau_0 < \tau_1 < \tau_2$.

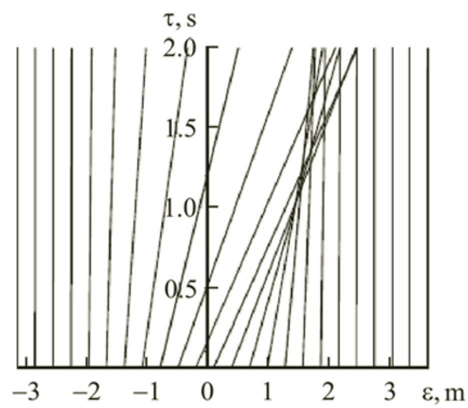


Fig. 2. Characteristic Riemann curves.

$$\xi = \chi + a\tau \exp(-\chi^2), \quad W = \exp(-(\xi - aW\tau)^2),$$

where $a = \dfrac{\rho_{22}N}{\varepsilon c(\rho_{11}\rho_{22} - \rho_{12}^2)}$. We consider the wave that runs in the positive direction of the *Ox* axis.

The initial condition assigns the wave profile in the form of a bell at the zero instant of time. As the perturbation propagates, the wave profile undergoes distortion. It is seen on the graphs of Fig. 1 that with time, on increase in the distance covered by the wave, the forward front facing in the direction of motion becomes steeper, whereas the back one becomes more sloping. The different portions of the wave profile run with different velocities. The breaking of the wave will occur when the characteristics intersect for the first time at the moment $t^* = \sqrt{\dfrac{e}{2a^2}}$ (Fig. 2). The wave surface is depicted in Fig. 3.

We will consider the case where the viscosity is present in the medium. In system (2), we introduce new symbols $W = \dfrac{\partial U}{\partial \xi}$ and $G = \dfrac{\partial V}{\partial \xi}$ and subtract the second equation of the system from the first one to obtain

$$\begin{cases} \varepsilon c(\rho_{11} + \rho_{12}) \dfrac{\partial W}{\partial \tau} + \varepsilon c(\rho_{12} + \rho_{22}) \dfrac{\partial G}{\partial \tau} + NW \dfrac{\partial W}{\partial \xi} = 0, \\ cb(\rho_{22} + \rho_{12})W - cb(\rho_{22} + \rho_{12})G + 2\varepsilon c(\rho_{11}\rho_{22} - \rho_{12}^2) \dfrac{\partial W}{\partial \tau} + 2N\rho_{22}W \dfrac{\partial W}{\partial \xi} = 0. \end{cases} \tag{4}$$

The second equation of system (4) is obtained by summing up the first and second equations of system (2) with account for the new symbols multiplied by ρ_{22} and ρ_{12}, respectively.

System (4) can be reduced to one equation for the function of the longitudinal deformation of the skeleton *W*:

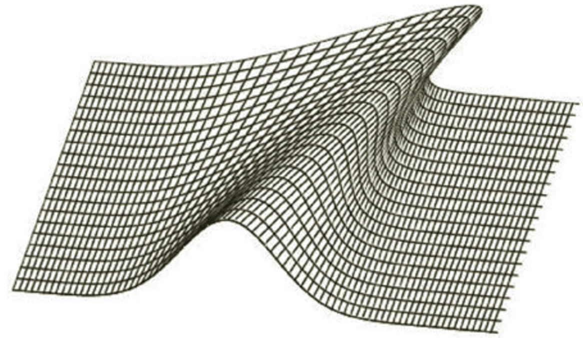


Fig. 3. Riemann wave surface in a medium without viscosity.

$$\varepsilon bc(\rho_{11} + \rho_{22} + 2\rho_{12}) \frac{\partial W}{\partial \tau} + 2\varepsilon^2 c(\rho_{11}\rho_{22} - \rho_{12}^2) \frac{\partial^2 W}{\partial \tau^2} + 2\varepsilon N \rho_{22} \frac{\partial}{\partial \tau}\left(W \frac{\partial W}{\partial \xi} \right) + bNW \frac{\partial W}{\partial \xi} = 0 \,. \tag{5}$$

It is seen that with b tending to zero with subsequent integration over τ Eq. (5) goes over into Eq. (3). Considering another limiting case where b is the infinitely large quantity, we also obtain the Riemann equation.

The classical equation for describing waves in a nonlinear nondispersing medium with dissipation is, as is known, the Burgers equation. From the obtained equation (5) it is seen that dispersion in the considered medium is absent. Equation (5) differs from the Burgers equation by the second derivative in time and by the presence of one another quadratic-nonlinear term (time derivative from the quadratic nonlinearity available in the Burgers equation). The dissipative terms in Eq. (5) are present explicitly and implicitly. Only one dissipative term enters explicitly into the equation. The additional (relative to the Burgers equation) nonlinear term demonstrates itself partially as a dissipative one, which is seen from the linear approximation for small perturbations. The presence of nonlinear and dissipative terms in Eq. (5) allows one to make the assumption about the possibility of the existence of stationary shock waves in the medium.

Let us consider a stationary wave $W = W(\chi)$, where the running coordinate $\chi = \xi - v\tau$. In this case, Eq. (5) takes the form

$$v^2 \frac{d^2 W}{d\chi^2} - a_1 v \frac{d}{d\chi}\left(W \frac{dW}{d\chi} \right) + 2a_2 bW \frac{dW}{d\chi} - a_3 bv \frac{dW}{d\chi} = 0$$

or

$$\frac{d}{d\chi}\left(v^2 \frac{dW}{d\chi} - a_1 vW \frac{dW}{d\chi} + a_2 bW^2 - a_3 bvW \right) = 0 \,, \tag{6}$$

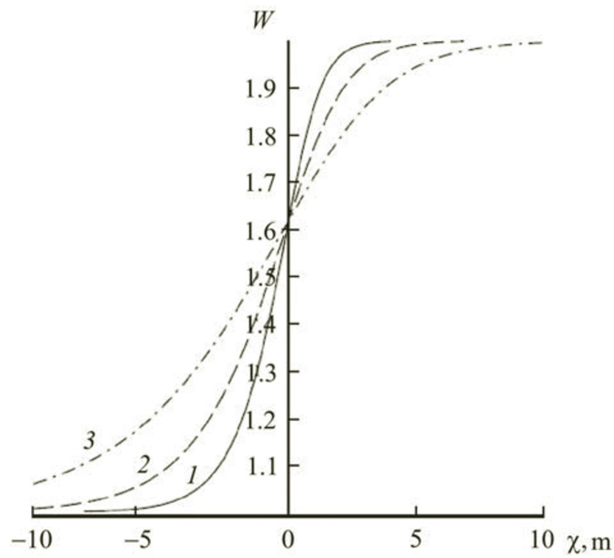
where $a_1 = \dfrac{N\rho_{22}}{\varepsilon c(\rho_{11}\rho_{22} - \rho_{12}^2)}$, $a_2 = \dfrac{N}{4\varepsilon^2 c(\rho_{11}\rho_{22} - \rho_{12}^2)}$, and $a_3 = \dfrac{\rho_{11} + \rho_{22} + 2\rho_{12}}{2\varepsilon(\rho_{11}\rho_{22} - \rho_{12}^2)}$. The last equation bears the resemblance to the equation given in [40] with accuracy up to the coefficients. The difference is in the presence of the last term in Eq. (6) which has the solution in the form of a stationary shock wave. Integrating the equations once over the running coordinate with account for the boundary conditions

$$W(\chi) = \begin{cases} W_1 \,, & \chi \to -\infty \,, \\ W_2 \,, & \chi \to +\infty \,, \end{cases} \tag{7}$$

we obtain an expression for the velocity of the nonlinear wave:

$$v = \frac{a_2}{a_3}(W_1 + W_2) \,. \tag{8}$$

The equation obtained after single integration is integrated again with account for the shock wave velocity (8) by the method of separation of variables. In the course of repeated integration we adopt that the integration constant is zero. We obtain the solution in an implicit form:



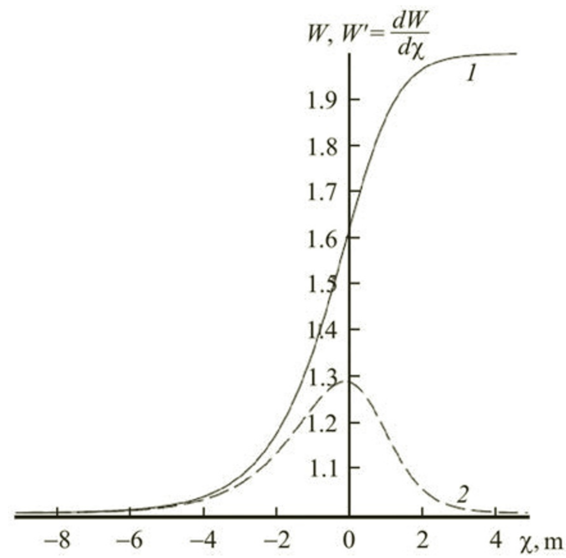


Fig. 4. Profile of a stationary shock wave $W(\chi)$ at different values of the viscosity coefficient b_1 (1), b_2 (2), b_3 (3), $b_3 < b_2 < b_1$.

Fig. 5. The $W(\chi)$ (1) and $W'(\chi)$ (2) curves. The graph of $W'(\chi)$ is displaced upward on the ordinate axis by W_1.

$$\chi = \frac{(W_1 + W_2)}{a_3^2 b(W_2 - W_1)} [(-a_1 a_3 W_1 + a_2(W_1 + W_2)) \ln (W - W_1) + (a_1 a_3 W_2 - a_2(W_1 + W_2)) \ln (W_2 - W)] . \tag{9}$$

The derivative has the form

$$\frac{dW}{d\chi} = \frac{a_3^2 b(W_2 - W)(W - W_1)}{(W_1 + W_2)(a_2(W_1 + W_2) - a_1 a_3 W)} ,$$

where W is determined from Eq. (9).

The profile of the solution $W(\chi)$ at different values of the viscosity coefficient b and derivative $W'(\chi)$ are presented in Figs. 4 and 5. It is seen from the figures that the profile of the solution of Eq. (9) has the form of the nonsymmetrical kink relative to the inflection point. On decrease in the viscosity coefficient the profile of the wave becomes more sloping, i.e., the width of the wave front increases.

The parameters of the shock wave appearing as a result of the mutual compensation of the nonlinearity and dissipation effects are related as

$$\left(a_1 - \frac{v}{A}\right) \frac{v}{a_2 b \Delta} = \text{const} , \tag{10}$$

where $A = W_2 - W_1$ is the shock wave amplitude, Δ is the characteristic width of the shock wave front, the shock wave velocity v is defined by expression (8). It is seen from relation (10) that the relationship between the wave front width and the fluid viscosity is inversely proportional. The wave front width depends inversely proportionally on the amplitude at $a_1 < \frac{v}{A}$ and directly proportionally at $a_1 > \frac{v}{A}$. The relationship between the shock wave parameters differs from that for the stationary shock wave of the Burgers equation. In Burgers′ classical equation the front width depends directly proportionally on viscosity.

Thus, this work presents a mathematical model that describes the propagation of a plane longitudinal wave in a porous liquid-saturated medium with account for the geometrical nonlinearity of the medium skeleton. Evolution equations have been obtained for the displacements of the medium skeleton and of the liquid in pores. It is shown that if the liquid flows freely from pores to pores, the system of evolution equations is reduced to one equation of a simple wave, i.e., the propagation of a plane longitudinal wave in a porous medium is described by the well-known equation of nonlinear wave dynamics, i.e., by the Riemann equation. If the liquid is held in the pores, the propagation of the wave is described by the equation that has the solution in the form of a stationary shock wave originating as a result of the mutual compensation of the effects of nonlinearity and dissipation. The dependence of the shock wave front width on the viscosity of the fluid saturating the pores and the shock wave amplitude has been determined. On increase in the viscosity coefficient the wave front width decreases. On increase of the wave amplitude the front width may both increase and decrease depending on the remaining parameters of the original system.

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NOTATION

A, elastic constant, Pa; a, a_1, a_2, a_3, used to reduce notation; *b*, viscosity coefficient, Pa·s/m^2; *c*, wave velocity, m/s; *e* = exp (1); *G*, longitudinal deformation (there is viscosity in the medium), 1; K_{per}, permeability coefficient, m^2; m_1, m_2, m_3, used to reduce notation; *N*, *Q*, *R*, elastic constants, Pa; *t*, time, s; *U*, *V*, displacements of the skeleton and of liquid along the coordinate *x*, m; *v*, velocity of nonlinear wave, m/s; W, longitudinal deformation (there is no longitudinal deformation in the medium), 1; *x*, coordinate m; ε, small parameter, 1; η, dynamic viscosity of the fluid, Pa·s; ρ_{11}, efficient density of the skeleton, kg/m^3; ρ_{22}, effective density of liquid in pores, kg/m^3; ρ_{12}, coefficient of mass connection between the fluid and the solid phase, kg/m^3; τ, new variable, slow time, s; Φ, porosity factor, 1; χ, running coordinate, m.

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