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INTEGRAL MODELS OF RHEOLOGICALLY NONSTATIONARY FLUIDS

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Consideration has been given to the status and prospects for development of integral models of rheologically nonstationary (viscoelastic) fluids. The predominant field of application of these models are media with a fairly long relaxation time (concentrated polymer solutions and melts, biological tissues, foams, and others), for which taking account of the preceding states assumes substantial importance. The emphasis has been placed on today's integral models developed on the basis of a classical Kaye–Bernstein–Kearsely–Zapas model through singling out viscous and purely elastic components in the potential function of the medium's stored energy (so-called factorization of the K-BKZ model). An analysis has been made of the thixotropy and rheopexy phenomena caused by the lag of the reaction of response of rheologically nonstationary fluids to external action. The authors have shown the expediency of the loop test used in determining the integral parameters of motion of viscoelastic fluids (velocity, flow rate, pressure difference, etc.), and also the possibility of using a flow curve representing the averaging of the ascending and descending branches of a hysteresis loop in the case of its small width.

Keywords: *mathematical models, viscoelastic fluids, thixotropy, rheopexy.*

Introduction. The present work is a continuation of [1] in which an analysis of differential mathematical models of rheologically nonstationary fluids was made. The general expression of an integral rheological model follows from the Boltzmann theory of linear viscoelasticity [2]:

$$\mathbf{T}(t) = \int_{-\infty}^{t} G(t - t')\mathbf{D}(t')dt' . \tag{1}$$

Here $G(t - t')$ is the relaxation modulus, a positive exponentially decreasing function of the time interval $(t - t')$, Pa. According to the principle of fading memory, the farther the quantity t' from the running time t, the smaller the influence of it on the modulus G. The quantity $\mathbf{T}(t)$ is determined by the strain history and allows computing the stress for each instant of time t and the known $G(t - t')$ function. The lower limit of integration in (1) can also be taken to be zero when by this we mean the instant of initial strain.

Expression (1) yields that the integral model takes account of all the preceding states of a viscoelastic fluid, however distant from the running instant of time they may be. Such models are used when the influence of memory effects is significant and the relaxation time is long (theoretically, infinite). Conversely, differential models used in describing polymer flows assume a relatively fast relaxation of the structure of a viscoelastic fluid.

Integral Models. The *Green–Rivlin model*, as noted in [3], was historically the first rheological model of simple materials with fading memory. According to Noll, the term "simple material" means a viscoelastic material the stress at whose given point is dependent on just the prehistory described by the first strain gradient and is independent of higher-order gradients [4]. The model is based on the hypothesis that the stress $\mathbf{T}(t)$ is caused by displacement gradients (strain-rate tensors $\mathbf{D}$) at the instant of time t and in N of the previous instants from 0 to t. When N tends to infinity the stress–strain relation is expressed by the sum of multiple integrals [5]

$$\mathbf{T}(t) = \int_{0}^{t} G_1(t - t_1)\mathbf{D}(t_1)dt_1 + \int_{0}^{t}\int_{0}^{t} G_2(t - t_1, t - t_2)\mathbf{D}(t_1)\mathbf{D}(t_2)dt_1dt_2 + \dots . \tag{2}$$

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The moduli G_i characterize the stress relaxation and they are determined by the material′s properties. The instants of time t_i ($i = 1, 2, \dots, N$) correspond to the variation in the deforming action. The material satisfying Eq. (2) is called a viscoelastic material of the integral type of order N. The shorter the strain history, the lower the order of the rheological model (in actual practice, use is made of the models up to the third order inclusive).

In the Green–Rivlin model, the stress is dependent on strain and time nonlinearly. In the analysis of the overall status of modeling of destruction of thermoplastics and composites (rubbers and foams), it is noted that their nonlinear properties are disregarded in the main or are replaced by linear models, which distorts the reaction of load removal. This approach is due to the fact that commercially used models for calculation of materials are often restricted to the linear viscoelasticity and cannot take account of experimental data because of the difficulty with identifying the parameters of a nonlinear model. In [6], foam is presented by a material subdivided into the linear superelastic and nonlinear viscoelastic parts. Here, a modified equation (2) in final form contains just the linear convergent integrals. The possibility to reliably model in this manner a number of nonlinear viscoelastic phenomena, e.g., air saturation of Confor CF-45-type polyurethane foams, has been shown.

The Green–Rivlin model is successfully used to describe viscoelastic properties of biological materials, in particular, brain tissues [7, 8]. The nonlinearity of their rheological characteristics is of decisive importance in studying the action of finite strain and of multiple shock accelerations appearing in the case of craniocerebral injuries. Experimental data of investigation of the viscoelastic properties of bovine-brain samples have shown that at a shear strain higher than 30%, the Green–Rivlin equation of third order describes quite accurately the nonlinearity of the reaction of brain tissue to strains [7]. The reaction of samples of human and bovine brain to stepwise strains has been studied in [8]. It has been shown experimentally that under small strains, the reactions of the samples′ response are satisfactorily described by linear and nonlinear models, with the nonlinear Green–Rivlin model being expedient at strains over 50%.

Pulsating motion of a viscoelastic-fluid flow in straight pipes with an arbitrary cross section has been investigated in [9–11]. This flow occurs during the operation of volumetric pumps producing cyclic pressure pulsations, which is characteristic of biological and technical systems alike (motion of the blood in aortas, the stimulation of oil production by feeding periodically the working fluid to the injection well, etc.). Clearly, the parameters of pulsating viscoelastic-fluid flow depend to the greatest extent on pressure values that are the nearest to a given instant of time. The Green–Rivlin model reflecting the stress relaxation of the viscoelastic fluid for each acting function by the relevant relaxation modulus G_i is capable of describing most accurately the characteristic features of such flow.

The *Kaye–Bernstein–Kearsley–Zapas (K-BKZ) model*, according to the initial version, relates stresses and strains in an incompressible viscoelastic fluid by the dependence of the form [12, 13]

$$\mathbf{T} - P \;\; = 2\int_{-\infty} \left(\frac{\partial \Omega}{\partial J}\,\mathbf{B} - \frac{\partial \Omega}{\partial J}\,\mathbf{B}^{-} \right) dt' \,, \tag{3}$$

where J_1 and J_2 are the first and second invariants of the Finger strain tensor $\mathbf{B}$ and $\Omega(J_1, J_2, (t - t'))$ is the function of the invariants J_1 and J_2 and of the instants of time t and t'. The function Ω is the stored-energy potential function [13, 14]; by it, there is determined the rate of damping of the stresses (Pa/s) from the action preceding the instant of time t. In the second part of [12], in considering a number of concrete viscoelastic-fluid flows, the function Ω was assigned by the sum of the exponents

$$\Omega = C_1 \exp\left[-K_1(t - t')\right](J_1 - 3) + C_2 \exp\left[-K_2(t - t')\right](J_2 - 3) \,. \tag{4}$$

Here C_1 and C_2 represent physically the relaxation moduli of the fluid (analog of the function G_i in the Green–Rivlin model), and $1/K_1$ and $1/K_2$ are the periods of stress relaxation which correspond to the values of the invariants J_1 and J_2.

A further development of the K-BKZ model was "factorization" of the Ω function, i.e., singling out, from it, of the time-dependent viscous part $m(t - t')$ (memory function) and of the time-independent purely elastic part $h(J_1, J_2)$ (damping function) determined by experiments respectively in the regions of linear and nonlinear strains of a viscoelastic fluid [15, 16]:

$$\mathbf{T} - P\delta = \int_{-\infty}^{t} m(t - t')[h_1(J_1, J_2)\mathbf{B}(t - t') + h_2(J_1, J_2)\mathbf{B}^{-1}(t - t')]dt' \,. \tag{5}$$

We consider the best known forms of a factorized K-BHZ model.

The *Rivlin–Sawyers model* [17]. Within the framework of this model, the damping functions h_1 and h_2 are the scalar functions of the invariants J_1 and J_2, whose expressions may be different. The memory function in Eq. (5) is defined as

$$m(t - t') = \sum_i \frac{\mu_i}{\lambda_i^2} \exp\left(-\frac{t - t'}{\lambda_i}\right), \tag{6}$$

where μ_i and λ_i are the experimental values of the shear viscosity (Pa·s) and of the relaxation period (s) at the *i*th rate of strain of the viscoelastic fluid.

The *Wagner model* combines the functions h_1 and h_2 into one [18]:

$$\mathbf{T} - P\delta = \int_{-\infty}^{t} m(t - t')[h(J_1, J_2)\mathbf{B}(t - t')]dt' . \tag{7}$$

The relationship of the damping function "after Wagner" and the potential function of the original K-BKZ model is determined by the relations

$$h(J_1, J_2) = \frac{\partial\Omega(J_1, J_2)}{\partial J_2} \quad \text{and} \quad \frac{\partial\Omega(J_1, J_2)}{\partial J_1} = 0 , \tag{8}$$

and its direct expression is of the form [18]

$$h = \exp\,[-n(J_2 - 3)^{1/2}] , \tag{9}$$

where $n = 1/\gamma_s = 2/\gamma_N$ is the constant damping function (γ_s and γ_N are the shear strains at maximum shear stresses τ_{12} and the difference $\tau_{11} - \tau_{22}$ of normal stresses).

A more general Wagner model contains two measures of finite strains [19]

$$\mathbf{T} - P\delta = \int_{-\infty}^{t} m(t - t')h(J_1, J_2)[(1 - \beta)\mathbf{B}^{-1}(t - t') - \beta\mathbf{B}(t - t')]dt' . \tag{10}$$

Here the expression of the damping function ($0 < a < 1$ is the empirical constant) is as follows:

$$h(J_1, J_2) = [1 + a(J_1 - 3)(J_2 - 3)]^{-1} . \tag{11}$$

The memory function in the Wagner model is expressed by the sum of the exponents [18]

$$m(t - t') = -\frac{\partial G(t - t')}{\partial t} = \sum_i \frac{G_i}{\lambda_i} \exp\left(-\frac{t - t'}{\lambda_i}\right). \tag{12}$$

The parameters of the memory function G_i and λ_i for the *i*th strain rate are determined experimentally as in the Rivlin–Sawyers model. A practically sufficient number of experiments in the linear region of strain of the medium is $i = 6$–8; with growth in *i*, the G_i values decrease fast, tending to zero [20].

The *Papanastasiou–Scriven–Macosko (PSM) model* was proposed to describe viscoelastic flows of polymer melts. Here the damping function is expressed by the relation [21]

$$h = \frac{\alpha}{(\alpha - 3) + \beta J_1 + (1 - \beta)J_2} , \tag{13}$$

where α and β are the parameters determined in shear and elongational or extensional flow of a viscoelastic medium. The quantity α ($\alpha > 1$) depends on the spectral width of the periods of relaxation λ_i and increase as the branching of the polymer chain grows. The parameter β ($0 < \beta < 1$) characterizes the degree of extension (straightening) of a polymer chain [21].

More recently, Luo and Tanner developed the PSM model by introducing the parameter θ into it, which takes account of the nonzero difference of the normal stresses N_1 and N_2 [22]

$$\mathbf{T} - P\delta = \frac{1}{1 - \theta} \int_{-\infty}^{t} m(t - t') \frac{\alpha}{(\alpha - 3) + \beta J_1 + (1 - \beta)J_2} [\mathbf{B}(t - t') + \theta\mathbf{B}^{-1}(t - t')]dt' , \tag{14}$$

where $\theta = \dfrac{N_2}{N_1 + N_2} = \dfrac{\tau_{22} - \tau_{33}}{\tau_{11} - \tau_{33}}$.

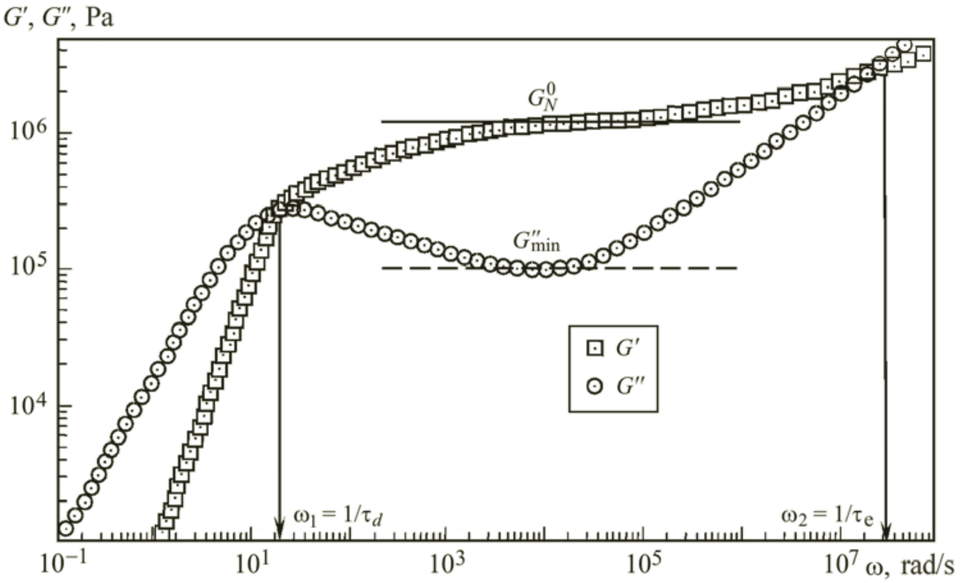

Fig. 1. Determination of parameters of the Doi–Edwards model from the moduli G' and G'' as functions of the strain frequency ω [24, Fig. 1].

The *Doi–Edwards (DE) model.* Viscoelastic-fluid flow is considered as a set of macromolecular motions in the tubes of linkages formed by neighboring macromolecules (tube model) [23]. The essence of this model is that an individual macromolecule is placed in a tube of the same configuration as the macromolecule itself, and the macromolecules forming the tube restrict its motion which is due to the snake motion, i.e., reptation, similarly to the motion of a snake (reptile) in a bunch of similar creatures. This yields another name of the model: a reptation model.

According to this model, a macromolecule of length L executes motion along the axis of a tube of diameter d, but its movements in the perpendicular direction are limited by the tube radius. The model predicts two different relaxation processes: a fast process occurring inside the tube with a relaxation time τ_e (so-called "equilibrium time") and a slow one associated with the variation in the configuration of the tube itself with a relaxation (reptation) time τ_d. Physically, τ_e is the time between successive "linkages" of the macromolecule inside the tube, and τ_d is the time of "crawling out" of the polymer molecule of length L from the tube of diameter d [24, 25].

Corresponding to the general formula (5) of the factorized K-BKZ model, the DE model differs by the expressions of the memory and damping functions which are defined on the basis of the reptation theory [26, 27]

$$m(t - t') = \frac{8G_N^0}{\pi^2 \tau_d} \sum_{k=0}^{\infty} \exp\left[-(2k + 1)^2 \frac{(t - t')}{\tau_d}\right], \tag{15}$$

$$h_1 = \frac{5}{J_1 - 1 + 2\sqrt{J_2 + 13/4}}, \quad h_2 = -\frac{h_1}{\sqrt{J_2 + 13/4}}. \tag{16}$$

The DE-model′s parameters (G_N^0 and τ_d) involved in expression (15) for a concrete viscoelastic fluid are found experimentally by taking the dependences of the elastic G' and loss G'' moduli on the frequency ω [24]. Figure 1 gives the plots of $G'(\omega)$ and $G''(\omega)$ for monodisperse polybutadiene with a molar weight of 99,060 g/mole at a temperature of 40°C. As we can see from the plots, the polymer in question, even when monodisperse, is not a Maxwellian fluid whose loss-modulus curve $G''(\omega)$ has the characteristic dome shape ([1], Fig. 2). Here the loss modulus has the local minimum $G''_{\min}$. The parameter G_N^0 (plateau modulus) involved in the equations of the DE model is the value of the storage modulus G' of the viscoelastic fluid at the loss modulus $G'' = G''_{\min}$, and the reptation time τ_d is the reciprocal of the frequency ω_1 at which the $G'(\omega)$ and $G''(\omega)$ curves intersect for the first time ($\tau_d = 1/\omega_1$). The second intersection of these curves determines the time of fast relaxation $\tau_e = 1/\omega_2$.

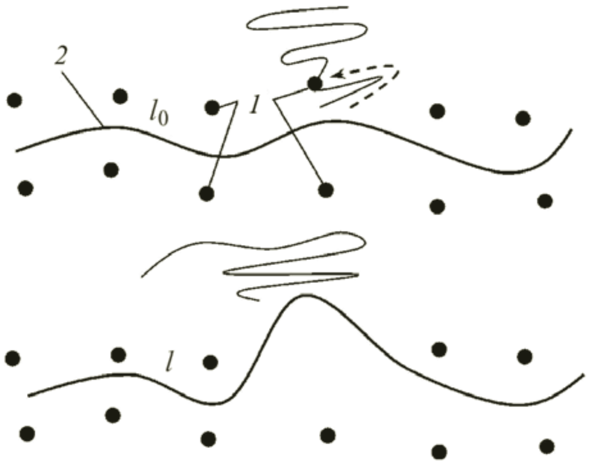

Fig. 2. Diagram of release of the polymer chain from the linkage tube: 1) macromolecules forming the linkage tube and 2) polymer chain [29, Fig. 1].

The *Mead–Larson–Doi (MLD) model* [28] takes account of the effects of stretching of a linkage tube and the release of a polymer chain from the linkage tube due to convective motion (convective constraint release (CCR)), which tends to change the shape of the polymer chain. As a result of the above effect, the equilibrium polymer-chain length l_0 corresponding to a quiescent state increases to the quantity l [29] (Fig. 2).

The equation of the MLD model in the adopted notation is of the form [30]

$$\mathbf{T} - P\boldsymbol{\delta} = 5G_N^0\varepsilon^2(t)\int_{-\infty}^{t} m(t-t')\mathbf{Q}(t-t')dt' \ , \tag{17}$$

where $\varepsilon(t) = l/l_0$ is the coefficient of elongation of the polymer chain and $\mathbf{Q}(t-t') = \dfrac{1}{J_1 - 1 + 2\sqrt{J_2 + 13/4}} \times \left(\mathbf{B} - \dfrac{\mathbf{B}^{-1}}{\sqrt{J_2 + 13/4}}\right)$ is the partial orientation tensor. A comparison of the expression $\mathbf{Q}(t - t')$ with expressions (16) shows that the damping functions of the MLD models are analogous to the functions h_1 and h_2 of the DE model. The memory function of the MLD model differs from an analogous functions of the DE model by the presence of the parameter ε^2 which is the function of time.

According to the MLD model, the total relaxation (reptation) time τ is [31]

$$\frac{1}{\tau} = \frac{1}{\varepsilon^2\tau_d} + \frac{1}{\varepsilon}\left[\frac{2}{\tau_e}\,\frac{\varepsilon - 1}{\varepsilon + 1}\right], \tag{18}$$

where τ_e and τ_d are the times of internal relaxation and reptation of the polymer chain. Formula (18) yields that at $\varepsilon = 1$, the relaxation time is $\tau = \tau_d$, and at $\varepsilon > 1$, $\tau > \tau_d$, i.e., in the absence of the extension of the polymer chain, reptation times in the DE and MLD models turn out to be identical, and with increase in the degree of extension it becomes longer than the analogous time in the DE model. This is an agreement with the physics of the process of motion of a polymer chain: the relaxation time grows with its elongation.

In [32], the reptation and convective release of polymer chains from the linkage tube in shear and biaxial extension of two samples of bidisperse polystyrene was investigated on the basis of a comparison of their damping functions h_i determined experimentally and calculated according to the MLD model. Experimental values of the damping functions were determined as the ratio of the relaxation moduli in the nonlinear $G(t, \gamma)$ and linear $G(t)$ regions of strain of the polystyrene samples. The obtained experimental and calculated values of the damping function turned out to be in good mutual agreement.

Thixotropy and Rheopexy in Viscoelastic Fluids. The phenomenon of thixotropy involves the reduction in the effective viscosity of a fluid with time at a constant strain rate and is caused by its destructuring. In the case of rheopexy, strains tend to structurize the fluid and to increase its viscosity. The time of establishment of the final value of the viscosity of fluids may reach tens and hundreds of minutes [33, 34]. Thixotropy and rheopexy are revealed in a loop test, i.e., taking successively the ascending and descending branches of a fluid-flow curve which form a hysteresis loop [35–37]. The amplitude of the

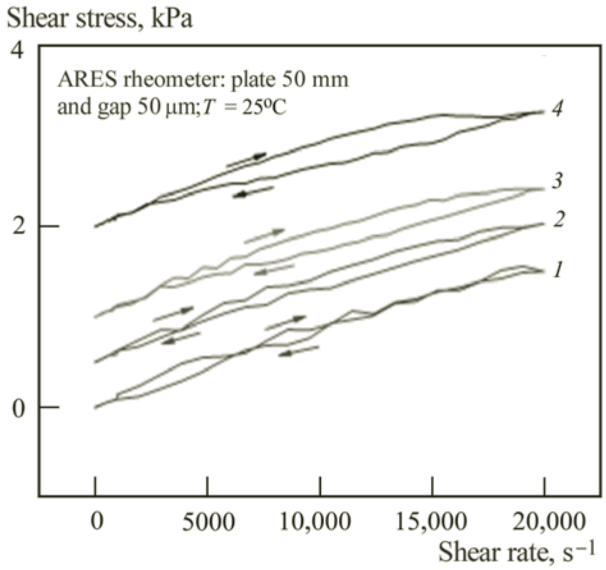

Fig. 3. Thixotropic loops at different durations t_L (loop time) of taking the flow curves: $t_L = 4$ (1), 10 (2), and 20 (3), and 40 (4) [38, Fig. 7]. Curves 2–4 are shifted for visual clarity vertically by 500, 1000, and 2000 Pa respectively.

hysteresis loop (difference of the stresses of the ascending and descending branches of the flow curve at a given shear rate) is affected by the duration of the stress measurement. We can give, as an example, results of the loop test of silicone oil DMS T25 [38] (Fig. 3): increase in the measurement time from 4 to 40 s noticeably expands the hysteresis loop, which is due to the ordering of the internal structure of the oil and to the decline in its final viscosity which is proportional to the length of the action of shear stresses.

The procedure of taking the hysteresis loop is carried out by a controlled-shear-rate rheometer (CR-rheometer) with linear increase in the shear rate for the time t_1 and its reduction for the time t_2 [39]. At small t_1 and t_2 values, when the fluid′s structure is not fully destroyed, a cycle of rise and reduction in the shear rate is implemented to determine the maximum amplitude of the hysteresis loop, with each subsequent cycle beginning from the point of completion of the previous one (Fig. 4a). To the full destructuring of the fluid, there corresponds the coincidence of the ascending and descending branches of the fluid-flow curves of the last cycle (the third in Fig. 4a). Another variant is that upon the rise in the shear rate, it remains maximum for the same t_3 until the fluid′s structure is fully destroyed, and thereafter the shear rate is reduced to zero (Fig. 4b). The ascending and descending branches of the flow curve, which form the maximum-amplitude hysteresis loop, determine the upper and lower values of effective viscosity of the viscoelastic fluid for each shear rate. In this case the loop test is meant to be an assessment for the range of variation in the effective viscosity of such a fluid, which is sufficient for determining the basic integral parameters of its motion (velocity, flow rate, pressure difference, etc.).

The area of the hysteresis loop is proportional to the energy consumed by the destructuring of the fluid. The size of this area obtained when the fluid′s structure is fully destroyed is the basic characteristic of its thixotropic properties. To assess the degree of thixotropy, the so-called thixotropic index Ti has proposed which is the ratio of the initial μ_I and steady-state μ_S viscosity at a constant shear rate to the value of the steady-state viscosity [36]

$$\mathrm{Ti} = \frac{\mu_I - \mu_S}{\mu_S} . \tag{19}$$

As follows from formula (19), the value of the index is positive for a thixotropic fluid, negative for a rheopexic fluid, and equal to zero for a rheologically stationary one. At a small absolute value of the index (small width of the hysteresis loop), the rheological properties of the medium can be described by a flow curve which is the averaging of the ascending and descending branches of the hysteresis loop. Notice that to assess the degree of pseudoplasticity and dilatancy of rheologically stationary fluids, use is made of an analogous ratio of the initial μ_0 and final μ_∞ viscosities with variation in the shear rate from the infinitely small rate to a limiting one at which μ_∞ becomes constant.

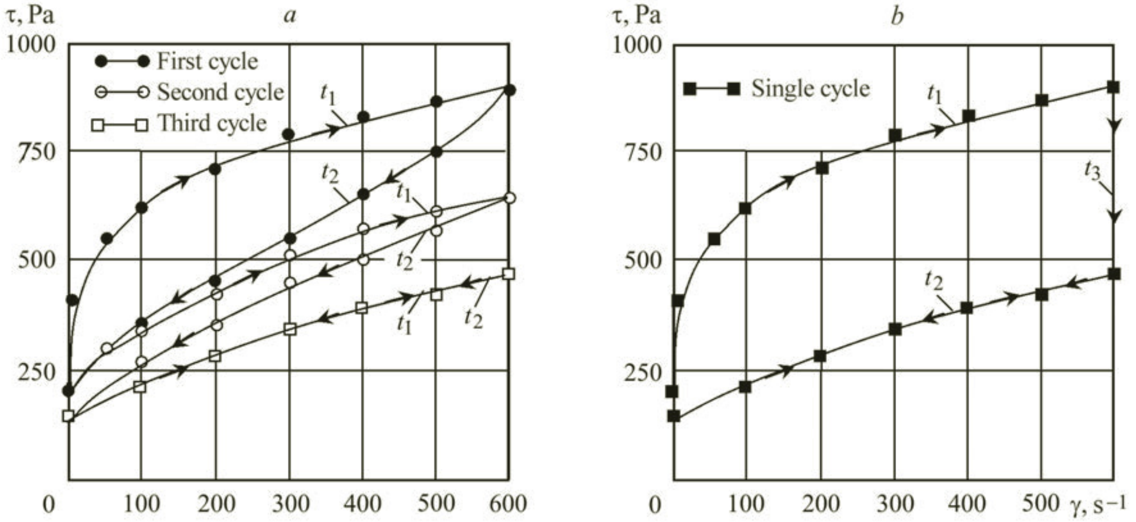

Fig. 4. Thixotropic test with the CR rheometer: a) multiple rise (t_1) and reduction (t_2) in the shear rate (three successive cycles) and b) single rise (t_1), the action of the maximum shear rate (t_3), and the reduction (t_2) in the shear rate [39, Fig. 116].

One of the most fruitful approaches to description of the rheological properties of the thixotropic and rheopexic system is the use of a structural-kinetics model according to which the degree of structuring is characterized by a scalar parameter ξ ranging from zero (absence or a full destruction of the internal structure) to unity (maximum development of the structure of the fluid′s volume). Equations of the model include the dependences $\tau(\gamma)$ at $\xi = \text{const}$ and $\xi = f(t)$ [40]:

$$\tau = (\tau_0 + \tau_1) + (K_0 + \xi K_1)\gamma^n \ , \tag{20}$$

$$\frac{d\xi}{dt} = a(1 - \xi) - b\xi\gamma^{\varepsilon} \ . \tag{21}$$

Here τ_0 and K_0 are the initial shear stress and the measure of consistency of the fluid, τ_1 and K_1 are the components of the shear stress and of the measure of consistency of the fluid, which are linearly dependent on the quantity ξ, and a, b, and ε are the kinetic constants of the fluid [41–43]. Equation (21) is similar to the equation of a reversible chemical reaction: the terms $a(1 - \xi)$ and $b\xi\gamma^{\varepsilon}$ characterize the rate of formation and destruction of the fluid′s structure.

The structural-kinetics model was used to describe the viscosity of volcanic lava representing a porous silicate melt [44]. The parameter ξ was taken to be equal to the ratio of the minimum diameter of gas pores to the diameter of an undeformed sphere with an equivalent volume. It has been established that the manifestations of thixotropy appear as the temperature and crystallization of the lava melt decrease (at a temperature higher than 1120$^{\text{o}}$C, basaltic lavas are characterized by the regular Newtonian viscosity [45]). Another, probably, a more significant reason for the thixotropy of volcanic lava is the strain of its internal pores (bubbles). Assuming, under the stresses, the shape of ellipsoids, the bubbles stretch streamwise, which decreases the lava viscosity irrespective of the concentration of crystallizing particles.

Conclusions. Among the integral models of rheologically nonstationary fluids, a leading role has been played by the K-BKZ model for longer than half the century. The number of citations of the pioneering studies [12] and [13] has only grown thus far: thus, in 2010, they were cited 220 times, which is more than in any of the earlier years [46]. The model′s idea of subdivision of the function characterizing the physical properties of a viscoelastic medium into two parts, one of which reflects the dependence of these properties on time (memory function), and the other, on the amplitude of acting strain (damping function), ensured the possibility for the fairly accurate calculations of viscoelastic-fluid flows in the region of nonlinear strain.

The trend of recent decades has been the "enrichment" of the initially phenomenological K-BKZ model with propositions of a molecular-kinetic theory of polymer (primarily, with the reptation theory). On this basis, we have the development of the K-BKZ model through the creation of its numerous modifications with various expressions of memory and damping functions giving the distinctive features of the morphology and movement of polymer chains. This close connection of the development of rheological models to the polymer theory is attributed to the fact that establishing a quantitative relationship between the molecular structure of a polymer and the manifestation of its rheological properties is of great practical importance for many technological processes.

Note that in contrast to the integral models known earlier, in the K-BKZ model and its modifications, the relaxation properties of the viscoelastic fluid are expressed by a single integral, which makes the numerical calculations much simpler and faster. This is precisely why the model has become widespread at present and is used, e.g., by the producers of software to model the molding of plastics, rubber mixes, and others.

A characteristic property of viscoelastic fluids is thixotropy or a more rare rheopexy. These properties are caused by the lag of the reaction of response of the viscoelastic fluids to the external action and are physically explained by the finite rate of destruction and restoration of their supramolecular structure. The basic characteristic of thixotropic and rheopexic properties is the maximum amplitude of a hysteresis loop, which is obtained as a result of the loop test using the CR rheometer. The above characteristic is meant to be a quantitative measure of the most important parameters of flow of viscoelastic fluids.

NOTATION

$\mathbf{B}$ and $\mathbf{B}^{-1}$, direct and inverse Finger strain tensors; $\mathbf{D}$, strain-rate tensor, s^{-1}; G, dynamic (complex) modulus of the viscoelastic fluid, Pa; G' and G'', storage (elastic) and energy-loss (dissipation) moduli of the viscoelastic fluid, Pa; G_N^0, storage modulus of the viscoelastic fluid on a horizontal portion of the $G'(\omega)$ curve (plateau modulus), Pa; N_1 and N_2, first and second differences of normal stresses of the viscoelastic fluid, Pa; P, pressure, Pa; $\mathbf{T}$, stress tensor, Pa; t, time, s; $\boldsymbol{\delta}$, Kronecker symbol (unit tensor); λ, relaxation constant of the viscoelastic fluid, s; ω, angular velocity (frequency), s^{-1}.

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