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AN INVERSE PROBLEM OF ACOUSTIC FLOW

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A one-dimensional mathematical model is suggested for nonstationary incompressible flow in a cylindrical tube under the action of a sonic wave propagating in it. Within the framework of this model, a problem of determining the acoustic energy density at the beginning of the tube from the given volumetric flow rate of the fluid in the tube is posed. This problem relates to the class of inverse problems associated with the restoration of the dependence of the right-hand sides of parabolic equations on time. A computational algorithm is proposed for solving the problem posed.

Keywords: *sonic wave, acoustic energy density, radiation pressure gradient, inverse problem.*

Introduction. It is well known that propagation of intense sonic waves, and especially of ultrasonic ones, in liquid and gaseous media frequently leads to the appearance of nonperiodic motions of a medium called acoustic flows. The reason for the occurrence of acoustic flows in liquid and gaseous media stems from the irreversible losses of energy and momentum of an acoustic wave in them. The impulse transported by the acoustic wave is transferred to the medium when the wave is absorbed in it and causes its motion [1–5]. Acoustic flows attract interest as they are of great importance in various technological processes associated with the effect of intense sonic and ultrasonic waves on a medium.

It is obvious that the hydrodynamic characteristics of acoustic flow caused by a sonic wave are determined by the acoustic characteristics of the wave. Numerous theoretical and experimental works are devoted to the study of the hydrodynamic characteristics of different types of acoustic flows causes by a sonic wave with given characteristics. Stationary acoustic flow of radial structure in a cylindrical tube with rigid walls was investigated analytically in [6]. In [4, 7, 8], stationary and nonstationary acoustic flows in cylindrical tubes were investigated by analytical methods on the basis of one-dimensional mathematical models. In [9, 10], methods of numerical simulation were used to study acoustic flows in various media.

It should be mentioned, however, that for practical application of acoustic flows in different areas, especially for pumping-over fluids, of great importance is determination of the parameters of an acoustic wave that caused the fluid flow with a given hydrodynamic characteristic. In the present work, the problem of determining the acoustic wave characteristics from the given fluid flow in a cylindrical tube is presented as an inverse problem for a one-dimensional equation of nonstationary acoustic flow of incompressible viscous fluid.

Formulation of the Problem. We consider nonstationary flow of a viscous incompressible fluid in a cylindrical tube of radius R with rigid walls. The flow is induced by the radiation pressure gradient produced in the fluid by an ultrasonic beam. The ultrasonic beam fills the tube completely and is oriented along its axis, with the ends of the tube being permeable for the fluid. It is assumed that the 0z axis is directed along the tube axis, and the fluid propagates along this axis so that only one of the three components of the flow velocity (u_r, u_φ, and u_z) remains: $u_z \neq 0$, whereas $u_r = 0$ and $u_\varphi = 0$. The fluid flow is assumed to be axisymmetric. The complete system of differential equations describing this flow has the form [11]

$$\frac{\partial u_z}{\partial t} + u_z \frac{\partial u_z}{\partial z} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) - \frac{1}{\rho}\frac{\partial P}{\partial z}\,,$$
$$\frac{\partial u_z}{\partial z} = 0\,, \quad \frac{\partial u_z}{\partial \varphi} = 0\,, \quad \frac{1}{\rho}\frac{\partial P}{\partial r} = 0\,, \quad \frac{1}{\rho}\frac{\partial P}{\partial \varphi} = 0\,. \tag{1}$$

It is seen from the second and third equations of system (1) that u_z is a function of only r and t and that the last two equations yield the independence of the pressure P of r and φ, i.e., $u_z = u_z(r, t)$ and $P = P(z, t)$. From system (1) we come then to the following equation of nonstationary viscous incompressible fluid flow in a tube:

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$$\frac{\partial u_z}{\partial t} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) - \frac{1}{\rho}\frac{\partial P}{\partial z} . \tag{2}$$

We assume that the radiation pressure $P(z, t)$ in the given sonic beam is equal to the acoustic energy density $E(z, t)$ [5]: $P(z, t) = E(z, t)$. If we take into account that the following relation is valid for the acoustic energy density:

$$E(z, t) = E(0, t)e^{-2\alpha z} \approx E(0, t)(1 - 2\alpha z) ,$$

then for the radiation pressure gradient we have

$$\frac{\partial P}{\partial z} = -2\alpha E(0, t) .$$

Assuming then that $u(r, t) = u_z(r, t)$ and $E_0(t) = E(0, t)$, we represent Eq. (2) in the form

$$\frac{\partial u}{\partial t} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{2\alpha}{\rho} E_0(t) . \tag{3}$$

Let the following initial condition be valid for Eq. (3):

$$u\big|_{t=0} = 0 , \tag{4}$$

as well as the natural boundary condition of the boundedness of the solution at $r = 0$, which is equivalent to the condition

$$\left.\frac{\partial u}{\partial r}\right|_{r=0} = 0 , \tag{5}$$

and the no-slip condition holds on the tube wall:

$$u\big|_{r=R} = 0 . \tag{6}$$

It is obvious that in assigning the law of the change in the acoustic energy density at the beginning of the tube in time $E_0(t)$, by solving problem (3)–(6) we may find the flow velocity distribution over the tube cross section and the law of change in the volumetric fluid flow rate in the tube in time:

$$Q(t) = \int_0^R 2\pi r u dr . \tag{7}$$

We assume now that the law of variation of $Q(t)$ is known and it is necessary to find such a law of the change of $E_0(t)$ which could ensure the assigned fluid flow in the tube. Thus, the problem resides in determining the functions $u(r, t)$ and $E_0(t)$ that would satisfy Eq. (3) and conditions (4)–(7). The posed problem relates to the class of inverse problems associated with the restoration of the dependence of the right-hand sides of parabolic equations on time [12]. The statements and numerical methods of solving inverse problems on the restoration of the dependence of the right-hand sides of parabolic and hyperbolic equations on time are considered in [13]. It should be noted that in the problem posed, condition (7) is not the classical local condition for Eq. (3).

Method for Solving the Problem. Let us reduce problem (3)–(7) to a problem with local conditions [14]. We multiply both sides of Eq. (3) by r and integrate the result on the segment $[0, r]$ over the variable r. Performing the integration by parts with account for condition (5), we obtain

$$\frac{\partial}{\partial t}\int_0^r \xi u d\xi = \nu r\frac{\partial u}{\partial r} + \int_0^r \xi \frac{2\alpha}{\rho} E_0(t) d\xi .$$

Denoting

$$\int_0^r u\xi d\xi = w(r, t) , \tag{8}$$

we write the last integral relation in the form

$$\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial r^2} - \frac{\nu}{r} \frac{\partial w}{\partial r} + \frac{\alpha}{\rho} r^2 E_0(t) . \tag{9}$$

In this case, the initial and boundary conditions for Eq. (9) will be determined by the equations

$$w\big|_{t=0} = 0 , \tag{10}$$

$$w\big|_{r=0} = 0 , \tag{11}$$

$$\left.\frac{\partial w}{\partial r}\right|_{r=R} = 0 , \tag{12}$$

and integral relation (7) takes the form

$$w\big|_{r=R} = \frac{Q(t)}{2\pi} . \tag{13}$$

We discretize Eq. (9) in time t. For this purpose we introduce a uniform difference grid in the domain $[0 \le t \le T]$ relative to the variable t:

$$\bar{\omega}_\tau = \left\{t_j = j\,\Delta t , \quad j = \overline{0, m}\right\} ,$$

with the step $\Delta t = \dfrac{T}{m}$. We approximate the derivative $\dfrac{\partial w}{\partial t}$ in Eq. (9) at t_j, $j = \overline{1, m}$, by the backward difference:

$$\left.\frac{\partial w}{\partial t}\right|_{(r,t_j)} \approx \frac{w(r, t_j) - w(r, t_{j-1})}{\Delta t} .$$

We introduce the notation $w^j(r) \approx w(r, t_j)$ and, using it, we write Eq. (9) and conditions (10)–(13) as

$$\frac{w^j(r) - w^{j-1}(r)}{\Delta t} = \nu \frac{d^2 w^j}{dr^2} - \frac{\nu}{r} \frac{dw^j}{dr} + \frac{\alpha}{\rho} r^2 E_0{}^j , \tag{14}$$

$$w^0(r) = 0 , \tag{15}$$

$$w^j\big|_{r=0} = 0 , \tag{16}$$

$$\left.\frac{dw^j}{dr}\right|_{r=R} = 0 , \tag{17}$$

$$w^j\big|_{r=R} = \frac{Q^j}{2\pi} , \tag{18}$$

where $Q^j = Q(t_j)$, $E_0^j \approx E_0(t_j)$, $j = 1, 2, \ldots, m$. Solution of problem (14)–(18) on each time layer $j = 1, 2, \ldots, m$ can be presented in the form [12]

$$w^j(r) = \theta^j(r) + E_0^j \varphi(r) , \tag{19}$$

where $\theta^j(r)$ and $\varphi(r)$ are unknown functions. Substituting Eq. (19) into Eq. (14), we obtain

$$\frac{\theta^j(r) + E_0^j\varphi(r) - w^{j-1}(r)}{\Delta t} = \nu \frac{d^2\theta^j}{dr^2} - \frac{\nu}{r} \frac{d\theta^j}{dr} + \nu E_0^j \frac{d^2\varphi}{dr^2} - E_0^j \frac{\nu}{r} \frac{d\varphi}{dr} + \frac{\alpha}{\rho} r^2 E_0^j .$$

From this we obtain the following boundary-value problems for the unknown functions $\theta^j(r)$ and $\varphi(r)$:

$$\frac{\theta^j(r) - w^{j-1}(r)}{\Delta t} = \nu \frac{d^2\theta^j}{dr^2} - \frac{\nu}{r}\frac{d\theta^j}{dr}, \quad \theta^j\Big|_{r=0} = 0, \quad \frac{d\theta^j}{dr}\bigg|_{r=R} = 0. \tag{20}$$

$$\frac{\varphi(r)}{\Delta t} = \nu \frac{d^2\varphi}{dr^2} - \frac{\nu}{r}\frac{d\varphi}{dr} + \frac{\alpha}{\rho}r^2, \quad \varphi\Big|_{r=0} = 0, \quad \frac{d\varphi}{dr}\bigg|_{r=R} = 0, \quad j = 1, 2, \ldots, m. \tag{21}$$

Substitution of (19) into the additional condition (18) yields

$$\theta^j(R) + E_0^j\varphi(R) = \frac{Q^j}{2\pi}, \quad E_0^j = \frac{Q^j/2\pi - \theta^j(R)}{\varphi(R)}, \quad j = 1, 2, \ldots, m. \tag{22}$$

Thus, to solve problem (14)–(18) for determining the functions $w^j(r)$ and E_0^j, $j = 1, 2, \ldots, m$, it is necessary first to determine the function $\varphi(r)$ from the solution of problem (21) and after this, from the solution of problem (20) to determine the function $\theta^j(r)$, the value of E_0^j from formula (22), and finally the function $w^j(r)$ from formula (19) successively for each time layer $j = 1, 2, \ldots, m$. For numerical solution of problems (20) and (21) the method of finite differences can be used. We introduce the difference grid which is uniform in the variable r in the region $[0 \le r \le R]$:

$$\bar{\omega}_h = \{r_i = i\Delta r, i = \overline{0, n}\},$$

where $\Delta r = \dfrac{R}{n}$. Using the integral method, it is possible to present the discrete analogs of problems (20) and (21) on the difference grid $\bar{\omega}_h$ in the form

$$\begin{gathered}\frac{\theta_i^j - w_i^{j-1}}{\Delta t} = \nu \frac{\theta_{i+1}^j - 2\theta_i^j + \theta_{i-1}^j}{\Delta r^2} - \frac{\nu}{r_i}\frac{\theta_i^j - \theta_{i-1}^j}{\Delta r}, \quad i = 1, 2, 3, \ldots, n-1, \\ \theta_0^j = 0, \quad \frac{\theta_n^j - \theta_{n-1}^j}{\Delta r} = 0,\end{gathered} \tag{23}$$

$$\begin{gathered}\frac{\varphi_i}{\Delta t} = \nu \frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{\Delta r^2} - \frac{\nu}{r_i}\frac{\varphi_i - \varphi_{i-1}}{\Delta r} + \frac{\alpha}{\rho}r_i^2, \quad i = 1, 2, 3, \ldots, n-1, \\ \varphi_0 = 0, \quad \frac{\varphi_n - \varphi_{n-1}}{\Delta r} = 0, \quad j = 1, 2, \ldots, m,\end{gathered} \tag{24}$$

where $w_i^j \approx w^j(r_i)$, $\theta_i^{j-1} \approx \theta^j(r_i)$, and $\varphi_i \approx \varphi(r_i)$. The difference problems (23) and (24) represent a linear system of algebraic equations with a tridiagonal matrix in which the approximate values of the sought functions $\theta^j(r)$ and $\varphi(r)$ in the internal nodes of the difference grid act as unknowns, i.e., θ_i^j and φ_i, $i = \overline{1, n-1}$. To solve the difference problems (23) and (24), the Thomas algorithm can be used (matching method) [12].

Results of Numerical Calculations. Based on the proposed computational algorithm, numerical experiments were carried out for model problems by the followings scheme. The density of the acoustic energy at the beginning of the tube $E_0(t)$ was assigned, and the solution of direct problem (9)–(12) was found. Next, using the formula $Q(t) = 2\pi w\big|_{r=R}$, the volumetric liquid flow rate in the tube was determined, and the dependence found was used to restore $E_0(t)$.

The first series of calculations was performed with the use of nonperturbed data. The results of the numerical experiment carried out for $R = 0.02$ m, $\nu = 10^{-6}$ m^2/s, $\alpha = 5\cdot10^{-5}$ m^{-1}, $\rho = 1000$ kg/m^3, $E_0(t) = 100 - 20\sin 3t$ J/m^3, $\Delta t = 0.05$, 2, and 10 s, and $\Delta t = 0.001$ m are presented in Table 1. The results of the numerical experiment show that with the use of nonperturbed input data the sought function $E_0(t)$ is restored on all computational grids in time (3rd, 4th, and 5th columns of the table). The exact and calculated values of the function $E_0(t)$ at the nonperturbed data coincide absolutely.

The second series of calculations was performed with the following perturbations being imposed on $Q(t_j)$:

$$Q_\delta(t_j) = Q(t_j) + \delta Q(t_j)(2\sigma_j - 1),$$

TABLE 1. Results of Numerical Experiment

t, s	E_0^t, J/m^3	$\bar{E}_0$, J/m^3		
		Δt = 0.05 s	Δt = 2 s	Δt = 10 s
10	119.76	119.76	119.76	119.76
20	106.10	106.10	106.10	106.10
30	82.12	82.12	82.12	82.12
40	88.39	88.39	88.39	88.39
50	114.30	114.30	114.30	114.30
60	116.02	116.02	116.02	116.02
70	90.65	90.65	90.65	90.65
80	81.09	81.09	81.09	81.09
90	103.52	103.52	103.52	103.52
100	120.00	120.00	120.00	120.00
110	102.65	102.65	102.65	102.65
120	80.82	80.82	80.82	80.82
130	91.44	91.44	91.44	91.44
140	116.54	116.54	116.54	116.54
150	113.67	113.67	113.67	113.67
160	87.68	87.68	87.68	87.68
170	82.53	82.53	82.53	82.53
180	106.93	106.93	106.93	106.93
190	119.61	119.61	119.61	119.61
200	99.12	99.12	99.12	99.12

TABLE 2. Results of Numerical Experiment

t, s	E_0^t, J/m^3	$\tilde{E}_0$, J/m^3		
		Δt = 0.5 min	Δt = 1 min	Δt = 5 min
10	119.76	109.03	119.44	114.28
20	106.10	85.49	99.84	101.72
30	82.12	72.87	74.46	84.43
40	88.39	97.39	75.42	85.02
50	114.30	119.99	116.09	113.56
60	116.02	115.03	110.60	116.44
70	90.65	93.39	82.57	95.34
80	81.09	85.73	87.04	79.97
90	103.52	125.18	112.67	100.93
100	120.00	113.14	124.25	117.10

where σ_j is a random variable on the segment [0, 1] modeled with the aid of a random-number generator and δ is the maximum relative perturbation. The results of the numerical experiment carried out for Δt = 0.5, 1, and 5 min and δ = 0.05 are presented in Table 2.

TABLE 3. Results of Numerical Calculations

$Q \cdot 10^5$, m^3/s	$\bar{E}_0$, J/m^3		
	$R = 0.05$ m	$R = 0.1$ m	$R = 0.2$ m
1.0	225.73	14.11	0.9
2.0	180.58	11.29	0.72
3.0	135.44	8.46	0.54
4.0	90.29	5.64	0.36
5.0	45.15	2.82	0.18

In using perturbed input data in which the error is of fluctuational character, the sought function $E_0(t)$ was restored with definite uncertainty. The errors in the input data manifested themselves to a greater degree on decrease in the time step (at $\Delta t = 0.5$ min the maximum error with which the solution was obtained was equal to 24%). However, with increase of the time step the accuracy of calculations increased (at $\Delta t = 5$ min the maximum error of finding the solution was equal to 6%).

An analysis of the result of numerical experiment indicate that due to the use of computational grids, rough in time, it is possible to reduce the influence of the error of input data on the accuracy of restoration of the function $E_0(t)$. In the proposed computational algorithm, the effect of regularization was attained by selecting a difference time grid.

Based on the proposed numerical method, the densities of the acoustic energy were also determined form the given liquid flow for different tubes. The results of numerical calculations carried out for $\nu = 10^{-6}$ m^2/s, $\rho = 1000$ kg/m^3, and $\alpha = 5 \cdot 10^{-5}$ m^{-1} are presented in Table 3. It follows from this table that with increase in the tube radius the density of the acoustic energy needed for the formation of the given flow of liquid decreases.

Thus, the proposed numerical method allows one to use an explicit formula for determining the sonic energy density needed for providing the assigned acoustic liquid flow in a tube.

Conclusions. The problem of determining the acoustic energy density at the beginning of the tube that provides the flow of liquid at an assigned flow rate is considered. The proposed method of simulation can find application in the case of acoustic effect on oil pools.

NOTATION

$E(z, t)$, density of acoustic energy, J/m^3; $E_0(t)$, density of acoustic energy at the beginning of the tube, J/m^3; E_0^t, precise values of function $E_0(t)$; $\bar{E}_0$, calculated values of function $E_0(t)$ at nonperturbed data; $\tilde{E}_0$, calculated values of function $E_0(t)$ at perturbed data; P, radiation pressure, Pa; $Q(t)$, volumetric liquid flow rate, m^3/s; r, radial coordinate, m; R, tube radius, m; t, time, s; α, coefficient of energy absorption, m^{-1}; δ, maximum relative perturbation; ν, kinematic viscosity of liquid, m^2/s; ρ, liquid density, kg/m^3.

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