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# GENERALIZED SOLUTION OF THE MIXED HEAT-CONDUCTION PROBLEM BY THE WEIGHTED TEMPERATURE METHOD

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*On the basis of the weighted temperature method, an algorithm of generalized solution of boundary-value problems on the heat conduction in bodies canonical in shape with boundary conditions of general form has been constructed. It is shown that this problem is equivalent, in the limit, to the infinite system of identities including n-fold integral operators for the temperature function, initial and boundary conditions, and internal heat source as well as an additional boundary function (the temperature at one of the boundary points or its derivative with respect to the coordinate of this point). High approximation accuracy of the approach proposed is demonstrated by the example of solving a number of boundary-value problems on nonstationary heat conduction with nonsymmetric and mixed boundary conditions.*

***Keywords:*** *boundary-value problem, nonstationary heat conduction, analytical methods, approximation, weighted temperature method.*

**Introduction.** The present work is a logical continuation of the works [1–4] in which, on the basis of introduction of $n$-fold integral operators, a boundary function, and boundary characteristics, mathematically equivalent sequences of integral identical equalities including a complete set of initial data on the structure of the differential heat-conduction equation, the properties of this equation, and its initial and boundary conditions have been constructed. We propose to solve the mixed heat-conduction problem by the weighted temperature method (WTM), in accordance with which a solution is constructed on the basis of systems of linear algebraic equations following from the integral identities including a weighted temperature (temperature function). The WTM is a highly efficient method because it involves preliminary determination of the boundary function representing the temperature at a boundary point of the computational region or its derivative with respect to the coordinate of this point. It is shown that the WTM is much simplier in obtaining an approximate solution with a better approximation as compared to the Kantorovich, Galerkin, and Tsoi methods and the method of additional boundary conditions [5–10]. Therefore, of interest is the use of the weighted temperature method for obtaining a generalized approximate solution of the mixed problem on the heat conduction in the nonsymmetric region $y \in [R_1, R_2]$ with boundary conditions of general form. Without question this problem deserves attention [10–14].

**1. Mathematical Formulation of the Problem.** We consider a one-dimensional boundary-value problem on the nonstationary heat conduction in the nonsymmetric region $y \in [R_1, R_2]$ with variable internal heat sources, an inhomogeneous initial condition, and boundary conditions of general form. The general formulation of this problem for a plate ($m = 0$), a hollow cylinder ($m = 1$), and a hollow sphere ($m = 2$) has, respectively, the form

$$c\rho \frac{\partial \overline{T}}{\partial \overline{t}} = \frac{\lambda}{\overline{y}^m} \frac{\partial}{\partial \overline{y}} \left( \overline{y}^m \frac{\partial \overline{T}}{\partial \overline{y}} \right) + \overline{Q}(\overline{y}, \overline{t})(\overline{R}_1, \overline{R}_2) \times (0, \infty) , \quad m = 0, 1, 2 , \tag{1.1}$$

$$\left. \left( \overline{\alpha}_1 \frac{\partial \overline{T}}{\partial \overline{y}} + \overline{\beta}_1 \overline{T} \right) \right|_{\overline{R}_1} = \overline{\gamma}_1(\overline{t}) , \quad \left. \left( \overline{\alpha}_2 \frac{\partial \overline{T}}{\partial \overline{y}} + \overline{\beta}_2 \overline{T} \right) \right|_{\overline{R}_2} = \overline{\gamma}_2(\overline{t}) , \quad \overline{\alpha}_1^2 + \overline{\beta}_1^2 \neq 0 , \quad \overline{\alpha}_2^2 + \overline{\beta}_2^2 \neq 0 , \quad \overline{t} > 0 , \tag{1.2}$$

$$\overline{T}(\overline{y}, 0) = \overline{\Phi}(\overline{y}) , \tag{1.3}$$

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where $\overline{T}(\overline{y}, 0) = \overline{\Phi}(\overline{y})$ is the initial temperature distribution in a body, $\overline{y}$ is the coordinate of a point in the computational region, $\overline{y} = \overline{R}_1$ and $\overline{y} = \overline{R}_2$ are the surfaces (shells) bounding the segment $[\overline{R}_1, \overline{R}_2]$, $\overline{t}$ is the dimensional time, $\lambda$, $c$, and $\rho$ are the heat-conduction coefficient, the specific heat capacity, and the density of the body, respectively, and $\overline{Q}(\overline{y}, \overline{t})$ is the specific density of the volume heat sources in the body.

We introduce the following dimensionless variables into consideration: $y = \dfrac{\overline{y}}{R}$, $t = \dfrac{\overline{t}}{\tau}$, $\tau = \dfrac{R^2}{\kappa}$, $T = \dfrac{\overline{T} - T^*}{\Delta T}$, $R_1 = \dfrac{\overline{R}_1}{R}$, and $R_2 = \dfrac{\overline{R}_2}{R}$, where $\kappa = \lambda/(c\rho)$ is the thermal diffusivity, $\Delta T$ is the temperature scale, $T^*$ is the reference temperature, $R$ is the length scale, and $\tau$ is the time scale. In this case, problem (1.1)–(1.3) takes the form

$$\frac{\partial T}{\partial t} = \frac{1}{y^m}\frac{\partial}{\partial y}\left(y^m \frac{\partial T}{\partial y}\right) + Q(y, t)\,, \tag{1.4}$$

$$\left[\alpha_1 \frac{\partial T}{\partial y} + \beta_1 T\right]_{R_1} = \gamma_1(t)\,, \quad \left[\alpha_2 \frac{\partial T}{\partial y} + \beta_2 T\right]_{R_2} = \gamma_2(t)\,, \quad \alpha_1^2 + \beta_1^2 \neq 0\,, \quad \alpha_2^2 + \beta_2^2 \neq 0\,, \quad t > 0\,, \tag{1.5}$$

$$T(y, 0) = \Phi(y)\,, \tag{1.6}$$

where $Q(y, t) = \dfrac{R^2}{\lambda \Delta T}\,\overline{Q}(y, t)$, $\Phi(y) = \dfrac{\overline{\Phi}(y) - T^*}{\Delta T}$, $\alpha_j = \dfrac{\overline{\alpha}_j}{R}$, $\beta_j = \overline{\beta}_j$, $\gamma_j(t) = \dfrac{\overline{\gamma}_1(t) - \overline{\beta}_1 T^*}{\Delta T}$ $(j = 1, 2)$.

**2. Generalized Identical Equalities.** The first term on the right of Eq. (1.4) represents the differential operator

$$L_0 \equiv \frac{1}{y^m}\frac{\partial}{\partial y}\left(y^m \frac{\partial}{\partial y}\right) \equiv \frac{1}{y^m} L\,, \tag{2.1}$$

for which the scalar-product rule is true: $(u, L\upsilon) = (Lu, \upsilon)$. In this case, the operator $L$ adheres to the Green formula:

$$(u, L\upsilon) - (Lu, \upsilon) = \left[y^m \left(u \frac{\partial \upsilon}{\partial y} - \upsilon \frac{\partial u}{\partial y}\right)\right]_{R_1}^{R_2}\,, \tag{2.2}$$

where $(u, \upsilon)$ is the scalar product of the functions $u = u(y)$ and $\upsilon = \upsilon(y)$ in the segment $y \in [R_1, R_2]$. We introduce, into consideration, the integral operators

$$\mathcal{L}_n(\cdot) \equiv ((\cdot), \mathcal{K}_n) = \int_{R_1}^{R_2} (\cdot)\mathcal{K}_n \partial y\,, \quad \forall n \in \mathbb{Z}_+\,. \tag{2.3}$$

In this case, the following integrals will be functionals with respect to $T = T(y, t)$, $\Phi = \Phi(y)$, and $Q = Q(y, t)$:

$$\mathcal{L}_n T \equiv (T, \mathcal{K}_n) = \int_{R_1}^{R_2} T\mathcal{K}_n \partial y\,, \quad \Phi_n \equiv (\Phi, \mathcal{K}_n) = \int_{R_1}^{R_2} \Phi\mathcal{K}_n \partial y\,,$$
$$Q_n \equiv \left(Q, \mathcal{K}_n\right) = \int_{R_1}^{R_2} q\mathcal{K}_n \partial y\,, \quad \forall n \in \mathbb{Z}_+\,, \tag{2.4}$$

where $\mathcal{K}_n = \mathcal{K}_n(y)$ are any (unknown) weighting functions (kernels). Let us write Eq. (1.4) in operator form ($D_t = \partial/\partial t$):

$$D_t T = \frac{1}{y^m} LT + Q\,. \tag{2.5}$$

*Sequences of integral relations formal in form*. Let us apply the operator $\mathcal{L}_n$ to Eq. (2.5):

$$\mathcal{L}_n(D_t T) = \mathcal{L}_n\left(\frac{1}{y^m} LT\right) + \mathcal{L}_n(Q)\,. \tag{2.6}$$

Using the Green formula (2.2) and introducing the function $\mathcal{M}_n = \mathcal{M}_n(y) = \mathcal{K}_n / y^m$ into consideration, we obtain the following relation for the first term on the right of Eq. (2.6):

$$\mathcal{L}_n\left(\frac{1}{y^m}\,LT\right) \equiv \left(\frac{1}{y^m}\,LT,\,\mathcal{K}_n\right) = \left(LT,\,\frac{\mathcal{K}_n}{y^m}\right) = \left(T,\,L\left(\frac{\mathcal{K}_n}{y^m}\right)\right) + P_n = (T,\,L\mathcal{M}_n) + P_n\;, \tag{2.7}$$

where

$$P_n = \left[y^m\left(\frac{\mathcal{K}_n}{y^m}\right)\frac{\partial T}{\partial y} - y^m T\,\frac{d}{\partial y}\left(\frac{\mathcal{K}_n}{y^m}\right)\right]_{R_1}^{R_2} = \left[y^m\left(\mathcal{M}_n\,\frac{\partial T}{\partial y} - \frac{d\mathcal{M}_n}{dy}\,T\right)\right]_{R_1}^{R_2}. \tag{2.8}$$

In view of Eqs. (2.4), (2.6), and (2.7) and the Leibnitz theorem, instead of Eq. (2.5), we write

$$D_t(\mathcal{L}_n T) = (T,\,L\mathcal{M}_n) + Q_n + P_n\;, \tag{2.9}$$

and arrive at the following sequence of integral relations:

$$\left\{D_t(\mathcal{L}_n T) = (T,\,L\mathcal{M}_n) + Q_n + P_n\right\}_n\;,\quad \forall n \in \mathbb{Z}_+\;. \tag{2.10}$$

In (2.10), unknowns are the temperature function $T(y, t)$ and the quantities $T(R_j, t)$ and $\partial T(R_j, t)/\partial y$ $(j = 1, 2)$ involved in the function $P_n$. The functions $\mathcal{M}_n$, $\forall n \in \mathbb{Z}_+$ (the kernels $\mathcal{K}_n = y^m \mathcal{M}_n$) and their values at the boundary points $\mathcal{M}_n(R_j)$ and $d\mathcal{M}_n(R_j)/dy$ $(j = 1, 2)$ are also unknown.

We now formulate the intermediate problem on determination of the boundary conditions for the functions $\mathcal{M}_n$ at which, in (2.10), the temperature $T(R_j, t)$ and its derivative $\partial T(R_j, t)/\partial y$ $(j = 1 \vee 2)$ at one of the boundary points are eliminated. We first rewrite expression (2.8) for the function $P_n$:

$$P_n = \left[y^m\left(\mathcal{M}_n\,\frac{\partial T}{\partial y} - T\,\frac{d\mathcal{M}_n}{dy}\right)\right]_{R_1}^{R_2} = \left[-y^m\,\frac{d\mathcal{M}_n}{dy}\left(\left(-\frac{\mathcal{M}_n}{d\mathcal{M}_n/dy}\right)\frac{\partial T}{\partial y} + T\right)\right]_{R_1}^{R_2} = \left[-y^m\,\frac{d\mathcal{M}_n}{dy}\,\Omega_n\right]_{R_1}^{R_2}, \tag{2.11}$$

where $\Omega_n = \left(-\dfrac{\mathcal{M}_n}{d\mathcal{M}_n/dy}\right)\dfrac{\partial T}{\partial y} + T$ and $\Omega_n(R_j) = \left[\left(-\dfrac{\mathcal{M}_n}{d\mathcal{M}_n/dy}\right)\dfrac{\partial T}{\partial y} + T\right]_{y=R_j}$, $j = 1, 2$.

Let us represent the boundary conditions (1.5) in the form

$$\left[\frac{\alpha_j}{\beta_j}\,\frac{\partial T}{\partial y} + T\right]_{y=R_j} = \frac{\gamma_j(t)}{\beta_j}\,,\quad j = 1,\,2\;\;(\beta_j \neq 0)\,. \tag{2.12}$$

Comparing (2.11) and (2.12), we arrive at the boundary conditions for the function $\mathcal{M}_n$:

$$\left[-\frac{\mathcal{M}_n}{\partial\mathcal{M}_n/\partial y}\right]_{y=R_j} = \frac{\alpha_j}{\beta_j} \;\Rightarrow\; \left[\alpha_j\,\frac{d\mathcal{M}_n}{dy} + \beta_j\mathcal{M}_n\right]_{y=R_j} = 0\,,\quad j = 1,\,2\,. \tag{2.13}$$

In the case where (2.12) is fulfilled, in view of (2.11)–(2.13) we write

$$\left[\left(-\frac{\mathcal{M}_n}{d\mathcal{M}_n/dy}\right)\frac{\partial T}{\partial y} + T\right]_{y=R_j} = \left[\frac{\alpha_j}{\beta_j}\,\frac{\partial T}{\partial y} + T\right]_{y=R_j} = \frac{\gamma_j(t)}{\beta_j}\,,\quad j = 1,\,2\,. \tag{2.14}$$

Then we will use the homogeneous boundary conditions for the function $\mathcal{M}_n$ at which, in Eqs. (2.10), the temperature $T(R_j, t)$ at the boundary point $y = R_j$ $(j = 1 \vee 2)$ and its derivative with respect to the coordinate of this point $\partial T(R_j, t)/\partial y$ are eliminated. In this case, the following three variants are possible:

$$\text{I) } \alpha_1 \frac{d\mathcal{M}_n(R_1)}{dy} + \beta_1 \mathcal{M}_n(R_1) = 0 , \quad \alpha_2 \frac{d\mathcal{M}_n(R_2)}{dy} + \beta_2 \mathcal{M}_n(R_2) = 0 , \tag{2.15.1}$$

$$\text{II) } \left\{ \alpha_j \frac{d\mathcal{M}_n(R_j)}{dy} + \beta_j \mathcal{M}_n(R_j) = 0, \ \mathcal{M}_n(R_l) = 0 \vee \frac{d\mathcal{M}_n(R_l)}{dy} = 0 \right\} \left( \frac{j = 1, \ l = 2}{j = 2, \ l = 1} \right) , \tag{2.15.2}$$

$$\text{III) } \mathcal{M}_n(R_j) = 0 , \quad \frac{d\mathcal{M}_n(R_j)}{dy} = 0 , \quad j = 1 \vee 2 . \tag{2.15.3}$$

Variant I. Using (2.11) and (2.15.1), we represent $P_n$ in two alternative forms:

$$P_n = \left[ -y^m \frac{d\mathcal{M}_n}{dy} \Omega \right]_{R_1}^{R_2} = \frac{R_1{}^m}{\beta_1} \frac{d\mathcal{M}_n(R_1)}{dy} \gamma_1(t) - \frac{R_2{}^m}{\beta_2} \frac{d\mathcal{M}_n(R_2)}{dy} \gamma_2(t) \ (\beta_1 \neq 0, \beta_2 \neq 0) , \tag{2.16.1}$$

$$P_n = R_2^m \mathcal{M}_n(R_2) \frac{\gamma_2(t)}{\alpha_2} - R_1^m \mathcal{M}_n(R_1) \frac{\gamma_1(t)}{\alpha_1} = \frac{\mathcal{K}_n(R_2)}{\alpha_2} \gamma_2(t) - \frac{\mathcal{K}_n(R_1)}{\alpha_1} \gamma_1(t) \ (\alpha_1 \neq 0, \alpha_2 \neq 0) . \tag{2.16.2}$$

In view of (2.16.1) and (2.16.2), from (2.10) we obtain, respectively, the sequences

$$\left\{ D_t(\mathcal{L}_n T) = (T, L\mathcal{M}_n) + Q_n + \frac{R_1^m}{\beta_1} \frac{d\mathcal{M}_n(R_1)}{dy} \gamma_1(t) - \frac{R_2^m}{\beta_2} \frac{d\mathcal{M}_n(R_2)}{dy} \gamma_2(t) \right\}_n , \quad \forall \ n \in \mathbb{Z}_+ , \tag{2.17.1}$$

$$\left\{ D_t(\mathcal{L}_n T) = (T, L\mathcal{M}_n) + Q_n + \frac{\mathcal{K}_n(R_2)}{\alpha_2} \gamma_2(t) - \frac{\mathcal{K}_n(R_1)}{\alpha_1} \gamma_1(t) \right\}_n , \quad \forall n \in \mathbb{Z}_+ . \tag{2.17.2}$$

Variant II. In the case where expression (2.15.2) is true and the condition $\mathcal{M}_n(R_2) = 0$ or the condition $d\mathcal{M}_n(R_l)/dy = 0$ is fulfilled, for the function $P_n$ defined by (2.11) at $j = 1$ and $l = 1$ we obtain, respectively, the expressions

$$\mathcal{M}_n^{(1)}(R_2) = 0 \quad \rightarrow \quad P_n = \frac{R_1^m}{\beta_1} \frac{d\mathcal{M}_n^{(1)}(R_1)}{dy} \gamma_1(t) - R_2^m \frac{d\mathcal{M}_n^{(1)}(R_2)}{dy} T(R_2) , \tag{2.18.1}$$

$$\frac{d\mathcal{M}_n^{(1)}(R_2)}{dy} = 0 \quad \rightarrow \quad P_n = \frac{R_1^m}{\beta_1} \frac{d\mathcal{M}_n^{(1)}(R_1)}{dy} \gamma_1(t) + R_2^m \mathcal{M}_n^{(1)}(R_2) \frac{\partial T(R_2)}{\partial y} . \tag{2.18.2}$$

In a similar manner we find the function $P_n$ for $j = 1$ and $l = 1$:

$$\mathcal{M}_n^{(2)}(R_1) = 0 \quad \rightarrow \quad P_n^{(2)} = -R_2^m \frac{d\mathcal{M}_n^{(2)}(R_2)}{dy} \frac{\gamma_2(t)}{\beta_2} + R_1^m \frac{d\mathcal{M}_n^{(2)}(R_1)}{dy} T(R_1) , \tag{2.19.1}$$

$$\frac{d\mathcal{M}_n^{(2)}(R_1)}{dy} = 0 \quad \rightarrow \quad P_n^{(2)} = -R_2^m \frac{d\mathcal{M}_n^{(2)}(R_2)}{dy} \frac{\gamma_2(t)}{\beta_2} - R_1^m \mathcal{M}_n^{(2)}(R_1) \frac{\partial T(R_1)}{\partial y} . \tag{2.19.2}$$

Generalization of (2.18.1)–(2.19.2) gives

$$\mathcal{M}_n(R_l) = 0 \quad \rightarrow \quad P_n^{(j)} = (-1)^l \left( \frac{R_j{}^m}{\beta_j} \frac{d\mathcal{M}_n(R_j)}{dy} \gamma_j(t) - R_l{}^m \frac{d\mathcal{M}_n(R_l)}{dy} T(R_l) \right) \left( \frac{j = 1, \ l = 2}{j = 2, \ l = 1} \right) , \tag{2.20}$$

$$\frac{d\mathcal{M}_n(R_l)}{dy} = 0 \quad \rightarrow \quad P_n^{(j)} = (-1)^l \left( \frac{R_j^m}{\beta_j} \frac{d\mathcal{M}_n(R_j)}{dy} \gamma_j(t) + R_l^m \mathcal{M}_n(R_l) \frac{\partial T(R_l)}{\partial y} \right) \left( \frac{j = 1, \ l = 2}{j = 2, \ l = 1} \right) . \tag{2.21}$$

In the long run, for the cases where the condition $\mathcal{M}_n(R_l) = 0$ or the condition $d\mathcal{M}_n(R_l)/dy = 0$ is fulfilled, from (2.10) we obtain, respectively, the sequences

$$\left\{ \begin{array}{c} D_t(\mathcal{L}_n T) = (T, L\mathcal{M}_n^{(j)}) + Q_n \\ + (-1)^l \left( \dfrac{R_j^m}{\beta_j} \dfrac{d\mathcal{M}_n^{(j)}(R_j)}{dy} \gamma_j(t) - R_l^m \dfrac{d\mathcal{M}_n^{(j)}(R_l)}{dy} T(R_l) \right) \end{array} \right\}_n \quad \forall n \in \mathbb{Z}_+ \left( \begin{array}{c} j = 1, l = 2 \\ j = 2, l = 1 \end{array} \right), \tag{2.22}$$

$$\left\{ \begin{array}{c} D_t(\mathcal{L}_n T) = (T, L\mathcal{M}_n^{(j)}) + Q_n^{(j)} \\ + (-1)^l \left( \dfrac{R_j^m}{\beta_j} \dfrac{d\mathcal{M}_n^{(j)}(R_j)}{dy} \gamma_j(t) + R_l^m \mathcal{M}_n^{(j)}(R_l) \dfrac{\partial T(R_l)}{\partial y} \right) \end{array} \right\}_n \quad \forall n \in \mathbb{Z}_+ \left( \begin{array}{c} j = 1, l = 2 \\ j = 2, l = 1 \end{array} \right). \tag{2.23}$$

Variant III. Using (2.15.3) and a substitution following from the boundary conditions (1.5), we write the following expressions for the function $P_n$:

$$T\Big|_{R_l} = \frac{\gamma_l(t)}{\beta_l} - \frac{\alpha_l}{\beta_l} \frac{\partial T}{\partial y}\Big|_{R_l}, \quad \frac{\partial T}{\partial y}\Big|_{R_l} = \frac{\gamma_l(t)}{\alpha_l} - \frac{\beta_l}{\alpha_l} T\Big|_{R_l} \quad (l = 1 \vee 2). \tag{2.24}$$

In this case,

$$\left\{ \begin{array}{c} \mathcal{M}_n(R_1) = 0 \\ \dfrac{d\mathcal{M}_n(R_1)}{dy} = 0 \end{array} \right\} \to \left\{ \begin{array}{l} P_n = R_2^m \left[ -\dfrac{d\mathcal{M}_n(R_2)}{dy} \dfrac{\gamma_2(t)}{\beta_2} + \left( \dfrac{\alpha_2}{\beta_2} \dfrac{d\mathcal{M}_n(R_2)}{dy} + \mathcal{M}_n(R_2) \right) \dfrac{\partial T(R_2)}{\partial y} \right] \\ P_n = R_2^m \left[ \mathcal{M}_n(R_2) \dfrac{\gamma_2(t)}{\alpha_2} - \left( \dfrac{d\mathcal{M}_n(R_2)}{dy} + \dfrac{\beta_2}{\alpha_2} \mathcal{M}_n(R_2) \right) T(R_2) \right] \end{array} \right., \tag{2.25}$$

$$\left\{ \begin{array}{c} \mathcal{M}_n(R_2) = 0 \\ \dfrac{d\mathcal{M}_n(R_2)}{dy} = 0 \end{array} \right\} \to \left\{ \begin{array}{l} P_n = R_1^m \left[ \dfrac{d\mathcal{M}_n(R_1)}{dy} \dfrac{\gamma_1(t)}{\beta_1} - \left( \dfrac{\alpha_1}{\beta_1} \dfrac{d\mathcal{M}_n(R_1)}{dy} + \mathcal{M}_n(R_1) \right) \dfrac{\partial T(R_1)}{\partial y} \right] \\ P_n = R_1^m \left[ -\mathcal{M}_n(R_1) \dfrac{\gamma_1(t)}{\alpha_1} + \left( \dfrac{d\mathcal{M}_n(R_1)}{dy} + \dfrac{\beta_1}{\alpha_1} \mathcal{M}_n(R_1) \right) T(R_1) \right] \end{array} \right.. \tag{2.26}$$

In view of (2.25) and (2.26), we write the function $P_n$ in the general form:

$$\left\{ \begin{array}{c} \mathcal{M}_n(R_j) = 0 \\ \dfrac{d\mathcal{M}_n(R_j)}{dy} = 0 \end{array} \right\} \to \left\{ \begin{array}{l} P_n = (-1)^l R_l^m \left[ \left( \dfrac{\alpha_l}{\beta_l} \dfrac{d\mathcal{M}_n(R_l)}{dy} + \mathcal{M}_n(R_l) \right) \dfrac{\partial T(R_l)}{\partial y} - \dfrac{d\mathcal{M}_n(R_l)}{dy} \dfrac{\gamma_l(t)}{\beta_l} \right] \\ P_n = (-1)^l R_l^m \left[ -\left( \dfrac{d\mathcal{M}_n(R_l)}{dy} + \dfrac{\beta_l}{\alpha_l} \mathcal{M}_n(R_l) \right) T(R_l) + \mathcal{M}_n(R_l) \dfrac{\gamma_l(t)}{\alpha_l} \right] \end{array} \right.. \tag{2.27}$$

On the basis of (2.10) and (2.27), we obtain, respectively, the dependences

$$\left\{ \begin{array}{c} D_t(\mathcal{L}_n T) = (T, L\mathcal{M}_n^{(j)}) + Q_n \\ + (-1)^l R_l^m \left[ \left( \dfrac{\alpha_l}{\beta_l} \dfrac{d\mathcal{M}_n(R_l)}{dy} + \mathcal{M}_n(R_l) \right) \dfrac{\partial T(R_l)}{\partial y} - \dfrac{d\mathcal{M}_n(R_l)}{dy} \dfrac{\gamma_l(t)}{\beta_l} \right] \end{array} \right\}_n \quad \forall n \in \mathbb{Z}_+ \left( \begin{array}{c} j = 1, l = 2 \\ j = 2, l = 1 \end{array} \right), \tag{2.28}$$

$$\left\{ \begin{array}{c} D_t(\mathcal{L}_n T) = (T, L\mathcal{M}_n^{(j)}) + Q_n^{(j)} \\ + (-1)^l R_l^m \left[ -\left( \dfrac{d\mathcal{M}_n(R_l)}{dy} + \dfrac{\beta_l}{\alpha_l} \mathcal{M}_n(R_l) \right) T(R_l) + \mathcal{M}_n(R_l) \dfrac{\gamma_l(t)}{\alpha_l} \right] \end{array} \right\}_n \quad \forall n \in \mathbb{Z}_+ \left( \begin{array}{c} j = 1, l = 2 \\ j = 2, l = 1 \end{array} \right). \tag{2.29}$$

It should be noted that the equations forming sequences (2.22), (2.23), (2.28), and (2.29) involve only one unknown (not counting the weighting functions $\mathcal{M}_n^{(j)}$): $T(R_l)$ or $\partial T(R_l)/\partial y$ ($l = 1 \vee 2$), and Eqs. (2.17.1) and (2.17.2) include the function $T(R_l)$ or the function $\partial T(R_l)/\partial y$ which are already involved in the term $(T, L\mathcal{M}_n)$. We will call them, in accordance with the mathematical definition of the boundary points of the computational region, the boundary functions and represent them in the following form:

$$\varphi_q(t) = \frac{\partial T(R_l, t)}{\partial x}, \quad \varphi_T(t) = T(R_l, t) \qquad (l = 1 \vee 2)\,. \tag{2.30}$$

*Integral equalities of the first order (n = 1).* The equation with $n = 1$ of sequence (2.10) has the form

$$D_t(\mathcal{L}_1 T) = (T, L\mathcal{M}_1) + Q_1 + P_1\,. \tag{2.31}$$

Eliminating the differentiation operation $L\mathcal{M}_1$ in (2.31) and assuming that

$$(T, L\mathcal{M}_1) = 0\,, \tag{2.32}$$

we obtain

$$D_t(\mathcal{L}_1 T) = Q_1 + P_1\,. \tag{2.33}$$

From (2.32) the homogeneous differential equation

$$\frac{d}{dy}\left(y^m \frac{d}{dy}\mathcal{M}_1\right) = 0 \tag{2.34}$$

follows. To obtain a nontrivial solution of Eq. (2.34), we formulate two boundary conditions for the segment $y \in [R_1, R_2]$, one of which will be inhomogeneous. As a homogeneous boundary condition determined by the boundary function, we will use one of relations (2.13). In this case, we will have two variants in which a nontrivial solution of the differential equation (2.33) is obtained with conservation of the boundary function:

$$1)\ \langle j = 1,\ l = 2\rangle \ \to\ \left.\left(\alpha_1 \frac{d\mathcal{M}_1}{dy} + \beta_1\mathcal{M}_1\right)\right|_{y=R_1} = 0\,, \quad \left.\left(\alpha_2 \frac{d\mathcal{M}_1}{dy} + \beta_2\mathcal{M}_1\right)\right|_{y=R_l} \neq 0\,, \tag{2.35.1}$$

$$2)\ \langle j = 2,\ l = 1\rangle \ \to\ \left.\left(\alpha_2 \frac{d\mathcal{M}_1}{dy} + \beta_2\mathcal{M}_1\right)\right|_{y=R_2} = 0\,, \quad \left.\left(\alpha_1 \frac{d\mathcal{M}_1}{dy} + \beta_1\mathcal{M}_1\right)\right|_{y=R_l} \neq 0\,. \tag{2.35.2}$$

They in the general form are as follows:

$$\left.\left(\alpha_j \frac{d\mathcal{M}_1}{dy} + \beta_j\mathcal{M}_1\right)\right|_{y=R_j} = 0\,, \quad \left.\left(\alpha_l \frac{d\mathcal{M}_1}{dy} + \beta_l\mathcal{M}_1\right)\right|_{y=R_l} \neq 0\,, \quad \frac{j = 1,\ l = 2}{j = 2,\ l = 1}\,. \tag{2.36}$$

Integration of Eq. (2.34) with one of the homogeneous boundary conditions (2.36) gives

$$\mathcal{M}_1^{(j)} = C\left(\int_{R_j}^{y} \frac{dy}{y^m} - \frac{\alpha_j}{\beta_j}\frac{1}{R_j^m}\right), \quad j = 1 \vee 2\,, \tag{2.37}$$

where $C$ is an integration constant. We can assume that $C = 1$. In this case,

$$\mathcal{M}_1^{(j)} = \int_{R_j}^{y} \frac{dy}{y^m} - \frac{\alpha_j}{\beta_j}\frac{1}{R_j^m}\,, \quad \mathcal{K}_1^{(j)} = y^m \int_{R_j}^{y} \frac{dy}{y^m} - \frac{\alpha_j}{\beta_j}\frac{y^m}{R_j^m}\,, \quad j = 1 \vee 2\,. \tag{2.38}$$

It follows from (2.38) that $C = 1$ corresponds to the relation

$$\frac{d\mathcal{M}_1^{(l)}(R_l)}{dy} = \frac{1}{R_l^m} \quad (l = 1,\ 2)\,. \tag{2.39}$$

Thus, at the boundary conditions (2.35.1) and (2.35.2), we have two variants of solving the differential equation (2.34):

$$1)\ \langle j=1,\ l=2\rangle \quad\to\quad \left\{\begin{array}{l}\left.\left(\alpha_1\dfrac{d\mathcal{M}_1^{(1)}}{dy}+\beta_1\mathcal{M}_1^{(1)}\right)\right|_{y=R_1}=0\\ \dfrac{d\mathcal{M}_1^{(1)}(R_2)}{dy}=\dfrac{1}{R_2^m}\end{array}\right\} \quad\to\quad \mathcal{M}_1^{(1)}=\int_{R_1}^{y}\frac{dy}{y^m}-\frac{\alpha_1}{\beta_1}\frac{1}{R_1^m}, \tag{2.40.1}$$

$$2)\ \langle j=2,\ l=1\rangle \quad\to\quad \left\{\begin{array}{l}\left.\left(\alpha_2\dfrac{d\mathcal{M}_1^{(2)}}{dy}+\beta_2\mathcal{M}_1^{(2)}\right)\right|_{y=R_2}=0\\ \dfrac{d\mathcal{M}_1^{(2)}(R_1)}{dy}=\dfrac{1}{R_1^m}\end{array}\right\} \quad\to\quad \mathcal{M}_1^{(2)}=\int_{R_2}^{y}\frac{dy}{y^m}-\frac{\alpha_2}{\beta_2}\frac{1}{R_2^m}. \tag{2.40.2}$$

Expanding the right side of Eq. (2.31), in view of (2.11) we obtain

$$\begin{aligned} D_t(\mathcal{L}_1T) &= Q_1+P_1=Q_1+R_1^m\frac{d\mathcal{M}_1(R_1)}{dy}\Omega_1(R_1)-R_2^m\frac{d\mathcal{M}_1(R_2)}{dy}\Omega_1(R_2)\\ &= Q_1-R_2^m\frac{d\mathcal{M}_1(R_2)}{dy}\left[\left(-\frac{\mathcal{M}_1}{d\mathcal{M}_1/dy}\right)\frac{\partial T}{\partial y}+T\right]_{y=R_2}+R_1^m\frac{d\mathcal{M}_1(R_1)}{dy}\left[\left(-\frac{\mathcal{M}_1}{d\mathcal{M}_1/dy}\right)\frac{\partial T}{\partial y}+T\right]_{y=R_1}. \end{aligned} \tag{2.41}$$

In the case where relations (2.14), (2.40.1), and (2.40.2) are fulfilled, Eq. (2.41) is divided into two equations corresponding to (2.40.1) and (2.40.2):

$$\begin{aligned} 1)\ D_t(\mathcal{L}_1^{(1)}T) &= Q_1^{(1)}+R_2^m\left[\mathcal{M}_1^{(1)}\frac{\partial T}{\partial y}-\frac{d\mathcal{M}_1^{(1)}}{dy}T\right]_{y=R_2}+R_1^m\frac{d\mathcal{M}_1^{(1)}(R_1)}{dy}\frac{\gamma_1(t)}{\beta_1}\\ &= Q_1^{(1)}+R_2^m\mathcal{M}_1^{(1)}(R_2)\frac{\partial T(R_2,t)}{\partial y}-T(R_2,t)+\frac{\gamma_1(t)}{\beta_1}, \end{aligned} \tag{2.42.1}$$

$$\begin{aligned} 2)\ D_t(\mathcal{L}_1^{(2)}T) &= Q_1^{(2)}-R_2^m\frac{d\mathcal{M}_1^{(2)}(R_2)}{dy}\frac{\gamma_2(t)}{\beta_2}+R_1^m\left[\frac{d\mathcal{M}_1^{(2)}}{dy}T-\mathcal{M}_1^{\text{II}}\frac{\partial T}{\partial y}\right]_{y=R_1}\\ &= Q_1^{(2)}-R_1^m\mathcal{M}_1^{(2)}(R_1)\frac{\partial T(R_1,t)}{\partial y}+T(R_1,t)-\frac{\gamma_2(t)}{\beta_2}, \end{aligned} \tag{2.42.2}$$

where $\mathcal{L}_1^{(1)}T\equiv(T,\mathcal{K}_1^{(1)})$, $\mathcal{L}_1^{(2)}T\equiv(T,\mathcal{K}_1^{(2)})$, $\mathcal{K}_1^{(1)}=y^m\mathcal{M}_1^{(1)}$, and $\mathcal{K}_2^{(2)}=y^m\mathcal{M}_2^{(2)}$. Each of Eqs. (2.42.1) and (2.42.2) involves two unknown functions: $T(R_{2,T})$ and $\partial T(R_{2,T})/\partial y$ in (2.42.1) and $T(R_{1,T})$ and $\partial T(R_{1,T})/\partial y$ in (2.42.2). Let us reduce Eqs. (2.42.1) and (2.42.2) to the form where only one boundary function is present. To do this, it will suffice to use one of substitution (2.25). For the boundary function $\varphi_q(t)$ ($l = 1, 2$) we have the relations

$$1)\ \left\langle\frac{j=1}{l=2}\right\rangle \quad\to\quad D_t(\mathcal{L}_1^{(1)}T)=\frac{\gamma_1(t)}{\beta_1}-\frac{\gamma_2(t)}{\beta_2}+Q_1^{(1)}+\frac{\alpha_2}{\beta_2}\left(1+R_2^m\mathcal{M}_1^{(1)}(R_2)\frac{\beta_2}{\alpha_2}\right)\frac{\partial T(R_2,t)}{\partial y}, \tag{2.43.1}$$

$$2)\ \left\langle\frac{j=1}{l=2}\right\rangle \quad\to\quad D_t(\mathcal{L}_1^{(2)}T)=\frac{\gamma_1(t)}{\beta_1}-\frac{\gamma_2(t)}{\beta_2}+Q_1^{(2)}-\frac{\alpha_1}{\beta_1}\left(1+R_1^m\mathcal{M}_1^{(2)}(R_1)\frac{\beta_1}{\alpha_1}\right)\frac{\partial T(R_1,t)}{\partial y}, \tag{2.43.2}$$

or, in the general form, the relation

$$D_t(\mathcal{L}_1^{(j)}T) = Q_1^{(j)} + \frac{\gamma_1(t)}{\beta_1} - \frac{\gamma_2(t)}{\beta_2} + p_q^{(j)}\varphi_q(t)\left(\frac{j = 1,\, l = 2}{j = 2,\, l = 1}\right), \tag{2.44}$$

where $p_q = (-1)^l(\alpha_l/\beta_l + R_l^m\mathcal{M}_1^{(j)}(R_l))$. For the boundary function $\varphi_T(t)$, we obtain the relations

$$1)\ \left\langle\frac{j = 1}{l = 2}\right\rangle \quad \to \quad D_t(\mathcal{L}_1^{(1)}T) = \frac{\gamma_1(t)}{\beta_1} + R_2^m\mathcal{M}_1^{(1)}(R_2)\,\frac{\gamma_2(t)}{\alpha_2} + Q_1 - \left(1 + R_2^m\mathcal{M}_1^{(1)}(R_2)\,\frac{\beta_2}{\alpha_2}\right)T(R_2,\, t)\,, \tag{2.45.1}$$

$$2)\ \left\langle\frac{j = 2}{l = 1}\right\rangle \quad \to \quad D_t(\mathcal{L}_1^{(2)}T) = -R_1^m\mathcal{M}_1^{(2)}(R_1)\,\frac{\gamma_1(t)}{\alpha_1} - \frac{\gamma_2(t)}{\beta_2} + Q_1 + \left(1 + R_1^m\mathcal{M}_1^{(2)}(R_1)\,\frac{\beta_1}{\alpha_1}\right)T(R_1,\, t)\,, \tag{2.45.2}$$

or, in the general form, the relation

$$D_t(\mathcal{L}_1^{(j)}T) = Q_1 + (-1)^l\left(\frac{\gamma_j(t)}{\beta_j} + R_l^m\mathcal{M}_1^{(1)}(R_l)\,\frac{\gamma_l(t)}{\alpha_l} - p_T\varphi_T(t)\right)\left(\frac{j = 1,\, l = 2}{j = 2,\, l = 1}\right), \tag{2.46}$$

where $p_T = 1 + R_l^m\mathcal{M}_1^{(j)}(R_l)\beta_l/\alpha_l$.

Integration of Eqs. (2.44) and (2.46) gives

$$\mathcal{L}_1^{(j)}T = \int\limits_0^t Q_1\,dt + \frac{1}{\beta_1}\int\limits_0^t \gamma_1(t)\,dt - \frac{1}{\beta_2}\int\limits_0^t \gamma_2(t)\,dt + p_1\int\limits_0^t \varphi_q(t)dt + C\left(\frac{j = 1,\, l = 2}{j = 2,\, l = 1}\right), \tag{2.47}$$

$$\mathcal{L}_1^{(j)}T = \int\limits_0^t Q_1\,dt + (-1)^l\left(\frac{1}{\beta_j}\int\limits_0^t \gamma_j(t)\,dt + \frac{R_l^m\mathcal{M}_1^{(1)}(R_l)}{\alpha_l}\int\limits_0^t \gamma_l(t)\,dt - p_2\int\limits_0^t \varphi_T(t)dt\right) + C\left(\frac{j = 1,\, l = 2}{j = 2,\, l = 1}\right). \tag{2.48}$$

Using the initial condition (1.6), we obtain the following relation for the integration constant $C$:

$$C = \mathcal{L}_1^{(j)}T(y,\, 0) = \mathcal{L}_1^{(j)}\Phi(y) = (\Phi,\, \mathcal{K}_1^{(j)}) = \Phi_1^{(j)}\ \ (j = 1,\, 2)\,. \tag{2.49}$$

In this case, instead of (2.47) and (2.48), we finally write

$$\mathcal{L}_1^{(j)}T = \int\limits_0^t Q_1\,dt + \frac{1}{\beta_1}\int\limits_0^t \gamma_1(t)\,dt - \frac{1}{\beta_2}\int\limits_0^t \gamma_2(t)\,dt + p_1\int\limits_0^t \varphi_q(t)dt + \Phi_1^{(j)}\ \left(\frac{j = 1,\, l = 2}{j = 2,\, l = 1}\right), \tag{2.50}$$

$$\mathcal{L}_1^{(j)}T = \int\limits_0^t Q_1\,dt + (-1)^l\left(\frac{1}{\beta_j}\int\limits_0^t \gamma_j(t)\,dt + \frac{R_l{}^m\mathcal{M}_1^{(1)}(R_l)}{\alpha_l}\int\limits_0^t \gamma_l(t)\,dt - p_2\int\limits_0^t \varphi_T(t)dt\right) + \Phi_1^{(j)}\ \left(\frac{j = 1,\, l = 2}{j = 2,\, l = 1}\right). \tag{2.51}$$

*Integral relations of the order n = 2, 3, … .* We now turn to Eqs. (2.17.1) and (2.17.2) with $n = 2, 3, \ldots$ and construct a recursion equation in the form

$$L\mathcal{M}_n^{(j)} = \mathcal{K}_{n-1}^{(j)} \ (n = 2, 3, \ldots) \tag{2.52}$$

or in the form

$$\frac{d}{dy}\left(y^m \frac{d\mathcal{M}_n^{(j)}}{dy}\right) = \mathcal{M}_{n-1}^{(j)} y^m \ (n = 2, 3, \ldots) . \tag{2.53}$$

The solution of Eq. (2.53) represents the sum of the general solution of the homogeneous equation and a partial solution of the inhomogeneous equation. Since the general solution of the system of equations

$$\frac{d}{dy}\left(y^m \frac{d\mathcal{M}_n^{(j)}}{dy}\right) = \mathcal{K}_{n-1}^{(j)}, \quad \alpha_j \frac{d\mathcal{M}_n(R_j)}{dy} + \beta_j \mathcal{M}_n(R_j) = 0 \ (n = 2, 3, \ldots) \tag{2.54}$$

is coincident with the solution for $\mathcal{M}_1$ with an accuracy to a constant multiplier, we have

$$\mathcal{M}_n^{(j)} = \int_{R_j}^{y} \frac{dy}{y^m} \int_{R_j}^{y} \mathcal{K}_{n-1}^{(j)} \, dy + C\mathcal{M}_1^{(j)} , \quad j = 1 \vee 2 , \quad (n = 2, 3, \ldots) . \tag{2.55}$$

Hence we obtain a recursion formula for the weighted function $\mathcal{K}_n^{(j)}$ ($n = 2, 3, \ldots$):

$$\mathcal{K}_n^{(j)} = y^m \int_{R_j}^{y} \frac{dy}{y^m} \int_{R_j}^{y} \mathcal{K}_{n-1}^{(j)} \, dy + C\mathcal{K}_1^{(j)} , \quad n = 2, 3, \ldots . \tag{2.56}$$

It will suffice to put $C = 1$ in (2.56). The constant $C$ can be also determined from other considerations.

Variant I. Substitution of (2.56) into the boundary condition

$$\alpha_l \left.\frac{d\mathcal{M}_n^{(j)}}{dx}\right|_{R_l} + \beta_l \left.\mathcal{M}_n^{(j)}\right|_{R_l} = 0 \quad (l = 1 \vee 2) \tag{2.57}$$

gives

$$C = (-1)^j \left(\frac{\alpha_l}{R_l^m} + \beta_l \mathcal{M}_1^{(j)}(R_l)\right)^{-1} \int_{R_1}^{R_2} \left(\frac{\alpha_l}{R_l^m} \mathcal{K}_{n-1}^{(j)} + \frac{\beta_l}{y^m} \int_{R_j}^{y} \mathcal{K}_{n-1}^{(j)} dy\right) dy \quad \left(\begin{matrix} j = 1, \, l = 2 \\ j = 2, \, l = 1 \end{matrix}\right) . \tag{2.58}$$

Thus, we have equations with $n = 1$ of sequences (2.17.1) and (2.17.2). Then, because of the equivalence of these sequences, we will dwell on one of them, e.g., (2.17.1).

In accordance with (2.56) ($C = 1$), the derivatives $d\mathcal{M}_n^{(j)}(R_{j,l})/dy$ take the form

$$\frac{d\mathcal{M}_n^{(j)}(R_j)}{dy} = \frac{1}{R_j^m} , \quad \frac{d\mathcal{M}_n^{(j)}(R_l)}{dy} = \frac{1}{R_l^m} \int_{R_j}^{R_l} \mathcal{M}_{n-1}^{(j)} y^m dy + \frac{1}{R_l^m} , \quad j = 1 \vee 2 \quad (n = 2, 3, \ldots) . \tag{2.59}$$

In this case, (2.17.1) is divided into two sequences:

$$\left\{ D_t(\mathcal{L}_n^{(j)} T) = (T, L\mathcal{M}_n^{(j)}) + Q_n^{(j)} + (-1)^l \left(\frac{\gamma_j(t)}{\beta_j} - \omega_n^{(j)} \frac{\gamma_l(t)}{\beta_l}\right)\right\}_n , \quad \forall n \in \mathbb{Z}_+ \left(\begin{matrix} j = 1, \, l = 2 \\ j = 2, \, l = 1 \end{matrix}\right) , \tag{2.60}$$

where $\omega_n^{(1)} = 1 + \int_{R_j}^{R_l} \mathcal{M}_{n-1}^{(j)} y^m dy$. Integration of Eq. (2.60) with account of the equality $(T, \mathcal{K}_0) = 0$ and the recursion formula (2.52) gives

$$\left\{ \mathcal{L}_n^{(j)} T = \int_0^t (T, \mathcal{K}_{n-1}^{(j)})\, dt + \int_0^t Q_n^{(j)} dt + (-1)^l \left( \int_0^t \frac{\gamma_j(t)}{\beta_j}\, dt - \omega_n^{(j)} \int_0^t \frac{\gamma_l(t)}{\beta_l}\, dt \right) + A_n^{(j)} \right\}_n \quad \forall n \in \mathbb{Z}_+ \,. \tag{2.61}$$

Determining the integration constants $A_n^{(j)} = (\mathcal{L}_n T(x, t = 0)$ and $\mathcal{K}_n^{(j)}) = (\Phi, \mathcal{K}_n^{(j)}) = \Phi_n^{(j)}$, we obtain

$$\left\{ \mathcal{L}_n^{(j)} T = \int_0^t (\mathcal{L}_{n-1}^{(j)} T)\, dt + W_n^{(j)}(t) \right\}_n \,, \quad \forall n \in \mathbb{Z}_+ \left( \frac{j = 1,\, l = 2}{j = 2,\, l = 1} \right), \tag{2.62}$$

where

$$W_n^{(j)}(t) = \int_0^t Q_n^{(j)} dt + (-1)^l \left( \int_0^t \frac{\gamma_j(t)}{\beta_j}\, dt - \omega_n^{(j)} \int_0^t \frac{\gamma_l(t)}{\beta_l}\, dt \right) + \Phi_n^{(j)} \quad (n = 1, 2, \ldots) \,. \tag{2.63}$$

Expanding sequence (2.62), we have

$$\begin{gathered} \mathcal{L}_1 T = W_1(t) \\ \mathcal{L}_2 T = \int_0^t \left( \mathcal{L}_1 T \right) dt + W_2(t) \\ \mathcal{L}_3 T = \int_0^t \left( \mathcal{L}_2 T \right) dt + W_3(t) \end{gathered} \,. \tag{2.64}$$

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Substituting the right side of the first equation of (2.64) into the integrand of the second equation of (2.64), instead of $(\mathcal{L}_1 T)$, we obtain the identical equality

$$\mathcal{L}_2 T \equiv \int_0^t W_1(t)\, dt + W_2(t) \,. \tag{2.65}$$

In the same way, substituting the right side of Eq. (2.65) into the integrand of the third equation of (2.64), instead of $(\mathcal{L}_2 T)$, we obtain

$$\mathcal{L}_3 T \equiv \int_0^t \left( \int_0^t W_1(t)\, dt + W_2(t) \right) dt + W_3(t) = \int_0^t dt \int_0^t W_1(t)\, dt + \int_0^t W_2(t)\, dt + W_3(t) \,. \tag{2.66}$$

Making analogous manipulations, we arrive at the general identity

$$\mathcal{L}_n T \equiv \underbrace{\int_0^t \ldots \int_0^t}_{n-1} W_1(t)\, dt^{(n-1)} + \underbrace{\int_0^t \ldots \int_0^t}_{n-2} W_2(t)\, dt^{(n-2)} + \ldots + \int_0^t dt \int_0^t W_{n-2}(t)\, dt^{(2)} + \int_0^t W_{n-1}(t)\, dt + W_n(t) \,. \tag{2.67}$$

Then, using (2.64)–(2.67) and introducing the integral operator $\mathcal{L}_k^t \equiv \underbrace{\int_0^t \ldots \int_0^t}_{k} (\cdot)\, dt^{(k)}$ into consideration, we obtain the sequence

$$\left\{ \mathcal{L}_n T \equiv \sum_{i=1}^{n} \mathcal{L}_{n-i}^t W_i(t) \right\}_n \,, \quad n \in \mathbb{Z}_+ \,. \tag{2.68}$$

Variant II. Depending on the boundary function selected, one of sequences (2.22) or (2.23) can be used but only at $n \geq 2$. Let us find $\mathcal{M}_n^{(j)}$ defined by (2.55). To do this, it will suffice to determine the integration constant $C$ with the use of one of the homogenous boundary conditions: $\mathcal{M}_n^{(j)}(R_l) = 0$ or $\partial\mathcal{M}_n^{(j)}(R_l)/\partial y = 0$. At $\mathcal{M}_n^{(j)}(R_l) = 0$, from (2.22) we obtain the sequence

$$\left\{\begin{aligned} & D_t(\mathcal{L}_n T) = (T,\, L\mathcal{M}_n^{(j)}) + Q_n \\ & + (-1)^l\left(\frac{C}{\beta_j}\gamma_j(t) - \left(C + \int_{R_j}^{R_l}\mathcal{K}_{n-1}^{(j)}dy\right)\varphi_T\right)\end{aligned}\right\}_n \quad \forall n \in \mathbb{Z}_+ \left(\frac{j = 1,\, l = 2}{j = 2,\, l = 1}\right), \tag{2.69}$$

where $C = -\left(\int_{R_j}^{R_l}\frac{dy}{y^m} - \frac{\alpha_j}{\beta_j}\frac{1}{R_j^m}\right)^{-1}\left(\int_{R_j}^{R_l}\frac{dy}{y^m}\int_{R_j}^{y}\mathcal{K}_{n-1}^{(j)}\,dy\right)$. Integration of the equations involved in (2.69) with account of (2.52), the equality $(T,\ \mathcal{K}_0) = 0$, and the initial condition (1.6) gives

$$\left\{\begin{aligned} & \mathcal{L}_n T = \int_0^t (T,\, \mathcal{K}_{n-1}^{(j)})\,dt + \int_0^t Q_n dt \\ & + (-1)^l\left(\frac{C}{\beta_j}\int_0^t\gamma_j(t)\,dt - \left(C + \int_{R_j}^{R_l}\mathcal{K}_{n-1}^{(j)}\,dy\right)\int_0^t\varphi_T(t)dt\right) + \Phi_n^{(j)}\end{aligned}\right\}_n , \quad \forall\, n \in \mathbb{Z}_+ \left(\frac{j = 1,\, l = 2}{j = 2,\, l = 1}\right). \tag{2.70}$$

By analogy with variant I, we introduce the designation

$$W_n^{(j)}(t) = \int_0^t Q_n^{(j)}dt + (-1)^l\left(\frac{C}{\beta_j}\int_0^t\gamma_j(t)\,dt - \left(C + \int_{R_j}^{R_l}\mathcal{K}_{n-1}^{(j)}dy\right)\int_0^t\varphi_T(t)dt\right) + \Phi_n^{(j)}\ (n = 1, 2, \ldots)\,. \tag{2.71}$$

Then, on the basis of (2.70) and (2.71), we arrive at the sequence of identities of the form of (2.68), including the integral boundary characteristics $\Upsilon_T^n(t) = \underbrace{\int_0^t dt \ldots \int_0^t}_{n}\varphi_T(t)dt$ $(n = 1, 2, \ldots)$. On the other hand, at the boundary condition $\partial\mathcal{M}_n^{(j)}(R_l)/\partial y = 0$, we have the sequence

$$\left\{D_t(\mathcal{L}_n T) = (T,\, L\mathcal{M}_n^{(j)}) + Q_n^{(j)} + (-1)^l\left(w_n^{(j)}\varphi_q(t) - \frac{1}{\beta_j}\int_{R_j}^{R_l}\mathcal{K}_{n-1}^{(j)}\,dy\gamma_j(t)\right)\right\}_n \forall n \in \mathbb{Z}_+\left(\frac{j = 1,\, l = 2}{j = 2,\, l = 1}\right), \tag{2.72}$$

where $w_n^{(j)} = \frac{\alpha_j}{\beta_j}\int_{R_j}^{R_l}\mathcal{K}_{n-1}^{(j)}\,dy - R_l^m\int_{R_j}^{R_l}\frac{dy}{y^m}\int_{y}^{R_l}\mathcal{K}_{n-1}^{(j)}\,dy$. In this case, for the function $\mathcal{M}_n^{(j)}$ we obtain, in accordance with (2.55), $C = -\int_{R_j}^{R_l}\mathcal{K}_{n-1}^{(j)}\,dy$. Integration of (2.72) gives

$$\left\{\begin{aligned} & \mathcal{L}_n T = \int_0^t (T,\, \mathcal{K}_{n-1}^{(j)})\,dt + \int_0^t Q_n^{(j)}dt + (-1)^l\left(w_n^{(j)}\int_0^t\varphi_q(t)dt\right. \\ & \left.- \frac{1}{\beta_j}\int_{R_j}^{R_l}\mathcal{K}_{n-1}^{(j)}\,dy\int_0^t\gamma_j(t)dt\right) + \Phi_n^{(j)}\end{aligned}\right\}_n \quad \forall n \in \mathbb{Z}_+\,, \tag{2.73}$$

where $w_n^{(j)} = (-1)^l \left( -\int\limits_{R_j}^{R_l} \mathcal{K}_{n-1}^{(j)}\, dy \frac{\gamma_j(t)}{\beta_j} + \left( \frac{\alpha_j}{\beta_j} \int\limits_{R_j}^{R_l} \mathcal{K}_{n-1}^{(j)}\, dy - R_l^m \int\limits_{R_j}^{R_l} \frac{dy}{y^m} \int\limits_{y}^{R_l} \mathcal{K}_{n-1}^{(j)}\, dy \right) \varphi_q \right)$. Introducing the designation

$$W_n^{(j)} = \int\limits_0^t Q_n^{(j)} dt + (-1)^l \left( w_n^{(j)} \int\limits_0^t \varphi_q(t) dt - \frac{1}{\beta_j} \int\limits_{R_j}^{R_l} \mathcal{K}_{n-1}^{(j)}\, dy \int\limits_0^t \gamma_j(t) dt \right) + \Phi_n^{(j)} \tag{2.74}$$

and performing operations similar to (2.69)–(2.71), we obtain the same (in the formal sense) sequence of identical equalities (2.68) including (in an explicit form) the integral boundary characteristics $\Upsilon_q^n(t) = \underbrace{\int\limits_0^t dt \ldots \int\limits_0^t}_{n} \varphi_q(t) dt$ $(n = 1, 2, \ldots)$.

Variant III. Let us consider the system of equations

$$\frac{d}{dy}\left( y^m \frac{d\mathcal{M}_n^{(j)}}{dy} \right) = \mathcal{K}_{n-1}^{(j)}, \quad \mathcal{M}_n(R_j) = 0, \quad \frac{d\mathcal{M}_n(R_j)}{dy} = 0 \quad (n = 2, 3, \ldots), \tag{2.75}$$

whose solution has the form

$$\mathcal{M}_n^{(j)} = \int\limits_{R_j}^{y} \frac{dy}{y^m} \int\limits_{R_j}^{y} \mathcal{K}_{n-1}^{(j)} dy, \quad j = 1 \vee 2 \ (n = 2, 3, \ldots). \tag{2.76}$$

For this variant we have two sequences of Eqs. (2.28) and (2.29) with $n \geq 2$. In view of (2.75), instead of (2.28) and (2.29), we obtain, respectively, the following equations with $j = 1$, $l = 2 \vee j = 2$, $l = 1$:

$$\left\{ \begin{array}{l} D_t(\mathcal{L}_n T) = \mathcal{L}_{n-1} T + Q_n \\ + (-1)^l \left[ \varphi_q(t) \int\limits_{R_j}^{R_l} \left( \frac{\alpha_l}{\beta_l} \mathcal{K}_{n-1}^{(j)} + \frac{R_l^m}{y^m} \int\limits_{R_j}^{R_l} \mathcal{K}_{n-1}^{(j)} dy \right) dy - \frac{\gamma_l(t)}{\beta_l} \int\limits_{R_j}^{R_l} \mathcal{K}_{n-1}^{(j)} dy \right] \end{array} \right\}_n \quad (n = 2, 3, \ldots), \tag{2.77}$$

$$\left\{ \begin{array}{l} D_t(\mathcal{L}_n T) = \mathcal{L}_{n-1} T + Q_n^{(j)} \\ + (-1)^l \left[ \gamma_l(t) \frac{R_l^m}{\alpha_l} \int\limits_{R_j}^{R_l} \frac{dy}{y^m} \int\limits_{R_j}^{R_l} \mathcal{K}_{n-1}^{(j)} dy - \varphi_T(t) \int\limits_{R_j}^{R_l} \left( \mathcal{K}_{n-1}^{(j)} + \frac{\beta_l}{\alpha_l} \frac{R_l^m}{y^m} \int\limits_{R_j}^{R_l} \mathcal{K}_{n-1}^{(j)} dy \right) dy \right] \end{array} \right\}_n \quad (n = 2, 3, \ldots). \tag{2.78}$$

Integration of Eqs. (2.77) and (2.78) gives

$$\left\{ \mathcal{L}_n^{(j)} T = \int\limits_0^t (\mathcal{L}_{n-1}^{(j)} T) dt + \int\limits_0^t Q_n\, dt + (-1)^l \left[ \varpi_q^{(j)} \int\limits_0^t \varphi_q(t)\, dt - p_q^{(j)} \int\limits_0^t \gamma_l(t)\, dt + \Phi_n^{(j)} \right] \right\}_n \quad (n = 2, 3, \ldots), \tag{2.79}$$

$$\left\{ \mathcal{L}_n^{(j)} T = \int\limits_0^t (\mathcal{L}_{n-1}^{(j)} T) dt + \int\limits_0^t Q_n\, dt + (-1)^l \left( -\varpi_T^{(j)} \int\limits_0^t \varphi_T(t) dt + p_T^{(j)} \int\limits_0^t \gamma_l(t) dt \right) + \Phi_n^{(j)} \right\}_n \quad (n = 2, 3, \ldots), \tag{2.80}$$

where

$$\varpi_q^{(j)} = \int\limits_{R_j}^{R_l} \left( \frac{\alpha_l}{\beta_l} \mathcal{K}_{n-1}^{(j)} + \frac{R_l^m}{y^m} \int\limits_{R_j}^{R_l} \mathcal{K}_{n-1}^{(j)} dy \right) dy, \quad p_q^{(j)} = \frac{1}{\beta_l} \int\limits_{R_j}^{R_l} \mathcal{K}_{n-1}^{(j)} dy,$$

$$\varpi_T^{(j)} = \int_{R_j}^{R_l} \left( \mathcal{K}_{n-1}^{(j)} + \frac{\beta_l}{\alpha_l} \frac{R_l^m}{y^m} \int_{R_j}^{R_l} \mathcal{K}_{n-1}^{(j)} dy \right) dy \,, \quad p_T^{(j)} = \frac{R_l^m}{\alpha_l} \int_{R_j}^{R_l} \frac{dy}{y^m} \int_{R_j}^{R_l} \mathcal{K}_{n-1}^{(j)} dy \,.$$

Introducing the following designation into the equations of sequence (2.79):

$$W_n^{(j)} = \int_0^t Q_n \, dt + (-1)^l \left[ \varpi_q^{(j)} \int_0^t \varphi_q(t) dt - p_q^{(j)} \int_0^t \gamma_l(t) \, dt + \Phi_n^{(j)} \right] \tag{2.81}$$

and then performing the operations analogous to those for (2.69)–(2.71), we arrive at the sequence of identical equalities identical formally to (2.68). Note that the equalities of this sequence involve the integral characteristics $\Upsilon_l^n(t) = \underbrace{\int_0^t dt \ldots \int_0^t}_{n} \gamma_l(t) \, dt$ and the boundary integral characteristics $\mathcal{F}_q^n(t) = \underbrace{\int_0^t dt \ldots \int_0^t}_{n} \varphi_q(t) \, dt$. Note that we will arrive at the sequence formally identical to (2.68) in the same way on the basis of Eqs. (2.80) with the use of the designation

$$W_n^{(j)} = \int_0^t Q_n \, dt + (-1)^l \left( -\varpi_T^{(j)} \int_0^t \varphi_T(t) \, dt + p_T^{(j)} \int_0^t \gamma_l(t) \, dt \right) + \Phi_n^{(j)} \,. \tag{2.82}$$

The identities obtained in this case form the sequence identical formally to (2.68), and they involve the integral characteristics $\Upsilon_l^n(t)$ and the boundary integral characteristics $\mathcal{F}_T^n(t) = \underbrace{\int_0^t dt \ldots \int_0^t}_{n} \varphi_T(t) \, dt$.

Thus, it has been established that the boundary-value problem for the generalized equation of heat conduction with boundary conditions of general form corresponds to the sequence of integral identities involving $n$-fold integral operators for the temperature function, initial and boundary conditions, and internal heat sources as well as a boundary function (temperature of heat flow) introduced additionally for a boundary point of the computational region.

**3. General Algorithm for Solving the Problem on Nonstationary Heat Conduction by the Weighted Temperature Method.** Generalized solution of the mixed heat-conduction problem includes the following operations. The desired solution of this problem is represented in the form of the polynomial

$$T(x, t) = \sum_{j=0}^{N} a_j(t) \, x^j \,. \tag{3.1}$$

In view of the boundary conditions of the boundary-value problem and expression (3.1) we may write two equations for polynomial coefficients. Substitution of (3.1) into the definition of the boundary function $\varphi(t)$ ($\varphi_T(t)$ or $\varphi_q(t)$) gives one more additional equation. The remaining $N-2$ equations are constructed on the basis of polynomial (3.1) and $N-2$ linear algebraic equations for one of the corresponding sequences (infinite systems) of identities. Substitution of (3.1) with the coefficients found into the heat-balance integral

$$\frac{d}{dt} \int_0^1 \left( \sum_{j=0}^{N} a_j(t) x^j \right) dx = \frac{\partial T(1, t)}{\partial x} - \frac{\partial T(0, t)}{\partial x} \tag{3.2}$$

gives the constitutive integro-differential equation

$$F \left( \varphi'(t), \{\mathcal{F}_k(t)\}_{k=0}^{N-2}, \{\Upsilon_1^k(t)\}_{k=0}^{N-2}, \{\Upsilon_2^k(t)\}_{k=0}^{N-2} \right) = 0 \,. \tag{3.3}$$

Introduction, into consideration, of the function

$$\nu(t) = \mathcal{F}_{N-2}(t) \tag{3.4}$$

transforms Eq. (3.3) into the ordinary differential equation of the $(N-1)$ order

$$F\left(\nu(t), \nu'(t), \nu''(t), \nu'''(t), \ldots, \nu^{(N-1)}(t), \{\Upsilon_1^k(t)\}_{k=0}^{N-2}, \{\Upsilon_2^k(t)\}_{k=0}^{N-2}, \{\Upsilon_1^k(t)\}_{k=0}^{N-2}, \{\Upsilon_1^k(t)\}_{k=0}^{N-2}\right) = 0 \tag{3.5}$$

with the zero initial conditions

$$\nu(0) = \nu'(0) = \nu''(0) = \ldots = \nu^{(N-2)}(0) = 0 . \tag{3.6}$$

Solution of the Cauchy problem (3.5), (3.6) gives the function

$$\nu(t) = f\left(\{\Upsilon_1^k(t)\}_{k=0}^{N-2}, \{\Upsilon_2^k(t)\}_{k=0}^{N-2}, \{\Upsilon_1^k(t)\}_{k=0}^{N-2}, \{\Upsilon_1^k(t)\}_{k=0}^{N-2}\right) . \tag{3.7}$$

Performing the subsequent $(N-2)$ fold differentiation of the function $\nu(t)$, we obtain the boundary function

$$\varphi(t) = \frac{d^{N-2}}{dt^{N-2}} \nu(t) . \tag{3.8}$$

Substitution of the boundary function $\varphi(t)$ into the polynomial coefficients involved in (3.1) leads to the desired temperature function $T(x, t)$.

**4. Examples of Solving the Problem with Nonsymmetrical Boundary Conditions.** We dwell on test problems on the nonstationary heat conduction in a lengthy plate ($m = 0$). The problem on such conduction in a spherical body ($m = 2$) will not be considered because it is reduced through the substitution $T(y, t) = U(y, t)/y$ to the Cartesian coordinate system. The boundary-value problem on the nonstationary heat conduction in a cylindrical space ($m = 1$) deserves, in our opinion, a separate consideration. We consider the first and third boundary-value problems on the nonstationary heat conduction in a lengthy plate using the algorithm for generalized solving such problems described in Par. 3.

*The first boundary-value problem.* Let us write mathematical formulation of the problem on the nonstationary heat conduction in a lengthy plate in the dimensionless form:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} , \tag{4.1}$$

$$T(0, t) = \gamma_1(t) , \quad T(1, t) = \gamma_2(t) , \tag{4.2}$$

$$T(x, 0) = 0 . \tag{4.3}$$

Here, $x = \overline{x}/R$, $t = \overline{t}/\tau$, $\tau = R^2/\kappa$, and $R$ is the thickness of the plate. The exact solution of this problem has the form

$$T(x, t) = (1 - x)\gamma_1(t) + x\gamma_2(t) - \sum_{n=1}^{N} \frac{2 \sin (n\pi x)}{\pi n} \Bigg( (\gamma_1(t) - (-1)^n \gamma_2(t)) \\ + \pi^2 n^2 \sin (n\pi x) \exp (-n^2\pi^2 t) \int_0^t (\gamma_1(t) - (-1)^n \gamma_2(t)) \exp (n^2\pi^2 t)\, dt \Bigg) . \tag{4.4}$$

The approximate solution of problem (4.1)–(4.3) is defined by the polynomial

$$T(x, t) = (1 - x)\gamma_1(t) + x\gamma_2(t) + x(1 - x) \sum_{j=0}^{N-2} a_j(t) x^j . \tag{4.5}$$

The polynomial coefficients are determined using the integral identities (2.68), (2.79), and (2.81) on the assumption that $\alpha_1 = \alpha_2 = 0$. In the case where internal heat sources are absent in the plate and the initial condition is homogeneous, these identical equalities take the form

$$\int_0^1 T(x, t)\, x^{2k-1}\, dx \equiv \Upsilon_1^k(t) - \sum_{i=1}^{k} \frac{(2k-1)\,!}{(2k-2i)\,!}\, \Upsilon_2^{2k-2i}(t) - \sum_{i=1}^{k} \frac{(2k-1)\,!}{(2k-2i+1)\,!}\, \mathcal{F}_{2k-2i+1}(t)\,, \quad k \in \mathbb{Z}_+\,, \tag{4.6}$$

where $\Upsilon_1^k(t) = \underbrace{\int_0^t dt \ldots \int_0^t}_{k} \gamma_1(t)\, dt$, $\Upsilon_2^{2k-2i}(t) = \underbrace{\int_0^t dt \ldots \int_0^t}_{2k-2i} \gamma_2(t)\, dt$, $\mathcal{F}_{2k-2i+1}(t) = \underbrace{\int_0^t dt \ldots \int_0^t}_{2k-2i+1} \varphi\, dt$, $\varphi(t) = \varphi_q(t) = \dfrac{\partial T(1, t)}{\partial x}$.

In this variant of solution of the problem, the derivative $\partial T(1, t)/\partial x$ characterizing the heat flow at the point $x = 1$ was used as a boundary function. One of the equations for the coefficients $a_j(t)$, $j = \overline{0, N-2}$, will be written using the definition of the boundary function $\varphi(t) = \partial T(1, t)/\partial x$. Substitution of (4.5) into it gives the equation

$$\sum_{j=0}^{N-2} a_j(t) = \gamma_2(t) - \gamma_1(t) - \varphi(t)\,. \tag{4.7}$$

The remaining $N - 2$ equations follow from (4.6). Substitution of polynomial (4.5) into them gives the system of linear algebraic equations

$$\left\{ \frac{1}{2} \sum_{j=0}^{N-2} \frac{a_j(t)}{p_{j+1} + (2j+3)k + 2k^2} = \Upsilon_1^k(t) - \sum_{i=1}^{k} \frac{(2k-1)\,!}{(2k-2i)\,!}\, \Upsilon_2^{2k-2i}(t) \right.$$
$$\left. - \frac{\gamma_1(t)}{2k(1+2k)} - \frac{\gamma_2(t)}{1+2k} - \sum_{i=1}^{k} \frac{(2k-1)\,!}{(2k-2i+1)\,!}\, \mathcal{F}_{2k-2i+1}(t) \right\}_{k=1}^{N-2} . \tag{4.8}$$

Solving this system, we find the coefficients $a_j(t)$, $j = \overline{0, N-2}$. Then we may use the heat-balance integral

$$\frac{d}{dt} \int_0^1 T(x, t)\, dx = \varphi(t) - \frac{\partial T(0, t)}{\partial x} \tag{4.9}$$

or the equality (4.6) of the $(N-1)$ order that, in view of (4.2), (4.3), and (4.6), takes the form

$$\frac{1}{2} \sum_{j=0}^{N-2} \frac{a_j(t)}{p_{j+1} + (2j+3)(N-1) + 2(N-1)^2} = \Upsilon_1^{N-1}(t) - \sum_{i=1}^{N-1} \frac{(2N-3)\,!}{(2N-2-2i)\,!}\, \Upsilon_2^{2(N-i-1)}(t)$$
$$- \frac{\gamma_1(t)}{2(2N^2-3N+1)} - \frac{\gamma_2(t)}{2N-1} - \sum_{i=1}^{N-1} \frac{(2N-3)\,!}{(2N-2i-1)\,!}\, \mathcal{F}_{2N-2i-1}(t)\,. \tag{4.10}$$

From (4.9) or (4.10) we can obtain a constitutive equation for the boundary function $\varphi(t)$. Let us consider two particular examples.

Example 1. We assume that $\gamma_1(t) = \sin(2\pi t)$ and $\gamma_2(t) = 0$ and represent the temperature profile in the form of the fifth-degree polynomial

$$T(x, t) = (1-x)\sin 2\pi t + x(1-x) \sum_{j=0}^{3} a_j(t)\, x^j\,. \tag{4.11}$$

We will determine the three coefficients $a_j(t)$, $j = \overline{1, 3}$, for one of the above described variants (Par. 2). In particular, on the basis of (4.6), we write the system of equations

$$\int_0^1 T(x,t)x\,dx = \mathcal{F}_1(t) + \frac{\sin(\pi t)^2}{\pi},$$

$$\int_0^1 T(x,t)\left(x + \frac{x^3}{6}\right)dx = \frac{7}{6}\mathcal{F}_1(t) + \mathcal{F}_2(t) + \frac{\sin(\pi t)^2}{\pi} - \frac{\sin(2\pi t) - 2\pi t}{4\pi^2},$$

$$\int_0^1 T(x,t)\left(x + \frac{x^3}{6} + \frac{x^5}{120}\right)dx = \frac{47}{40}\mathcal{F}_1(t) + \frac{7}{6}\mathcal{F}_2(t) + \mathcal{F}_3(t) \tag{4.12}$$

$$+ \frac{\sin(\pi t)^2}{\pi} - \frac{\sin(2\pi t) - 2\pi t}{4\pi^2} + \frac{\cos(2\pi t) + 2\pi^2 t^2 - 1}{8\pi^3}.$$

Substitution of (4.11) into this system gives three linear algebraic equations. We will construct the fourth equation on the basis of the equality $\varphi(t) = \partial T(1, t)/\partial x$, from which it follows that

$$\sum_{j=0}^{3} a_j(t) = \varphi(t) + \sin(2\pi t). \tag{4.13}$$

Solving system (4.12), (4.13), we find the polynomial coefficients (not presented for brevity). The unknown boundary function $\varphi(t)$ is determined on the basis of the heat-balance integral (4.9). As a result, we arrive at the integro-differential equation

$$\varphi'(t) + 603\varphi(t) + 91{,}350\mathcal{F}_1(t) + 3{,}371{,}760\mathcal{F}_2(t) + 24{,}948{,}000\mathcal{F}_3(t) = \frac{3{,}118{,}500 - \pi^2 7245}{\pi^3}$$

$$- \frac{7560(825t - 52)t}{\pi} - \frac{3{,}118{,}500 - 7245\pi^2 + 5\pi^4}{\pi^3}\cos(2\pi t) + \frac{210\pi^2 - 196{,}560}{\pi^2}\sin(2\pi t). \tag{4.14}$$

Introducing the function $v = \mathcal{F}_3(t)$ into consideration, instead of (4.14), we obtain the Cauchy problem in the form of the fourth-order differential equation

$$v^{(4)} + 603v^{(3)} + 91{,}350v'' + 3{,}371{,}760v' + 24{,}948{,}000v = \frac{3{,}118{,}500 - \pi^2 7245}{\pi^3}$$

$$- \frac{7560(825t - 52)t}{\pi} - \frac{3{,}118{,}500 - 7245\pi^2 + 5\pi^4}{\pi^3}\cos(2\pi t) + \frac{210\pi^2 - 196{,}560}{\pi^2}\sin(2\pi t), \tag{4.15}$$

with the obvious initial conditions $v(0) = v'(0) = v''(0) = v^{(3)}(0) = 0$. Solution of (4.15) gives the function $v$. Since $v''' = \varphi$ is an identical equality, from (4.15) we obtain

$$\varphi = v''' = 0.68725\cos(2\pi t) - 0.46492\sin(2\pi t) - 0.90553e^{-9.8685t}$$

$$+ 0.29626e^{-39.841t} - 0.17667e^{-162.28t} + 0.09870e^{-391.01t}. \tag{4.16}$$

Determination of the boundary function $\varphi$ involved in the polynomial coefficients completes, in essence, the solution of the problem. The complete solution of the problem is not presented for brevity. Figure 1 shows the temperature profiles corresponding to (4.11) and (4.16). It is seen from this figure that the approximate and exact solutions are completely coincident.

*Third boundary-value problem.* It is assumed that a medium with a temperature $\gamma_1(t)$ flows over one side of the lengthy plate having a heat-transfer coefficient $h_1$ and a medium with a temperature $\gamma_2(t)$ flows over the opposite side of the plate having a heat-transfer coefficient $h_2$. The problem on the nonstationary heat conduction in this plate is defined by the heat-conduction equation (4.1) with the initial condition (4.3) and the boundary conditions

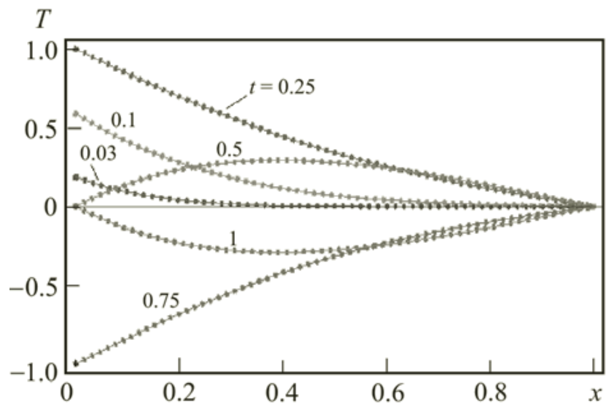


Fig. 1. Temperature distribution in a lengthy plate at different instants of time at $\gamma_1(t) = \sin(2\pi t)$ and $\gamma_2(t) = 0$: full lines) exact solution; dotted lines) solution by the WTM at $N = 5$.

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} - \mathrm{Bi}_1 \left. T\right|_{x=0} = -\mathrm{Bi}_1\,\gamma_1(t)\,, \quad \left.\frac{\partial T}{\partial x}\right|_{x=1} + \mathrm{Bi}_2 \left. T\right|_{x=1} = \mathrm{Bi}_2\,\gamma_2(t)\,. \tag{4.17}$$

To pass to the boundary conditions of the form of (1.5), we write

$$-\frac{1}{\mathrm{Bi}_1}\left.\frac{\partial T}{\partial x}\right|_{x=0} + \left. T\right|_{x=0} = \gamma_1(t)\,, \quad \frac{1}{\mathrm{Bi}_2}\left.\frac{\partial T}{\partial x}\right|_{x=1} + \left. T\right|_{x=1} = \gamma_2(t)\,. \tag{4.18}$$

Hence we have $\alpha_1 = -\mathrm{Bi}_1^{-1}$, $\beta_1 = 1$, $\alpha_2 = -\mathrm{Bi}_2^{-1}$, and $\beta_2 = 1$. Then we can use the infinite systems of identities obtained in Par. 2.

Example 2. Assuming that $\mathrm{Bi}_1 = 2$, $T_1(t) = t/2$, $\mathrm{Bi}_2 = 1.5$, and $T_2(t) = t^3/3$, we obtain $\alpha_1 = -1/2$, $\beta_1 = 1$, $\gamma_1 = t/2$, $\alpha_2 = -2/3$, $\beta_2 = 1$, and $\gamma_2 = t^2/3$. The temperature field in the plate is defined by the polynomial

$$T(x, t) = \sum_{j=0}^{5} a_j(t)\, x^j\,. \tag{4.19}$$

Two equations are obtained from the boundary conditions

$$2a_0(t) - a_1(t) = t\,, \quad \sum_{j=0}^{5} (2j + 3)\, a_j(t) = t^2\,. \tag{4.20}$$

The third equation is constructed on the basis of the equality $\varphi(t) = \partial T(1, t)/\partial x$, from which it follows that

$$\sum_{j=1}^{5} j a_j(t) = \varphi(t)\,. \tag{4.21}$$

The remaining three equations are obtained from the infinite system of identities (4.6):

$$\begin{gathered}
\int_0^1 T(x, t)\left(x + \frac{1}{2}\right) dx = \frac{13}{6}\,\mathcal{F}_1(t) + \frac{t^2}{4} - \frac{t^3}{9}\,, \\
\int_0^1 T(x, t)\left(\frac{x^2}{4} + \frac{x^3}{6}\right) dx = \frac{13}{12}\,\mathcal{F}_1(t) + \frac{13}{6}\,\mathcal{F}_2(t) - \frac{t^3}{36} - \frac{t^4}{36}\,, \\
\int_0^1 T(x, t)\left(\frac{x^4}{48} + \frac{x^5}{120}\right) dx = \frac{9}{80}\,\mathcal{F}_1(t) + \frac{13}{12}\,\mathcal{F}_2(t) + \frac{13}{6}\,\mathcal{F}_3(t) - \frac{t^3}{72} - \frac{t^4}{144} - \frac{t^5}{180}\,.
\end{gathered} \tag{4.22}$$

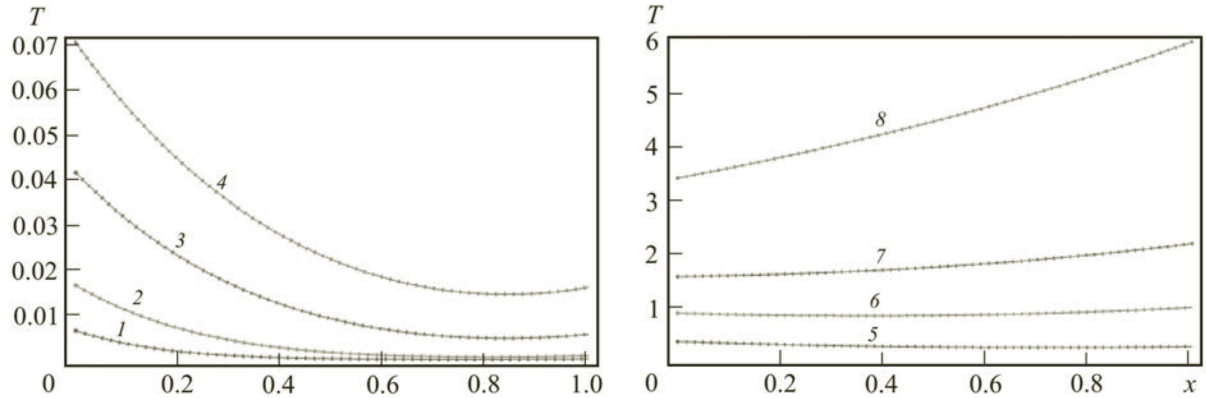


Fig. 2. Temperature distributions in the plate under the conditions of nonsymmetric convective heat exchange at $\gamma_1(t) = t/2$, $\mathrm{Bi}_1 = 2$, $\gamma_2(t) = t^2/3$, $\mathrm{Bi}_2 = 1.5$, and $t = 0.05$ (1), 0.1 (2), 0.2 (3), 0.3 (4), 1 (5), 2 (6), 3 (7), and 5 (8): full lines) exact solution; dotted lines) solution by the WTM.

Equations (4.20)–(4.22) form a closed system for $a_j(t)$, $j = \overline{1, 5}$. Then we find the function $\varphi(t)$, involved in $a_j(t)$, using the heat-balance integral (4.9). Substitution of polynomial (4.19) into it gives the equation

$$78{,}854\varphi'(t) + 14{,}350{,}938\varphi(t) + 689{,}851{,}260\mathcal{F}_1(t) + 8{,}809{,}426{,}080\mathcal{F}_2(t) + 18{,}810{,}792{,}000\mathcal{F}_3(t)$$
$$= 114{,}296t + 5{,}017{,}890t^2 + 113{,}511{,}720t^3 + 52{,}650{,}360t^4 + 48{,}232{,}800t^5 - 91 \,. \tag{4.23}$$

As noted above, a possible variant of solving this equation is introduction, into consideration, of the function $v(t) = \mathcal{F}_3(t)$ transforming Eq. (4.23) into the third-order ordinary differential equation. Hence we write

$$78{,}854v^{(4)}(t) + 14{,}350{,}938v^{(3)}(t) + 689{,}851{,}260v''(t) + 8{,}809{,}426{,}080v'(t) + 18{,}810{,}792{,}000v(t)$$
$$= 114{,}296t + 5{,}017{,}890t^2 + 113{,}511{,}720t^3 + 52{,}650{,}360t^4 + 48{,}232{,}800t^5 - 91 \,. \tag{4.24}$$

Solution of Eq. (4.24) with the zero boundary conditions $v(0) = v'(0) = v''(0) = v^{(3)}(0) = 0$ and subsequent threefold differentiation of the function $v(t)$ give

$$\varphi(t) = 0.0609467 - 0.0769231t + 0.153846t^2 - 0.0691\mathrm{e}^{-2.65865}$$
$$+ 0.00907\mathrm{e}^{-15.7581t} - 0.001262\mathrm{e}^{-50.240t} + 0.0003057\mathrm{e}^{-113.3377t} \,. \tag{4.25}$$

Figure 2 shows the kinetics of change in the temperature field within the plate with nonsymmetric conditions of heat exchange. It is seen from this figure that the approximate and exact solutions are in good agreement.

**5. Examples of Solving the Problem with Mixed Boundary Conditions.** Problems on nonstationary heat conduction with mixed boundary conditions occupy an important place in the analytical theory of heat conduction [10–13]. As an example of approximate solving such problems on the basis of the weighted temperature method, we will solve the problem on the nonstationary heat conduction in a lengthy plate in the case where one of the surfaces of the plate exchanges heat with the environment by the Newton law (the third-kind boundary condition) and its other surface has a constant temperature (the first-kind boundary condition) or is subjected to the action of a heat flow (the second-kind boundary condition). To obtain approximate solutions, we will use the general algorithm described in Par. 3 and represent the boundary conditions (1.5) at $\alpha_i \neq 0$ and $\beta_i \neq 0$ in the form

$$\frac{\partial T(0, t)}{\partial x} = \frac{\beta_1}{\alpha_1}\left(\frac{\gamma_1(t)}{\beta_1} - T(0, t)\right), \quad \frac{\partial T(1, t)}{\partial x} = \frac{\beta_2}{\alpha_2}\left(\frac{\gamma_2(t)}{\beta_2} - T(1, t)\right). \tag{5.1}$$

Then, passing to the traditional third-kind boundary conditions, we obtain $\beta_1/\alpha_1 = \mathrm{Bi}_1$ ($\alpha_1 < 0$), $\beta_2/\alpha_2 = \mathrm{Bi}_2$, $\gamma_1(t)/\beta_1 = T_1(t)$, and $\gamma_2(t)/\beta_2 = T_2(t)$, where $T_1(t)$ and $T_2(t)$ are the temperatures of the media on the side of the surfaces $x = 0$ and $x = 1$, respectively. On the basis of identities (4.6), we write the first three equations for the boundary conditions (5.1):

$$\int_0^1 T(x,t)\left(\frac{\alpha_1}{\beta_1} - x\right)dx \equiv \frac{\Upsilon_1^{(2)}(t)}{\beta_2} - \frac{\Upsilon_1^{(1)}(t)}{\beta_1} + \left(\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} - 1\right)\mathcal{F}_1(t)\,,$$

$$\int_0^1 T(x,t)\left(\frac{\alpha_1}{\beta_1} - \frac{x}{3}\right)\frac{x^2}{2}\,dx \equiv \left(\frac{1}{2} - \frac{\alpha_1}{\beta_1}\right)\frac{\Upsilon_1^{(2)}(t)}{\beta_2} + \frac{\Upsilon_2^{(2)}(t)}{\beta_2} - \frac{\Upsilon_2^{(1)}(t)}{\beta_1}$$

$$+\left(\frac{\alpha_1\alpha_2}{\beta_1\beta_2} + \frac{\alpha_1}{2\beta_1} - \frac{\alpha_2}{2\beta_2} - \frac{1}{6}\right)\mathcal{F}_1(t) + \left(\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} - 1\right)\mathcal{F}_2(t)\,, \tag{5.2}$$

$$\int_0^1 T(x,t)\left(\frac{\alpha_1}{\beta_1} - \frac{x}{5}\right)\frac{x^4}{24}\,dx \equiv \left(\frac{1}{4} - \frac{\alpha_1}{\beta_1}\right)\frac{\Upsilon_1^{(2)}(t)}{6\beta_2} + \left(\frac{1}{2} - \frac{\alpha_1}{\beta_1}\right)\frac{\Upsilon_2^{(2)}(t)}{\beta_2} + \frac{\Upsilon_3^{(2)}(t)}{\beta_2} - \frac{\Upsilon_3^{(1)}(t)}{\beta_1}$$

$$+\frac{1}{6}\left(\frac{\alpha_1\alpha_2}{\beta_1\beta_2} + \frac{\alpha_1}{4\beta_1} - \frac{\alpha_2}{4\beta_2} - \frac{1}{20}\right)\mathcal{F}_1(t) + \left(\frac{\alpha_1\alpha_2}{\beta_1\beta_2} + \frac{\alpha_1}{2\beta_1} - \frac{\alpha_2}{2\beta_2} - \frac{1}{6}\right)\mathcal{F}_2(t) + \left(\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} - 1\right)\mathcal{F}_3(t)\,,$$

where $\Upsilon_n^{(1)}(t) = \underbrace{\int_0^t dt \ldots \int_0^t}_{n} \gamma_1(t)\,dt$, $\Upsilon_n^{(2)}(t) = \underbrace{\int_0^t dt \ldots \int_0^t}_{n} \gamma_2(t)\,dt$, $\mathcal{F}_n(t) = \underbrace{\int_0^t dt \ldots \int_0^t}_{n} \varphi(t)\,dt$ ($n = 1, 2, \ldots$). Then we direct our attention to concrete problems and their solution on the basis of the weighted temperature method.

Example 3. Let us set the boundary conditions

$$T(0,t) = 1\,, \quad \frac{\partial T(1,t)}{\partial x} + \mathrm{Bi}\,T(1,t) = 0\,. \tag{5.3}$$

The exact solution of the problem has the form [15]

$$T(x,t) = \frac{1 + \mathrm{Bi}\,(1-x)}{1+\mathrm{Bi}} - 2\sum_{n=1}^{\infty}\frac{(\mu_n^2 + \mathrm{Bi})\sin(\mu_n x)}{\mu_n(\mathrm{Bi} + \mathrm{Bi}^2 + \mu_n^2)}\exp(-\mu_n^2 t)\,, \tag{5.4}$$

where $\mu_n$ are roots of the characteristic equation $\mu_n \cot \mu_n + \mathrm{Bi} = 0$.

In accordance with the weighted function method and the algorithm presented in Par. 3, we define the desired temperature profile by the power polynomial ($N = 5$)

$$T(x,t) = 1 + \sum_{j=1}^{5} a_j(t)x^j\,. \tag{5.5}$$

For determining the polynomial coefficients $a_j(t)$, we have the second boundary condition (5.3), three Eqs. (5.2), and the relation $\partial T(1,t)/\partial x = \varphi(t)$. Let us assume that Bi = 1. In this case, in view of (5.1), the identical equalities (5.2) take the form

$$\int_0^1 T(x,t)x\,dx \equiv t + 2\mathcal{F}_1(t)\,,$$

$$\int_0^1 T(x,t)\frac{x^3}{6}\,dx \equiv \frac{t^2}{2} + \frac{2}{3}\mathcal{F}_1(t) + 2\mathcal{F}_2(t)\,, \tag{5.6}$$

$$\int_0^1 T(x,t)\frac{x^5}{120}\,dx \equiv \frac{t^3}{6} + \frac{1}{20}\mathcal{F}_1(t) + \frac{2}{3}\mathcal{F}_2(t) + 2\mathcal{F}_3(t)\,.$$

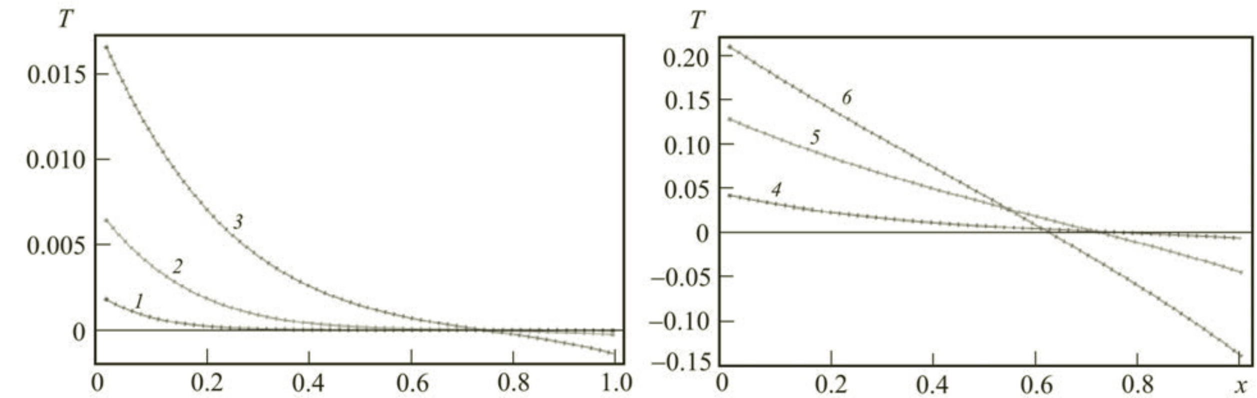


Fig. 3. Temperature distributions in the plate under the conditions of nonsymmetric convective heat exchange at $Bi_1 = 2$, $\gamma_1(t) = t/2$, $Bi_2 = 3/2$, and $\gamma_2(t) = 5/3t^2$ at the instant of time $t = 0.02$ (1), 0.05 (2), 0.1 (3), 0.2 (4), 0.5 (5), and 0.8 (6): full lines) exact solution; dotted lines) solution by the WTM.

Substitution of (5.5) into (5.6) leads to the system of linear algebraic equations ($\mathbf{A}a_j = \mathbf{B}$)

$$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{30} & \frac{1}{36} & \frac{1}{42} & \frac{1}{48} & \frac{1}{54} \\ \frac{1}{840} & \frac{1}{960} & \frac{1}{1080} & \frac{1}{1200} & \frac{1}{1320} \end{pmatrix} \begin{pmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \\ a_5(t) \end{pmatrix} = \begin{pmatrix} -1 \\ \varphi(t) \\ \frac{t-1}{2} + 2\mathcal{F}_1(t) \\ \frac{t^2}{2} - \frac{1}{24} + \frac{2}{3}\mathcal{F}_1(t) + 2\mathcal{F}_2(t) \\ \frac{t^3}{6} - \frac{1}{720} + \frac{1}{20}\mathcal{F}_1(t) + \frac{2}{3}\mathcal{F}_2(t) + 2\mathcal{F}_3(t) \end{pmatrix}. \quad (5.7)$$

The solution of (5.7) has the form $a = \mathbf{A}^{-1}\mathbf{B}$. Substituting the coefficients $a_j(t)$ ($j = \overline{1, 5}$) determined from (5.7) into (4.19) and then into the heat-balance integral (4.9), we obtain the integro-differential equation

$$\begin{aligned} &29\varphi'(t) + 9528\varphi(t) + 752{,}220\mathcal{F}_1(t) + 15{,}059{,}520\mathcal{F}_2(t) + 49{,}896{,}000\mathcal{F}_3(t) \\ &= 210 - 14{,}490t + 393{,}120t^2 - 4{,}158{,}000t^3 . \end{aligned} \quad (5.8)$$

Introduction of the function $v = \mathcal{F}_3(t)$ into (5.8) transforms this equation into the fourth-order ordinary differential equation:

$$\begin{aligned} &29\varphi^{(4)}(t) + 9528v^{(3)}(t) + 752{,}220v''(t) + 15{,}059{,}520v'(t) + 49{,}896{,}000v(t) \\ &= 210 - 14{,}490t + 393{,}120t^2 - 4{,}158{,}000t^3 \end{aligned} \quad (5.9)$$

with the initial conditions $v(0) = v'(0) = v''(0) = v^{(3)}(0) = 0$. Solution of (5.9) with subsequent threefold differentiation of the function $v$ gives the solution

$$\varphi(t) = 0.7396e^{-4.11585t} - 0.3696e^{-24.056t} + 0.2103e^{-78.22t} - 0.08032e^{-222.16t} - 0.5 . \quad (5.10)$$

Substitution of (5.10) into the polynomial coefficients involved in (5.5) gives the final solution of the problem (not presented for brevity). Results of solving the problem are presented in Fig. 3. Here, the temperature dependences corresponding to the exact solution (5.4) are also shown. It is seen from this figure that the approximate and exact temperature profiles are practically completely coincident.

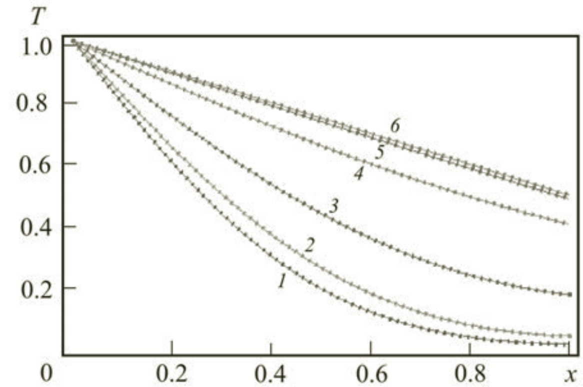


Fig. 4. Temperature distributions in the plate at the boundary conditions of the first kind ($x = 0$) and third kind at the instants of time $t = 0.075$ (1), 0.1 (2), 0.2 (3), 0.5 (4), 1 (5), and ∞ (6): full lines) exact solution; dotted lines) solution by the WTM.

Example 4. We now find a solution of the problem at the boundary conditions

$$-\frac{\partial T(0,t)}{\partial x} = 1\,, \quad \frac{\partial T(1,t)}{\partial x} + \mathrm{Bi}\,T(1,t) = 0\,. \tag{5.11}$$

The exact solution has the form [16]

$$T(x,t) = 1 - x + \frac{1}{\mathrm{Bi}} - 2\sum_{n=1}^{\infty} \frac{(\mu_n^2 + \mathrm{Bi}^2)\cos(\mu_n x)}{\mu_n^2(\mathrm{Bi} + \mathrm{Bi}^2 + \mu_n^2)} \exp(-\mu_n^2 t)\,, \tag{5.12}$$

where $\mu_n$ are roots of the equation $\mu \tan \mu = \mathrm{Bi}$.

The desired approximate solution is represented in the form of the polynomial

$$T(x,t) = \sum_{j=0}^{5} a_j(t) x^j\,. \tag{5.13}$$

In accordance with (5.2), for the problem being considered we have the system of identities

$$\int_0^1 T(x,t)\,dx \equiv t + \mathcal{F}_1(t)\,, \quad \int_0^1 T(x,t)\frac{x^2}{2}\,dx \equiv \frac{t^2}{2} + \frac{3}{2}\mathcal{F}_1(t) + \mathcal{F}_2(t)\,,$$
$$\int_0^1 T(x,t)\frac{x^4}{24}\,dx \equiv \frac{t^3}{6} + \frac{5}{24}\mathcal{F}_1(t) + \frac{3}{2}\mathcal{F}_2(t) + \mathcal{F}_3(t)\,. \tag{5.14}$$

On the basis of the boundary conditions (5.11), the condition $\partial T(1,t)/\partial x = \varphi(t)$, and the three equations (5.14), we obtain the system of algebraic equations in matrix form

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{8} & \frac{1}{10} & \frac{1}{12} & \frac{1}{14} & \frac{1}{16} \\ \frac{1}{120} & \frac{1}{144} & \frac{1}{168} & \frac{1}{192} & \frac{1}{216} & \frac{1}{240} \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} a_0(t) \\ a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \\ a_5(t) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ t + \mathcal{F}_1(t) \\ \frac{t^2}{2} + \frac{3}{2}\mathcal{F}_1(t) + \mathcal{F}_2(t) \\ \frac{t^3}{6} + \frac{5}{24}\mathcal{F}_1(t) + \frac{3}{2}\mathcal{F}_2(t) + \mathcal{F}_3(t) \\ \varphi(t) \end{pmatrix}. \tag{5.15}$$

Determining the polynomial coefficients from (5.15) and using the heat-balance integral (4.9), we obtain the equation

$$\frac{13}{5}\,\varphi'(t)+\frac{845}{2}\,\varphi(t)+17{,}073\mathcal{F}_1(t)+155{,}400\mathcal{F}_2(t)+105{,}840\mathcal{F}_3(t)\;=\;1-63t+1680t^2-17{,}640t^3\;. \tag{5.16}$$

Then we will perform the substitution $v(t) = \mathcal{F}_3(t)$, which transforms Eq. (5.16) into an ordinary differential equation of fourth order:

$$\frac{13}{5}\,v^{(4)}(t)+\frac{845}{2}\,v^{(3)}(t)+17{,}073v''(t)+155{,}400v'(t)+105{,}840v(t)\;=\;1-63t+1680t^2-17{,}640t^3\;. \tag{5.17}$$

For the initial conditions, we write the obvious equalities $v(0) = v'(0) = v''(0) = v^{(3)}(0) = 0$. Solution of the Cauchy problem with subsequent threefold differentiation give the boundary function

$$\varphi(t)\;=\;1.1191e^{-0.74017t}\;-\;0.1514e^{-11.7364t}\;+\;0.0463e^{-44.3412t}\;-\;0.0141e^{-105.682t}\;-1\;. \tag{5.18}$$

Determination of the boundary function $\varphi(t)$ completes the solution of the problem. The complete solution of the problem is not given for brevity. Figure 4 presents the exact temperature profile and the approximate temperature profiles calculated by (5.13) and (5.18) for different instants of time. It is seen from this figure that the exact and approximate solutions are in very good agreement.

**Conclusions.** A radically new scheme of approximate solving boundary-value problems on heat conduction has been developed. In accordance with the approach proposed, the heat-conduction equation is transformed into the infinite system of identities involving the integral operators for the temperature function, initial and boundary conditions, and internal heat sources as well as an additional boundary function in the form of the temperature or the heat flow at a boundary point of the computational region. The solution of the problem on the basis of the weighted temperature method is represented in the form of power polynomials, and it is maximally adopted for engineering calculations. It was shown by the example of solving a number of problems on nonstationary heat conduction with nonsymmetrical and mixed boundary condition that this method is highly efficient, highly accurate, and, at the same time, simple enough.

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