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ON ONE METHOD OF SOLVING NONSTATIONARY BOUNDARY-VALUE PROBLEMS

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An exact analytical solution of the problem on the heat conduction in an infinite plate with the first-kind symmetric boundary conditions has been obtained using the integral method of heat balance with an additional desired function and additional boundary conditions. The solution of the partial differential equation was reduced to the integration of the ordinary differential equation for the additional desired function. It is shown that the fulfillment of the differential equation at the boundary points of the computational region is equivalent to its fulfillment within this region. In the approach proposed there is no need to integrate the indicated equation with respect to the space variable because of the fulfillment of the integral condition of heat balance, which allows this approach to be applied to the solution of problems that are difficult to solve with the use of classical exact analytical methods.

Keywords: nonstationary heat conduction, integral method of heat balance, additional desired function, additional boundary conditions.

Introduction. The advantage of the methods based on the construction of the heat-balance integral is that they make it possible to obtain approximate analytical solutions simple in form. The main limitation of these methods is their low accuracy [1–12]. In them, the velocity of propagation of heat is assumed to be finite despite the fact that they are realized with the use of the parabolic heat-conduction equation suggesting that this velocity is infinite. In this case, the heat-conduction process in a plate is divided into two time stages, at the first of which the front of a temperature disturbance moves from the surface of the plate to its center. Here, the depth of the heated layer is taken as an additional desired function. At the second stage, the temperature of the plate changes across the whole its width, and an additional temperature function characterizing the change in the temperature at the center of the plate with time is introduced into consideration. Investigations of solutions of the boundary-value problems on heat conduction in a plate [1, 10–12] has shown that, with increase in the number of approximations n, the calculated time Fo_1 of movement of a temperature-disturbance front to the center of the plate at the first stage of the heat-conduction process decreases and, in the limit, tends to zero: $\mathrm{Fo}_1 \to 0$ at $n \to \infty$. Consequently, the solution of such a problem approximately defines the process of heat transfer with an infinite velocity. In this way, the contradiction associated with the supposition that the velocity of heat propagation is finite is cleared. As the number of approximations increases, at the first stage of the heat-conduction process, the time range in which the velocity of heat propagation is determined decreases, and this time increases at the second stage. Therefore, in the case where a large number of approximations is used in solving the problem on the heat conduction in a plate, only the temperatures for small values of the time and space variables can be determined from the solutions for the first stage. In this case, the role of the second stage in determining the temperature state of the plate increases.

In the present work, an approach is proposed to the analytical solution of problems on nonstationary heat conduction, which allows one to avoid the consideration of the first stage of the heat-conduction process in the integral method of heat balance. In accordance with this approach, the solution of the problem on nonstationary heat conduction in a plate is represented in the form of a trigonometric (algebraic) polynomial whose unknown coefficients are determined using an additional desired function prescribed at the center of the plate and additional boundary conditions, the fulfillment of which by the desired solution is equivalent to the fulfillment of the differential equation at the boundary points of the computational region. Note that the methods of solving heat-conduction problems, based on the fulfillment of heat-conduction equation at the boundary points, were used in works [13–15].

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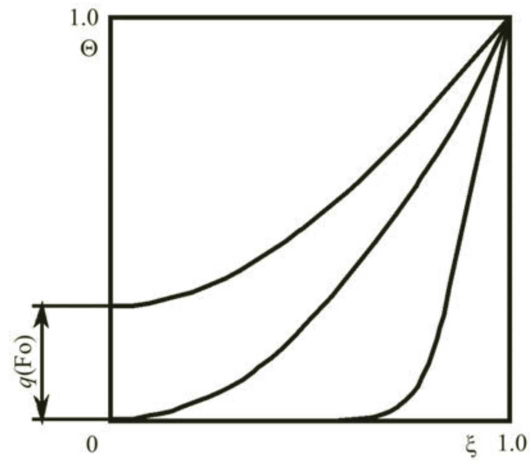

Fig. 1. Scheme of heat exchange in the plate.

Mathematical Formulation of the Problem and Method of Its Solution. We consider the main provisions of the method proposed by the example of solving the boundary-value problem on heat conduction in an infinite plate with the symmetric first-kind boundary conditions (Fig. 1)

$$\frac{\partial \Theta(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{\partial^2 \Theta(\xi, \mathrm{Fo})}{\partial \xi^2}\,, \quad \mathrm{Fo} > 0\,, \quad 0 < \xi < 1\,, \tag{1}$$

$$\Theta(\xi, 0) = 0\,, \tag{2}$$

$$\frac{\partial \Theta(0, \mathrm{Fo})}{\partial \xi} = 0\,, \tag{3}$$

$$\Theta(1, \mathrm{Fo}) = 1\,. \tag{4}$$

Let us introduce the desired function

$$q(\mathrm{Fo}) = \Theta(0, \mathrm{Fo})\,, \tag{5}$$

defining the change in the temperature at the center of the plate with time. The temperature at the point $\xi = 0$ begins to change as soon as the first-kind boundary conditions are applied to the surface of the plate because the velocity of heat propagation is infinite. Since the temperature at the center of the plate is a desired quantity of problem (1)–(4), the introduction of the function $q(\mathrm{Fo})$ does not change this problem and only makes the process of obtaining its solution simplier. The solution of problem (1)–(5) is sought in the form

$$\Theta(\xi, \mathrm{Fo}) = 1 + \sum_{k=1}^{n} b_k(q)\varphi_k(\xi)\,, \quad k = \overline{1, n}\,, \tag{6}$$

where $b_k(q)$ ($k = \overline{1, n}$) are unknown coefficients and $\varphi_k(\xi) = \cos(r\pi\xi/2)$ ($r = 2k - 1$) are coordinate functions. Relation (6) satisfies the boundary conditions (3) and (4). The coefficients $b_k(q)$ ($k = \overline{1, n}$) are determined from relation (5) with the use of additional boundary conditions set at the points $\xi = 0$ and $\xi = 1$.

To obtain additional boundary conditions as applied to the point $\xi = 0$, we differentiate relation (5) and the boundary condition (3) with respect to the variable Fo:

$$\frac{\partial \Theta(0, \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{dq(\mathrm{Fo})}{\partial \mathrm{Fo}}\,, \tag{7}$$

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \Theta(0,\ \mathrm{Fo})}{\partial \mathrm{Fo}}\right) = 0\ . \tag{8}$$

Comparing relation (7) with Eq. (1), we obtain the first additional boundary condition

$$\frac{\partial^2 \Theta(0,\ \mathrm{Fo})}{\partial \xi^2} = \frac{dq(\mathrm{Fo})}{d\mathrm{Fo}}\ . \tag{9}$$

In view of Eq. (1), relation (8) is brought to the form of the additional boundary condition

$$\partial^3 \Theta(0,\ \mathrm{Fo})/\partial \xi^3 = 0\ . \tag{10}$$

Differentiating relations (9) and (10) with respect to the variable Fo and using Eq. (1), we find the additional boundary conditions

$$\frac{\partial^4 \Theta(0,\ \mathrm{Fo})}{\partial \xi^4} = \frac{d^2 q(\mathrm{Fo})}{d\mathrm{Fo}^2}\ , \tag{11}$$

$$\frac{\partial^5 \Theta(0,\ \mathrm{Fo})}{\partial \xi^5} = 0\ . \tag{12}$$

Differentiating relations (11) and (12) with respect to the variable Fo and using Eq. (1), we obtain the additional boundary conditions

$$\frac{\partial^6 \Theta(0,\ \mathrm{Fo})}{\partial \xi^6} = \frac{\partial^3 q(\mathrm{Fo})}{d\mathrm{Fo}^3}\ , \tag{13}$$

$$\frac{\partial^7 \Theta(0,\ \mathrm{Fo})}{\partial \xi^7} = 0\ . \tag{14}$$

Analysis of relations (9)–(14) shows that the following general formulas can be written for the additional boundary conditions at the point $\xi = 0$:

$$\partial^{2i} \Theta(0,\ \mathrm{Fo})/\partial \xi^{2i} = \partial^i q(\mathrm{Fo})/\partial \mathrm{Fo}^i\ , \quad i = 1,\ 2,\ 3,\ \dots\ , \tag{15}$$

$$\partial^i \Theta(0,\ \mathrm{Fo})/\partial \xi^i = 0\ , \quad i = 3,\ 5,\ 7,\ \dots\ . \tag{16}$$

Because of the use of the trigonometric coordinate functions in solution (6), this solution satisfies conditions (16), which is equivalent to the fulfillment of Eq. (1) at the point $\xi = 0$ (via the equality of its left and right sides to zero in the limiting case).

Now we find additional boundary conditions as applied to the point $\xi = 1$. Differentiating the boundary condition (4) with respect to the variable Fo and comparing the relation obtained with Eq. (1), we obtain the first additional boundary condition

$$\partial^2 \Theta(1,\ \mathrm{Fo})/\partial \xi^2 = 0\ . \tag{17}$$

Differentiation of relation (17) with respect to the variable Fo gives, in view of Eq. (1), the second additional boundary condition

$$\partial^4 \Theta(1,\ \mathrm{Fo})/\partial \xi^4 = 0\ . \tag{18}$$

In a similar way, differentiating the previous additional boundary condition with respect to the variable Fo and using Eq. (1), one can find any number of additional boundary conditions at the point $\xi = 1$. The general formula for them has the form

$$\partial^i \Theta(1,\ \mathrm{Fo})/\partial \xi^i = 0\ , \quad i = 2,\ 4,\ 6,\ \dots\ . \tag{19}$$

Note that, in view of the accepted system of coordinate functions, solution (6) satisfies conditions (19) in any approximation, which is equivalent to the fulfillment of Eq. (1) at the point $\xi = 1$. Thus, of all the additional boundary conditions of the form (15), (16), and (19), only conditions (15) remain unfulfilled by solution (6). In what follows conditions (15) in combination with condition (5) will be used for determining the unknown coefficients $b_k(\text{Fo})$ ($k = \overline{1, n}$) in relation (6). The direct fulfillment of the additional boundary conditions, obtained by formulas (16) and (19), in solution (6) is due to the linearity of the differential equation (1) and, as a consequence, the simplicity of the additional boundary conditions obtained on its basis. For any other more complex boundary-value problems, e.g., problems with nonlinear differential operators, the additional boundary conditions will be more complex, and, in the cases where these conditions are not fulfilled by the desired solution *a priori*, they should be fulfilled through the determination of the unknown coefficients of the solution.

To obtain the solution of problem (1)–(4) in the first approximation, we substitute one term of series (6) into (5) and, as a result, obtain an algebraic linear equation for the unknown coefficient $b_1(q)$. Solution of this equation gives $b_1(q) = q(\text{Fo}) - 1$. Relation (6) with the determined coefficient $b_1(q)$ takes the form

$$\Theta(\xi, \text{Fo}) = 1 + (q(\text{Fo}) - 1) \cos (\pi\xi/2) . \tag{20}$$

To determine the unknown function $q(\text{Fo})$ involved in relation (20), we require the fulfillment of the heat-balance integral (Eq. (1) averaged over the thickness of the plate) of the form

$$\int_0^1 \frac{\partial\Theta(\xi, \text{Fo})}{\partial \text{Fo}} \, d\xi = \int_0^1 \frac{\partial^2\Theta(\xi, \text{Fo})}{\partial\xi^2} \, d\xi . \tag{21}$$

Substituting (20) into (21) and integrating the expression obtained, we find

$$dq(\text{Fo})/d\text{Fo} = -\pi^2 (q(\text{Fo}) - 1)/4 . \tag{22}$$

Integration of Eq. (22) gives

$$q(\text{Fo}) = 1 + C_1 \exp (-\pi^2 \text{Fo}/4) , \tag{23}$$

where C_1 is an integration constant determined from the initial condition $q(0) = 0$. From (23) we find $C_1 = -1$.

Substituting (23) into (20) and using the determined value of the integration constant, we obtain the solution of problem (1)–(4) in the first approximation

$$\Theta(\xi, \text{Fo}) = 1 - \exp (-\pi^2 \text{Fo}/4) \cos (\pi\xi/2) . \tag{24}$$

Relation (24) exactly satisfies the boundary conditions (3) and (4) and the heat-balance integral (21). Note that, due to the use of the additional boundary conditions, the solution of the problem obtained even in the first approximation exactly satisfies the initial differential equation (1). In this case, only the initial condition (2) is fulfilled approximately (in the first approximation). Consequently, by increasing the number of terms in series (6) one can refine the fulfillment of the initial condition of the boundary-value problem. In the subsequent approximations, the unknown coefficients of the solution will be determined from the condition (5) and the additional boundary conditions (15). In particular, in the second approximation, substituting two terms of (6) into (5) and (15) at $i = 1$, we obtain a system of two linear algebraic equations for the coefficients $b_1(q)$ and $b_2(q)$. On determination of these coefficients, relation (6) takes the form

$$\Theta(\xi, \text{Fo}) = 1 + [(4q' + 9\pi^2 (q - 1)) \cos (\pi\xi/2) + (4q' + \pi^2 (q - 1)) \cos (3\pi\xi/2)]/(8\pi^2) , \tag{25}$$

where $q = q(\text{Fo})$ and $q' = dq(\text{Fo})/d\text{Fo}$. Substitution of (25) into (21) gives

$$\frac{4}{3\pi^3} q'' + \frac{10}{3\pi} q' + \frac{3\pi}{4} q - \frac{3\pi}{4} = 0 , \tag{26}$$

where $q'' = d^2q(\text{Fo})/d\text{Fo}^2$. The initial condition for Eq. (26) has the form

$$q(0) = 0 . \tag{27}$$

Integration of Eq. (26) gives

$$q(\mathrm{Fo}) = -C_1 \exp(-\pi^2\mathrm{Fo}/4) - C_2 \exp(-9\pi^2\mathrm{Fo}/4) + 1 , \tag{28}$$

where C_1 and C_2 are integration constants determined from the initial condition (27). Substituting (28) into (27), we find

$$C_1 + C_2 = 1 . \tag{29}$$

Since the coefficients C_1 and C_2 have not yet been determined, we represent relation (29) as the expansion of unity in a Fourier series in terms of cosines in the range $0 \le \xi \le 1$:

$$C_1 \cos(\pi\xi/2) + C_2 \cos(3\pi\xi/2) = 1 . \tag{30}$$

Relation (30), as well as relation (29), represents a residual of the initial condition (27). To find the expansion coefficients C_1 and C_2, we require that the residual be orthogonal to the coordinate functions $\cos(j\pi\xi/2)$ ($j = 1, 3$):

$$\int_0^1 [C_1 \cos(\pi\xi/2) + C_2 \cos(3\pi\xi/2)] \cos(j\pi\xi/2) d\xi = \cos(j\pi\xi/2) , \quad j = 1, 3 . \tag{31}$$

Because of the orthogonality of the cosines in (31), this relation is brought to a system of two linear algebraic equations in which the unknown coefficients C_1 and C_2 are separated (each equation involves only one of these coefficients). Solving these equations, we find $C_1 = 4/\pi$ and $C_2 = -4/3\pi$. On substitution of (28) into (25), the solution of problem (1)–(4) in the second approximation with the determined integration constants C_1 and C_2 takes the form

$$\Theta(\xi, \mathrm{Fo}) = 1 + \sum_{k=1}^{2} C_k \exp(\nu_k \mathrm{Fo}) \cos(r\pi\xi/2) , \tag{32}$$

where $C_k = 4(-1)^{k+1}/(r\pi)$, $\nu_k = r^2\pi^2/4$, $r = 2k - 1$, and $k = 1, 2$. Solution (32) exactly satisfies Eq. (1) and the boundary conditions (3)–(4), and it approximately satisfies (in the second approximation) the initial condition (2). Analysis of the results of calculations by formula (2) allows the conclusion that the discrepancy between the solution obtained and the exact analytical solution of the problem (1)–(4) [16, 17] in the range of change in the Fourier number $0.1 \le \mathrm{Fo} < \infty$ decreased from 8% (in the first approximation) to 3% (in the second approximation).

Before proceeding to finding the solutions of the problem in the subsequent approximations, we consider one more method of determining the integration constants C_1 and C_2. Substitution of (28) into (25) gives

$$\Theta(\xi, \mathrm{Fo}) = 1 - C_1 \exp(-\pi^2\mathrm{Fo}/4) \cos(\pi\xi/2) - C_2 \exp(-9\pi^2\mathrm{Fo}/4) \cos(3\pi\xi/2) . \tag{33}$$

Constructing the residual of the initial condition (2) and requiring that it be orthogonal to the coordinate functions $\cos(\pi\xi/2)$ and $\cos(3\pi\xi/2)$, we obtain relation (31). Thus, determining the integration constants through the fulfillment of the initial conditions for the functions $q(0) = 0$ and $\Theta(\xi, 0) = 0$, we arrive at the identical results. The unknown coefficients of the third approximation $b_k(q)$ ($k = 1, 2, 3$) are obtained from relation (5) with two additional boundary conditions determined by the general formula (15) at $i = 1, 2$. In this case, we obtain the third-order ordinary differential equation for the unknown function $q(\mathrm{Fo})$

$$\frac{4}{15\pi^5} q''' + \frac{7}{3\pi^3} q'' + \frac{259}{60\pi} q' + \frac{15\pi}{16} q = \frac{15\pi}{16} . \tag{34}$$

Integration of this equation gives

$$q(\mathrm{Fo}) = -C_1 \exp\left(-\frac{\pi^2}{4}\mathrm{Fo}\right) - C_2 \exp\left(-\frac{9\pi^2}{4}\mathrm{Fo}\right) - C_3 \exp\left(-\frac{25\pi^2}{4}\mathrm{Fo}\right) + 1 , \tag{35}$$

where C_1, C_2, and C_3 are integration constants determined from the initial condition (27). Substituting (35) into (27), we obtain

$$C_1 + C_2 + C_3 = 1 . \tag{36}$$

Let us represent (36) in the form of the expansion of unity in a Fourier series in terms of cosines in the range $0 \le \xi \le 1$:

$$C_1 \cos(\pi\xi/2) + C_2 \cos(3\pi\xi/2) + C_3 \cos(5\pi\xi/2) = 1 \tag{37}$$

and require that the residual of the initial condition (37) be orthogonal to the coordinate functions $\cos(j\pi\xi/2)$ ($j = 1, 3, 5$):

$$\int_0^1 [C_1 \cos(\pi\xi/2) + C_2 \cos(3\pi\xi/2) + C_3 \cos(5\pi\xi/2) - 1] \cos(j\pi\xi/2)\, d\xi = \cos(j\pi\xi/2) , \quad j = 1, 3, 5 . \tag{38}$$

Because of the orthogonality of the cosines in (38), this relation is brought to the individual three algebraic equations for C_1, C_2, and C_3, from the solution of which we find $C_1 = 4/\pi$, $C_2 = -4/3\pi$, and $C_3 = 4/5\pi$. The coefficients C_k and the eigenvalues ν_k (the coefficients under the sign of the exponents in relation (35)) can be determined from the general formulas

$$C_k = 4(-1)^{k+1}/(r\pi) , \quad \nu_k = r^2\pi^2/4 , \tag{39}$$

where $r = 2k - 1$ and $k = 1, 2, 3$. In view of (35) and (39), relation (33) representing the solution of problem (1)–(4) in the third approximation takes the form

$$\Theta(\xi, \text{Fo}) = 1 + \sum_{k=1}^{n} C_k \exp(\nu_k \text{Fo}) \cos(r\pi\xi/2) , \quad r = 2k - 1 , \quad n = 3 . \tag{40}$$

Investigation of the solutions of the problem for the subsequent approximations allows the conclusion that they are defined by relation (40) at all the values of $k = \overline{1, \infty}$ in which the coefficients C_k and the eigenvalues ν_k determined by the general formulas (39) are identical to those determined by the exact formulas. Consequently, formula (40) at $n \to \infty$ is identical to the exact analytical solution of problem (1)–(4) [16].

Thus, using the additional boundary conditions, we have obtained the exact analytical solution of the boundary-value problem (1)–(4) on condition that Eq. (1) is fulfilled only at the boundaries of the computational region $\xi = 0$ and $\xi = 1$. The fulfillment of Eq. (1) averaged over the coordinate ξ (the integral of heat balance (21)) makes it possible to obviate its integration with respect to the spatial coordinate and reduce the solution of the problem to the integration of the ordinary differential equation only with respect of the time variable.

Application of the Method. As a concrete example of the application of the above-described method to the solution of more complex problems, we refer to the problem on the heat conduction in an infinite plate with variable physical properties (a heat-conduction coefficient changing by the exponential law depending on the space coordinate) in the following mathematical formulation:

$$c\rho \frac{\partial T(x, \tau)}{\partial \tau} = \frac{\partial}{\partial x}\left[\lambda(x) \frac{\partial T(x, \tau)}{\partial x}\right] , \quad \tau > 0 , \quad 0 < x < \delta , \tag{41}$$

$$T(x, 0) = T_0 , \quad \partial T(0, \tau)/\partial x = 0 , \quad T(\delta, \tau) = T_\text{w} , \tag{42}$$

where $\lambda(x) = \lambda_0 \exp(-mx)$, m is a coefficient, and λ_0 is the heat-conduction coefficient at $x = 0$. With the use of the accepted designations, problem (41)–(42) takes the form

$$\frac{\partial \Theta(\xi, \text{Fo})}{\partial \text{Fo}} = \frac{\partial}{\partial \xi}\left[e^{-\nu\xi} \frac{\partial \Theta(\xi, \text{Fo})}{\partial \xi}\right] , \quad \text{Fo} > 0 , \quad 0 < \xi < 1 , \tag{43}$$

$$\Theta(\xi, 0) = 0 , \tag{44}$$

$$\frac{\partial\Theta(0,\ \mathrm{Fo})}{\partial\xi} = 0\ , \tag{45}$$

$$\Theta(1,\ \mathrm{Fo}) = 1\ , \tag{46}$$

where $\nu = m\delta$.

As above, we introduce an additional desired function in the form of (5) into consideration. The solution of problem (43)–(46) with the use of (5) is sought in the form

$$\Theta(\xi,\ \mathrm{Fo}) = \sum_{k=0}^{n} b_k(q)\varphi_k(\xi)\ , \quad k = \overline{0,\ n}\ , \tag{47}$$

where $\varphi_k(\xi) = \xi^k$ ($k = \overline{0,\ n}$) are algebraic coordinate functions. Note that we use the algebraic functions because the additional boundary conditions are difficult to apply to the problem being considered and, therefore, cannot be fulfilled *a priori* through the use of trigonometric functions. Algebraic coordinate functions are more preferable in this case because they produce chain systems of algebraic linear equations for the unknown coefficients $b_k(q)$, which allows one to find their solutions with practically any number of approximations.

To obtain a solution of problem (43)–(46), we substitute three terms of series (47) into (5), (45), and (46). As a result, we will have three linear algebraic equations for the unknown coefficients $b_k(q)$ ($k = 0, 1, 2$). On determination of the coefficients $b_k(q)$, relation (47) takes the form

$$\Theta(\xi,\ \mathrm{Fo}) = q(\mathrm{Fo}) + (1 - q(\mathrm{Fo}))\xi^2\ . \tag{48}$$

The integral of heat balance for Eq. (43) is written in the form

$$\int_0^1 \frac{\partial\Theta(\xi,\ \mathrm{Fo})}{\partial \mathrm{Fo}} d\xi = \int_0^1 \frac{\partial}{\partial\xi}\left[e^{-\nu\xi}\ \frac{\partial\Theta(\xi,\ \mathrm{Fo})}{\partial\xi}\right] d\xi\ . \tag{49}$$

Substitution of (48) into (49) gives

$$q' - 3(1 - q(\mathrm{Fo}))\ \exp(-\nu) = 0\ , \tag{50}$$

where $q' = dq(\mathrm{Fo})/d\mathrm{Fo}$. Integrating Eq. (50), we obtain

$$q(\mathrm{Fo}) = 1 + C_1 e^{-3\mathrm{Fo}\ \exp(-\nu)}\ , \tag{51}$$

where C_1 is an integration constant. Substitution of (51) into (48) gives

$$\Theta(\xi,\ \mathrm{Fo}) = 1 + C_1 e^{-3\mathrm{Fo}\ \exp(-\nu)}\psi(\xi)\ , \tag{52}$$

where $\psi(\xi) = 1 - \xi^2$. To determine the integration constant, we construct the residual of the initial condition (44) and require the orthogonality of this residual to the function $\psi(\xi)$ representing, in essence, an eigenfunction:

$$\int_0^1 [1 + C_1(1 - \xi)^2]\ (1 - \xi^2) d\xi = 0\ . \tag{53}$$

Integrating (53), we find $C_1 = -3/2$. Relation (52) with the determined value of the integration constant C_1 represents the solution of problem (43)–(46) in the first approximation. This solution exactly satisfied the boundary conditions (45)–(46) and the heat-balance integral (49) (the averaged Eq. (43)), and it approximately satisfies (in the first approximation) Eq. (43) and the initial condition (44).

To increase the accuracy of the solution of the problem, it is necessary to increase the number of terms in series (47) whose unknown coefficients are determined from the main boundary conditions (45) and (46), relation (5), and certain additional boundary conditions by the above-described method. In particular, for solving problem (43)–(46) in the second approximation, these conditions have the form

$$2\nu \frac{\partial^2\Theta(0,\ \mathrm{Fo})}{\partial\xi^2} - \frac{\partial^3\Theta(0,\ \mathrm{Fo})}{\partial\xi^3} = 0\ , \tag{54}$$

$$\frac{\partial^2\Theta(0,\ \mathrm{Fo})}{\partial\xi^2} = \frac{dq(\mathrm{Fo})}{d\mathrm{Fo}}\ , \tag{55}$$

$$\frac{\partial^2\Theta(1,\ \mathrm{Fo})}{\partial\xi^2} - \nu\,\frac{\partial\Theta(1,\ \mathrm{Fo})}{\partial\xi} = 0\ . \tag{56}$$

Substituting six terms of series (47) into relations (5), (45), (46), and (54)–(56), we have a chain system of six algebraic linear equations for $b_k(q)$ ($k = 0, 1, 2, \ldots, 5$). On determination of the coefficients $b_k(q)$ ($k = 0, 1, 2, \ldots, 5$) in (47), this relation takes the form

$$\Theta(\xi,\ \mathrm{Fo}) = q + q'\xi^2/2 + \nu q'\xi^3/3 + (\mu_1(1-q) + \mu_2 q')\xi^4/\mu_3 + (\mu_4(1-q) + \mu_5 q')\xi^5/(\mu_3/2)\ , \tag{57}$$

where $\mu_1 = 30(\nu - 4)$, $\mu_2 = 54 + \nu(19 - 4\nu)$, $\mu_3 = 6(\nu - 8)$, $\mu_4 = 12(\nu - 3)$, and $\mu_5 = 15 + \nu(3 - \nu)$. Substituting (57) into the heat-balance integral (49) and integrating the relation obtained with respect to the unknown function $q(\mathrm{Fo})$, we obtain the second-order ordinary differential equation

$$r_1 q'' + r_2 q' + q - 1 = 0\ , \tag{58}$$

where $q' = dq(\mathrm{Fo})/d\mathrm{Fo}$, $q'' = d^2q(\mathrm{Fo})/d\mathrm{Fo}^2$, $r_1 = e^{\nu}(6\nu - \nu^2 + 66)/3600$, and $r_2 = -(2\nu e^{\nu} - 18e^{\nu} - 2\nu - 9)/60$. Integration of Eq. (58) gives

$$q(\mathrm{Fo}) = 1 + C_1 \exp(z_1\mathrm{Fo}) + C_2 \exp(z_2\mathrm{Fo})\ , \tag{59}$$

where $z_{1,2} = \left(-r_2 \pm \sqrt{r_2^2 - 4r_1}\right)/(2r_1)$, and C_1 and C_2 are integration constants. Substituting (59) into (57), we find

$$\Theta(\xi,\ \mathrm{Fo}) = 1 + C_1\psi_1(\xi)e^{z_1\mathrm{Fo}} + C_2\psi_2(\xi)e^{z_2\mathrm{Fo}}\ , \tag{60}$$

where

$$\psi_k(\xi) = 1 + R_{1,k}\xi^2 + R_{2,k}\xi^3 + R_{3,k}\xi^4 - R_{4,k}\xi^5\ ,$$

$$R_{1,k} = z_k/2\ , \quad R_{2,k} = z_k\nu/3\ , \quad R_{3,k} = (z_k\mu_2 - \mu_1)/\mu_3\ ,$$

$$R_{4,k} = 2(z_k\mu_5 - \mu_4)/\mu_3\ , \quad k = 1,\ 2\ .$$

Substituting (60) into the initial condition (44) and requiring the orthogonality of the residual obtained to the functions $\psi_1(\xi)$ and $\psi_2(\xi)$, we obtain a system of two algebraic linear equations for the integration constants C_1 and C_2:

$$\int_0^1 [(1 + C_1\psi_1(\xi) + C_2\psi_2(\xi)]\psi_j(\xi)d\xi = 0\ , \quad j = 1,\ 2\ . \tag{61}$$

Solving the system of equations (61), we find

$$C_k = \frac{a_1 z_k - a_2}{(z_1 - z_2)a_3} \quad (k = 1,\ 2)\ , \tag{62}$$

where

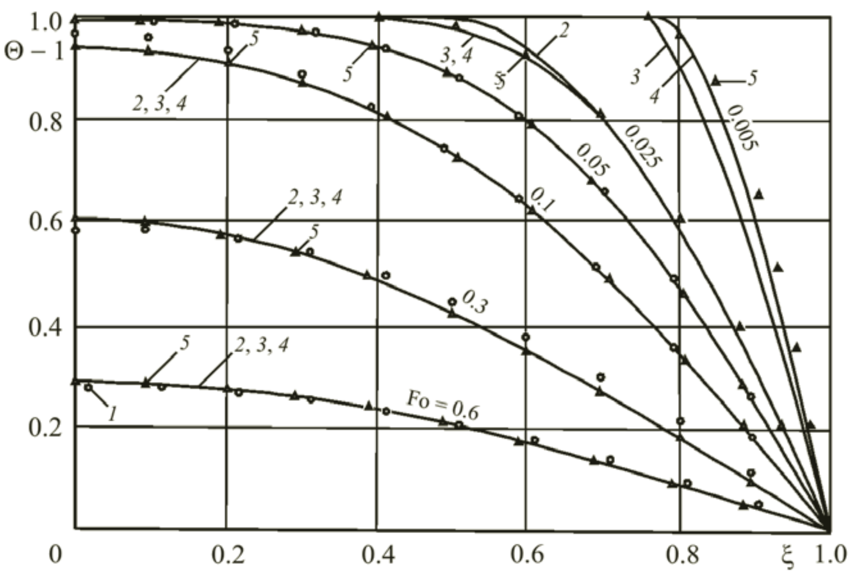

Fig. 2. Temperature distribution in the plate at ν = 0.01 calculated by formula (3.278) from [18] (1) and by formula (47) in the second (2), third (3), and fourth (4) approximations: 5) numerical solution.

$$a_1 = 618{,}084 + 157{,}848\nu - 8304\nu^2 - 2508\nu^3 + 179\nu^4 ,$$

$$a_2 = 4{,}254{,}480 - 44{,}640\nu - 178{,}200\nu^2 + 10{,}920\nu^3 ,$$

$$a_3 = (30{,}501{,}684 + 7{,}409{,}208\nu - 515{,}664\nu^2 - 121{,}548\nu^3 + 8869\nu^4)/42 .$$

If $\nu = 0$, relations (59) and (60) are brought to the form

$$q(\mathrm{Fo}) = 1 - 1.2572e^{-2.47097\mathrm{Fo}} + 0.4061e^{-22.0745\mathrm{Fo}} , \tag{63}$$

$$\Theta(\xi, \mathrm{Fo}) = 1 + (-1.2572 + 1.5533\xi^2 - 0.35183\xi^4 + 0.055767\xi^5)e^{-2.47097\mathrm{Fo}} + (0.40612 - 4.4823\xi^2 + 9.0703\xi^4 - 4.9941\xi^5)e^{-22.0745\mathrm{Fo}} . \tag{64}$$

Note that the coefficients under the sign of the exponents in relations (63) and (64) differ insignificantly from the exact eigenvalues $z_1 = 2.4674$ and $z_2 = 22.2066$ of the Sturm–Liouville problem. The coefficients C_k ($k = 1, 2$), with which the exponents of relations (63) are prefixed, differ insignificantly from the coefficients of the classical exact analytical solution, obtained from the fulfillment of the initial condition of the boundary-value problem, whose exact values are $C_1 = 1.2732$ and $C_2 = 0.4241$. Comparison of the results of calculations by formula (64) at $\nu = 0$ with the classical exact analytical solution of problem (43)–(46) [18] shows that, in the range of $0.1 \le \mathrm{Fo} < \infty$, the discrepancy between them does not exceed 1%. The solutions for other approximations can be found analogously. In finding these solutions, no significant difficulties appear, and only the volume of computational work increases because of the complexity of the problem being solved, which is, *per se*, nonlinear (it is characterized by the second-kind nonlinearity due to the dependence of the physical properties of the plate on the space variable).

The results of calculations of the temperature of the plate by formula (60) at $\nu = 0.01$, the results of calculations of this temperature in the first approximation in [18], and the results of analogous calculations by the method of finite differences are compared in Fig. 2. Analysis of these data allows the conclusion that, in the range $0.05 \le \mathrm{Fo} < \infty$, the results of calculations by formula (60) are practically identical to the results of calculations by the numerical method (the step method). In Fig. 2, the results of calculations of the temperature of the plate by formula (47) in the third ($k = \overline{0, 8}$) and fourth ($k = \overline{0, 11}$) approximations are also presented. Analysis of these data shows that the accuracy of solving the problem being considered with the use of additional boundary conditions determined by the above-described method increases substantially with increase in the number of terms in series (47), which is evidence of the convergence of this method.

Conclusions. An exact analytical solution of the problem on the heat conduction in an infinite plate with the symmetric first-kind boundary conditions has been obtained using the integral method of heat balance with an additional desired function and additional boundary conditions. The use of the additional desired function is caused by the need for taking into account the fact that the velocity of heat propagation is infinite in accordance with the parabolic heat-conduction equation. With this function, the solution of the partial differential equation can be reduced to the integration of the ordinary differential equation. Using the method proposed, we also obtained a high-accuracy approximate analytical solution of the problem on the heat conduction in an infinite plate with an exponential dependence of the heat-conduction coefficient on the space variable in the simple form of the product of the experimental time function and the exponential coordinate, dependent on only the space variable. It was shown that the accuracy of solving the problem substantially increases with increase in the number of approximations used, which is evidence of the convergence of the method proposed in the case where it is used for solving boundary-value problems defined by complex differential operators.

The construction of additional boundary conditions is based on the use of the differential equation and the main boundary conditions of the boundary-value problem. They are found in such a form that their fulfillment in relation to the desired solution is equivalent to the fulfillment of the differential equation at the boundary points of the computational region. It was shown that the fulfillment of this equation at the boundary points provides its fulfillment throughout the range of change in the time and space variables.

Since the process of obtaining an analytical solution of a boundary-value problem including a partial differential equation is reduced to the integration of the ordinary differential equation, the method proposed can be used for solving problems defined by equations with complex differential operators, e.g., nonlinear problems and problems with variable physical properties (which were solved in the present work), problems with time-dependent initial conditions and boundary conditions, problems with heat sources, and others.

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NOTATION

a, thermal diffusivity; c, heat capacity; $\mathrm{Fo} = a\tau/\delta^2$, Fourier number; T, temperature; $\Delta T = T_\mathrm{w} - T_0$; T_0, initial temperature; T_w, temperature of the wall at $x = \delta$; x, coordinate; δ, thickness of the plate; $\Theta = (T - T_0)/\Delta T$, dimensionless temperature; λ, heat-conduction coefficient; $\xi = x/\delta$, dimensionless coordinate; ρ, density; τ, time. Subscript: w, wall.

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