DOI 10.1007/s10891-016-1435-3

Journal of Engineering Physics and Thermophysics, Vol. 89, No. 3, May, 2016

DYNAMIC STABILITY OF A CYLINDRICAL SHELL REINFORCED BY LONGITUDINAL RIBS AND A HOLLOW CYLINDER UNDER THE ACTION OF AXIAL FORCES

V. N. Bakulin,[a] E. N. Volkov,[b] and A. Ya. Nedbai[b]

UDC 629.7.023:531.391.5

The dynamic stability of a cylindrical orthotropic shell reinforced by longitudinal ribs and a hollow cylinder under the action of axial forces changing harmonically with time was investigated with regard for the axial contact interaction of the shell with the ribs. A solution of the differential equations defining this process has been obtained in the form of trigonometric series in the angular and time coordinates. A two-term approximation of the Mathieu–Hill equations of motion was used for construction of the main region of instability of the shell. As a result, the problem was reduced to a system of algebraic equations for components of displacements of the shell at the locations of the ribs. The problem for uniformly spaced ribs was solved in the explicit form. A numerical example of this solution is presented.

Keywords: *dynamic instability, cylindrical orthotropic shell, longitudinal ribs, stability region, hollow cylinder.*

Introduction. At present, the provision of dynamic stability (parametric resonance) of members of flying vehicles is a pressing problem to designers of airplanes and rockets, which is explained by the wide use of composite materials in the load-bearing elements of such apparatus, the high flight speed of them, and the complex conditions of their use.

The most typical sources of vibrations of members of flying vehicles are acoustic effects determined by the aerodynamics of an aircraft, its buffet, and the atmospheric turbulence as well as the mechanical effects that are due to the pulsations of the trust of the aircraft and the work of its engines and pneumatic hydraulic units. Even though the level of the loads formed by these actions is most often small, they, at any geometrical parameters of an aircraft and properties of the materials of its members, can destroy the load-bearing elements of the aircraft.

A number of works [1–4] are devoted to the problems of the dynamic stability of cylindrical shells that are used in the majority of aircrafts as load-bearing elements. However, the class of shells made of composite materials, which are reinforced by stiffening ribs, has practically not been investigated [5–9].

Formulation of the Problem. The dynamic stability of a cylindrical orthotropic shell reinforced by longitudinal ribs and an elastic cylinder is investigated with regard for the axial forces of the contact interaction of the shell with the ribs. A two-term approximation of the Mathieu–Hill equations is used in the calculations for increasing their accuracy (by 7%). The problem for uniformly spaced ribs is solved in the explicit form.

The most typical situation, where the indicated shell is reinforced by ribs positioned symmetrically relative to the vertical diameter of the shell, is considered. It is assumed that, in the subcritical state, the axial deformations of the shell and of the ribs are equal. The ribs and the shell are subjected to the action of the external pressure and the axial compressive force that has a constant and a variable components. The calculations are carried out with regard for all the interactions between the shell and the ribs in the plane of the ribs. The tangential and inertial forces of the shell are not taken into account. The action of the cylinder on the shell is estimated by the bedding coefficient, which is determined from the three-dimensional elasticity theory.

Let us introduce the dimensionless system of cylindrical coordinates in which the linear sizes are related to the radius of the middle surface of the shell. In this case, the movement of the shell is defined by the equations [10]

[a]Institute of Applied Mechanics of the Russian Academy of Sciences, 7 Leningrad Ave., Moscow, 125040, Russia; email: vbak@yandex.ru; [b]Moscow Institute of Heat Engineering, 10 Birch Alley, Moscow, 127273, Russia; email: eugeenw1@mail.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 89, No. 3, pp. 742–747, May–June, 2016. Original article submitted March 23, 2015.

$$l_{11}u + l_{12}w = -a_{10}L_{22}\sum_{i=1}^{M}\left[\frac{\partial^2 u_i}{\partial\alpha^2} - a_{13}\frac{\partial^2 u_i}{\partial t^2} + a_{14}\frac{\partial^3 w_i}{\partial\alpha^3}\right]\delta(\beta-\beta_i)\,,$$

$$l_{21}u + l_{22}w = -a_{10}L_{22}\sum_{i=1}^{M}\left[a_{14}\frac{\partial^3 u_i}{\partial\alpha^3} + a_{12}\frac{\partial^4 w_i}{\partial\alpha^4} + a_{13}\frac{\partial^2 w_i}{\partial t^2}\right. \tag{1}$$

$$\left. + a_{11}\left(T_0 + T_1\cos\omega t\right)\frac{\partial^2 w_i}{\partial\alpha^2}\right]\delta(\beta-\beta_i)\,,$$

where

$$l_{11} = L_{11}L_{22} - L_{12}^2\,,\quad l_{12} = L_{13}L_{22} - L_{12}L_{23}\,,\quad l_{21} = l_{12}\,,\quad l_{22} = L_{22}L_{33} - L_{23}^2\,,$$

$$L_{11} = \frac{\partial^2}{\partial\alpha^2} + a_1\frac{\partial^2}{\partial\beta^2} - a_7\frac{\partial^2}{\partial t^2}\,,\quad L_{12} = a_2\frac{\partial^2}{\partial\alpha\partial\beta}\,,\quad L_{13} = \nu_\beta\frac{\partial}{\partial\alpha} + a_3\left[a_1\frac{\partial^3}{\partial\alpha\partial\beta^2} - \frac{\partial^3}{\partial\alpha^3}\right],$$

$$L_{22} = a_1\frac{\partial^2}{\partial\alpha^2} + a_4\frac{\partial^2}{\partial\beta^2}\,,\quad L_{23} = a_4\frac{\partial}{\partial\beta} - a_3a_5\frac{\partial^3}{\partial\alpha^2\partial\beta}\,,$$

$$L_{33} = a_4 + a_9 + a_3\left[\frac{\partial^4}{\partial\alpha^4} + 2a_6\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + a_4\left(\frac{\partial^2}{\partial\beta^2}+1\right)^2\right] + a_7\frac{\partial^2}{\partial t^2}$$

$$+ a_8(T_0 + T_1\cos\omega t)\frac{\partial^2}{\partial\alpha^2} + a_{15}\frac{\partial^2}{\partial\beta^2}\,,$$

$$a_1 = \frac{G_{\alpha\beta}(1-\nu_{\alpha\beta})}{E_\alpha}\,,\quad a_2 = a_1 + \nu_\beta\,,\quad a_3 = \frac{h^2}{12R^2}\,,\quad a_4 = \frac{E_\beta}{E_\alpha}\,,\quad a_5 = 3a_1 + \nu_\beta\,,$$

$$a_6 = 2a_1 + \nu_\beta\,,\quad a_7 = B\rho_0 h\,,\quad a_8 = \frac{a_{16}B}{(2\pi Ra_{16} + M)R^2}\,,\quad a_9 = BK_0\,,\quad a_{10} = \frac{BEF}{R^3}\,,$$

$$a_{11} = \frac{1}{(2\pi Ra_{16} + M)EF}\,,\quad a_{12} = \frac{I}{FR^2}\,,\quad a_{13} = \frac{\rho_1 R^2}{E}\,,\quad a_{14} = \frac{\xi}{R}\,,\quad a_{15} = \frac{Bp}{R}\,,\quad a_{16} = \frac{E_\alpha h}{EF}\,,$$

$$B = \frac{R^2(1-\nu_\alpha\nu_\beta)}{E_\alpha h}\,,\quad u_i = u(\alpha,\beta_i)\,,\quad w_i = w(\alpha,\beta_i)\,.$$

Analytical Solution. Equation (1) is solved in the following form:

$$u = \cos\gamma\alpha\sum_{n=0}^{\infty}\varphi_n(t)\cos n\beta\,,\quad w = \sin\gamma\alpha\sum_{n=0}^{\infty}\psi_n(t)\cos n\beta\,, \tag{2}$$

where $\gamma = \dfrac{m\pi}{\alpha_0}$, $\alpha_0 = \dfrac{L}{R}$.

Expanding the delta function in a trigonometric series and substituting (2) into (1), we arrive at a system of ordinary differential Mathieu–Hill equations for φ_n and ψ_n (the argument t will be omitted in what follows):

$$\left(b_3 \frac{d^2}{dt^2} + b_1\right) \varphi_n - b_2 \psi_n = -\frac{(2-\delta_{0n}) b_6}{2\pi} \sum_{i=1}^{M} \left[\left(a_{13} \frac{d^2}{dt^2} + \gamma^2\right) u_{mi} + a_{14}\gamma^3 w_{mi}\right] \cos n\beta_i \ ,$$

$$b_2 \varphi_n - \left[b_3 \frac{d^2}{dt^2} - b_4 (T_0 + T_1 \cos \omega t) + b_5\right] \psi_n = \frac{(2-\delta_{0n}) b_6}{2\pi} \sum_{i=1}^{M} \left\{ a_{14}\gamma^3 u_{mi} \right. \tag{3}$$

$$\left. + \left[a_{13} \frac{d^2}{dt^2} - a_{11}\gamma^2 (T_0 + T_1 \cos \omega t) + a_{12}\gamma^4\right] w_{mi} \right\} \cos n\beta_i \ ,$$

where $b_0 = a_1\gamma^2 + a_4 n^2$, $b_1 = (\gamma^2 + a_1 n^2) b_0 - a_2^2 \gamma^2 n^2$,

$$b_2 = \left[\nu_\beta \gamma + a_3(\gamma^3 - a_1 \gamma n^2)\right] b_0 + a_2 \gamma n^2 (a_4 + a_3 a_5 \gamma^2) \ , \quad b_3 = b_0 a_7 \ , \quad b_4 = b_0 a_8 \gamma^2 \ ,$$

$$b_5 = \left\{a_4 + a_9 + a_3 \left[\gamma^4 + 2a_6 \gamma^2 n^2 + a_4 (n^2 - 1)^2\right] - a_{15} n^2\right\} b_0 + (a_4 + a_3 a_5 \gamma^2)^2 n^2 \ ,$$

$$b_6 = b_0 a_{10} \ .$$

The bedding coefficient K_0 involved in b_5 is determined as [10]

$$K_0 = \frac{2\mu}{R} \frac{\Delta}{\psi} \ .$$

Here, $\psi = \sum_{j=1}^{6} F_j D_{6j}$, $F_1 = -\frac{n^2}{\gamma} I_n(\gamma)$, $F_3 = \frac{(\lambda+\mu)}{2(\lambda+2\mu)} \gamma \left(\frac{n^2}{\gamma^2} + 1\right) I_n(\gamma)$, and $F_5 = -I_n'(\gamma)$ (to obtain F_j with even indices j, it is necessary to change the modified Bessel function of the first kind to the modified Bessel function of the second kind), D_{6i} and Δ are respectively the complement and determinant of the matrix with the elements

$$C_{11} = \frac{n^2}{x} I_n(x) \ , \quad C_{13} = -I_n'(x) - \frac{\lambda+\mu}{\lambda+2\mu} \left(\frac{n^2}{x^2} + 1\right) x I_n(x) \ , \quad x = Z_0 \gamma,$$

$$C_{55} = \frac{1}{\gamma}\left[I_n'(\gamma) - \frac{1}{\gamma} I_n(\gamma)\right], \quad C_{61} = \frac{n^2}{\gamma}\left[\frac{1}{\gamma} I_n(\gamma) - I_n'(\gamma)\right],$$

where $I_n(x)$ and $I_n(\gamma)$ are modified Bessel functions of the first kind of the order n, and the quantities denoted by the prime are derivatives with respect to the corresponding argument. Equations (3) are solved in the following form:

$$\{\varphi_n, \psi_n, u_{mi}, w_{mi}\} = \sum_{k=1,3,\ldots}^{\infty} \left\{A_k^{(n)}, A_{k+1}^{(n)}, A_{ki}, A_{(k+1)i}\right\} \sin \frac{k\omega t}{2}$$

$$+ \left\{B_k^{(n)}, B_{k+1}^{(n)}, B_{ki}, B_{(k+1)i}\right\} \cos \frac{k\omega t}{2} \ . \tag{4}$$

Substituting the first sum of (4) into (3) and equating the coefficients of the identical quantities $\sin \frac{k\omega t}{2}$, we obtain a system of inhomogeneous algebraic equations. According to [1], it will suffice to use one term of a series; however, we will use two expansion terms because the use of the two-term approximation makes it possible to increase the accuracy of calculations by 7%. As a result, we have

$$
\begin{aligned}
c_{11}A_1^{(n)} + c_{12}A_2^{(n)} &= \frac{(2-\delta_{0n})b_6}{2\pi}\sum_{i=1}^{M}(d_1A_{1i} + d_2A_{2i})\cos n\beta_i\ ,\\
c_{21}A_1^{(n)} + c_{22}A_2^{(n)} + c_{24}A_4^{(n)} &= \frac{(2-\delta_{0n})b_6}{2\pi}\sum_{i=1}^{M}(d_3A_{1i} + d_4A_{2i} + d_5A_{4i})\cos n\beta_i\ ,\\
c_{33}A_3^{(n)} + c_{34}A_4^{(n)} &= \frac{(2-\delta_{0n})b_6}{2\pi}\sum_{i=1}^{M}(d_6A_{3i} + d_7A_{4i})\cos n\beta_i\ ,\\
c_{42}A_2^{(n)} + c_{43}A_3^{(n)} + c_{44}A_4^{(n)} &= \frac{(2-\delta_{0n})b_6}{2\pi}\sum_{i=1}^{M}(d_8A_{2i} + d_9A_{3i} + d_{10}A_{4i})\cos n\beta_i\ ,
\end{aligned}
\tag{5}
$$

where

$$c_{11} = b_1 - \frac{b_3\omega^2}{4}\,, \quad c_{12} = -c_{21} = c_{34} = -c_{43} = -b_2\,, \quad c_{22} = \frac{b_3\omega^2}{4} + b_4T_0 - b_5 \pm \frac{b_4T_1}{2}\,,$$

$$c_{24} = c_{42} = \frac{b_4T_1}{2}\,, \quad c_{33} = b_1 - 9\,\frac{b_3\omega^2}{4}\,, \quad c_{44} = 9\,\frac{b_3\omega^2}{4} + b_4T_0 - b_5\,,$$

$$c_{13} = c_{14} = c_{23} = c_{31} = c_{32} = c_{41} = 0\,,$$

$$d_1 = \frac{a_{13}\omega^2}{4} - \gamma^2\,, \quad d_2 = -d_3 = d_7 = -d_9 = -a_{14}\gamma^2\,,$$

$$d_4 = a_{12}\gamma^4 - a_{11}\gamma^2T_0 - \frac{a_{13}\omega^2}{4} \mp \frac{a_{11}\gamma^2T_1}{2}\,, \quad d_5 = d_8 = -\frac{a_{11}\gamma^2T_1}{2}\,,$$

$$d_6 = 9\,\frac{a_{13}\omega^2}{4} - \gamma^2\,, \quad d_{10} = a_{12}\gamma^4 - a_{11}\gamma^2T_0 - 9\,\frac{a_{13}\omega^2}{4}\,.$$

Solving system (5) for $A_j^{(n)}$, we obtain

$$
\begin{aligned}
A_j^{(n)} = \frac{(2-\delta_{0n})b_6}{2\pi\Delta}\sum_{i=1}^{M}\Big[&\big(D_{1j}d_1 + D_{2j}d_3\big)A_{1i} + \big(D_{1j}d_2 + D_{2j}d_4 + D_{4j}d_8\big)A_{2i}\\
&+ \big(D_{3j}d_6 + D_{4j}d_9\big)A_{3i} + \big(D_{2j}d_5 + D_{3j}d_7 + D_{4j}d_{10}\big)A_{4i}\Big]\cos n\beta_i\,, \quad j = 1, 2, 3, 4\,,
\end{aligned}
\tag{6}
$$

where Δ and D_{kj} are, respectively, the determinant and complement of the matrix c_{kj}.

Since, at the locations of the ribs, the relation $A_{kj} = \sum_{n=0}^{\infty} A_j^{(n)}\cos n\beta_k$ is true, substituting (6) into this expression, we obtain a system of $4M$ algebraic equations:

$$
\begin{aligned}
A_{jk} = \frac{1}{2\pi}\sum_{i=1}^{M}\sum_{n=0}^{\infty}\frac{(2-\delta_{0n})b_6}{\Delta}\Big[&\big(D_{1j}d_1 + D_{2j}d_3\big)A_{1i} + \big(D_{1j}d_2 + D_{2j}d_4 + D_{4j}d_8\big)A_{2i}\\
&+ \big(D_{3j}d_6 + D_{4j}d_9\big)A_{3i} + \big(D_{2j}d_5 + D_{3j}d_7 + D_{4j}d_{10}\big)A_{4i}\Big]\cos n\beta_i\,\cos n\beta_k\,,\\
&j = 1, 2, 3, 4,\ \ k = 1, 2, \ldots, M\,.
\end{aligned}
\tag{7}
$$

When the determinant of this system is equal to zero, a characteristic equation of critical frequencies is obtained.

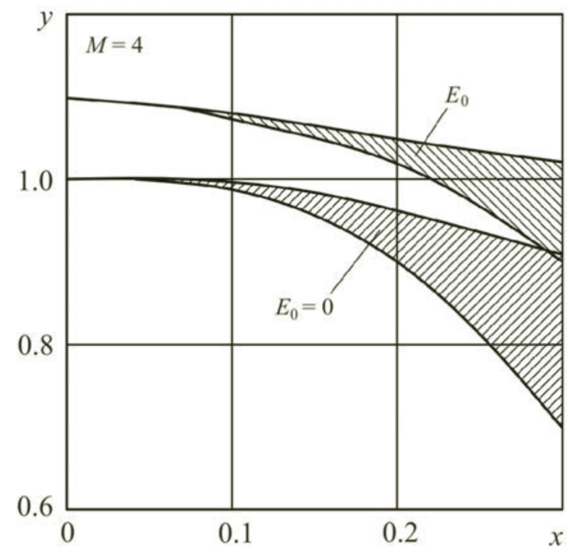

Fig. 1. Instability regions of the hollow shell (1) and the shell with a cylinder (2).

Substituting the second sum of (4) into (3), we obtain a characteristic equation of the type of (7) in which the unknowns A_{1i}–A_{4i} should be changed to B_{1i}–B_{4i}, respectively, and the coefficients c_{22} and d_4 should be used with the signs "+" and "–", respectively. In the case of uniform disposition of the ribs, system (7) is solved in the following form:

$$A_{jk} = A_j \cos \frac{2\pi ks}{M}, \quad 0 \le s \le \frac{M}{2}. \tag{8}$$

Substituting (8) into (7), we obtain a system of four algebraic equations:

$$\frac{M}{2\pi} \sum_N \frac{b_6}{\Delta} \Big[\big(D_{1j} d_1 + D_{2j} d_3 \big) A_1 + \big(D_{1j} d_2 + D_{2j} d_4 + D_{4j} d_8 \big) A_2$$
$$+ \big(D_{3j} d_6 + D_{4j} d_9 \big) A_3 + \big(D_{2j} d_5 + D_{3j} d_7 + D_{4j} d_{10} \big) A_4 \Big] - A_j = 0, \quad j = 1, 2, 3, 4, \tag{9}$$

where the summation is over the quantity N taking the values

$$N = rM + s, \quad r = 0, 1, 2, \dots;$$
$$N = rM - s, \quad r = 1, 2, 3, \dots.$$

A characteristic equation is obtained in the case where the determinant of system (9) is equal to zero. Assigning different integer values to m and s, we find a critical frequency.

Numerical Examples. As an example, we will consider a shell reinforced by a cylinder and identical ribs spaced uniformly. The basic parameters of the shell, cylinder, and the ribs are as follows:

$$\frac{L}{R} = 4.0, \quad \frac{h}{R} = 0.015, \quad \frac{H}{R} = 0.05, \quad \frac{b}{R} = 0.03, \quad \frac{E_\beta}{E_0} = 11.5 \cdot 10^3, \quad \frac{\xi}{R} = -0.032, \quad \frac{p}{E_0} = 0,$$

$$\frac{R_0}{R} = 0.4, \quad \frac{E_\alpha, E}{E_0} = 7.5 \cdot 10^3, \quad \frac{G_{\alpha\beta}}{E_0} = 1.25 \cdot 10^3, \quad \nu_\alpha = 0.15, \quad \nu_\beta = 0.23, \quad \nu_0 = 0.49.$$

Here, H and b are the height and width of the ribs, and R_0, E_0, and ν_0 are the radius of the cylinder, the modulus of elasticity of the cylinder material, and its Poisson coefficient. The shell and the ribs are subjected to the action of the axial force whose constant component is equal to the half the calculated critical force: $T_0 = \sqrt{E_\alpha E_\beta} h^2$.

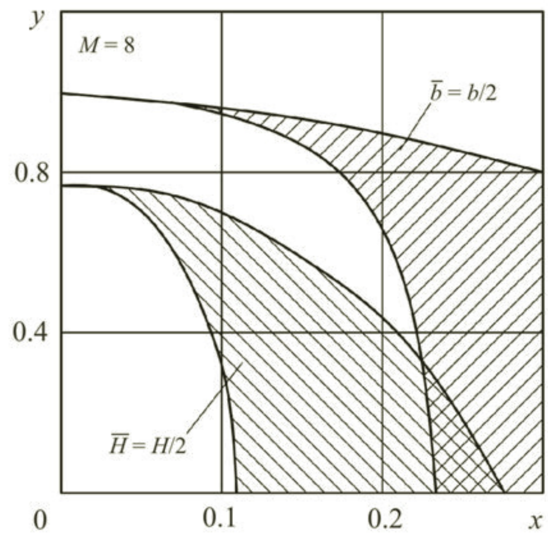

Fig. 2. Instability regions of the shells with ribs of decreased height (1) and decreased width (2).

Figure 1 shows the instability regions of the shell with four ribs, reinforced by a cylinder and without it. On the axis of ordinates the dimensionless values of the critical frequencies of vibration of the shell ($y = \frac{\omega}{\omega_0}$, where ω_0 is the eigenfrequency of the hollow shell with ribs) are plotted, and the abscissa is the dimensionless amplitude of the variable component of the axial force $x = T_1/T_0$.

Figure 2 shows the instability regions of the shell reinforced by a cylinder and eight ribs. In the first variant of calculations, the basic height of the ribs $\bar{H}$ is decreased by two times, and, in the second variant of calculations, the width of the ribs $\bar{b}$ is decreased. For both variants, the total sectional area of the eight ribs is equal to that of the basic four ribs.

It is seen from the above-presented examples that the cylinder reinforcing the shell narrows the instability region of the shell and heightens its boundaries by 10%. In this case, a decrease in the height of the ribs by two times decreases the dynamic-stability limit of the shell by four times, and a decrease in the basic width of the ribs by two times decreases its stability limit by two times, which points to the fact that the parameters of the ribs and their locations substantially influence the value and boundaries of the instability region of the shell.

Conclusions. A methodology of calculating the dynamic stability of a composite cylindrical shell with regard for the reinforcing influence of an elastic cylinder and discretely spaced longitudinal ribs has been developed. This methodology allows one to avoid, even at the design stage, the main parametric resonance in the frequency range with a maximum energy level and to increase the reliability of the shell.

This work was carried out with financial support from the Russian Foundation for Basic Research (grant No. 14-08-01026-a).

NOTATION

E and E_0, moduli of elasticity of the material of the ribs and the material of the cylinder; E_α and E_β, moduli of elasticity of the shell in the axial and circumferential directions; F, area of a rib; $G_{\alpha\beta}$, shear modulus; h, thickness of the shell; I, moment of inertia of a rib; L, length of the shell; m, parameter of wave formation; M, number of ribs; p, external pressure; R, radius of a shell; s, integer characterizing the loss in the stability of the shell; T_0, constant component of the axial force; T_1, amplitude of the variable component of the axial force; u, axial movement of the shell; w, radial movement of the shell; α, dimensionless coordinate along the generatrix; β, dimensionless coordinate in the circumferential direction; $\delta(\beta)$, delta function; δ_{0n}, Kronecker symbol; ν_0, Poisson coefficient of the material of the cylinder; ν_α and ν_β, Poisson coefficients of the shell in the axial and circumferential directions; ξ, distance from the axial line of a rib to the middle surface of the shell, assumed to be positive in the case where the axial line of the rib is positioned inside it; ρ_0 and ρ_1, densities of the materials of the shell and the ribs; ω, frequency of pulsations.

REFERENCES

1. V. V. Bolotin, *Dynamic Stability of Elastic Systems* [in Russian], GITTL, Moscow (1956).
2. A. S. Vol′mir, *Nonlinear Dynamics of Plates and Shells* [in Russian], Nauka, Moscow (1972).
3. P. M. Ogibalov and M. A. Koltunov, *Shells and Plates* [in Russian], Izd. MGU, Moscow (1971).
4. A. A. Mailybaev and A. P. Seiranyan, Multiparameter problems of stability, in: *Theory and Applications in Mechanics* [in Russian], Fizmatlit, Moscow (2009).
5. V. V. Bagdasar′yan, O. V. Kuznetsov, and I. S. Malyutin, On parametric resonance of a cylindrical shell reinforced by lengthwise ribs, in: *Problems in Mathematical Physics and Oscillation Theory*, Issue 3, 89–96 (1975).
6. B. Kh. Éshmatov, Nonlinear vibrations and dynamic stability of a viscoelastic circular cylindrical shell with account for the shear strain and the rotational inertia, *Izv. Ross. Akad. Nauk*, *Mekh. Tverd. Tela*, No. 3, 102–117 (2009).
7. M. V. Serov, V. N. Mozgunov, and A. A. Zubarev, Dynamic stability of a cylindrical shell reinforced by a rigid cylinder under external pressure, in: *Proc. 23rd Int. Sci. Conf.*, June 2–29, 2010, SGTU, Saratov (2010), Vol. 5, Section 5, pp. 47–49.
8. R. A. Iskenderov, Stability of a ribbed cylindrical shell with a filler under dynamic loading, *Vestn. Bakinsk. Univ*., *Ser. Fiz.-Mat. Nauk*, No. 2, 61–66 (2009).
9. V. N. Bakulin, E. N. Volkov, and A. Ya. Nedbai, Flutter of a lamellar cylindrical shell reinforced by circular ribs and loaded with axial forces, *Dokl. Akad. Nauk*, **463**, No. 4, 414–417 (2015).
10. Yu. S. Solomonov, V. P. Georgievskii, A. Ya. Nedbai, and V. A. Andryushin, *Methods for Calculating Cylindrical Shells Made from Composite Materials* [in Russian], Fizmatlit, Moscow (2009).