DOI 10.1007/s10891-015-1344-x
Journal of Engineering Physics and Thermophysics, Vol. 88, No. 6, November, 2015

ANALYTICAL ESTIMATION OF VELOCITY AND TEMPERATURE FIELDS IN A CIRCULAR PIPE ON THE BASIS OF STOCHASTIC EQUATIONS AND EQUIVALENCE OF MEASURES

A.V. Dmitrenko UDC 537.3

A stream of nonisothermal Newtonian liquid in a circular smooth pipe is considered on the basis of systems of stochastic equations and of the physical law of equivalence of measures between laminar and turbulent motions. Analytical expressions were previously obtained for isothermal flows for the first and second critical Reynolds numbers, critical point, indices of velocity profiles, second-order correlation moments, correlation functions, and spectral functions depending on the parameters of initial turbulence. Analytical expressions, obtained with the use of the earlier derived formulas for the critical Reynolds numbers and the critical points, are presented for the indices of velocity and temperature profiles as functions of the initial turbulence parameters as well as of the Eckert and Prandtl numbers.

Keywords: *equivalence of measures, stochastic equations, turbulence.*

Introduction. Investigations [1–23] were devoted to the search for equations and invariants that could determine the start of transition from a deterministic motion to a turbulent one. An analysis of these works shows that the theory of measure in A. N. Kolmogorov′s and A. Ya. Khinchin′s works was used for the development of the statistical theory of developed turbulence represented as a stationary random process for which a theoretical-probabilistic measure and, correspondingly, a multidimensional probability density, allowing one to determine statistical and theoretical-probabilistic average quantities, are determinable. The statistical theory was further developed in the works of A. M. Obukhov and W. Heisenberg on turbulence generation [18, 24–29], but no critical numbers have been determined. Note that the well-known Orr–Sommerfeld equation provides a possibility of numerical integration with subsequent determination of the critical numbers. However, as follows from the literature, we failed to obtain solutions, e.g., analytical dependences for the velocity field, in the case of the further development of turbulence. J. Taylor′s attempt at establishing the dependence of the critical Reynolds numbers on the initial turbulence parameters ended only with deviation of a semiempirical formula for a circular cylinder without any other results for other flow parameters. As a whole, the advances in the statistical theory resulted in the development of such numerical methods as the RANS (initially suggested by A. A. Fridman and L. V. Keller in 1925) and LES [24–29].

The development of the theory of strange attractors and construction of dynamic systems are based on the results of the theory of measure obtained in the works of A. N. Kolmogorov and Ya. G. Sinai in deriving the entropy formula (the Kolmogorov–Sinai entropy). This led to the application of the theory of measure in obtaining a generalized expression for the entropy of a dynamic system. It should be noted that sometimes the A. Renyi entropy is applied as a generalization of K. Shannon′s information entropy [8–16]. Entropy relations are applied in determining the correlation dimension of the attractor and of the number of the degrees of freedom of a system. However, the theory of attractors studies the temporal, other than spatial, development of instability at a specific arbitrarily chosen point. No analytical dependences for estimating critical numbers and subsequently determining the turbulence process fields have been obtained. Attempts to explain and calculate turbulent flows on the basis of the theory of solutions, in particular the Korteweg–de Vries equation, or only on the fractality hypothesis, have not led to the expected results, for example, to the determination of critical numbers [8–16].

Yu. L. Klimontovich′s works [14], based on the analysis of M. A. Leontovich′s and M. Sato′s equations, are related to the application of his theorem and qualitative description of transition to turbulence based on the analysis and comparison of entropy in a laminar and turbulent states without the possibility of determining the critical parameters. A special place is occupied by L. D. Landau′s qualitative theory that determines the occurrence of turbulence by an infinite number of duplication

National Research Nuclear University (MIFI), 31 Kashirskoe Highway, Moscow 115409, Russia; Moscow State University of Railways (MIIT), 9 Obraztsov Str., Moscow, 127994, Russia; email: AVDmitrenko@mephi.ru, ammsv@yandex.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 88, No. 6, pp. 1512–1520, November–December, 2015. Original article submitted September 7, 2015.

of the frequencies of the perturbation already present in the flow. However, the Reynolds number in Landau′s theory is the "controlling" parameter and its critical value is not determined. The latest experimental investigations of the process of transition show that at the very beginning the process develops by a scheme close to Landau′s scenario (a sequential, possibly fractal, increase in frequencies — a sequence of bifurcations — occurs several times) followed by a "catastrophic" increase in the number of frequencies and formation of a continuous spectrum. The research is made, however, in a selected rather than determined region (point) of space. Meanwhile it is difficult not to accept the validity of O. Reynolds′ opinion expressed by him more than a century ago that developed turbulence is formed in a flow as a result of energy transfer from the main motion into random fluctuations. It is important to note an experimentally confirmed fact that random fluctuations are always present in a flow (initial turbulence) and the lower this degree of initial turbulence, the higher the values of the Reynolds number at which this transition takes place. These well-known facts have not been explained theoretically up to now to the extent when, using a single theoretical formulation of the problem, it is possible to calculate a majority of turbulence parameters of interest for the practice and theory. The development of the DNS numerical methods has led to the necessity of "stochastization" of the Navier–Stokes equation or to its transformation to a form of the type of P. Langevin′s equation [8–13] by supplementing its right-hand side with a free term that determines perturbation, albeit without corresponding terms in the continuity and energy equations [16, 23–30].

Thus, a real experimentally confirmed pattern of flow is determined by a flux of a continuous medium with fluctuations available in it, and therefore the start of transition to turbulence can be determined as the start of the interaction of these fluctuations with the main flow.

In [17–20], it was shown, for an isothermal flow, that the basic parameters of turbulence can be calculated theoretically on the basis of a system of stochastic equations and equations for the law of the equivalence of measures between a deterministic and a random motions. Note that the stochastic equations, derived in [17–19], on their right-hand side include free terms of gradient and nongradient structures. For this purpose, the following space-time domains were determined in [17–23]: 1) the start of turbulence generation; 2) turbulence generation; 3) diffusion; 4) turbulence dissipation. With the use of the law of the equivalence of measures, each of the domains has its own system of stochastic equations within the framework of the general system of stochastic equations for mass, momentum, and energy. For a nonisothermal medium in domain 1, analytical expressions were presented in [21–23] for calculating the critical point and Reynolds number depending on the initial turbulence of flow in a pipe and on a flat plate. In the present paper, analytical formulas for the indices of velocity and temperature profiles as functions of the parameters of initial turbulence and the Eckert and Prandtl numbers were obtained for domain 2 on the basis of the stochastic system of equations for energy, momentum, and mass.

System of Equations. The general system of stochastic conservation equations for isothermal and nonisothermal media was obtained in [17–23]. It includes:

equation of mass (continuity)

$$\frac{d(\rho)_{\mathrm{col,st}}}{d\tau} = -\frac{\rho_{\mathrm{st}}}{\tau_{\mathrm{cor}}} - \frac{d\rho_{\mathrm{st}}}{d\tau}\,; \tag{1}$$

momentum equation

$$\frac{d(\rho u_i)_{\mathrm{col,st}}}{d\tau} = \mathrm{div}\,(\tau_{i,j})_{\mathrm{col,st}} + \mathrm{div}\,(\tau_{i,j})_{\mathrm{st}} - \frac{(\rho u_i)_{\mathrm{st}}}{\tau_{\mathrm{cor}}} - \frac{d(\rho u_i)_{\mathrm{st}}}{d\tau}\,; \tag{2}$$

energy equation

$$\frac{dE_{\mathrm{col,st}}}{d\tau} = \mathrm{div}\left(\lambda\,\frac{\partial T}{\partial x_j} + u_i\tau_{i,j}\right)_{\mathrm{col,st}} + \mathrm{div}\left(\lambda_i\,\frac{\partial T}{\partial x_j} + u_i\tau_{i,j}\right)_{\mathrm{st}} - \frac{E_{\mathrm{st}}}{\tau_{\mathrm{cor}}} - \frac{dE_{\mathrm{st}}}{d\tau}\,. \tag{3}$$

Here and subsequently, τ, ρ, $\mathbf{U}$, E, T, and $\tau_{i,j}$ are the time, density, velocity vector, energy, and temperature, the stress tensor $\tau_{i,j} = P + \sigma_{i,j}$, $\sigma_{i,j} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\left(\xi - \frac{2}{3}\mu\right)\frac{\partial u_l}{\partial x_l}$, the subscripts $i, j, l = 1, 2, 3$, and the parameters ν, μ, and ξ are the kinematic, dynamic, and second viscosities. The quantities u_i, u_j, u_l, x_i, x_j, and x_l are the velocities and coordinates corresponding to i, j, and l; $\delta_{ij} = 1$ at $i = j$; $\delta_{ij} = 0$ at $i \neq j$; P is the liquid or gas pressure; λ is the thermal conductivity; c_p and c_v are the specific heats at constant pressure and volume.

In [17–20], for the transfer of the substantial quantity Φ (mass (density ρ), momentum ($\rho\mathbf{U}$), energy (E)) of the deterministic (laminar) motion into a random (turbulent) one, for domain 1 of the start of turbulence generation, the pair

$(N, M) = (1, 0)$, with the equivalence of measures being written as $(d\Phi_{\text{col,st}})_{1,0} = -R_{1,0}(\Phi_{\text{st}})$ and $\left(\frac{d\Phi_{\text{col,st}}}{d\tau}\right)_{1,0} = -R_{1,0}\left(\frac{\Phi_{\text{st}}}{\tau_{\text{cor}}}\right)$. Applying the correlator $D_{N,M}(r_c, m_{ci}, \tau_c) = D_{1,1}(r_c, m_{ci}, \tau_c)$ obtained in [17–20], the equivalence relation for the pair $(N, M) = (1, 1)$ was defined as $(d\Phi_{\text{col,st}})_{1,1} = -R_{1,1}(d\Phi_{\text{st}})$, $\left(\frac{d\Phi_{\text{col,st}}}{d\tau}\right)_{1,1} = -R_{1,1}\left(\frac{d\Phi_{\text{st}}}{d\tau}\right)$, where $R_{1,0}$ and $R_{1,1}$ are fractal coefficients, $\Phi_{\text{col,st}}$ is part of the field of Φ, namely, its deterministic component (subscript col,st) the stochastic component of the measure of which is zero; Φ_{st} is part of Φ, namely, the proper stochastic component (subscript st). The relations of the equivalence for momentum and mass (density) have been determined in the same way. For example, to obtain new analytical relations, the fractal coefficients $R_{1,0}$ and $R_{1,1}$ are taken equal to unity, and the indices "cr" or "c" relate to the critical point, $r\,(x_{\text{cr}}, \tau_{\text{cr}})$ or τ_c. The critical point is the space-time point of the start of interaction between the deterministic and random fields. It was shown in [17–20] that for the pair $(N, M) = (1, 1)$ for domain 2 of turbulence generation, $r_{c1}(x_c + \Delta x_0 + \Delta x_1, \tau_c + \Delta\tau_0 + \Delta\tau_1)$, we have the following system:

$$\left(\frac{d\rho_{\text{col,st}}}{d\tau}\right)_{1,1} = -\frac{d\rho_{\text{st}}}{d\tau},$$

$$\left(\frac{d\,(\rho\mathbf{U})_{\text{col,st}}}{d\tau}\right)_{1,1} = -\left(\frac{d\,(\rho\mathbf{U})_{\text{st}}}{d\tau}\right),\ \operatorname{div}\left(\tau_{i,j}\right)_{\text{col,st2}} = \frac{d\,(\rho u_i)_{\text{st}}}{d\tau}, \tag{4}$$

$$\frac{d\left(E_{\text{col,st}}\right)_{1,1}}{d\tau} = -\left(\frac{dE_{\text{st}}}{d\tau}\right)_{1,1},\ \operatorname{div}\left(\lambda\frac{\partial T}{\partial x} + u_i\tau_{i,j}\right)_{\text{col,st2}} = \left(\frac{dE_{\text{st}}}{d\tau}\right)_{1,1}.$$

The subscript (col,st2) relates to the pair $(N, M) = (1, 1)$ and the subscript (col,st1), to the pair $(N, M) = (1, 0)$.

Determination of Velocity and Temperature Profiles in a Pipe. As is known, experimental investigations of the averaged characteristics of developed turbulence have shown that the velocity and temperature profiles have the affine similarity. Thus, for example, equations are obtained for the classical flow in a pipe. In the case of flow with constant physical properties, equations for laminar motion have the form $u_1 = U_0\left(\frac{x_2}{R}\right)^2$, $\left[\frac{T - T_w}{T_0 - T_w} = \left(\frac{x_2}{R}\right)^4\right]$ and for turbulent flow, accordingly, $\left[\left(\frac{x_2}{R}\right)^{1/n} = \left(\frac{u_1}{U_0}\right)\right]$, $\left[\frac{T - T_w}{T_0 - T_w} = \left(\frac{x_2}{R}\right)^{1/n_T}\right]$, $T = T_w + (T_0 - T_w)\left(\frac{x_2}{R}\right)^{1/n_T}$. Here T_0 and T_w are the temperatures on the pipe axis and on the wall; R, U_0, and u_1 are the pipe radius and velocities on the axis and along x_1; x_1 and x_2 are the longitudinal and transverse coordinates. Analogously to [17–20] and in accord with system (4), we may write

$$\frac{\left(\operatorname{div}\left(u_i\tau_{i,j}\right)_{\text{col,st2}}\right)_{\tau^1_{\text{cor1}UP}} + \left(\operatorname{div}\left(\lambda_i\frac{\partial T}{\partial x_j}\right)_{\text{col,st2}}\right)_{\tau^1_{\text{cor1}T}}}{\left(\operatorname{div}\left(u_i\tau_{i,j}\right)_{\text{col,st2}}\right)_{\tau^0_{\text{cor1}UP}} + \left(\operatorname{div}\left(\lambda_i\frac{\partial T}{\partial x_j}\right)_{\text{col,st2}}\right)_{\tau^0_{\text{cor1}T}}} = \frac{\left(E_{\text{st}}\right)_{\tau^1_{\text{cor}}}}{\left(E_{\text{st}}\right)_{\tau^0_{\text{cor}}}}. \tag{5}$$

To determine the velocity and temperature profiles, similar to [17–20], for the deterministic (laminar) pipe flow and initial times $\tau^0_{\text{cor1}UP}$ and $\tau^0_{\text{cor1}T}$ we write

$$\operatorname{div}\left(u_i\tau_{i,j}\right)_{\text{col,st2}} = \frac{\partial}{\partial x_2}\left\{\left[U_0\left(\frac{x_2}{R}\right)^2\right]\frac{\partial}{\partial x_2}\mu\left[U_0\left(\frac{x_2}{R}\right)^2\right]\right\} = 6\mu\left(\frac{U_0}{R}\right)^2\left(\frac{x_2}{R}\right)^2,$$

$$\operatorname{div}\left(\lambda\frac{\partial T}{\partial x_i}\right)_{\text{col,st2}} = 12\lambda\left(\frac{T_0 - T_w}{R^2}\right)\left(\frac{x_2}{R}\right)^2. \tag{6}$$

Correspondingly for turbulent pipe flow and finite times $\tau^1_{\text{cor1}UP}$ and $\tau^1_{\text{cor1}T}$, to determine the velocity and temperature profiles, we write

$$\operatorname{div}\left(u_i\tau_{i,j}\right)_{\tau^1_{\text{cor}}} = \frac{\partial}{\partial x_2}\left\{\left[U_0\left(\frac{x_2}{R}\right)^{1/n}\right]\frac{\partial}{\partial x_2}\mu\left[U_0\left(\frac{x_2}{R}\right)^{1/n}\right]\right\} = \frac{1}{n}\frac{2-n}{n}\mu\left(\frac{U_0}{R}\right)^2\left(\frac{x_2}{R}\right)^{2/n-2},$$

$$\left[\operatorname{div}\left(\lambda\frac{\partial T}{\partial x_i}\right)\right]_{\tau^1_{\text{cor}}} = \frac{1-n_T}{n_T n_T}\lambda\left(\frac{T_0-T_{\text{w}}}{R^2}\right)\left(\frac{x_2}{R}\right)^{1/n_T-2}. \tag{7}$$

We then obtain

$$\left|\frac{\frac{1}{n}\frac{2-n}{n}\mu\left(\frac{U_0}{R}\right)^2\left(\frac{x_2}{R}\right)^{2/n-2}+\frac{1-n_T}{n_T^2}\lambda\left(\frac{T_0-T_{\text{w}}}{R^2}\right)\left(\frac{x_2}{R}\right)^{1/n_T-2}}{6\mu\left(\frac{U_0}{R}\right)^2\left(\frac{x_2}{R}\right)^2+12\lambda\left(\frac{T_0-T_{\text{w}}}{R}\right)^2\left(\frac{x_2}{R}\right)^2}\right| = \frac{(E_{\text{st}})_{\tau^1_{\text{cor}}}}{(E_{\text{st}})_{\tau^0_{\text{cor}}}}. \tag{8}$$

We determine now the right-hand sides of Eqs. (5) and (8):

$$\frac{(E_{\text{st}})_{\tau^1_{\text{cor}}}}{(E_{\text{st}})_{\tau^0_{\text{cor}}}} = \frac{(E_{\text{st}})_{\tau^1_{\text{cor}U}}+(E_{\text{st}})_{\tau^1_{\text{cor}T}}}{(E_{\text{st}})_{\tau^0_{\text{cor}U}}+(E_{\text{st}})_{\tau^0_{\text{cor}T}}}.$$

Applying the theorem of the mean, we write the following equation:

$$\int_{\tau^0_{\text{cor}UP}}^{\tau^1_{\text{cor}UP}}\operatorname{div}\left(u_i\tau_{i,j}\right)_{\text{col,st2}}d\tau+\int_{\tau^0_{\text{cor}T}}^{\tau^1_{\text{cor}T}}\left(\lambda\frac{\partial T}{\partial x_j}\right)_{\text{col,st2}}d\tau = \int_{\tau^0_{\text{cor}UP}}^{\tau^1_{\text{cor}UP}}\frac{d\,(E_{\text{st}})_{U,P}}{d\tau}d\tau+\int_{\tau^0_{\text{cor}T}}^{\tau^1_{\text{cor}T}}\frac{d\,(E_{\text{st}})_T}{d\tau}d\tau. \tag{9}$$

According to [17–20], the expression for the velocity field has the form

$$\frac{(E_{\text{st}})_{\tau^1_{\text{cor}UP}}}{(E_{\text{st}})_{\tau^0_{\text{cor}UP}}} = \left|\left((\text{Re}_{\text{st}})_U-\frac{1}{(\text{Re}_{\text{st}})_U}\right)\right|. \tag{10}$$

The temperature field is represented as

$$\frac{(E_{\text{st}})_{\tau^1_{\text{cor}T}}}{(E_{\text{st}})_{\tau^0_{\text{cor}T}}} = \left|\sqrt{\frac{\left(u_j^2\right)_{\text{st}}}{\left(u_i^2\right)_{\text{st}}}}(\text{Re}_{\text{st}})_U-\frac{1}{\sqrt{\frac{\left(u_j^2\right)_{\text{st}}}{\left(u_i^2\right)_{\text{st}}}}(\text{Re}_{\text{st}})_U}\right|. \tag{11}$$

The final expression takes the form

$$\frac{(E_{\text{st}})_{\tau^1_{\text{cor}}}}{(E_{\text{st}})_{\tau^0_{\text{cor}}}} = \frac{\left\{\left|(\text{Re}_{\text{st}})_U\right|+2\frac{\text{T}_T}{\text{EcTu}^2}\left|F\,(\text{Re}_{\text{st}})_U\right|\right\}}{1+2\frac{\text{T}_T}{\text{EcTu}^2}},\quad F=\sqrt{\frac{\left(u_j^2\right)_{\text{st}}}{\left(u_i^2\right)_{\text{st}}}}. \tag{12}$$

Substituting the last expression and relations (10), (11) into (8), we obtain

$$\left|\frac{\left(\frac{2-n}{n^2}\left(\frac{x_2}{R}\right)^{2/n-4}\right)_U+\left(\frac{1-n_T}{2n_T^2}\right)_T\frac{1}{\Pr}\left(\frac{T_0-T_w}{U^2/2c_p}\right)\left(\frac{x_2}{R}\right)^{1/n_T-4}}{6+12\,\frac{1}{\Pr}\,\frac{T_0-T_w}{U^2/2c_p}}\right|=\left|(\mathrm{Re}_{st})_U\right|\frac{1+2\,\frac{F\mathrm{T}_T}{\mathrm{EcTu}^2}}{1+2\,\frac{\mathrm{T}_T}{\mathrm{EcTu}^2}}\,. \tag{13}$$

Equation (13) includes the indices of the velocity (n) and temperature (n_T) profiles. The index n can be determined with the use of the equation of motion in the stochastic system of equations (4):

$$\frac{\left(\mathrm{div}\left(\tau_{i,j}\right)_{\mathrm{col,st2}}\right)_{\tau^1_{\mathrm{cor1}UP}}}{\left(\mathrm{div}\left(\tau_{i,j}\right)_{\mathrm{col,st2}}\right)_{\tau^0_{\mathrm{cor1}UP}}}=\frac{\left((\rho U)_{\mathrm{st}}\right)_{\tau^1_{\mathrm{cor}}}}{\left((\rho U)_{\mathrm{st}}\right)_{\tau^0_{\mathrm{cor}}}}\,. \tag{14}$$

For the times of the beginning, $\tau^0_{\mathrm{cor1}UP}$, and end, $\tau^1_{\mathrm{cor1}UP}$, of the interaction of fields we write

$$\mathrm{div}\left(\tau_{i,j}\right)_{\mathrm{col,st1}}=\frac{\partial}{\partial x_2}\left\{\frac{\partial}{\partial x_2}\,\mu\left[U_0\left(\frac{x_2}{R}\right)^2\right]\right\}=2\mu\,\frac{U_0}{R^2}\,, \tag{15}$$

$$\mathrm{div}\left(\tau_{i,j}\right)_{\tau^1_{\mathrm{cor}}}=\frac{\partial}{\partial x_2}\left\{\frac{\partial}{\partial x_2}\,\mu\left[U_0\left(\frac{x_2}{R}\right)^{1/n}\right]\right\}=\frac{1}{n}\,\frac{1-n}{n}\,\mu\,\frac{U_0}{R^2}\left(\frac{x_2}{R}\right)^{1/n-2}\,. \tag{16}$$

We then obtain

$$\left|\frac{\frac{1-n}{n^2}\,\mu\,\frac{U_0}{R^2}\left(\frac{x_2}{R}\right)^{1/n-2}}{2\mu\,\frac{U_0}{R^2}}\right|=\frac{\left((\rho U)_{\mathrm{st}}\right)_{\tau^1_{\mathrm{cor}}}}{\left((\rho U)_{\mathrm{st}}\right)_{\tau^0_{\mathrm{cor}}}}\,. \tag{17}$$

As a result, with account for [18, 19], the expression for the index of the velocity profile has the form

$$\frac{\left((\rho U)_{\mathrm{st}}\right)_{\tau^1_{\mathrm{cor}}}}{\left((\rho U)_{\mathrm{st}}\right)_{\tau^0_{\mathrm{cor}}}}=\left|\mathrm{Re}_{\mathrm{st}}-\frac{1}{\mathrm{Re}_{\mathrm{st}}}\right|^{0.5}, \tag{18}$$

$$\left|\frac{1}{2n}\sqrt{\left(\frac{R}{x_2}\right)^4\left(\frac{x_2}{R}\right)^{2/n}}\right|=\left|\mathrm{Re}_{\mathrm{st}}-\frac{1}{\mathrm{Re}_{\mathrm{st}}}\right|^{0.5}. \tag{19}$$

According to [22, 23], the equality for the critical point has the form

$$\frac{x_2}{R}=\left[\frac{1}{4}\,\frac{(E_{\mathrm{st}})_{UP}}{U_0^2}\,\frac{R}{L_U}\,\frac{\mathrm{EcPr}}{1+\mathrm{EcPr}}\left(1+\frac{2\mathrm{T}_T}{\mathrm{Tu}^2\mathrm{Ec}}\right)\right]^{1/3}, \tag{20}$$

where $L=L_{UP}=L_U$ is the scale of turbulence; the subscripts UP and U relate to the velocity field; the subscript T relates to the temperature field; L_y is the scale of turbulence determined along the coordinate $x_2=y$; L_x is the scale of turbulence determined along the coordinate $x_1=x$. Here x_1 and x_2 are the coordinates directed along the wall and normally to it; $\mathrm{Ec}=\frac{U_0^2}{c_p(T_0-T_w)}$ is the Eckert number; $L_T=\frac{L}{\Pr}$; $\Pr=\frac{\rho\nu c_p}{\lambda}$ is the Prandtl number; according to [24–29] $\mathrm{T}_T=|T_{\mathrm{st}}|/(T_0-T_w)=5\cdot(10^{-5}–10^{-3})$, $\mathrm{Tu}=\sqrt{\frac{(u_i^2)_{\mathrm{st}}}{U_0^2}}=10^{-3}–10^{-2}$, $(E_{\mathrm{st}})_T/(E_{\mathrm{st}})_{UP}=\frac{c_pT_{\mathrm{st}}}{(E_{\mathrm{st}})_{UP}/\rho}=2\left(\frac{T_{\mathrm{st}}}{T_0-T_w}\right)\frac{U_0^2}{u_{\mathrm{st}}^2}\,\frac{c_p(T-T_w)}{U_0^2}=2\,\frac{\mathrm{T}_T}{\mathrm{EcTu}^2}$. Thus, we have three analytical equations (13), (19), and (20), i.e., the exponents n and n_T can be determined. Just as in [21–23], according to the experimental conditions in [24–28] with Ec = –(0.1–0.010), Pr = 0.72, and $\mathrm{Re_{st}}$ = 10–30, from (19) and (20) we have that $n \sim 7$ and from (13) that $n_T \sim 8$.

An Example of Calculations of the Indices of Profiles *n* and n_T. The profiles $\left[\left(\frac{x_2}{R}\right)^{1/n} = \left(\frac{u_1}{U_0}\right)\right]$, $\left[\frac{T - T_\mathrm{w}}{T_0 - T_\mathrm{w}} = \left(\frac{x_2}{R}\right)^{1/n_T}\right]$ are applicable to flows with constant physical properties. In this case, n and n_T can be estimated in the following order:

1) According to (20), we determine the critical point:

$$\frac{x_2}{R} = \left[\frac{1}{4}\frac{(E_\mathrm{st})_{UP}}{U_0^2}\frac{R}{L_U}\frac{\mathrm{EcPr}}{1+\mathrm{EcPr}}\left(1+\frac{2\mathrm{T}_T}{\mathrm{Tu}^2\mathrm{Ec}}\right)\right]^{1/3} = \left[\frac{1}{4}\frac{(E_\mathrm{st})_{UP}}{U_0^2}\frac{R}{L_U}\right]^{1/3}\left[\frac{\mathrm{EcPr}}{1+\mathrm{EcPr}} + \frac{2\mathrm{PrT}_T}{(1+\mathrm{EcPr})\mathrm{Tu}^2}\right]^{1/3}$$

$$= \left[\frac{1}{4}(0.01)^2\,(1/0.05)\right]^{1/3}\left[\frac{-(0.01\cdot 0.72)}{1-0.01\cdot 0.72} + \frac{0.72\cdot 0.0005}{(1-0.01\cdot 0.72)(0.01)^2}\right]^{1/3}$$

$$= 0.0005^{0.3333}(-0.0072 + 3.6)^{0.3333} \approx 0.07\cdot 1.5 \approx 0.105\,.$$

2) The index n of the velocity profile is estimated according to (19) from the conditions $\mathrm{Re_{st}} \approx$ 20–30 [23–29];

$$\left|\frac{1}{2n}\sqrt{\left(\frac{R}{x_2}\right)^4\left(\frac{x_2}{R}\right)^{2/n}}\right| = \left|\mathrm{Re_{st}} - \frac{1}{\mathrm{Re_{st}}}\right|^{0.5}.$$

The left-hand side $\left|\frac{1}{2n}\sqrt{\left(\frac{x_2}{\mathrm{R}}\right)^{2/n-4}}\right| = \left|\frac{1}{14}\sqrt{0.105^{2/7-4}}\right| = 0.072\sqrt{0.105^{2/7-4}} = 0.072\sqrt{4319} = 0.072\cdot 65.7 \approx 4.7$, which lies within the indicated limits $\sqrt{\mathrm{Re_{st}}} \approx$ 4.4–5.4 of the right-hand side of the equation, i.e., $n \approx 7$.

3) The index n_T is estimated by formula (13) with account for the values $n = 7$, $x_2/R = 0.105$, Ec = –0.01, and Pr = 0.72 determined in the first items and for the conditions [23–29] $\mathrm{Re_{st}} \approx 30$, $\mathrm{T}_T = |T_\mathrm{st}|/(T_0 - T_\mathrm{w}) = 5\cdot(10^{-5}–10^{-3})$, $\mathrm{Tu} = \sqrt{\frac{(u_i^2)_\mathrm{st}}{U_0^2}} = 10^{-3}–10^{-2}$. We then obtain

$$\left|\frac{\left(\frac{2-n}{n^2}\left(\frac{x_2}{R}\right)^{2/n-4}\right)_U + \left(\frac{1-n_T}{2n_T^2}\right)_T\frac{1}{\mathrm{Pr}}\left(\frac{T_0 - T_\mathrm{w}}{U^2/2c_p}\right)\left(\frac{x_2}{R}\right)^{1/n_T-4}}{6 + 12\,\frac{1}{\mathrm{Pr}}\left(\frac{T_0-T_\mathrm{w}}{U^2/2c_p}\right)}\right| = \left|(\mathrm{Re_{st}})_U\right|\frac{1+2\,\frac{F\mathrm{T}_T}{\mathrm{EcTu}^2}}{1+2\,\frac{\mathrm{T}_T}{\mathrm{EcTu}^2}}\,.$$

The right-hand side of the transformed equation (13) is equal to

$$\left(\frac{1-n}{2n^2}\right)_T (x_2/R)^{1/n_T} = (x_2/R)^4\,\frac{\mathrm{Pr\ Ec}}{2}\left((\mathrm{Re_{st}})_U\left(\frac{1+2F\,\frac{\mathrm{T}_T}{\mathrm{EcTu}^2}}{1+2\,\frac{\mathrm{T}_T}{\mathrm{EcTu}^2}}\right)\left(6+24\,\frac{1}{\mathrm{Pr\ Ec}}\right) - \left|\left(\frac{2-7}{49}\,(x_2/R)^{2/n-4}\right)\right|\right).$$

$$0.105^4(-0.0036)\left[30\cdot(351/501)\,(-3422) - \left(\frac{2-7}{49}\,4329\right)\right]$$

$$\approx (-0.000122)0.0036(-71874 + 440) = 0.000122\cdot 0.0036\cdot 714344 \approx 0.032\,,$$

the left-hand side, to

$$\frac{7}{128}\,0.105^{1/8} = 0.0546 \cdot 0.758 \approx 0.04 .$$

As we see, there is agreement between the values of the left-hand and right-hand sides of the equation, i.e., $n_T \approx 8$, and the estimate given shows good agreement with experimental data for the indices of the profiles n and n_T.

Conclusions. The calculated estimates showed that the values of the indices of velocity and temperature profiles for a pipe n and n_T agree with the experimental data of [24–29]. Therefore the total spectrum of the indices n and n_T can be obtained with the use of the analytical formulas and initial data presented in the paper for the initial turbulence. The results of calculations obtained agree with the classical experimental data regarding the fact that the values of n and n_T increase with the Reynolds number in a pipe in a turbulent regime. Analytical formulas show in this case that even in a nonisothermal flow with constant properties the temperature and velocity fields exert their influence on the flow not only depending on the Ec, Pr, and M numbers, but also depending on the initial turbulence of the flow. In the case of high temperature, high velocity, and heterogeneous flows, the essence of the process does not change, but the effects are more complex [24, 30, 31, 34–38]. Thus, for the process of nonisothermal liquid flow, based on the analytical dependences derived, satisfactory agreements are obtained between theoretical estimates and experimental data for both the critical Reynolds numbers [21–23] and the indices n and n_T of the velocity and temperature profiles depending on the initial turbulence. The results obtained show that as for isothermal, so also for nonisothermal flows, the reason for the process of transition from a deterministic state into a turbulent one is the physical law of the equivalence of measures between them.

REFERENCES

1. L. D. Landau, Toward the problem of turbulence, *Dokl. Akad. Nauk SSSR*, **44**, 339–342 (1944).
2. A. N. Kolmogorov, A new metric invariant of transitive dynamic systems and automorphisms of the Lebesgue spaces, *Dokl. Akad. Nauk SSSR*, **119**, No. 5, 861–864 (1958).
3. A. N. Kolmogorov, About the entropy per time unit as a metric invariant of automorphisms, *Dokl. Akad. Nauk SSSR*, **124**, No. 4, 754–755 (1959).
4. A. N. Kolmogorov, Mathematical models of turbulent motion of an incompressible viscous fluid, *Usp. Mat. Nauk*, **59**, Issue 1(355), 5–10 (2004).
5. E. N. Lorenz, Deterministic nonperiodic flow, *J. Atmos. Sci*., **20**, 130–141 (1963); doi: 10.1175/1520-0469.
6. D. Ruelle and F. Takens, On the nature of turbulence, *Commun. Math. Phys*., **20**, 167–192 (1971), also **23**, 343–344; http://dx.doi.org/10.1007/bf01646553.
7. M. Feigenbaum, The transition to aperiodic behavior in turbulent systems, *Commun. Math. Phys*., **77**, No. 1, 65–86 (1980).
8. M. I. Rabinovich, Stochastic self-oscillations and turbulence, *Usp. Fiz. Nauk*, **125**, 123–168 (1978).
9. A. S. Monin, On the nature of turbulence, *Usp. Fiz.Nauk*, **125**, 97–122 (1978).
10. M. I. Rabinovich and M. M. Sushchik, Coherent structures in turbulent flows, in: A. V. Gaponov and M. I. Rabinovich (Eds.), *Nonlinear Waves. Self-Organization* [in Russian], Nauka, Moscow (1983), pp. 58–84.
11. G. M. Zaslavskii, *Stochasticity of Dynamic Systems* [in Russian], Nauka, Moscow (1984).
12. V. V. Struminskii, Origination of turbulence, *Dokl. Akad. Nauk SSSR*, **307**, No. 3, 564–567 (1989).
13. A. A. Samarskii, V. I. Mazhukin, P. P. Matus, and I. A. Mikhailik, Z/2 conservative schemes for the Korteweg–de Vries equations, *Dokl. Akad. Nauk*, **357**, No. 4, 458–461 (1997).
14. Yu. L. Klimontovich, Problems of the statistical theory of open systems: criteria of the relative degree of the ordering of states in the self-organization processes, *Usp. Fiz. Nauk*, **158**, Issue 1, 59–91 (1989); doi: 10.1070/pu1999v042n01abeh000445.
15. K. R. Sreenivasan, Fractals and multifractals in fluid turbulence, *Ann. Rev. Fluid Mech.*, **23**, 539–600 (1991).
16. S. A. Orzag and L. C. Kells, Transition to turbulence in plane Poiseuille and plane Couette flow, *J. Fluid Mech*., **96**, No. 1, 159–205 (1980); doi.org/10.1017/s0022112080002066/.
17. A. V. Dmitrenko, Equivalence of measures and stochastic equations for turbulent flows, *Dokl. Akad. Nauk*, **450**, No. 6, 651–658 (2013); doi.org/10.1134/s1028335813060098.
18. A. V. Dmitrenko, Equivalent measures and stochastic equations for determination of the turbulent velocity fields and correlation moments of the second order, *Abstr. Int. Conf.* "Turbulence and Wave Processes,"Lomonosov Moscow State University, November 26–28, 2013, Moscow(2013),pp. 39–40; http://www.dubrovinlab.msu.ru/turbulencemdm100ru.

19. A. V. Dmitrenko, *Theory of Equivalent Measures and Sets with Repeating Denumerable Fractal Elements. Stochastic Thermodynamics and Turbulence. Determinacy–Randomness Correlator* [in Russian], Galleya-Print, Moscow (2013); http://search.rsl.ru/ru/catalog/record/6633402.
20. A. V. Dmitrenko, *Regular Coupling between Deterministic (Laminar) and Random (Turbulent) Motions — Equivalence of Measures*, Scientific Discovery Diploma No. 458, registration No. 583 of December 2, 2013.
21. A. V. Dmitrenko, Some analytical results of the theory of equivalence measures and stochastic theory of turbulence for non-isothermal flows, *Adv. Stud. Theor. Phys*., **8**, No. 25, 1101–1111 (2014); http://dx.doi.org/10.12988/astp.2014.49131.
22. A. V. Dmitrenko, The theory of equivalence measures and stochastic theory of turbulence for non-isothermal flow on the flat plate, *First Thermal and Fluids Engineering Summer Conference*, August 9–12, 2015, New York, NY, USA, "Thermal Fluids Engineering Addressing Grand Challenges"; http://www.astfe.org/tfesc/conference_program/.
23. A. V. Dmitrenko, Determination of critical Reynolds numbers for non-isothermal flows using stochastic theories of turbulence and equivalent measures, *Heat Transf. Res*., **46**, Issue 12, 1102–1112 (2015); DOI:10.1615/HeatTransRes.2015014191; http://dl.begellhouse.com/journals/46784ef93dddff27,forthcoming.html.
24. A. V. Dmitrenko, *Fundamentals of Heat and Mass Transfer and Hydrodynamics of Single-Phase and Two-Phase Media. Criteria, Integral and Statistical Methods, and Direct Numerical Simulation* [in Russian], Galleya-Print, Moscow (2008).
25. J. O. Hinze, *Turbulence*, 2nd edn., McGraw-Hill Book Company, New York (1975).
26. A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, Vols. 1 and 2, MIT Press, Cambridge, Massachusetts, USA (1971).
27. H. Schlichting, *Boundary-Layer Theory*, 6th edn., McGraw-Hill Book Company, New York (1968).
28. S. B. Pope, *Turbulent Flows*, Cambridge University Press, Cambridge, UK (2000); http://dx.doi.org/10.1017/cbo9780511840531.
29. P. A. Davidson, *Turbulence*, Oxford University Press, New York (2004).
30. A. V. Dmitrenko, Calculation of the pulsations of pressure of heterogeneous turbulent flows, *Dokl. Akad. Nauk*, **415**, No. 1, 44–47 (2007); http://dx.doi.org/10.1134/s1028335807120166.
31. A. V. Dmitrenko, Calculation of the boundary layer of atwo-phase medium, *High Temp.*, **40**, Issue 5, 706–715 (2002); http://dx.doi.org/10.1023/A:1020436720213.
32. V. G. Priymak, Splitting dynamics of coherent structures in a transitional round-pipe flow, *Dokl. Phys*., **58**, No. 10, 457–465 (2013).
33. A. V. Fursikov, Moment theory for Navier–Stokes equations with a random right-hand side, *Izv. Ross. Akad. Nauk*, **56**, No. 6, 1273–1315 (1992).
34. A. V. Dmitrenko, Heat and mass transfer and friction in injection to a supersonic region of the Laval nozzle, *Heat Transf. Res*., **31**, Issues 6–8, 338–399 (2000); http://dx.doi.org/:10.1615/HeatTrasRes.v31.i6-8.30.
35. A. V. Dmitrenko, Film cooling in nozzles with large geometric expansion using method of integral relation and second moment closure model for turbulence, *33th AIAA/ASME/SAE/ASEE Joint Propulsion Conference and Exhibit AIAA Paper 97-2911* (1997); doi: 10.2514/6.1997-2911. http://arc.aiaa.org/doi/abs/10.2514/6.1997-2911.
36. A. V. Dmitrenko, Heat and mass transfer in combustion chamber using a second-moment turbulence closure including an influence coefficient of the density fluctuation in film cooling conditions, *34th AIAA/ASME/SAE/ASEE Joint Propulsion Conference and Exhibit, AIAA Paper 98-3444* (1998); doi: 10.2514/6.1998-3444; http://arc.aiaa.org/doi/abs/10.2514/6.1998-3444.
37. A. V. Dmitrenko, Nonselfsimilarity of a boundary-layer flow of a high-temperature gas in a Laval nozzle, *Aviats. Tekh*., No. 1, 39–42 (1993).
38. A. V. Dmitrenko, Computational investigations of a turbulent thermal boundary layer in the presence of external flow pulsations, in: *Proc. 11th Conf. Young Scientists*, Moscow Physicotechnical Institute, Part 2, Moscow (1986), pp. 48–52. Deposited at VINITI 08.08.86, No. 5698-B86.