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MODELING NONSTATIONARY NONLINEAR-VISCOUS LIQUID FLOWS THROUGH A PIPELINE

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The process of nonstationary incompressible nonlinear-viscous liquid flow through a pipeline is considered. It is described by a parabolic-type one-dimensional nonlinear equation. The problem of determining the dependence of the pressure difference on time from the assigned volumetric flow rate in the given pipeline is formulated. Such a problem pertains to the class of inverse problems connected with the restoration of the dependence of the right-hand sides of parabolic equations on time. A computational algorithm for solving the problem has been suggested.

Keywords: *pipeline transportation, nonlinear-viscous fluid, pressure difference, inverse problem.*

Introduction. At the present time, pipeline transportation is the main form of conveying nonlinear-viscous fluids. Usually, in designing pipelines the following parameters are specified: the fluid flow rate, which is the basic characteristic of the pipeline capacity in conformity with its function, and the location of the pipeline beginning and end. Here, among the particular concerns of the present work, is the determination of the pressure difference needed for the transportation of the specified fluid flow through the given pipeline. In practice, for solving such a problem, use is mainly made of the differential equations of stationary nonlinear-viscous fluid flow through a pipeline [1–3]. However, for the pipeline transport operation it is important to carry out investigations on determination of the dependence of pressure difference on the fluid flow rate for transported nonstationary nonlinear-viscous fluid flows. This work presents a numerical method of determining such a dependence by solving an inverse problem for the equation of nonstationary nonlinear-viscous fluid flow in a pipeline.

Formulation of the Problem. Let there be a horizontal rigid-walled pipeline of radius R and length l with an incompressible nonlinear-viscous fluid pumped through it. The rheological equation of the fluid has the form

$$\tau = k\left|\frac{\partial u}{\partial r}\right|^{\gamma-1}\frac{\partial u}{\partial r}.$$

The 0z axis is assumed to lie along the pipeline axis, and the flow, to be directed along the pipe axis so that, of the three velocity components (u_r, u_φ, u_z), only the velocity u_z remains, whereas u_r and u_φ are equal to zero. The mathematical model of nonstationary nonlinear-viscous fluid flow in the pipeline can then be presented in the form [3]

$$\rho\frac{\partial u_z}{\partial t} + \rho u_z\frac{\partial u_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\left|\frac{\partial u_z}{\partial r}\right|^{\gamma-1}\frac{\partial u_z}{\partial r}\right) - \frac{\partial P}{\partial z}, \quad 0 < r < R, \quad 0 < t \le T,$$

$$\frac{\partial u_z}{\partial z} = 0, \quad \frac{1}{\rho}\frac{\partial P}{\partial r} = 0, \quad \frac{1}{\rho}\frac{\partial P}{\partial \varphi} = 0.$$

It follows from the second equation of this system that u_z is a function of only r and t and that the last two equations show the pressure P to be independent of r and φ, i.e., that $\frac{\partial P}{\partial z}$ is a function of time. Assuming that

$$u(r, t) = u_z(r, t), \; f(t) = -\frac{\partial P}{\partial z},$$

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from the previous system of equations we obtain the equation of nonstationary nonlinear-viscous incompressible fluid flow in a pipeline in the form

$$\rho \frac{\partial u_z}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk \left| \frac{\partial u_z}{\partial r} \right|^{\gamma-1} \frac{\partial u_z}{\partial r} \right) + f(t), \quad 0 < r < R, \quad 0 < t \le T . \tag{1}$$

Let the following initial condition hold for Eq. (1):

$$u(r, 0) = \varphi(r) , \tag{2}$$

and the natural boundary conditions:

$$\left.\frac{\partial u}{\partial r}\right|_{r=0} = 0 , \tag{3}$$

$$u(R, t) = 0 . \tag{4}$$

It is obvious that by specifying the law of pressure variation $f(t)$ in time and solving problem (1)–(4), we can find the fluid flow rate in the pipeline from the formula

$$Q(t) = \int_0^R 2\pi r u dr . \tag{5}$$

We assume now that the function of the change in the volumetric flow rate of fluid $Q(t)$ is known and we are to find such a law of the change in the pressure difference $f(t)$ that would ensure the passage of the given fluid volume through the pipeline. Thus, we are confronted with the problem of determining the functions $u(r, t)$ and $f(t)$ that would satisfy Eq. (1) and conditions (2)–(5). This problem relates to the class of inverse problems associated with the regeneration of the dependence of the right-hand sides of parabolic equations on time [4]. However, here as an additional condition we assign the integral characteristic of the investigated process.

Method of Solution. We multiply both sides of Eq. (1) by r and integrate the result over $[0, r]$ with respect to the variable r. Performing integration by parts and accounting for condition (3), we obtain

$$\rho \frac{\partial}{\partial t}\int_0^r u\eta d\eta = kr \left| \frac{\partial u}{\partial r} \right|^{\gamma-1} \frac{\partial u}{\partial r} + \int_0^r f\eta d\eta .$$

Denoting

$$\int_0^r u\eta \, d\eta = w(r, t) ,$$

we write the last integral relation in the form

$$\rho \frac{\partial w}{\partial t} = k \left| \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} \right|^{\gamma-1} \left(\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w}{\partial r} \right) + \frac{r^2}{2} f(t), \quad 0 < r < R, \quad 0 < t \le T . \tag{6}$$

Taking into account (2) and (4), for Eq. (6) we will seek the following initial condition:

$$w(r, 0) = \psi(r) , \tag{7}$$

and the boundary conditions:

$$w(0, t) = 0 , \tag{8}$$

$$\left.\frac{\partial w}{\partial r}\right|_{r=R} = 0 , \tag{9}$$

$$w(R, t) = \frac{Q}{2\pi}, \tag{10}$$

where $\psi(r) = \int_0^r \eta\varphi(\eta)d\eta$.

To numerically solve the inverse problem (6)–(10), we use the approach suggested in [5, 6]. For this purpose, we introduce a uniform difference grid

$$\omega_{h\tau} = \{(t_j, r_i) : r_i = i\Delta r, \quad t_j = j\Delta t, \quad i = 0, 1, 2, ..., N, \quad j = 0, 1, 2, ..., M\}$$

with the step $\Delta r = R/N$ for the variable r and the step $\Delta t = T/M$ in time t. We represent the difference analog of Eq. (6) on the grid $\omega_{h\tau}$ in the following form:

$$\rho \frac{w_i^{j+1} - w_i^j}{\Delta t} = k \left| \frac{1}{r_i} \frac{w_{i+1}^j - 2w_i^j + w_{i-1}^j}{\Delta r^2} - \frac{1}{r_i^2} \frac{w_i^j - w_{i-1}^j}{\Delta r} \right|^{\gamma-1}$$

$$+ \left(\frac{w_{i+1}^{j+1} - 2w_i^{j+1} + w_{i-1}^{j+1}}{\Delta r^2} - \frac{1}{r_i} \frac{w_i^{j+1} - w_{i-1}^{j+1}}{\Delta r} \right) + \frac{r_i^2}{2} f^{j+1},$$

$$i = 1, 2, 3, \dots, N-1, \quad j = 0, 1, 2, 3, \dots, M-1,$$

where $w_i^j \approx w(r_i, t_j)$. Approximating conditions (7)–(10), we obtain

$$w_i^0 = ø_i, \quad i = \overline{0, N},$$

$$w_0^{j+1} = 0,$$

$$\frac{w_N^{j+1} - w_{N-1}^{j+1}}{h} = 0,$$

$$w_N^{j+1} = \frac{Q^{j+1}}{2\pi},$$

where $\psi_i = \psi(r_i)$, $Q^{j+1} = Q(t_{j+1})$. We transform the resulting system of the difference equations as

$$a_i w_{i-1}^{j+1} - c_i w_i^{j+1} + b_i w_{i+1}^{j+1} = -(\rho\Delta r^2 w_i^j + 0.5\tau\Delta r^2 r_i^2 f^{j+1}), \quad i = \overline{1, N-1}, j = \overline{0, M-1}, \tag{11}$$

$$w_i^0 = \psi_i, \quad i = \overline{0, N}, \tag{12}$$

$$w_0^{j+1} = 0, \tag{13}$$

$$w_N^{j+1} = w_{N-1}^{j+1}, \tag{14}$$

$$w_N^{j+1} = \frac{Q^{j+1}}{2\pi}, \quad j = 0, 1, 2, ..., M-1, \tag{15}$$

where $a_i = \tau\lambda_i^j \left(1 + \frac{\Delta r}{r_i}\right)$, $b_i = \tau\lambda_i^j$, $c_i = a_i + b_i + \rho\Delta r^2$, $\lambda_i^j = \left| \frac{1}{r_i} \frac{w_{i+1}^j - 2w_i^j + w_{i-1}^j}{\Delta r^2} - \frac{1}{r_i^2} \frac{w_i^j - w_{i-1}^j}{\Delta r} \right|^{\gamma-1}$.

We represent the solution of problem (11)–(15) in the form

$$w_{i+1}^{j+1} = \alpha_{i+1} w_i^{j+1} + \beta_{i+1}, \quad i = 0,\ 1,\ 2, ..., N-1, \tag{16}$$

where α_{i+1}, β_{i+1} are the so far unknown coefficients. An analogous expression for w_i^{j+1} is

$$w_{i+1}^{j+1} = \alpha_{i+1} w_i^{j+1} + \beta_{i+1}.$$

Substituting the expressions for w_i^{j+1} and w_{i-1}^{j+1} into Eq. (11), we obtain the following nonlinear equations for determining the coefficients α_i and β_i:

$$\alpha_i = \frac{a_i}{c_i - b_i \alpha_{i+1}}, \quad \beta_i = \frac{b_i \beta_{i+1} + \rho \Delta r^2 w_i^j + 0.5\tau \Delta r^2 r_i^2 f^{j+1}}{c_i - b_i \alpha_{i+1}}, \quad i = N-1,\ N-2,\ \ldots, 1.$$

We find the initial values of the coefficients α_i and β_i from the requirement of the equivalence of condition (16) at $i = N-1$, i.e., $w_N^{j+1} = \alpha_N w_{N-1}^{j+1} + \beta_N$, to Eq. (14):

$$\alpha_N = 1, \qquad \beta_N = 0.$$

The nonlinear equations for β_i is transformed as

$$\beta_i = \frac{\alpha_i b_i}{a_i} \beta_{i+1} + \frac{\alpha_i \rho \Delta r^2}{a_i} w_i^j + \frac{0.5 \alpha_i \tau \Delta r^2 r_i^2}{a_i} f^{j+1}$$

or

$$\beta_i = s_i \beta_{i+1} + y_i + d_i f^{j+1}, \tag{17}$$

where $s_i = \dfrac{\alpha_i b_i}{a_i}$, $y_i = \dfrac{\alpha_i \rho \Delta r^2}{a_i} w_i^j$, $d_i = \dfrac{0.5 \alpha_i \tau \Delta r^2 r_i^2}{a_i}$.

Let us introduce the new variables $\tilde{\beta}_i$, $\tilde{d}_i$, $i = 1, 2, \ldots, N-1$ that satisfy the equations

$$\tilde{\beta}_i = s_i \tilde{\beta}_{i+1} + y_i,\ \tilde{d}_i = s_i \tilde{d}_{i+1} + d_i,\ \tilde{\beta}_N = \beta_N,\ \tilde{d}_N = 0.$$

With account for the newly introduced variables, Eq. (17) can be represented in the form of the recurrent relation

$$\tilde{\beta}_i = \tilde{\beta}_i + \tilde{d}_i f^{j+1}, \qquad i = N-1,\ \ N-2, ..., 1. \tag{18}$$

We will now find the relationship between w_N^{j+1} and w_0^{j+1} in explicit form. For this purpose, we write relation (16) at $i = N-1$:

$$w_N^{j+1} = \alpha_N w_{N-1}^{j+1} + \beta_N.$$

Substituting, into this equation, the expression for w_{N-1}^{j+1}, i.e., $w_{N-1}^{j+1} = \alpha_{N-1} w_{N-2}^{j+1} + \beta_{N-1}$, we obtain

$$w_N^{j+1} = \alpha_N \alpha_{N-1} w_{N-2}^{j+1} + \alpha_N \beta_{N-1} + \beta_N.$$

Next, using the expressions for w_{N-2}^{j+1}, w_{N-3}^{j+1}, …, w_1^{j+1} in the latter equation, we obtain the following equation that connects w_N^{j+1} and w_0^{j+1}:

TABLE 1. Results of Numerical Experiment

t	f^t	$\bar{f}$	$\tilde{f}$ at $\delta = 0.002$			
			$\tau = 0.05$	$\tau = 0.5$	$\tau = 1$	$\tau = 2$
10	6.01	6.01	3.00	5.66	6.03	6.03
20	6.75	6.75	5.67	6.12	6.57	6.76
30	7.00	7.00	10.42	6.70	6.79	6.85
40	6.70	6.70	8.90	6.92	6.39	6.61
50	5.94	5.94	5.92	6.04	5.91	5.91
60	4.91	4.91	2.62	5.02	4.76	4.80
70	3.91	3.91	–2.90	4.01	3.78	4.01
80	3.21	3.21	2.45	3.34	3.32	3.05
90	3.00	3.00	4.62	3.09	3.17	3.12
100	3.35	3.35	–0.03	3.05	3.40	3.33
110	4.14	4.14	3.68	4.26	4.08	4.22
120	5.18	5.18	5.72	5.43	5.23	5.10
130	6.16	6.16	6.63	6.33	6.18	6.09

$$w_N^{j+1} = w_0^{j+1} \prod_{i=1}^{N} \alpha_i + \sum_{i=1}^{N-1} \beta_i \prod_{n=i+1}^{N} \alpha_n + \beta_N .$$

This equation, subject to the recurrent relation (18), can be written as

$$w_N^{j+1} = w_0^{j+1} \prod_{i=1}^{N} \alpha_i + \sum_{i=1}^{N-1} \tilde{\beta}_i \prod_{n=i+1}^{N} \alpha_n + f^{j+1} \sum_{i=1}^{N-1} \tilde{d}_i \prod_{n=i+1}^{N} \alpha_n + \beta_N .$$

The latter equation, with Eqs. (13) and (15) taken into account, yields an approximate value of the sought function $f(t)$ at $t = t_{j+1}$:

$$f^{j+1} = \frac{\dfrac{Q^{j+1}}{2\pi} - \sum_{i=1}^{N-1} \tilde{\beta}_i \prod_{n=i+1}^{N} \alpha_n - \beta_N}{\sum_{i=1}^{N-1} \tilde{d}_i \prod_{n=i+1}^{N} \alpha_n} . \tag{19}$$

Determining f^{j+1} by Eq. (19), we can successively find w_1^{j+1}, w_2^{j+1}, ..., w_N^{j+1} from the recurrent formula (16). In going over to the next time layer, the computational procedure is repeated.

Thus, the proposed numerical method allows one in each time layer to successively determine the pressure difference and the fluid flow velocity distribution in the pipeline.

Results of Numerical Calculations. To elucidate whether the proposed computational algorithm can be efficiently applied in practice, numerical experiments were carried out for model problems. The schematic of a numerical experiment was as follows. The direct problem (6)–(9) was solved for the given functions $f(t)$ and $\psi(r)$. The resulting dependence $Q(t) = 2\pi w(R, t)$ was taken as that affording precise data for numerical solution of the inverse problem on the regeneration of $f(t)$.

The first series of calculations was carried out with the use of these unperturbed data. The second was carried out after imposing, on $Q(t)$, some function that models the error of experimental data:

$$\tilde{Q}(t) = Q(t) + \delta\sigma(t) ,$$

where $\sigma(t)$ denotes a random process modeled with the aid of a random-number generator.

The calculations were performed on a spatial-temporal difference grid with the steps $h = 0.02$; $\tau = 0.05$; 0.5; 1; 2. The results of the numerical experiment carried out for the case $\gamma = 0.8$, $\rho = 850$ kg/m^3; $R = 1.2$ m; $k = 0.2$ Pa·s$^\gamma$; $f(t) = 5 - 2\sin 10t$; $\psi(r) = 0.02r + 0.05$ with the use of unperturbed and perturbed input data are presented in Table 1. An error of the level $\delta = 0.002$ was used for the perturbation of input data.

As the results of the numerical experiment show, in using unperturbed input data the sought function $f(t)$ is regenerated exactly on all computational time grids (the 2nd and 3rd columns of Table 1). In the case of perturbed input data at which the error has a fluctuational character, the sought function $f(t)$ is regenerated with an error that is manifested more strongly on decrease in the time step ($\tau = 0.05$). However, the results obtained at large time steps ($\tau = 0.5$, 1, 2) point to the fact that an increase in the time step ensures the algorithm stability against the errors in the input data. An analysis of the results of numerical experiment indicates that in the proposed computational algorithm the regularization effect is realized by selecting the difference grid in time.

Conclusions. The problem on determining the pressure difference needed for the passage of the assigned volume of nonlinear-viscous fluid through the given pipeline has been considered. To solve the problem posed, a numerical method is suggested based on the use of a mathematical model of the process of nonstationary incompressible nonlinear-viscous fluid flow in a pipeline. The proposed method can be applied for investigating the flow of nonlinear-viscous fluids in pipelines.

NOTATION

$f(t)$, function describing a change of the pressure difference in time, Pa/m; $\bar{f}$, computed values of the function $f(t)$ at unperturbed data; $\tilde{f}$, computed values of the function $f(t)$ at perturbed data; f^t, precise values of the function $f(t)$; k, consistency parameter, Pa·s$^\gamma$; l, pipeline length, m; P, pressure, Pa; $Q(t)$, volumetric flow rate of fluid, m^3/s; R, pipe radius, m; r, radial coordinate, m; t, time, s; u_r, u_φ, u_z, fluid flow velocity vector components, m/s; γ, dimensionless parameter; δ, error level; ρ, fluid density, kg/m^3; $\sigma(t)$, random process modeled with the aid of a random-number generator; τ, shear stress, Pa.

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