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HEAT EXCHANGE BETWEEN A LIQUID FLOWING IN A TUBE AND AN EXTERNAL FLOW AROUND IT WITH INTENSE STIRRING

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A mathematical model of heat exchange between a tube (a coil) with a liquid flowing inside it and an external flow around it is considered. Limiting equations of the process with intense stirring are derived. Some solutions of the proposed equations are obtained that can be used in modeling the processes occurring in heat exchangers.

Keywords: *heat exchange, heat exchanger, heat-transfer agent, stirring, displacement.*

Introduction. It is usually assumed [1, 2] that in double-tube type heat exchangers there occurs an ideal displacement of a moving heat-transfer agent along the longitudinal coordinate (along the heat exchanger). However, if a heat exchanger contains a vessel where a heat-transfer agent is stirred intensely (for example, by means of a stirrer) and in this vessel there is a tube (a coil) in which a second heat-transfer agent flows, then, despite a certain similarity of the situations, one has to correct the theory. In principle, a unified approach to the description of the processes occurring in the vessel-coil apparatus and in a double-tube type heat exchanger is possible if one assumes that there is intensive stirring along the longitudinal coordinate in the intertube space.

In the present work, basic equations are derived for describing the operation of a double-tube type heat exchanger with intense longitudinal stirring in the intertube space. These equations are also suitable for the analysis of heat transfer in a vessel-coil apparatus. The selected technique for the derivation of these equations was attributable to the simpler analysis in the one-dimensional (with respect to coordinate) case. The direct derivation of corresponding equations in a vessel-coil apparatus requires an analysis of three-dimensional (with respect to coordinates) equations, which leads to the necessity of using a more complex mathematical technique. This does not entail fundamental difficulties, but calculations become more complicated. An example of this kind of three-dimensional analysis in a mathematically similar problem (an analogous algorithm of investigation) was considered in [3].

Formulation of the Problem. It should be noted that the following standard assumptions [1, 2] were made in our model: 1) one of the liquids flows inside a straight tube with another liquid flowing around it; 2) the liquid flow is considered as unidirectional (occurring along the x axis); 3) the stirring is full and is arranged in the direction perpendicular to the direction of flow motion, and therefore the velocity and temperature in any plane perpendicular to the x axis is averaged; 4) the stirring of the heat-transfer agent and heat transfer to the tube wall in the direction of the motion of media is not accounted for in the in-tube space but is taken into account in the intertube space; 5) the specific heats and densities of the liquid and wall in the region of the temperatures considered are taken to be constant; 6) the heat transfer coefficients are regarded as constant over the entire heat-exchanger surface; 7) heat losses into the environment are neglected; 8) the cross section and shape of the flow in the tube are taken to be invariable.

In view of the foregoing, the equation of heat transfer in the in-tube space is taken in its traditional form [2, 4]:

$$\rho_{\mathrm{T}} S_{\mathrm{T}} C_{\mathrm{T}} \frac{\partial T}{\partial t} + G_{\mathrm{T}} C_{\mathrm{T}} \frac{\partial T}{\partial x} = KP\left(\theta - T\right). \tag{1}$$

At the same time, in the intertube space the longitudinal stirring is taken into account:

$$\rho_{\theta} S_{\theta} C_{\theta} \frac{\partial \theta}{\partial t} + G_{\theta} C_{\theta} \frac{\partial \theta}{\partial x} + KP\left(\theta - T\right) = S_{\theta} \lambda \frac{\partial^2 \theta}{\partial x^2}. \tag{2}$$

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The quantities G, C, S, and ρ have the subscripts T or θ depending on whether they relate to the in-tube or intertube space. We assume that the thickness of the tube wall can be neglected just as its influence on the process of heat transfer (the equation of heat conduction in the wall is not used). Supplementary conditions for Eq. (1) have the form

$$T\big|_{x=0} = T_{x0}(t), \quad T\big|_{t=0} = T_{t0}(x). \tag{3}$$

They correspond to the assignment of temperature at the inlet to the in-tube space of the heat-exchanger and of the initial distribution of temperature in it. Supplementary conditions for Eq. (2) are

$$\theta\big|_{t=0} = \theta_0(x), \quad S_\theta \lambda \left.\frac{\partial\theta}{\partial x}\right|_{x=0} = G_\theta C_\theta \left[\theta - \theta_+(t)\right], \quad \left.\frac{\partial\theta}{\partial x}\right|_{x=L} = 0, \tag{4}$$

where the second condition represents the energy balance at the inlet into the intertube space of the heat exchanger ($\theta_+(t)$ is the temperature of the incoming flow), and the third condition represents the traditional Danckwerts condition.

Simplification of Problem (1)–(4) in the Case of Intense Stirring. Of main interest in the present work is the limiting case of intense stirring since it is this case that is implemented (or is desired to be implemented) in vessel-coil apparatuses. For asymptotic consideration of the problem, it is expedient to write it in dimensionless coordinates. The previous experience of derivation of the equations for the regime of ideal stirring [5–7] and a further analysis of the problem prompt the introduction of the following dimensionless variables and similarity parameters:

$$z = \frac{x}{L}, \quad \tau = \frac{KPt}{\rho_\theta S_\theta C_\theta}, \quad A = \frac{KPL}{G_\theta C_\theta}, \quad B = \frac{KPL}{G_T C_T}, \quad \mathrm{Pe} = \frac{LG_\theta C_\theta}{S_\theta \lambda}, \quad \varepsilon = \frac{\rho_T S_T KPL^2}{\rho_\theta C_\theta G_T W}, \quad W = S_\theta L, \tag{5}$$

where W determines the volume of the intertube space. In the variables of Eq. (5) Eqs. (1) and (2) are written as

$$\varepsilon \frac{\partial T}{\partial \tau} + \frac{\partial T}{\partial z} = B(\theta - T), \tag{6}$$

$$\mathrm{Pe}\left[A\frac{\partial\theta}{\partial\tau} + \frac{\partial\theta}{\partial z} + A(\theta - T)\right] = \frac{\partial^2\theta}{\partial z^2}. \tag{7}$$

The regime of intense (ideal) stirring is realized as a limiting variant of problem (6), (7) with corresponding supplementary conditions for $\mathrm{Pe} \to 0$ [5–7]. To carry out a limiting analysis of Eq. (7), we also write out the dimensionless supplementary conditions for it:

$$\left.\frac{\partial\theta}{\partial z}\right|_{z=0} = \mathrm{Pe}(\theta - \theta_+), \quad \left.\frac{\partial\theta}{\partial z}\right|_{z=1} = 0. \tag{8}$$

The initial conditions for Eqs. (6) and (7) retain their form also after the substitution of variables with dimensionless τ and z.

We will seek the solution of problem (6)–(8), (3) subject to initial condition (4) for the function θ in the form of the perturbation method series [8, 9]:

$$\theta = g_0 + \mathrm{Pe}g_1 + \mathrm{Pe}^2 g_2 + \ldots, \quad T = G_0 + \mathrm{Pe}G_1 + \mathrm{Pe}^2 G_2 + \ldots. \tag{9}$$

Here we limit ourselves to the basic approximation g_0, G_0. In such a case an analysis of Eq. (7) is actually sufficient. Substituting expansion (9) into Eqs. (7) and (8), and grouping terms of the same order in the Pe number, we obtain the following problem in the basic approximation:

$$\partial^2 g_0/\partial z^2 = 0; \quad \partial g_0/\partial z\,|_{z=0;1} = 0,$$

whence we find that the function g_0 depends only on τ. We use this fact in the problem of a first approximation in the Pe number:

$$\frac{\partial^2 g_1}{\partial z^2} = A\left(\frac{dg_0}{d\tau} + g_0 - G_0\right), \quad \left.\frac{\partial g_1}{\partial z}\right|_{z=0} = g_0 - \theta_+, \quad \left.\frac{\partial g_1}{\partial z}\right|_{z=1} = 0. \tag{10}$$

After integrating Eq. (10) with respect to z over (0, 1) and taking the boundary conditions into account, we obtain the following expression:

$$\frac{dg_0}{d\tau} + g_0 - \langle G_0 \rangle = \frac{\theta_+ - g_0}{A}, \tag{11}$$

where the angular brackets designate the average value of some quantity:

$$\langle f \rangle = \int_0^1 f\,(z, ...)\,dz\,. \tag{12}$$

In this equation, certain parameters can be put in lieu of the three periods. If required (insufficiently intense stirring, or a not very small Pe as in the present case), correction equations for the basic approximation can be obtained, i.e., equations for the functions g_1 and G_1, etc., in expansion (9) by the method described in [3, 5–7].

If the function g_0 (9) depends only on time, the question remains open as to how one should use the initial condition (4) present in the initial formulation of the problem. The thing is that the time scale $\rho_\theta S_\theta C_\theta / KP$ selected in (5) is large enough. It cannot be used for describing the initial stage of the process, i.e., the specific features of stirring within the framework of the diffusional model (7). The problem turns out to be singularly perturbed for Pe → 0 [8, 9], and for its investigation it is necessary to construct another supplementing expansion (9) with a smaller (than in (5)), time scale. It is interesting to note that traditionally (for example, in [1]) the equation of type (11) are called the equation of the model of ideal stirring: they do not describe large time scale, but intense, stirring.

Let us introduce the "compressed" time $\zeta = \tau/\mathrm{Pe}$ and substitute it into Eq. (7). Next, instead of (9) we determine the "internal" expansion:

$$\theta = \overline{g_0} + \mathrm{Pe}\overline{g_1} + \mathrm{Pe}^2\overline{g_2} + ... \tag{13}$$

for Eq. (7), designing the terms of the internal expansion by overbars. In the basic internal expansion, realizable at Pe → 0, in the "internal" equation we obtain the problem

$$\frac{\partial^2 \overline{g_0}}{\partial z^2} = A\frac{\partial \overline{g_0}}{\partial \zeta}, \quad \overline{g_0}\Big|_{\zeta=0} = \theta_0\,(z), \quad \frac{\partial \overline{g_0}}{\partial z}\Big|_{z=0;1} = 0\,. \tag{14}$$

The solution of (14) can be easily found by the standard methods of mathematical physics, for example, with the aid of the Fourier method [10]. It is just problem (14) that describes the details of stirring, i.e., the reduction of the solution to a uniformly distributed temperature over the vessel′ space within the framework of the adopted diffusional model.

For the purposes of this work, i.e., the matching of expansion (13) with the solution of the external problem, which is of basic interest for practice, it will be sufficient to determine the average value of the function $\overline{g}_0$. Applying the averaging procedure (12) to Eq. (14) and taking into account the supplementary conditions, we have

$$A\frac{d\langle \overline{g_0} \rangle}{d\zeta} = 0\,, \quad \Rightarrow \langle \overline{g_0} \rangle = \mathrm{const}\,(\zeta) = \langle \theta_0\,(z) \rangle\,. \tag{15}$$

According to the principle of limiting matching in the case of its applicability [8, 9], the following limiting equality must hold subject to (15):

$$g_0\,(0) = \lim_{\tau\to 0} g_0\,(\tau) = \lim_{\tau\to 0} \langle g_0\,(\tau) \rangle = \lim_{\zeta\to\infty} \langle \overline{g_0} \rangle = \langle \theta_0 \rangle\,. \tag{16}$$

Relation (16) is actually the initial condition for Eq. (11). The fact that, as a result, the initial condition for the "external" problem has been derived simply by averaging the initial condition agrees naturally with the intuitive viewpoint. The prediction of an analog of condition (16) for the highest approximations of expansion (9) just becomes nontrivial [3, 5–7].

Example of Calculation of the Vessel-Coil Apparatus on the Basis of Model (6), (11). Now we will be concerned with problem (6), (3), (16), and (11) in the limiting case $A \to \infty$, i.e.,

$$\frac{d\theta}{d\tau} = \langle T \rangle - \theta\,. \tag{17}$$

Within the framework of the basic approximation adopted here, limiting ourselves only to the zero-order terms with respect to the Re number and for the brevity of notation, we can replace the functions g_0 and G_0, respectively, by the functions θ and T from Eq. (11) and (6), precisely which is done in Eq. (17). Equation (6), after substitution of expansion (9) into it, will essentially be the equation of the basic approximation in the Pe number. The analysis performed herein shows that the function θ in this equation depends now only on the dimensionless time τ.

It is worthwhile to simplify Eq. (11) because we consider a noncontinuous-flow apparatus, i.e., $G_\theta = 0 \Rightarrow A \to \infty$. Note that for $A \to \infty$ the large value of A is multiplied by the small Pe in the asymptotic analysis of Eq. (7). Actually, for correct deviation of Eq. (17) it is necessary that $\mathrm{Pe}A = KPL^3/\mathrm{W}\lambda << 1$. We see that the latter expression does not contain the quantity G_θ that would tend to zero (it is involved in A and Pe separately), so that we can derive the limiting equation (17) anew, and so the situation will even be simpler.

For the nonessential simplification we will assume that $T_{t0} = \text{const}\ (x)$ and $T_{x0} = \text{const}\ (t)$ in the supplementary conditions (3). It should be noted that in the simplified variant ($\varepsilon = 0$ in Eq. (6)) the problem was investigated in [11] and, in essence, was described earlier in [12]. In the latter works the absence of the "nonstationary" terms was considered as self-evident. This seems to be due to the fact that a case like this is often encountered in practice. Moreover, the formulas obtained were confirmed experimentally. Considering the foregoing, it should be stressed that of practical interest is the investigation of this problem at small values of ε.

Asymptotic Analysis of Problem (6), (3), (16), (17) at $\varepsilon = 0$. Just as above, we limit ourselves only to the solution of the equations of the basic approximation in the parameter ε.

External solution. Assuming that $\varepsilon = 0$ in (6), we obtain an equation

$$\frac{\partial T}{\partial z} = B\left(\theta - T\right), \tag{18}$$

which together with (17) determines the progress of the process in a quasi-stationary approximation. The supplementary conditions for these equations are

$$\theta\left(0\right) = \theta_0 = \text{const},\ T(t, 0) = T_{z0} = \text{const}\ . \tag{19}$$

The second condition (in the variable t) for the equation of the external problem is not required. In a typical situation of interest there is the inequality $T_{z0} < \theta_0$ which is assumed to be satisfied, i.e., a cooling liquid flows inside the tube (coil).

Since the function θ is independent of z, it can be regarded as a constant in Eq. (18). Then the integral of this equation subject to the second condition of (19) has the form

$$T(\tau, z) = \theta(\tau) + \left[T_{z0} - \theta(\tau)\right] \exp\left(-Bz\right). \tag{20}$$

Equation (17) involves the average value of T. For this quantity, from relation (20) subject to (12), we have

$$\langle T(\tau)\rangle = \theta(\tau) + \kappa[T_{z0} - \theta(\tau)]\ , \tag{21}$$

where

$$\kappa = \left[1 - \exp\left(-B\right)\right]/B\ . \tag{22}$$

Substitution of (21) into (17) yields

$$\frac{d\theta}{d\tau} = \kappa\left(T_{z0} - \theta\right). \tag{23}$$

From Eq. (23), subject to the first condition of (19), we find the function $\theta(\tau)$:

$$\theta(\tau) = T_{z0} + (\theta_0 - T_{z0}) \exp\left(-\kappa\tau\right), \tag{24}$$

which precisely completes the construction of the external problem. It should be noted that, accurate to notations, such a solution is cited in [11]. Equations (18)–(24) can also yield us the results obtained in [12].

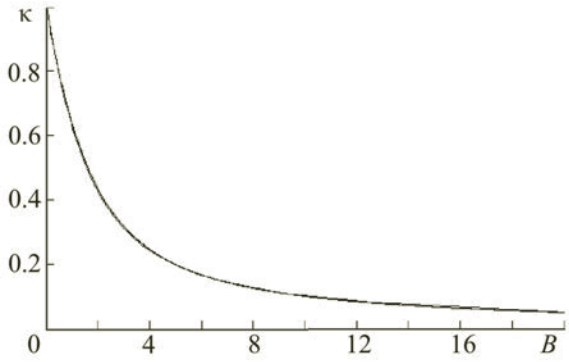

Fig. 1. Graphs of functions (22) and (25).

In processing the experimental data on the change of the functions of the problem in time, it is often convenient to determine the exponent in relations of type (24). It is natural that of basic interest is the parameter B connected with the heat transfer coefficient K. The quantity B is not expressed in an explicit form via the parameter κ from the transcendental equation (22). Though it is easy to find B from the known value of κ with the aid of standard numerical algorithms, it is more preferable to have a sufficiently accurate analytical expression. There exist methods for constructing approximation expressions that make use of the expertise of the asymptotic solutions of the problem. We avail ourselves here of the Padé approximation [13] that allows one to approximately "invert" dependence (22) as follows [14]:

$$B = \frac{2\left(1-\kappa\right)\left(1+1.4\kappa\right)}{\kappa\left(2+0.4\kappa\right)} . \tag{25}$$

Figure 1 presents the dependences of κ on B, i.e., the graph that illustrates the problem of "inversion" of (22), and the approximate dependence of function (25). The plotted curves turned out to be so close that it was meaningless to label them in the figure, i.e., the approximation of the dependence (22) by formula (25) is quite acceptable. The slightly noticeable difference in the graphs is evident only in the interval $B \in (1.0, 4.2)$. At $B > 4.2$ ($\kappa < 0.2$) the asymptotic ($\kappa \to 0$, $B \to \infty$) formula $B \approx 1/\kappa$ can be used, and also beyond Fig. 1.

Internal solution. To construct this expansion, we introduce the "internal" time $\xi = \tau/\varepsilon$. The system of equations (6), (17) in the internal variables ξ, z can be rewritten as

$$\frac{\partial T}{\partial \xi} + \frac{\partial T}{\partial z} = B\left(\theta - T\right), \tag{26}$$

$$\frac{d\theta}{d\xi} = \varepsilon\left(\langle T\rangle - \theta\right). \tag{27}$$

The supplementary conditions for system (26), (27) are the following relations:

$$\theta\left(0\right) = \theta_0 = \text{const}, \quad T(\xi, 0) = T_{z0} = \text{const}, \quad T\left(0, z\right) = T_{t0} = \text{const}. \tag{28}$$

It is seen from Eq. (27) that in the basic approximation in the ε number the solution of this equation is the constant $\theta(\xi) = \theta_0 = \text{const}$, i.e., in the small time scale in which system (26), (27) "is operating", the temperature in the intertube space has insufficient time to be altered. This simplifies Eq. (26), which can be solved, for example, by the operational method [10, 15]. As a result we obtain

$$T(\xi, z) = \theta_0 + (T_{z0} - \theta_0)\exp\left(-Bz\right)H(\xi - z) + (T_{t0} - \theta_0)\exp\left(-B\xi\right)H(z - \xi), \tag{29}$$

where $H(z) = 1$ at $z > 0$ and $H(z) = 0$ at $z < 0$ is the Heaviside function. This function shows that the solution in the tube is of wave character. The wave velocity in the dimensionless coordinates ξ, z is equal to unity and in the dimensional, to $U_T = G_T/(S_T\rho_T)$, i.e., to the heat-transfer agent velocity in the tube. After the passage of the wave "front" through the entire tube of

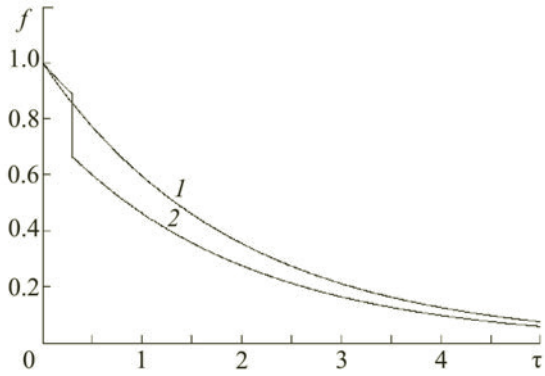

Fig. 2. Dimensionless temperatures vs. time: 1) $f = f_\theta$; 2) $f = f_T$.

length L in the dimensional time $t = L/U_T$, solution (29) corresponds, as is to be the case, to Eq. (20) of the external solution at $\tau = 0$.

Uniformly applicable solution of zero order in ε. When we have at hand the external and internal solutions of the corresponding equations of zero order in ε, we, using the standard methods of [8, 9], can construct a combined, uniformly applicable approximation of the solution. Strictly speaking, the uniformity of the approximation can be disturbed when $\tau \to \infty$. In the problem considered, to construct a combined expression, use can be made of the sum of the external and internal solutions minus their common part obtained by the matching procedure. The common part for the function θ is θ_0 and for the function T, $\theta_0 + (T_{z0} - \theta_0) \exp(-Bz)$. As a result, we arrive at a situation where Eq. (24) of the external solution will also be a uniformly acceptable approximation for the function θ. For the function T we find

$$T = (T_{z0} - \theta_0) \exp(-Bz) H(\xi - z) + (T_{t0} - \theta_0) \exp(-B\mathrm{x}) H(z - \xi) + [\theta(\tau) - \theta_0][1 - \exp(-Bz)] . \quad (30)$$

Example. As a simple example of calculation by the proposed relations, we analyze a practically conventional case with $T_{t0} = \theta_0 > T_{z0}$, i.e., the temperature in the entire system is the same up to the moment of flow "initiation" in the tube. Let us consider the temperature at the exit from the tube $T(\tau, 1)$ and the temperature in the vessel θ as functions of time. For graphical illustration of the solution, it is convenient to represent the above-mentioned temperatures in the form of dimensionless complexes:

$$f_\theta(\tau) = \frac{\theta - T_{z0}}{\theta_0 - T_{z0}} = \exp(-\kappa\tau) , \quad f_T(\tau) = \frac{T - T_{z0}}{\theta_0 - T_{z0}} = \exp(-\kappa\tau)\left[1 - \exp(-B)\right] + H(\varepsilon - \tau) \exp(-B) . \quad (31)$$

To be more specific, we adopt the following values of the parameters: $\varepsilon = 0.3$ and $B = 1.5$, which, according to Eq. (22), leads to $\kappa = 0.518$. The results are presented in Fig. 2. We see that the influence of the wave "front" (of the internal solution) is exerted within the framework of the model at hand as a jump of the function f_T at the moment $\tau = \varepsilon(t = L/U_T)$. In a practical situation, sharp changes in the function T can be smoothed by means of the longitudinal stirring ignored in the in-tube space. Moreover, when the value of ε is small enough in an experiment (more precisely, when the time L/U_T is small as compared to the observation time), the described effect will be barely discernible.

Attention should be paid to the fact that $f_T > f_\theta$ before the passage of the wave "front" and $f_T < f_\theta$ after its passage. This is explained as follows: after the initiation of flow in the tube, the indirect heat transfer through the liquid in the vessel has no time to exert a noticeable effect on the heat-transfer agent portions located near the exit from the tube. The temperature in the vessel first decreases and then starts to influence the mentioned heat-transfer agent portions in the tube. However, after the passage of the wave "front," the cooler (cooling) heat-transfer agent naturally remains cooler. This can be easily verified analytically. To do this, we subtract the function f_θ from f_T according to (31). As a result we obtain $f_T(\tau) - f_\theta(\tau) = \exp(-B)[H(\varepsilon - \tau) - \exp(-\kappa\tau)]$. According to what has been said above, the indicated difference between the functions at $\tau > 0$ is negative or positive depending on whether the Heaviside function is equal to zero or unity (whether the "front" arrived at the point $z = 1$ or not). It is also natural that, when $\tau \to \infty$, both dependences get closer and tend to the common value of temperature $\theta(\infty) = T(\infty, 1) = T_{z0}$ equal to that at the coil inlet.

Exact Solution of the Problem. In this section we do not use the perturbation method [8, 9], but analyze the solution of the problem on the basis of Eqs. (26) and (27) not assuming that the parameter ε is small. We consider the practically most interesting situation where $T_{t0} = \theta_0 > T_{z0}$ under conditions (28).

We analyze the problem with the aid of the Laplace transformation [10, 15] for the variable ξ:

$$f^*(p) = \int_0^\infty f(\xi) \exp(-p\xi)\, d\xi . \tag{32}$$

The quantities transformed in this way will be denoted by an asterisk *. Applying operation (22) to system (26), (27) with account for the supplementary conditions (28), we obtain

$$dT^*/dz + (p + B)T^* = \theta_0 + B\theta^* , \quad (p + \varepsilon)\theta^* + \theta_0 + \varepsilon \left\langle T^* \right\rangle , \quad T^*|_{z=0} = T_{z0}/p . \tag{33}$$

Next, from (33), after integration of the differential equation for the function T^*, we have

$$\begin{aligned} T^* &= \frac{\theta_0 + B\theta^*}{p + B} + \left(\frac{T_{z0}}{p} - \frac{\theta_0 + B\theta^*}{p + B} \right) \exp\left[-z\,(p + B) \right] , \\ \left\langle T^* \right\rangle &= \frac{\theta_0 + B\theta^*}{p + B} + \left(\frac{T_{z0}}{p} - \frac{\theta_0 + B\theta^*}{p + B} \right) \frac{\left[1 - \exp\left(-p - B \right) \right]}{(p + B)} . \end{aligned} \tag{34}$$

Substituting the expression for the Laplace-transformed average temperature in the tube $\left\langle T^* \right\rangle$ into the second relation of (33) and making some transformations, we obtain

$$\theta^* = \theta_0 \frac{R(p) - Q(p)}{pR(p) + BQ(p)} + T_{z0} \frac{(p + B)\, Q(p)}{p\left[pR(p) + BQ(p) \right]} , \tag{35}$$

where

$$R(p) = (p + B)(p + B + \varepsilon) , \quad Q(p) = \varepsilon[1 - \exp(-p - B)] .$$

In Eq. (35) the influence of the temperatures θ_0 and T_{z0} on the process is separated. The inversion of (35) is connected with the determination of its singular points (poles) and with the finding, at these points, of the residues of this expression multiplied by exp ($p\xi$) [10, 15]. Note that relation (35) does not have branching points.

It is evident that Eq. (35) has a singular point $p = 0$. The value of the corresponding residue is equal to T_{z0}, which conforms to the stationary solution for $\xi \to \infty$. In addition to the point $p = 0$, the expression in the denominator of Eq. (35)

$$F(p) = pR(p) + BQ(p) \tag{36}$$

yields only one real singular point (pole) for relation (35). In the general case, the function $F(p)$ has two real zeros, with one of them being the point $p = -B$. The point $p = -B$ will be a second-order zero if the function

$$\nu(\varepsilon, B) = 2\varepsilon - B(2 + \varepsilon) \tag{37}$$

is not equal to zero. Otherwise, it is a third-order zero. In the latter variant the point $p = -B$ is the pole of relation (35). Of importance is the fact that the numerators in Eq. (35) at $p = -B$ have second-order zeros; therefore, at $\nu \neq 0$ they "compensate" the second-order zeros of the denominator, i.e., the point $p = -B$ is not singular. It can be shown by the methods of differential calculus that there always exists a real root of the equation $F(p) = 0$ (except for $p = -B$) that at $\nu = 0$ combines with the root $p = -B$. Thus, there is only one pole at the real point: $p = -B$ at $\nu = 0$ and at some other point at $\nu \neq 0$. The situation is illustrated by Fig. 3 where the function F is plotted at several (characteristic) values of the parameters $\varepsilon > 0$ and $B > 0$. Note that the foregoing is related precisely to this, physically meaningful, variant of the positive values of ε and B.

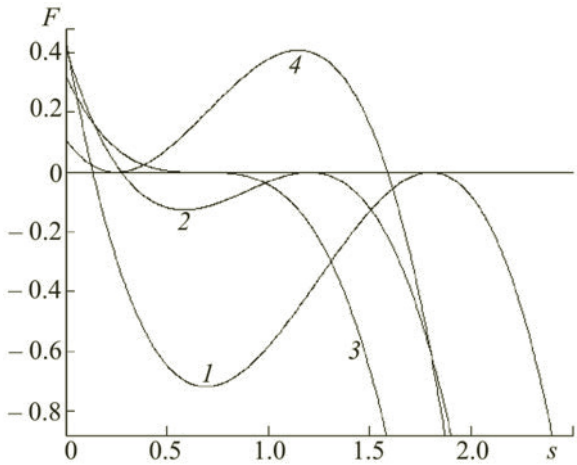

Fig. 3. Graphs of function F at: 1) $\varepsilon = 0.5$, $B = 1.8$; 2) 0.5, 1.2; 3) 2/3; 4) 2, 0.25.

For the sake of convenience, in Fig. 3 we actually "represented" the abscissa axis by replacing $p = -s$ in order to present the results in the right half-plane. Curves 1 and 2 correspond to the variant with $\nu < 0$. They intersect the abscissa axis at the points $s = 0.1337$ and 0.2746, respectively. In some sense, curve 3 is exclusive because $\nu = 0$. For curve 4 the value of the function ν in Eq. (37) is positive. Here the zero of the function F, which is also the pole of Eq. (35), is realized at $s = 1.5884$. Apart from the indicated roots of the equation $F(p) = 0$, it has an infinite number of complex roots, which follows from the generalized Picard theorem [16]. Indeed, let us consider the expression

$$F_\omega(p) = \omega[pR(p) + B\varepsilon] - B\varepsilon \exp(-p - B) . \tag{38}$$

At $\omega = 1$ Eq. (38) yields Eq. (36), and at $\omega = 0$ the relation $-B\varepsilon \exp(-p - B)$, which does not contain any root as a function of p [16]. Thus, according to the generalized Picard theorem, the value $\omega = 0$ is exclusive for the equation $F_\omega(p) = 0$, which means that at any complex (different from zero) values of the parameter ω, at $\omega = 1$ too, the equation $F_\omega(p) = 0$ has an infinite set of complex roots.

We denote the real root by the symbol p_0, and the complex roots, with the positive imaginary part for definiteness, by p_k ($k = 1, 2, 3, \ldots$). Applying the operation of complex conjugation to the equation $F(p_k) = 0$ and taking into account the fact that all the coefficients at the function F are real, we easily see that the quantities $\overline{p}_k$, complex conjugate of p_k, will also be roots of the equation $F(p) = 0$. Therefore in what follows, when writing the solution, it is expedient to combine terms corresponding to the complex conjugate points. As a result of the application of the theorem on residues and of other standard operations [10, 15], we can write the inversion of Eq. (35) in the form

$$\theta(\xi) = T_{z0} + \frac{\theta_0 \left\{\left[R(p_0) - Q(p_0)\right] + T_{z0}(p_0 + B) Q(p_0)/p_0\right\} \exp(p_0\xi)}{p_0 R'_p(p_0) + R(p_0) + BQ'_p(p_0)}$$
$$+ 2\,\mathrm{Re} \sum_{k=1}^{\infty} \frac{\theta_0 \left\{\left[R(p_k) - Q(p_k)\right] + T_{z0}(p_k + B) Q(p_k)/p_k\right\} \exp(p_k\xi)}{p_k R'_p(p_k) + R(p_k) + BQ'_p(p_k)} . \tag{39}$$

Here Re is the real part of the expression. Note that with $\varepsilon \to 0$ and changeover from the variable ξ to τ Eq. (39) can yield the limiting form of solution (24).

In (39) all of the complex roots p_k have a negative real part whose absolute value exceeds the real part p_0. This allows the assertion that the behavior of the function $\theta(\xi)$ is determined at large enough values of time by the first terms (outside the sum) of Eq. (39). We will show this in the case of $\nu < 0$. For this purpose, we use the argument that, in the given case, gives the number of zeros inside the contour γ traversed in the positive direction (the region bounded by the contour remains at the left when the contour is marched down) [15]:

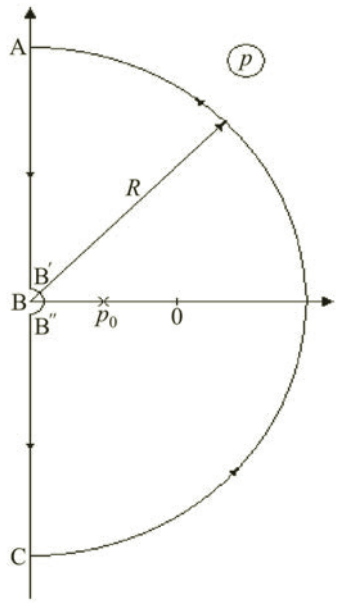

Fig. 4. Contour inside which the number of roots of the equation $F(p) = 0$ is determined.

$$N = \frac{1}{2\pi i} \oint_{\gamma} \frac{F'_p(p)}{F(p)}\, dp = \frac{1}{2\pi} \Delta_{\gamma} \arg F(p), \tag{40}$$

where i is the imaginary unit and $\Delta_\gamma \arg F(p)$ designates the full increment of the argument of the function $F(p)$ in traversing the contour γ. It is assumed here that the function $F(p)$ is analytic inside γ, continuous over γ, and does not become zero on the contour (each zero is counted as many times as its multiplicity). In our case the function $F(p)$ enters into the denominator of Eq. (35). The necessary contour is shown in Fig. 4. The point B (the root of $F(p)$) is traversed over an arc (semicircle) of small radius which is to be allowed to go to zero. The transition CA takes place along an arc of large radius that subsequently should be allowed to go to infinity. In the right half-plane Re $(p + B) > 0$, the asymptotic behavior of the function $F(p)$ is as follows: $|F(p)| \cong |p|^3$. This assures the absence of zeros of the function F on the large semicircle arc at its fairly large radius. Analogously we can easily see that at a rather small radius of the semicircle with the center at the point B ($p = -B$), the function F on the arc of this semicircle also does not have roots. Substituting the value of p on the ABC line ($p = -B + i\beta$) into the function $F(p)$, equating the real and imaginary parts of the resulting expression to zero, we can see that in the case of the presence of the root of $F(p)$ on the ABC line the variable β must satisfy the equation $\beta^2\{\beta^4 + [2\varepsilon B + (\varepsilon - B)^2]\beta^2 - \varepsilon\nu B\} = 0$ obtained after some transformations with the aid of dependence (37). At $\nu < 0$ it is seen from the latter equation that the sole preliminarily known root of the equation obtained is $\beta = 0$, since the expression in the curly brackets is always positive. The foregoing provides the possibility of using the argument principle [15] in the form formulated above. We can easily verify that on the AB′ line the change in the argument of the function $F(p)$ is equal to $\pi/2$ (to within an infinitely small quantity coupled with the arc radius near the point B) just as on the B″C line. In marching down the second-order zero on the small-radius arc the increment of the argument of the function $F(p)$ is approximately equal to 2π, whereas in marching down the large-radius arc with account for the asymptotic behavior of $|F(p)| \cong |p|^3$, the increment is approximately equal to 3π. In the limit, when the first radius tends to zero and the second to infinity, we find that the total increment of the argument of the function $F(p)$ is equal to 2π. Therefore, according to Eq. (40), inside the considered contour the function $F(p)$ has one root marked by a cross in Fig. 4. Its availability has already been proved above. Therefore all the other roots of the function $F(p)$ will have their real part smaller than $-B$. An analogous analysis of the case $\nu > 0$ will not be discussed here.

Solution of the form of (39) is especially convenient at sufficiently large values of time when the exponents under the summation sign become small; therefore, in the calculation we may limit ourselves to a small number of terms of the sum. At small values of time, it is worthwhile to suggest another form of solution that would be convenient for the present case. To do this, the function $F(p)$ that enters into the denominator of Eq. (35) should be represented in the form $F(p) = V(p)[1 +$

$B\varepsilon \exp(-p - B)/V(p)]$, where $V(p) = R(p) + B\varepsilon$. With the use of this representation one should expand Eq. (35) in exp $(-p)$. The structure of the resulting Laplace-transformed solution is

$$\theta^* (p) = R_0 (p) + \sum_{j=1}^{\infty} R_j (p) \exp(-pj), \tag{41}$$

where the function $R_j(p)$ ($j = 0, 1, 2, \ldots$) can easily be expressed through $V(p)$ and $R(p)$. The inversion of Eq. (41) has the form

$$\theta(\xi) = \sum_{k=0}^{3} \operatorname{res}\left[R_0 (p_k) \exp(p_k \xi)\right] + \sum_{j=1}^{\infty} \sum_{k=0}^{3} \operatorname{res}\left[R_j (p_k)\right] \exp\left[p_k \xi (1 - j)\right] H(\xi - j), \tag{42}$$

where $p_0 = 0$ and p_k ($k = 1, 2, 3$) are the roots of the third-degree polynomial $V(p) = 0$. Of importance is the fact that at any fixed moment $\xi = \xi_*$ relation (42) contains a finite number of terms, which is ensured by the functions $H(\xi - j)$ in it. The terms appearing anew on increase in the time ξ correspond to the wave character of the process (the multiplicity of the passage by a heat-transfer agent with a unit dimensionless velocity through the coil after the start of the process). It should also be noted that for $\varepsilon \to 0$ Eq. (42) may yield a "wave" solution in the form of (29). Determining the function $\theta(\xi)$ and using Eqs. (34), (39), (42), we can easily construct a solution also for the function $T(\xi, z)$. However, this construction, as well as the matching of the limiting variant of the function $T(\xi, z)$ for $\varepsilon \to 0$ with relations (29), (30) will not be discussed here. Actually, the basic stage of calculation is the determination of the roots of the third-degree equation $V(p) = 0$, which can first be derived analytically and then easily found using the modern program means such as MathCad.

CONCLUSIONS

1. Based on the model that takes into account the longitudinal stirring in the intertube space, an asymptotic model of heat exchange between the in-tube and intertube spaces with intense stirring has been constructed. The intertube space can be replaced by a vessel-coil apparatus, and the in-tube space, by a coil.

2. The quasi-stationary approximation of the asymptotic model has been investigated thoroughly.

3. An exact solution of the asymptotic model equations has been constructed.

NOTATION

A, B, similarity parameters (5); C, heat capacity of the heat-transfer agent, J/(kg·°C); $F(p) = pR(p) + BQ(p)$; $F_\omega(p)$, auxiliary function defined by Eq. (38); f, dimensionless temperature defined by Eq. (31); G, mass flow of the heat-transfer agent, kg/s; H, Heaviside function; K, heat transfer coefficient, W/(m^2·°C); L, tube length, m; N, number of zeros inside the contour γ; P, tube perimeter along which heat-transfer agents come into contact, m; Pe, Peclet number; p, parameter of Laplace transformation, see Eq. (32); $Q(p) = \varepsilon[1 - \exp(-p - B)]$; $R(p) = (p + B)(p + B + \varepsilon)$; res, residues of the corresponding function at singular points (poles); S, cross-sectional area of the channel, m^2; T, θ, temperatures in the in-tube and intertube space, respectively, °C; T_{x0}, temperature at the inlet of the heat exchanger in-tube space, °C; T_{t0}, initial distribution of temperature in the heat exchanger in-tube space, °C; t, time, s; $U_T = G_T/(S_T\rho_T)$, wave velocity, m/s; W, volume of intertube space, m^3; x, coordinate along the heat exchanger, m; $z = x/L$, dimensionless coordinate along the heat exchanger; β, imaginary part p on the ABC line (Fig. 4); ε, small parameter defined by Eq. (5); $\zeta = \tau/\mathrm{Pe}$, "compressed" dimensionless time; θ_0, initial distribution of temperature in the heat exchanger intertube space, °C; $\theta_+(t)$, temperature in the inlet flow into the heat exchanger intertube space, °C; κ, parameter defined by Eq. (37); λ, effective thermal conductivity, W/(m·°C); $\nu(\varepsilon, B)$, function defined by Eq. (37); $\xi = \tau/\varepsilon$, "internal" dimensionless time; ρ, heat-transfer agent density, kg/m^3; τ, dimensionless time; $\langle\ \rangle$, sign pertaining to the averaging of functions. Indices: T, in-tube space; θ, intertube space; 1, 2, ... , number of the function in expansions (9); *, refers to the functions Laplace-transformed according to Eq. (32).

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