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PLANE WAVES IN A TRANSVERSELY ISOTROPIC ROTATING MAGNETOTHERMOELASTIC MEDIUM

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The governing equations for a transversely isotropic rotating magnetothermoelastic medium are solved, giving a cubic velocity equation, which is indicative of three plane waves. Some limiting cases are considered: in the absence of anisotropy, rotation, and thermal and magnetic effects. The effects of the anisotropy, rotation, thermal and magnetic parameters on the speeds of plane waves are shown graphically.

Keywords: *anisotropic thermoelasticity, rotation, magnetic field, plane waves.*

Introduction. Schoenberg and Censor [1] studied the effect of rotation on the plane wave propagation in an isotropic medium. They considered the propagation of three plane waves in a rotating isotropic medium. According to their results, a longitudinal or transverse wave can exist only if the direction of propagation and the axis of rotation are either parallel or perpendicular. The wave propagation in a transversely isotropic solid has been studied in [2–8]. The effect of rotation does not increase the number of waves in a transversely isotropic medium, but significantly affects their speeds. Chandrasekharajah and Srinath [9, 10] have studied thermoelastic plane waves in a rotating isotropic solid. Ahmad and Khan [11] have discussed the theory of thermoelastic plane waves in a rotating isotropic material and have shown the existence of four waves in the medium. None of these waves is dilatational or transverse in character, unless the special propagation directions are considered. Problems on the propagation of waves in rotating isotropic and anisotropic bodies with electric, magnetic, and thermal effects have been studied in [12–16]. In the present paper, the Lord and Shulman theory of generalized thermoelasticity [17] is applied to study the propagation of plane waves in a transversely isotropic rotating magnetothermoelastic medium.

Formulation of the Problem and Its Solution. We consider an infinite, homogeneous, transversely isotropic, thermally and electrically conducting elastic medium with the reference temperature T_0 under the action of a primary magnetic field with the magnetic induction $\mathbf{B}_0$. The medium is uniformly rotating with an angular velocity $\mathbf{\Omega} = \Omega\mathbf{n}$, where $\mathbf{n}$ is the unit vector in the direction of the axis of rotation. The displacement equation in a rotating frame of reference includes two additional terms corresponding to the centripetal acceleration due to only time-varying motion $(\mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{u})$ and to the Coriolis acceleration $\left(2\Omega \times \frac{\partial \mathbf{u}}{\partial t}\right)$. We assume that the medium, is transversely isotropic in such a way that the planes of isotropy are perpendicular to the z axis. The origin of the frame is located on the plane surface, and the z axis is directed normally into the medium which is thus represented by $z \ge 0$. We restrict our analysis to the plane strain parallel to the xz plane with the displacement vector $\mathbf{u} = (u, 0, w)$ and temperature $T(x, z, t)$. We also assume that the half-space is rotating about the y axis with the angular velocity $\mathbf{\Omega} = (0, \Omega, 0)$. We consider the time-dependent dynamic solutions and the time-independent part of the centripetal acceleration, and all the body forces are neglected, except for the time-dependent part of the electromagnetic body force. Then, the displacement equations in an elastic solid with increase in the temperature T above the reference temperature T_0 are

$$c_{11}\frac{\partial^2 u}{\partial x^2} + (c_{13}+c_{44})\frac{\partial^2 w}{\partial x \partial z} + c_{44}\frac{\partial^2 u}{\partial z^2} - \beta_1\frac{\partial T}{\partial x} + (\mathbf{J}\times\mathbf{B})_1 = \rho\left(\frac{\partial^2 u}{\partial t^2} - \Omega^2 u + 2\Omega\frac{\partial w}{\partial t}\right), \tag{1}$$

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$$c_{44}\frac{\partial^2 w}{\partial x^2}+(c_{13}+c_{44})\frac{\partial^2 u}{\partial x\partial z}+c_{33}\frac{\partial^2 w}{\partial z^2}-\beta_3\frac{\partial T}{\partial x}+(\mathbf{J}\times\mathbf{B})_3=\rho\left(\frac{\partial^2 w}{\partial t^2}-\Omega^2 w-2\Omega\frac{\partial u}{\partial t}\right), \tag{2}$$

where $\mathbf{B} = \mu_e\mathbf{H}$. We set $\mathbf{H} = \mathbf{H}_0 + h(x, z, t)$, where $\mathbf{H}_0 = (0, H_0, 0)$ is the constant primary magnetic field strength. The perturbation of the magnetic field $\mathbf{h}$ is so small that the product of $\mathbf{h}$, $\mathbf{u}$, and their derivatives can be neglected in linearizing the field equations.

Following Lord and Shulman [17], we write the heat conduction equation as

$$K_1\frac{\partial^2 T}{\partial x^2}+K_3\frac{\partial^2 T}{\partial z^2}=\rho C_E\left(\frac{\partial T}{\partial t}+\tau_0\frac{\partial^2 T}{\partial t^2}\right)+\beta_1 T_0\left(\frac{\partial^2 u}{\partial t\partial z}+\tau_0\frac{\partial^3 u}{\partial t^2\partial z}\right)+\beta_3 T_0\left(\frac{\partial^2 w}{\partial t\partial z}+\tau_0\frac{\partial^3 w}{\partial t^2\partial z}\right), \tag{3}$$

where τ_0 is a short time required to establish a steady-state heat conduction when a temperature gradient is suddenly produced in a solid. This time is called the thermal relaxation time.

The electromagnetic field is governed by the Maxwell's equations in the absence of the displacement current and charge density [18]:

$$\operatorname{curl}\mathbf{h}=\mathbf{J},\quad \operatorname{curl}\mathbf{E}=-\mu_e\frac{\partial\mathbf{h}}{\partial t},\quad \operatorname{div}\mathbf{h}=0,\quad \operatorname{div}\mathbf{E}=0, \tag{4}$$

where $\mathbf{h} = \operatorname{curl}(\mathbf{u}\times\mathbf{H}_0)$. The generalized Ohm's law is

$$\mathbf{J}=\sigma\left[\mathbf{E}+\left(\frac{\partial\mathbf{u}}{\mathbf{h}t}\times\mathbf{B}\right)\right], \tag{5}$$

where the small effect of the temperature gradient on the conduction current $\mathbf{J}$ is neglected. With the help of Eqs. (4) and (5), Eqs. (1) and (2) become

$$c_{11}\frac{\partial^2 u}{\partial x^2}+(c_{13}+c_{44})\frac{\partial^2 w}{\partial x\partial z}+c_{44}\frac{\partial^2 u}{\partial z^2}-\beta_1\frac{\partial T}{\partial x}+\mu_e H_0^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 w}{\partial x\partial z}\right)=\rho\left(\frac{\partial^2 u}{\partial t^2}-\Omega^2 u+2\Omega\frac{\partial w}{\partial t}\right), \tag{6}$$

$$c_{44}\frac{\partial^2 w}{\partial x^2}+(c_{13}+c_{44})\frac{\partial^2 u}{\partial x\partial z}+c_{33}\frac{\partial^2 w}{\partial z^2}-\beta_3\frac{\partial T}{\partial x}+\mu_e H_0^2\left(\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 w}{\partial z^2}\right)=\rho\left(\frac{\partial^2 w}{\partial t^2}-\Omega^2 w-2\Omega\frac{\partial u}{\partial t}\right). \tag{7}$$

To solve Eqs. (3), (6), and (7), we assume

$$\{u, w, T\}=\{A, B, C\}\exp\{ik(x\sin\theta+z\cos\theta-vt)\}, \tag{8}$$

Using Eq. (8), we obtain from Eqs. (3), (6), and (7) the following set of three homogeneous equations for A, B, and C:

$$(D_1-\Omega^*\zeta)A+\left(D_2-2i\frac{\Omega}{\omega}\zeta\right)B+i\frac{\beta_1}{k}\sin\theta\, C=0, \tag{9}$$

$$\left(D_2+2i\frac{\Omega}{\omega}\zeta\right)A+(D_3-\Omega^*\zeta)B+i\frac{\beta_3}{k}\cos\theta\, C=0, \tag{10}$$

$$\varepsilon\zeta\sin\theta\, A+\overline{\beta}\,\varepsilon\zeta\cos\theta\, B+i\frac{\beta_1}{k}(D_5-\zeta)C=0, \tag{11}$$

where

$$\zeta = \rho v^2 , \quad D_1 = c_{11} \sin^2 \theta + c_{44} \cos^2 \theta + \mu_e H_0^2 \sin^2 \theta , \quad D_2 = \left(c_{13} + c_{44} + \mu_e H_0^2\right) \sin \theta \cos \theta ,$$

$$D_3 = c_{44} \sin^2 \theta + c_{33} \cos^2 \theta + \mu_e H_0^2 \cos^2 \theta , \quad D_4 = K_1 \sin^2 \theta + K_3 \cos^2 \theta , \quad D_5 = \frac{D_4}{\tau^* C_E} ,$$

$$\tau^* = \tau_0 + \frac{i}{\omega} , \quad \Omega^* = 1 + \left(\frac{\Omega}{w}\right)^2 , \quad \bar{\beta} = \frac{\beta_3}{\beta_1} , \quad \omega = kv , \quad \varepsilon = \frac{\beta_1^2 T_0}{\rho C_E} .$$

For the existence of a nontrivial solution of Eqs. (9)–(11), the determinant of the coefficients at A, B, and C must be equal to zero, i.e.,

$$\begin{vmatrix} D_1 - \Omega^* \zeta & D_2 - 2i \dfrac{\Omega}{\omega} \zeta & \sin \theta \\ D_2 + 2i \dfrac{\Omega}{\omega} \zeta & D_3 - \Omega^* \zeta & \bar{\beta} \cos \theta \\ \varepsilon \zeta \sin \theta & \bar{\beta}\, \varepsilon \zeta \cos \theta & D_5 - \zeta \end{vmatrix} = 0 ,$$

which presents a cubic equation relative to ζ:

$$A_0 \zeta^3 + A_1 \zeta^2 + A_2 \zeta + A_3 = 0 , \tag{12}$$

where

$$A_0 = 4 \left(\frac{\Omega}{\omega}\right)^2 - \Omega^{*2} ,$$

$$A_1 = \Omega^* \, (D_1 + D_3 + \varepsilon \sin^2 \theta + \bar{\beta}^{\,2} \varepsilon \cos^2 \theta) + D_5 \left(\Omega^{*2} - 4 \left(\frac{\Omega}{\omega}\right)^2\right),$$

$$A_2 = D_2^2 - D_1 D_3 - \Omega^* D_5 \, (D_1 + D_3) + \varepsilon \left(2 \bar{\beta} D_2 \sin \theta \cos \theta - \bar{\beta}^{\,2} D_1 \cos^2 \theta - D_3 \sin^2 \theta\right) ,$$

$$A_3 = D_5 \left(D_1 D_3 - D_2^2\right) .$$

The real parts of the three roots of Eq. (12) correspond to the speeds of propagation of the qP, qSV, and qT plane waves.

Limiting Cases. (i) In the absence of thermal effects, i.e., at $\varepsilon = 0$, $K_1 = K_3 = 0$, and $D_5 = 0$, Eq. (12) reduces to

$$L \zeta^2 + M \zeta + N = 0 , \tag{13}$$

where

$$L = 4 \left(\frac{\Omega}{\omega}\right)^2 - \Omega^{*2} , \quad M = \Omega^* \, (D_1 + D_3) , \quad N = D_2^2 - D_1 D_3 .$$

The two roots of quadratic equation (13) correspond to the speeds of the qP and qSV waves in a rotating transversely isotropic magnetoelastic medium.

(ii) For the isotropic thermoelastic case, we have

$$\beta_1 = \beta_3 = \beta \,, \quad \bar{\beta} = 1 \,, \quad K_1 = K_3 = K \,, \quad c_{11} = \lambda + 2\mu \,, \quad c_{13} = \lambda \,, \quad c_{44} = \mu \,, \quad c_{33} = \lambda + 2\mu \,.$$

Here, Eq. (12) reduces to

$$P\zeta^3 + Q\zeta^2 + R\zeta + S = 0 \,, \tag{14}$$

where

$$P = 4\left(\frac{\Omega}{\omega}\right)^2 - \Omega^{*2} \,, \quad Q = \Omega^* \left(D_1^* + D_3^* + \varepsilon^*\right) + D_5^* \left(\Omega^{*2} - 4\left(\frac{\Omega}{\omega}\right)^2\right),$$

$$R = \left(D_2^{*2} - D_1^* D_3^*\right) - \Omega^* D_5^* \left(D_1^* + D_3^*\right) + \varepsilon^* \left(2D_2^* \sin\theta\cos\theta - D_1^* \cos^2\theta - D_3^* \sin^2\theta\right) ,$$

$$S = D_5^* \left(D_1^* D_3^* - D_2^{*2}\right) ,$$

$$D_1^* = (\lambda + 2\mu)\sin^2\theta + \mu\cos^2\theta + \mu_e H_0^2 \sin^2\theta \,, \quad D_1^* = \left(\lambda + \mu + \mu_e H_0^2\right)\sin\theta\cos\theta \,,$$

$$D_3^* = \mu\sin^2\theta + (\lambda + 2\mu)\cos^2\theta + \mu_e H_0^2 \cos^2\theta \,, \quad D_5^* = \frac{K}{\tau^* C_E} \,, \quad \varepsilon^* = \frac{\beta^2 T_0}{\rho C_E} \,.$$

The three roots of cubic equation (14) correspond to the speeds of the qP, qSV, and qT waves in a rotating isotropic magnetothermoelastic medium.

(iii) In the absence of rotation and magnetic effects, i.e., at $H_0 = 0$ and $\Omega = 0$, Eq. (12) reduces to

$$\zeta^3 + G_1\zeta^2 + G_2\zeta + G_3 = 0 \,, \tag{15}$$

where

$$G_1 = -\left(D_1' + D_3' + D_5\,\varepsilon\sin^2\theta + \bar{\beta}^2\,\varepsilon\cos^2\theta\right) ,$$

$$G_2 = \left(D_1' D_3' - D_2'^2\right) + D_5\left(D_1' - D_3'\right) + \varepsilon\left(\bar{\beta}^2 D_1' \cos^2\theta + D_3' \sin^2\theta - 2\bar{\beta} D_2' \sin\theta\cos\theta\right) ,$$

$$G_3 = D_5\left(D_2'^2 - D_1' D_3'\right) ,$$

$$D_1' = c_{11}\sin^2\theta + c_{44}\cos^2\theta \,, \quad D_2' = (c_{13} + c_{44})\sin\theta\cos\theta \,, \quad D_3' = c_{44}\sin^2\theta + c_{33}\cos^2\theta \,.$$

The three roots of cubic equation (15) correspond to the speeds of the qP, qSV, and qT waves in a transversely isotropic thermoelastic medium.

Numerical Results and Discussion. To compute the speeds of the plane waves, the following relevant elastic and thermal constants are used [7]:

$$\rho = 7.14\cdot10^3\ \text{kg}\cdot\text{m}^{-3} \,, \quad c_{11} = 1.628\cdot10^9\ \text{N}\cdot\text{m}^{-2} \,, \quad c_{33} = 1.562\cdot10^9\ \text{N}\cdot\text{m}^{-2} \,,$$

$$c_{13} = 0.385\cdot10^9\ \text{N}\cdot\text{m}^{-2} \,, \quad c_{44} = 0.6215\cdot10^9\ \text{N}\cdot\text{m}^{-2} \,, \quad \beta_1 = 5.75\cdot10^6\ \text{N}\cdot\text{m}^{-2}\cdot\text{deg}^{-1} \,,$$

$$\beta_3 = 5.17\cdot10^6\ \text{N}\cdot\text{m}^{-2}\cdot\text{deg}^{-1} \,, \quad K_1 = 1.24\cdot10^2\ \text{W}\cdot\text{m}^{-1}\cdot\text{deg}^{-1} \,, \quad K_3 = 1.34\cdot10^2\ \text{W}\cdot\text{m}^{-1}\cdot\text{deg}^{-1} \,,$$

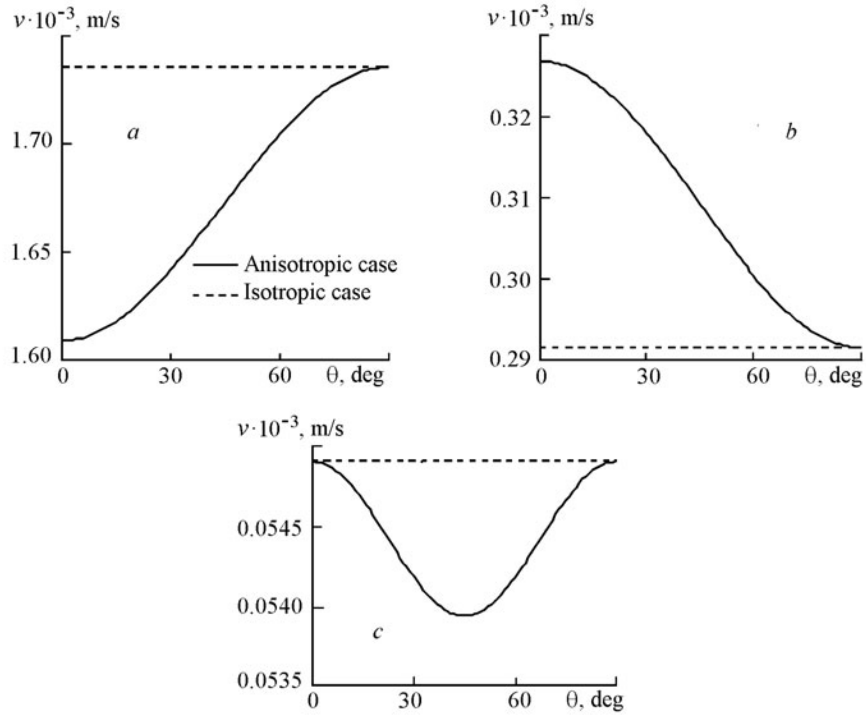

Fig. 1. Variations of the speeds of the qP (a), qSV (b), and qT (c) waves with the angle of propagation at $H_0 = 10$ A/m and $\frac{\Omega}{\omega} = 5$.

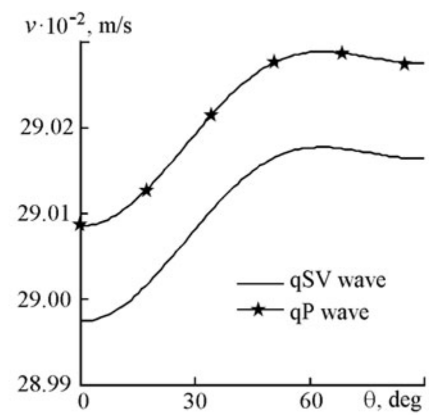

Fig. 2. Variations of the speeds of the qP and qSV waves with the angle of propagation in the absence of thermal effects at $H_0 = 10$ A/m and $\frac{\Omega}{\omega} = 5$.

$$C_E = 3.9 \cdot 10^2 \ \mathrm{J \cdot kg^{-1} \cdot deg^{-1}}, \quad T_0 = 296 \ \mathrm{K}, \quad \tau_0 = 0.05 \cdot 10^{-12} \ \mathrm{s}.$$

With the help of a Fortran program, the absolute values of the real speeds of various plane waves are computed versus the angle of propagation and magnetic parameter. These variations in the speeds of the plane waves are shown in Figs. 1–3.

The speed of the qP wave increases from a minimum value of $1.6 \cdot 10^3$ m/s at $\theta = 0^\circ$ to a maximum value of $1.735 \cdot 10^3$ m/s at $\theta = 90^\circ$ (Fig. 1a), whereas the speed of the qSV wave decreases from $0.327 \cdot 10^3$ m/s at $\theta = 0^\circ$ to $0.292 \cdot 10^3$ m/s at $\theta = 90^\circ$ (Fig. 1b). As to the qT wave, its maximum speed is equal to $0.0549 \cdot 10^3$ m/s at $\theta = 0^\circ$ and 90°, and the minimum speed — to $0.05395 \cdot 10^3$ m/s at $\theta = 45^\circ$ (Fig. 1c). In the absence of thermal effects, the cor-

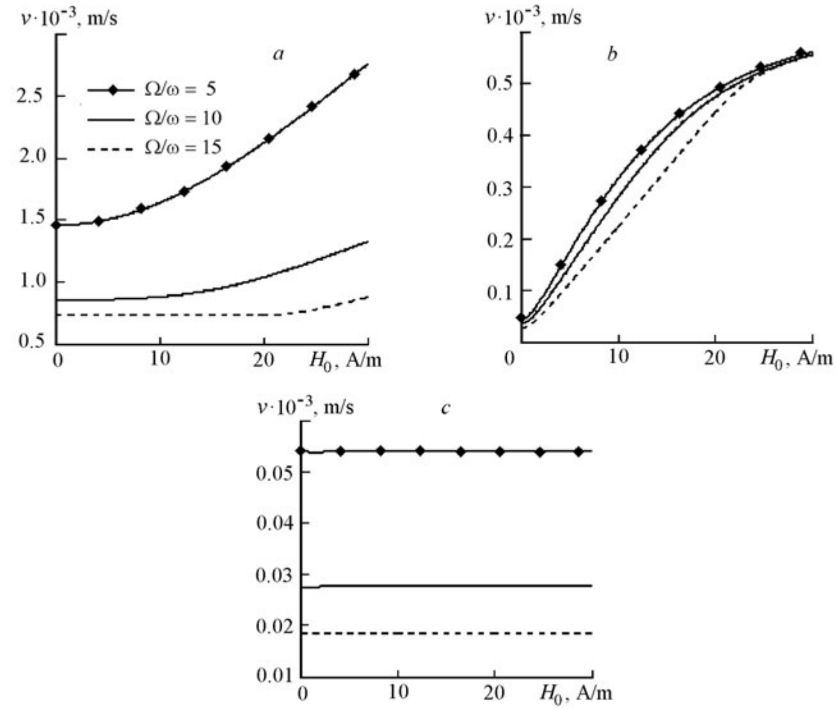

Fig. 3. Variations of the speeds of the qP (a), qSV (b), and qT (c) waves with the magnetic parameter at $= 30^{o}$ for different values of $\frac{\Omega}{\omega}$.

responding curves in Fig. 1 reduce to the curves in Fig. 2, where the qT wave disappears and the speeds of the qP and qSV waves increase with the angle of propagation.

The speeds of the qP, qT, and qSV waves are also computed against the magnetic parameter H_0 for different angles of propagation and three different values of the rotation parameter. As seen from Fig. 3, the speeds of the qP (Fig. 3a) and qSV (Fig. 3b) waves increase with the magnetic parameter, whereas the changes in the speed of the qT waves are insignificant (Fig. 3c). It is evident from the figure that an increase in the value of the rotation parameter leads to a decrease in the speed of each of the waves. It should be mentioned that the value of the angle θ slightly affects the values $v(H_0)$, so that we present this dependence only for $\theta = 30^{o}$.

Conclusions. The solutions for plane waves in a transversely isotropic rotating magnetothermoelastic medium show the existence of three waves, namely, the qP, qT, and qSV ones. The speeds of these waves change significantly with the angle of propagation, as well as with the transverse anisotropy, thermal, magnetic, and rotation parameters.

NOTATION

B, magnetic induction; c_{ij}, elastic constants; C_E, specific heat at constant strain; **E**, electric field strength; **H**, magnetic field strength; **h**, perturbation of magnetic field strength; **J**, current; K_1, K_3, thermal conductivities; k, wave number; **n**, unit vector; T, temperature; T_0, reference temperature; t, time; **u**, displacement vector; u, w, components of the displacement vector; v, wave speed; x, y, z, coordinates; β_1, β_3, thermal coefficients; θ, angle of propagation measured from the normal to the half-space; λ, μ, Lame's constants; μ_e, magnetic permeability; ρ, density; σ, electrical conductivity; τ_0, relaxation time; Ω, angular velocity; ω, circular frequency.

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