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Ranking opportunity sets in the space of functionings

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Abstract. We develop a ranking of compact, convex and comprehensive opportunity sets defined in the evaluative space of individual functionings. We suppose the existence of a target, that is a multi-dimensional bliss point in terms of functionings. This leads us to define concepts such as *essentiality* and *freedom* in a novel way. As a main result, we give an axiomatic characterization of the ranking obtained by minimizing the Euclidean distance between each opportunity set and the target.

Key words: essentiality, Euclidean distance, freedom, functionings, opportunity sets, set inclusion, target.

1. Introduction

This paper contributes to the strand of literature which contends the standard economic approach according to which income is regarded as the only source of inequality among individuals. As Amartya Sen remarked, "there may be some accentuation of inequality due to the coupling of (i) economic inequality and (ii) unequal advantages in converting incomes into capabilities, the two together intensifying the problem of inequality of opportunity-freedoms" (Sen, [6], p. 536).[1] Our research was stimulated by the empirical evidence supporting it. We performed some computations on the European Community Households Panel (ECHP) data for two EU countries (Italy and Germany), by combining the breakdown in quintiles of the households' income distribution with information about education and health. We found a significant correlation between the income level and the quality of both educational achievements and health conditions. Besides, we discovered dissimilar levels of these two functionings among individuals belonging to the same quintiles but living in different regions. The diversity of functionings across individuals endowed with the same income may easily be explained by Sen's capability theory, according to which *commodities* generate *characteristics* that are converted into *functionings*. In fact, this process: (i) is driven by heterogeous preferences and (ii) is affected by differences in opportunity sets. Here, we focus on

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the second claim. The remark by Sen that "differences in age, gender, talents, disability etc. can make different persons have quite divergent substantive opportunities even when they have the very same commodity bundle" (Foster and Sen, [2], p. 209) is presented as the formal problem of ranking individual opportunity sets in an economic environment. In so doing, we depart from the previous works on ranking finite opportunity sets in terms of *freedom of choice* (see, among others, Pattanaik and Xu [3], and Puppe [4]). We assume a perspective similar to Xu [9, 10], who suggests the comparison of opportunity sets in terms of the *standard of living* offered to an agent. We adopt a linear version of Sen's capability model [7] in order to provide a microeconomic foundation for compact, convex and comprehensive opportunity sets in the space of functionings. By assuming a sort of multi-dimensional poverty line (target) in terms of functionings, we rank the capability sets by their distance from it. As a main result, we provide an axiomatic characterization of the ranking induced by *minimization of the Euclidean distance* of different sets of functionings from the target. This ranking provides a suitable metric for comparing capability sets in a multi-dimensional setting, which has a *dual* interpretation in terms of cost minimization. Assuming a linear technology that transforms commodities into functionings, we devise a distance-induced ranking such that an opportunity set A is better than an opportunity set B if, for given income and prices, the minimum value of the resources needed to reach the target is lower for A than for B.

The paper is organized as follows. In Section 2, we introduce the basic notation and provide a microeconomic foundation for our set-up based on Sen's capabilities' theory. We then model the problem of comparing compact, convex and comprehensive opportunity sets with respect to a 'target'. In Section 3, our axiomatic framework is developed. In line with the relevant literature, we first introduce a *Set-Inclusion* Axiom as the most suitable property of a ranking between capability sets. However, since ranking based on this axiom excludes any couple of capability sets that mutually intersect or are disjoint, we add further axioms: *Scale Independence, Essentiality, Contraction* and *Freedom.* Two features of this axiomatic structure are worth noting. First, we introduce *Essentiality* by considering different vectors of functionings as essential (or otherwise) with respect to achievement of the target. Second, we allow individuals to dispose of options which guarantee a certain degree of liberty of choice *within* the essential part of their opportunity sets. Minimum-distance ranking is then introduced and characterized. Section 4 concludes the paper with suggestions for further developments. All proofs are in the Appendix.

2. The setup

2.1. NOTATION AND DEFINITIONS

Let $\mathbb{R}_{+}^{G}$ and $\mathbb{R}_{+}^{C}$ respectively be the spaces of G *goods* and C *functionings*. We denote the vectors of $\mathbb{R}_{+}^{G}$ by $\mathbf{g}_1, \mathbf{g}_2, \ldots$ and the vectors of $\mathbb{R}_{+}^{C}$ by $\mathbf{a}, \mathbf{b}, \ldots$. For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{+}^{C}$, $\mathbf{a} \geq \mathbf{b}$ if $a_j \geq b_j$ and $\mathbf{a} > \mathbf{b}$ if $a_j > b_j$ for $j = 1, \ldots, C$. We focus our attention on the family $\mathbb{V}$ of non-degenerate subsets of $\mathbb{R}_{+}^{C}$, with elements $A, B, \ldots$ having the following properties:

For any $A \in \mathbb{V}$,

1. *Compactness*: A is bounded and closed.
2. *Convexity*: for all $\mathbf{a}, \mathbf{b} \in A$, and all $\alpha \in [0, 1]$,

$$\alpha\mathbf{a} + (1 - \alpha)\mathbf{b} \in A.$$

3. *Comprehensiveness*: for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}_{+}^{C}$,

$$[\mathbf{a} \geq \mathbf{b} \text{ and } \mathbf{a} \in A] \Longrightarrow \mathbf{b} \in A.$$

We interpret $\mathbb{V}$ as the set of all compact, convex, comprehensive opportunity sets in the space of functionings $\mathbb{R}_{+}^{C}$.

Let $\succsim$ be a *quasi order* (that is, a reflexive, transitive binary relation) on $\mathbb{V}$, and let $\succ$ and $\backsim$ be the asymmetric and symmetric parts of $\succsim$, respectively. When $A \succsim B$, we say that A provides at least as many opportunities as B. We introduce the notion of *set-inclusion* as follows:

DEFINITION 1. For all $A, B \in \mathbb{V}$,

$B \subset A$ if, for all $\mathbf{b} \in B$, there exists a vector $\mathbf{a} \in A$ such that $\mathbf{a} > \mathbf{b}$.
$B \subseteq A$ if, for all $\mathbf{b} \in B$, there exists a vector $\mathbf{a} \in A$ such that $\mathbf{a} \geq \mathbf{b}$.

We say that $\mathbf{a}$ belongs to the *boundary* of A when $\mathbf{a} \in A$ and there is not any $\mathbf{a}' \in A$ such that $\mathbf{a}' > \mathbf{a}$.

In order to complete our set-up, let us introduce a reference point $\mathbf{t} \in \mathbb{R}_{++}^{C}$, a *target* in the space of functionings. We suppose that the target is exogenous: it can be a 'mean level' or a sort of 'multi-dimensional poverty line' in terms of functionings. We define then $\mathbb{V}'$ and $\mathbb{V}''$ as the subsets of $\mathbb{V}$, composed of opportunity sets that do not include the target as an interior point and opportunity sets for which the target is an exterior point, respectively. Henceforth, we suppose as given the reference point $\mathbf{t} \in \mathbb{R}_{++}^{C}$ and that $\mathbb{V}'$ and $\mathbb{V}''$ are non-empty.

DEFINITION 2. Given a target $\mathbf{t} \in \mathbb{R}_{++}^{C}$, we define:

$\mathbb{V}' \subseteq \mathbb{V}$ the family of all the opportunity sets A such that if $\mathbf{a} > \mathbf{t}$ then $\mathbf{a} \notin A$.
$\mathbb{V}'' \subseteq \mathbb{V}'$ the family of all the opportunity sets A such that if $\mathbf{a} \geq \mathbf{t}$ then $\mathbf{a} \notin A$.

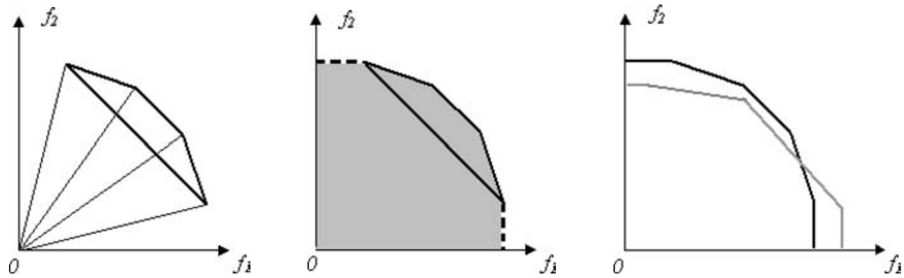


Figure 1. Opportunity sets in the functionings space.

Let us now define the *Euclidean distance* between an opportunity set and the target and the notion of *support hyperplane* for an opportunity set.

DEFINITION 3. Given $\mathbf{t} \in \mathbb{R}^C_{++}$, for any $A \subseteq \mathbb{V}$,

$$d(A, \mathbf{t}) = \inf\{\|\mathbf{a} - \mathbf{t}\| \mid \mathbf{a} \in A\},$$

where $\|\mathbf{a} - \mathbf{t}\|$ is the Euclidean distance between the vectors $\mathbf{a}$ and $\mathbf{t}$.

DEFINITION 4. Let $A \in \mathbb{V}$ and H be an opportunity set and a hyperplane, respectively. Then H is a support hyperplane of A, if H meets A and A lies in one of the closed half-spaces determined by H.

It will be useful to introduce the *nearest* point of A to $\mathbf{t}$, denoted by $\mathbf{a}^*$. The following lemma summarizes the main properties of $\mathbf{a}^*$.

LEMMA 1. *Given a target* $\mathbf{t}$ *and an opportunity set* $A \in \mathbb{V}'$, *there exists a unique point* $\mathbf{a}^*$ *belonging to the boundary of* A, *which is the nearest point of* A *to* $\mathbf{t}$. *Moreover, a hyperplane* H *that supports* A *and is orthogonal to the line* $\lambda\mathbf{t} + (1 - \lambda)\mathbf{a}^*$ *passes through* $\mathbf{a}^*$.

The family of opportunity sets $\mathbb{V}$ can be derived from Sen's capability model, as we show in the next subsection.

2.2. A MICROECONOMIC FOUNDATION OF THE MODEL

Following Sen ([7], pp. 6–7), let us suppose that *goods* generate *characteristics* and characteristics are transformed into *functionings*. Assuming twice linear technology, we designate by $\mathbf{s}_g \in \mathbb{R}^C_+$ as the maximal level of functionings obtained by a given (type of) individual who consumes one unit of good g. The vectors of functionings obtained by spending a fixed income y in good g (with price p_g) are denoted by $\mathbf{a}_g = \frac{y}{p_g}\mathbf{s}_g$, with $g = 1, \ldots, G$.[2] By purchasing different goods, the feasible set of functionings generated by income y is given by the convex hull of the vectors $\mathbf{a}_g$, denoted $co(\mathbf{a}_g \mid g = 1, \ldots, G)$ and represented in the bold contour of Figure 1(a). By construction, it is a compact set. As in standard production theory, we also assume *free disposal*. In other words, by using as many inputs as for the feasible vector $\mathbf{a}$ in the space of functionings, we allow 'production plans' generating a smaller or equal amount of functionings. We then derive the compact, convex, comprehensive set (Figure 1(b)), interpreted as the opportunity set in the space of functionings of a given (type of) individual, for a fixed level of income and prices. Note that, for given prices and consumption technology, an increase in income produces a proportional expansion of A. Figure 1(c) represents two capability sets in $\mathbb{R}^2_+$, produced by two individuals having different consumption technologies with four goods and a given income.

We now develop a general set-up to rank opportunity sets in the space of functionings.

3. On ranking opportunity sets

3.1. AXIOMS

In the first part of this section we describe some plausible properties that a ranking $\succsim$ on $\mathbb{V}$ should satisfy. The following axiom reflects the idea that a set contains more opportunities than its subsets.

AXIOM 1 (I) (Set-Inclusion). For all $A, B \in \mathbb{V}$,

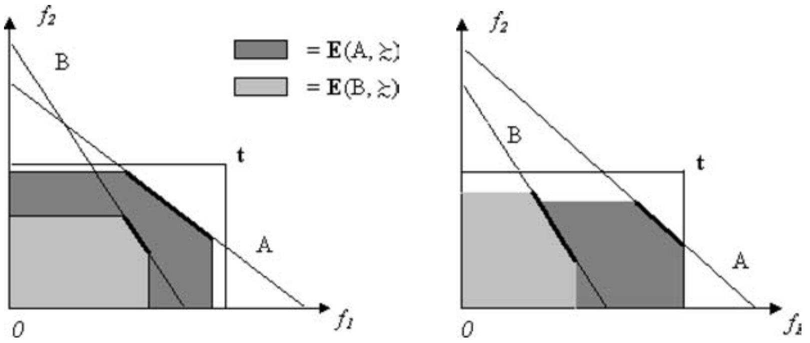
$$B \subset A \Longrightarrow A \succ B.$$

A problem arises when two sets cannot be compared by Axiom I. As a consequence, we introduce additional properties of $\succsim$, which depend on the target. We assume that the ranking between two opportunity sets is preserved after a proportional 'shrinking' of the opportunity sets.

AXIOM 2 (S) (Scale Independence). For any $A, B \in \mathbb{V}$, if $A \succ B$, then $\alpha A \succ \alpha B$ for all $\alpha \in (0, 1)$.

This property is intuitive in the microeconomic setting introduced above: it means that if a consumption technology generates 'better' opportunity sets for a given amount of income, this ranking is also preserved after income reductions. Notice that we do not require a similar property in the case of 'expansions' of opportunity sets, where any individual could reach a fixed target.

The main idea behind a target-based ranking is that not every point of an opportunity set has the same degree of importance for achievement of $\mathbf{t}$. Indeed, we assume that only some elements of an opportunity set matter in order to judge it as good as another one. Such elements are the 'essential part' of an opportunity set, formally defined as follows:



(a) (b)

Figure 2. Configurations of essential sets.

DEFINITION 5. For any $A \in \mathbb{V}'$ a subset of A is Essential according to $\succsim$ and it is denoted as $E(A, \succsim)$ if it is compact, convex, comprehensive and if:

(1) $\mathbf{a} \leq \mathbf{t}\ \forall \mathbf{a} \in E(A, \succsim)$,
(2) $E(A, \succsim) \backsim A$.

Let $\Xi(A, \succsim)$ denote the collection of all essential sets of A according to $\succsim$. The next axiom guarantees the existence of an essential set for all the opportunity sets of $\mathbb{V}$ that do not include the target as an interior point.

AXIOM 3 (E) (Essentiality w.r.t. a target). For all $A \in \mathbb{V}'$, $\Xi(A, \succsim) \neq \emptyset$.

Note that Axiom E differs from the notion of essentiality introduced by Puppe [4]. Here the essentiality of a vector of functionings is determined by the relation between the vector and the target $\mathbf{t}$. If, for instance, a vector contains a higher quantity of some functioning than $\mathbf{t}$, then it cannot be essential, just as a very high level of education cannot be considered of primary importance. Moreover, our notion of essential set is not defined for opportunity sets including the target as an interior point, in order to guarantee consistency between Axiom E and Axiom I. Further, if $\succsim$ satisfies Axiom I, then every set $E(A, \succsim)$ has at least one point on the boundary of A.

By combining the notions of set inclusion and essentiality, it is possible to compare some opportunity sets as shown in Figure 2(a).

The intersection of linear opportunity sets A and B occurs for a level of functioning 2 evaluated as not essential. On the contrary, over the *essential* part of individual functionings, the opportunity set A is clearly larger than B. In such a case, the *inclusion of essential sets* allows ranking of a couple of opportunity sets that the ranking induced by Axiom I does not. However, counterexamples arise where $B \subset A$ but $E(B, \succsim) \subset E(A, \succsim)$ does not hold (see Figure 2(b)). Additional properties of $\succsim$ can be useful. The next axiom is similar to Nash's IIA Axiom. It

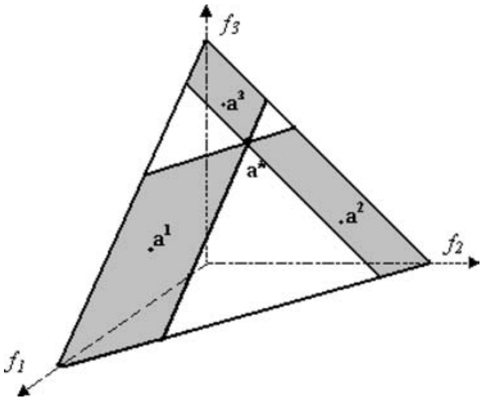


Figure 3. Opportunity set on the hyperplane H.

requires that whenever a vector of functionings relies on the boundary of A and is essential, it is also essential for all the subsets of A containing it.

AXIOM 4 (C) (Essentiality Contraction). For any $A \in \mathbb{V}'$, assume that **a** belongs to the boundary of A. If $\mathbf{a} \in E(A, \succsim)$ for some $E(A, \succsim) \in \Xi(A, \succsim)$, then $\forall B \subseteq A$, if $\mathbf{a} \in B$, then there exists $E(B, \succsim) \in \Xi(B, \succsim)$ such that $\mathbf{a} \in E(B, \succsim)$.

The target represents a desired level of functionings, then it should be suitable for all individuals of a society to converge towards **t**. In fact, the smaller the distance of each individual opportunity set from **t**, the smaller amount of income needed to reach the target. Nevertheless, people cannot be forced to achieve the target and 'minimal liberty' must be guaranteed. In this spirit, we express *freedom of choice* by requiring the presence of vectors of functionings that lie 'around' $\mathbf{a}^*$ in at least an essential set.

EXAMPLE 1. Let A be the opportunity set belonging to the hyperplane H represented in Figure 3.

Let $\mathbf{a}^* = (a_1^*, a_2^*, a_3^*)$ denote the projection of an exterior target on A. We require a weak property of symmetry of $\Xi(A, \succsim)$ with respect to $\mathbf{a}^*$: some essential set must contain at least three vectors as $\mathbf{a}^1 = (a_1^1, a_2^1, a_3^1)$, $\mathbf{a}^2 = (a_1^2, a_2^2, a_3^1)$ and $\mathbf{a}^3 = (a_1^3, a_2^3, a_3^3)$, such that

$$\begin{array}{ccc} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \\ a_1^1 > a_1^* & a_1^2 < a_1^* & a_1^3 < a_1^* \\ a_2^1 < a_2^* & a_2^2 > a_2^* & a_2^3 < a_2^* \\ a_3^1 < a_3^* & a_3^2 < a_3^* & a_3^3 > a_3^*. \end{array}$$

In other terms, there exists at least an 'essential point' in each of the three grey sectors of A.

We introduce the Freedom Axiom as follows:

AXIOM 5 (F) (Freedom). Suppose an opportunity set $A \in \mathbb{V}''$ given by the positive part of the hyperplane H of equation $h_0 + \mathbf{h} \cdot \mathbf{a} = 0$, with $\mathbf{h} \geq 0$. Let $\mathbf{a}^*$ be the projection of $\mathbf{t}$ on A. Then there exists an essential set $E(A, \succsim) \in \Xi(A, \succsim)$ containing at least C vectors $\mathbf{a}^z$, with $z = 1, \dots, C$, such that:

$$\begin{aligned} a_j^z &> a_j^* \quad \text{if } j = z, \\ a_j^z &< a_j^* \quad \text{if } j \neq z. \end{aligned} \tag{1}$$

Note that Axiom F is defined only for 'flat' opportunity sets separated from the target, despite the fact that elements of $\mathbb{V}''$ may have a more general form. An economic interpretation of Axiom F is that people can spend an equal time endowment in the production of different functionings such as education, health, etc. However, they are *free* to spend their time in activities improving one particular functioning (for instance health) by reducing *all* the activities which produce functionings different from that one. Axiom F considers as essential the fact that people must have the possibility of increasing the production of the functioning they like most. Moreover, the 'symmetry' of the essential set required in Axiom F is implied whenever the conditions (1) for $\mathbf{a}^z$ are replaced by:

$$\begin{aligned} a_j^z &> \theta_j a_j^* \quad \text{for some } \theta_j > 1 \text{ if } j = z, \\ a_j^z &< a_j^* \quad \text{if } j \neq z. \end{aligned} \tag{2}$$

In this case, a higher 'freedom of choice' is guaranteed, and with a different threshold for each functioning.

3.2. THE MINIMAL DISTANCE ORDERING

The relations induced by set inclusion or by set inclusion between essential sets are quasi orders. In order to obtain full comparability at least on the elements of $\mathbb{V}'$, we axiomatically characterize the relation $\succsim_{\min}$, which ranks the opportunity sets according to their distance from the target. It remains a quasi order on $\mathbb{V}$ since, if two opportunity sets contain the target as an interior point, we can rank them only applying the set Inclusion Axiom. Formally:

DEFINITION 6. Given a target $\mathbf{t} \in \mathbb{R}^C_{++}$, for all $A, B \in \mathbb{V}$, we define the relation $\succsim_{\min}$ over $\mathbb{V}$ as follows:

(i) $A \succsim_{\min} B$ if $d(A, \mathbf{t}) \leq d(B, \mathbf{t})$ whenever $\mathbf{t} \notin A \cap B$

(ii) $\left.\begin{array}{ll} A \succ_{\min} B & \text{if } B \subset A \\ A \text{ non-comparable with } B & \text{otherwise} \end{array}\right\}$ whenever $\mathbf{t} \in A \cap B$.

The next lemma shows under what conditions the point of an opportunity set with minimal distance from the target belongs to an essential set.

LEMMA 2. *If the quasi order* $\succsim$ *on* $\mathbb{V}$ *satisfies Axioms* I, E, C, F*, then, for any* $A \in \mathbb{V}'$*, there exists* $E(A, \succsim) \in \Xi(A, \succsim)$ *such that:*

$$\mathbf{a}^* \in E(A, \succsim).$$

The previous lemma brings our model close to a general problem of choice with respect to a reference point, solved by Rubinstein and Zhou [5] in terms of minimization of Euclidean distance. The next theorem provides a characterization of $\succsim_{\min}$.

THEOREM 1. *A quasi order* $\succsim$ *on* $\mathbb{V}$ *satisfies Axioms* I, S, E, C, F *if and only if* $\succsim = \succsim_{\min}$.

4. Conclusions

The axiomatic framework developed in this paper ranks different capability sets by their distance from a reference point. In order to make operationally appealing the proposed metric of opportunities, it would be useful to express the distance of opportunity sets from the target in terms of an *income-prices* metric. This requires exploration of the relationship between income level and functionings. Promising contributions have come from recent advances in microeconometrics, notably in the fields of nutrition, health and gender-discrimination problems (see Deaton [1], for instance). Analysis of *deprivation profiles* for different populations, or a given population in time, could enable 'welfarist' poverty and inequality comparisons, replacing the usual income-gaps with the value of the resources needed for each person to reach a target established in a multidimensional functionings' setting. Finally, the problem of correct measurement of the level of functionings should be tackled. This problem is closely related to the central question of how to provide rigorous foundations for a target that could apply to all individuals in a society. The UN Human Development Index is just one example of the increasing efforts of the international community towards precise measurement and assessment of official standards in terms of individual functionings.

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Notes

[1] We recall that Sen defines the capabilities as sets containing the various combinations of functionings a person can achieve, and a functioning as "command over the characteristics of goods", that is the extent to which an individual succeeds in using the characteristics of the various goods she has access to.

[2] The distance between the origin vector 0 and a_g is: $\frac{y}{p_g}\|s_g\|$. The ratio $p_g/\|s_g\|$ can be considered a subjective price of good g. The market price is 'adjusted' by individual consumption productivity, such that a loss in consumption productivity s_g has the same effect as an increase in the market price.

Appendix

Proof of Lemma 1. We know that $\mathbf{a}^*$ exists, is unique and that $(\mathbf{t}-\mathbf{a}^*)\cdot(\mathbf{a}-\mathbf{a}^*) \leq 0$ for every $\mathbf{a}$ in A (see Webster, [8], Theorem 2.4.1, p. 65). Moreover, the point $\mathbf{a}^*$ cannot be an *interior* point of A, because otherwise, there would be some ball in A containing $\mathbf{a}^*$ and other points of A different from $\mathbf{a}^*$ and nearer to $\mathbf{t}$. $\mathbf{a}^*$ is therefore on the boundary of A. It follows, there exists a hyperplane passing through $\mathbf{a}^*$ which supports A (see Webster, [8], Theorem 2.4.12, p. 71). Finally, we show that the hyperplane H passing through $\mathbf{a}^*$ and orthogonal to the line joining $\mathbf{t}$ and $\mathbf{a}^*$ supports A. Let: $h_0 + a_1h_1 + \cdots + a_Ch_C = 0$ be the equation of the hyperplane H, which can also be written: $h_0 + \mathbf{h}\cdot\mathbf{a} = 0$. We also have $h_i \geq 0$ for $i = 1, \ldots, C$, because A is a comprehensive set. Suppose by contradiction that H does not support A. Then, for some vector $\hat{\mathbf{a}} \in A$, we have: $h_0 + \mathbf{h}\cdot\hat{\mathbf{a}} > 0$, which gives: $\mathbf{h}\cdot(\hat{\mathbf{a}}-\mathbf{a}^*) > 0$. Since $\mathbf{a}^*$ is unique, it follows $(\mathbf{t}-\mathbf{a}^*)(\mathbf{a}-a^*) \leq 0$, $\forall\mathbf{a} \in A$. Since $\mathbf{t}$ is normal to H and external to it, then $(\mathbf{t}-\mathbf{a}^*) = \lambda\mathbf{h}$ for some positive scalar λ. By substitution, we obtain $\mathbf{h}\cdot(\hat{\mathbf{a}}-\mathbf{a}^*) \leq 0$, a contradiction. □

Proof of Lemma 2. If target $\mathbf{t}$ is on the boundary of A, by the Essentiality and the Inclusion Axioms, $\mathbf{t}$ coincides with the essential set of A. Suppose now that $\mathbf{t}$ is separated from A. Lemma 1 ensures the existence of a hyperplane H supporting A, perpendicular to the line joining $\mathbf{t}$ and $\mathbf{a}^* = (a_1^*, \ldots, a_n^*)$ (the point of A with minimum distance from $\mathbf{t}$). Let D be an opportunity set generated by the part of this hyperplane lying in the positive orthant of $\mathbb{R}^C$. By construction, we have $D \supseteq A$. The Axiom F guarantees the existence of a set $E(D, \succsim) \in \Xi(D, \succsim)$, containing at least C vectors $\mathbf{a}^z = (a_1^z, \ldots, a_C^z)$, with $z = 1, \ldots, C$, such that $a_j^z > a_j^*$ if $j = z$ and $a_j^z < a_j^*$ if $j \neq z$.

We now show that $\mathbf{a}^*$ belongs to $co(\mathbf{a}^1, \ldots, \mathbf{a}^C)$, which implies $\mathbf{a}^* \in E(D, \succsim)$.

Let $h_0 + h_1a_1 + \cdots + h_Ca_C = 0$ be the equation of the hyperplane H containing D and passing through a^*. This equation is satisfied by each point $\mathbf{a}^z$.

In order to simplify the problem and without loss of generality, we consider the case $\mathbb{R}^3$. Applying a translation which puts the point $\mathbf{a}^* = (a_1^*, a_2^*, a_3^*)$ in the origin, the set $E(D, \succsim)$ contains three vectors with coordinates:

$$\begin{aligned}
&(x_1, y_1, z_1) = (a_1^1 - a_1^*, a_2^1 - a_2^*, a_3^1 - a_3^*),\\
&\quad \text{with } x_1 > 0, y_1 < 0, z_1 < 0,\\
&(x_2, y_2, z_2) = (a_1^2 - a_1^*, a_2^2 - a_2^*, a_3^2 - a_3^*),\\
&\quad \text{with } x_2 < 0, y_2 > 0, z_2 < 0,\\
&(x_3, y_3, z_3) = (a_1^3 - a_1^*, a_2^3 - a_2^*, a_3^3 - a_3^*),\\
&\quad \text{with } x_3 < 0, y_3 < 0, z_3 > 0.
\end{aligned} \tag{3}$$

The equation of the hyperplane containing D becomes $xa + yb + zc = 0$, with $a, b, c > 0$.

We now prove that the origin can be expressed as a convex combination of the three points above. Using the equation of the hyperplane and denoting the elements of the simplex Δ^2 as (α, β, γ), we have to show that the following system only admits positive solutions:

$$\begin{cases} -\dfrac{y_1 b + z_1 c}{a}\alpha - \dfrac{y_2 b + z_2 c}{a}\beta - \dfrac{y_3 b + z_3 c}{a}(1 - \alpha - \beta) = 0,\\ y_1\alpha + y_2\beta + y_3(1 - \alpha - \beta) = 0,\\ z_1\alpha + z_2\beta + z_3(1 - \alpha - \beta) = 0. \end{cases}$$

The solutions are:

$$\begin{aligned}
\beta &= \frac{y_3 z_1 - y_1 z_3}{y_2 z_3 - y_3 z_2 + y_3 z_1 - y_1 z_3 - y_2 z_1 + y_1 z_2},\\
\alpha &= \frac{y_2 z_3 - y_3 z_2}{y_2 z_3 - y_3 z_2 + y_3 z_1 - y_1 z_3 - y_2 z_1 + y_1 z_2}.
\end{aligned}$$

Writing $y_3 z_1 - y_1 z_3 = S$ and $y_2 z_3 - y_3 z_2 = R$, we get:

$$\begin{aligned}
\beta &= \frac{S}{R + S + y_1 z_2 - y_2 z_1},\\
\alpha &= \frac{R}{R + S + y_1 z_2 - y_2 z_1}.
\end{aligned}$$

Since $x_2 = -\frac{y_2 b + z_2 c}{a} < 0$ and $x_3 = -\frac{y_3 b + z_3 c}{a} < 0$, we find $R > 0$, and by using (3) we obtain the result.

Finally, assume that $\mathbf{a}^*$ does not belong to $E(A, \succsim)$. From $\mathbf{a}^* \in E(D, \succsim)$ and $A \subseteq D$, we obtain a contradiction of the Axiom C. □

Proof of Theorem 1. It is easy to show that $\succsim_{\min}$ satisfies Axioms I, E, C, S and F. Then we show that if $\succsim$ on $\mathbb{V}$ satisfies Axioms I, E, C, S and F, then $\succsim = \succsim_{\min}$. Consider two opportunity sets A and $B \in \mathbb{V}$. Three cases have to be considered.

1. If both opportunity sets contain the target, no essential sets are defined and only the Inclusion Axiom holds, so the proof is trivial.

2. The target $\mathbf{t}$ is exterior to A and B. Let us denote the points of A and of B with minimal distance from $\mathbf{t}$ with $\mathbf{a}^*$ and $\mathbf{b}^*$ respectively. By contradiction, suppose that $A \succsim B$, but $d(a^*, \mathbf{t}) > d(b^*, \mathbf{t})$. By Lemma 2, for essential sets $E(A, \succsim) \in \Xi(A, \succsim)$ and $E(B, \succsim) \in \Xi(B, \succsim)$ we obtain: $a^* \in E(A, \succsim)$ and $b^* \in E(B, \succsim)$. For some scalar $\alpha > 1$, we therefore have $E(\alpha A, \succsim) \subseteq E(\alpha B, \succsim)$. Two sub-cases are then possible.
   (i) There exists $\bar{\alpha} > 1$ such that $E(\bar{\alpha} A, \succsim) \subset E(\bar{\alpha} B, \succsim)$. Then, by the Inclusion Axiom, $E(\bar{\alpha} B, \succsim) \succ E(\bar{\alpha} A, \succsim)$. By Axiom E and transitivity, $\bar{\alpha} B \succ \bar{\alpha} A$. Multiplying both opportunity sets by $1/\bar{\alpha}$, we conclude that $B \succ A$ by Axiom B. This contradicts $A \succsim B$.
   (ii) Suppose now there is no $\alpha > 1$ such that $E(\alpha A, \succsim) \subset E(\alpha B, \succsim)$. We multiply both A and B by the $min\ \alpha$ such that $d(B, \mathbf{t}) = 0$. We denote such a scalar $\hat{\alpha}$ (with $\hat{\alpha} > 1$ by construction). After expansion of the opportunity sets, the target $\mathbf{t}$ is on the boundary of $\hat{\alpha} B$, $\mathbf{t}$ therefore coincides with the only essential set of $\hat{\alpha} B$ and $E(\hat{\alpha} A, \succsim) \subseteq E(\hat{\alpha} B, \succsim)$. Let us now introduce the new opportunity set D which is the positive part of a hyperplane H which supports $\hat{\alpha} B$ at point $\mathbf{t}$. Since each vector of $E(\hat{\alpha} A, \succsim)$ has its elements less than or equal to the respective elements of $\mathbf{t}$, then $E(\hat{\alpha} A, \succsim)$ lies below D. More precisely, $E(\hat{\alpha} A, \succsim) \subset D$ and, by applying the set inclusion Axiom, $D \succ E(\hat{\alpha} A, \succsim)$. Since $E(D, \succsim) = \mathbf{t} = E(\hat{\alpha} B, \succsim)$, then by Axiom E and by transitivity we get $\hat{\alpha} B \succ \hat{\alpha} A$. Finally, multiplying both sets by $1/\hat{\alpha}$ and by applying Axiom S, we obtain the contradiction $B \succ A$.
3. $\mathbf{t} \in A$ and $\mathbf{t} \notin B$. If $\mathbf{E}(B, \succsim) \subset A$, by the Inclusion Axiom we obtain $A \succ E(B, \succsim)$ and by Axiom E we obtain $A \succ B$, with $d(A, \mathbf{t}) < d(B, \mathbf{t})$. If $\mathbf{E}(B, \succsim) \subseteq A$ we repeat the reasoning of point 2(ii) to complete the proof. □

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