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Reducibility of Linear Quasi-periodic Hamiltonian Derivative Wave Equations and Half-Wave Equations Under the Brjuno Conditions

Zhaowei Lou[1,2]

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Abstract
In this paper, we prove the reducibility for some linear quasi-periodic Hamiltonian derivative wave and half-wave equations under the Brjuno–Rüssmann non-resonance conditions. This is an extension of previous results of reducibility on Hamiltonian PDEs that required stronger (Diophantine) non-resonance conditions.

Keywords Reducibility · KAM · Brjuno–Rüssmann condition

Mathematics Subject Classification 37K55 · 35L05 · 35Q55

1 Introduction and Main Result

The reducibility theory of linear quasi-periodic systems is the generalization of the classical Floquet theory for linear periodic systems. It is important both in the linear problems (spectral analysis of operator, growth of Sobolev norms) and in the non-linear case(linear stability analysis of quasi-periodic solutions of non-linear systems). The first reducibility result via Kolmogorov–Arnold–Moser (KAM) theory was due to Bogoljubov, Mitropoliskii and Samoilenko [11], Dinaburg and Sinai [17] for finite degrees of freedom systems. Since then KAM theory has been a powerful tool to study reducibility theory. In the late 1980s and early 1990s, KAM theory was extended to non-linear partial differential equations (PDEs) by Kuksin [33] and Wayne [48]. See also [35, 42, 43] for further developments. As a corollary, these results imply the reducibility of the variational equations for quasi-periodic solutions of non-linear PDEs. In fact, "reducibility is not only an important outcome of KAM but also an essential ingredient in the proof" [20].

The first pure reducibility result for linear quasi-periodic PDEs was given by Bambusi and Graffi [5]. They proved the reducibility of linear Schrödinger equations with unbounded

✉ Zhaowei Lou
zwlou@nuaa.edu.cn

1 School of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China

2 MIIT Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles, Nanjing 211106, China

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perturbations. Eliasson and Kuksin [19] investigated the reducibility of higher dimensional linear quasi-periodic Schrödinger equations. Combining the pseudo-differential calculus, Baldi, Berti and Montalto [1, 2] obtained the reducibility of quasi-linear forced perturbations of Airy equation and quasi-linear KdV equation. Thereafter, these results are developed and extended widely. One could refer to [3, 4, 6–9, 28–30, 36, 37, 39] and the references therein.

Consider a linear quasi-periodic PDE of the form

$$\partial_t u = (A + P(\omega t))u, \quad \omega \in \mathbb{R}^n \setminus \{0\}, \tag{1.1}$$

where A is a positive self-adjoint operator and P is a operator-valued function with the basic frequencies ω. It is well known that KAM reducibility requires a lower bound on small divisors of the form

$$|k \cdot \omega + \lambda_i(\omega) - \lambda_i(\omega)|, \tag{1.2}$$

where $k \cdot \omega = \sum_{i=1}^{n} k_i \omega_i$ and $\{\lambda_i\}$ are the eigenvalues of the operator A. In all the above-mentioned papers, the lower bound of Diophantine type was used. Namely, the following non-resonance conditions holds: $|k \cdot \omega + \lambda_i(\omega) - \lambda_i(\omega)| \geq \frac{\gamma}{|k|^\tau}$, where the constants $\gamma > 0,\ \tau > n - 1$. On the other hand, thanks to the pioneering works of Brjuno [12], the Diophantine conditions can be weakened to the Brjuno conditions. To make it applicable in KAM scheme, Rüssmann [44, 45] introduced the notion of an approximation function to characterize the Brjuno conditions. Under such Brjuno–Rüssmann type conditions, Pöschel [40] proved the persistence of elliptic lower dimensional tori in finite dimensional Hamiltonian systems. In [41], Pöschel also proved the existence of infinite dimensional invariant tori in infinite dimensional Hamiltonian systems of the form $H = \omega \cdot I + P(\theta, I)$. Later on, Xu and You [49] and Chavaudret and Marmi [14] proved the reducibility of linear ODEs with almost periodic coefficients and quasi-periodic cocycles under such Brjuno–Rüssmann type conditions, respectively. See also [31, 46, 47] for nonlinear forced ODEs. We also mention some Brjuno type quasi-periodic results of Corsi and Gentile [15] and Gentile [27] for forced non-Hamiltonian ODEs without using approximation function.

To the best of our knowledge, there has been no Brjuno–Rüssmann type results in KAM theory for PDEs. In this paper, we establish a reducibility theorem for some linear Hamiltonian PDEs under Brjuno-Rüssmann non-resonance conditions. More precisely, we consider the following linear quasi-periodic derivative wave equations

$$\partial_{tt} u - \partial_{xx} u + mu + \epsilon V(\omega t, x)\mathbf{D}_m u = 0, \quad m \geq 0\ x \in [0, \pi] \tag{1.3}$$

and linear quasi-periodic half-wave equations

$$\mathrm{i}\partial_t u + \mathbf{D}_0 u + \epsilon V(\omega t, x)u = 0, \quad x \in [0, \pi], \tag{1.4}$$

under Dirichlet boundary conditions, where the Fourier multiplier $\mathbf{D}_m := \sqrt{-\partial_{xx} + m}$. The basic fequencies ω of the potential V satisfy the Brjuno–Rüssmann non-resonance conditions. The wave Eq. (1.3) covers the variational equation around any small amplitude quasi-periodic solutions of nonlinear Hamiltonian derivative wave equation $\partial_{tt}u - \partial_{xx}u + mu + f(\mathbf{D}_m u) = 0$, where $f(z) = az^3 + O(z^5),\ a \neq 0$. Quasi-periodic solutions with Diophantine frequencies of this nonlinear wave equation under periodic boundary conditions have been obtained in [10]. The half-wave Eq. (1.4) is an important class of PDEs arising in various physical problems [13, 18, 24, 32, 38]. There are two main difficulties when studying the reducibility theory of the Eqs. (1.3) and (1.4). The first one is the weak dispersion relation since the eigenvalues $\lambda_j \sim j,\ j \to \infty$. The second one is the bad smoothness of the perturbations. To overcome this, we introduce a simplified version of Töplitz–Lipschitz functions and Töplitz–Lipschitz matrices, which were first proposed by Eliasson and Kuksin [20] in KAM theory for the higher dimensional Schrödinger equations. Such simplified form is more suitable to the Eqs. (1.3) and (1.4) and it was also used in [25, 26]. Different from that in [25, 26], we characterize the Töplitz–Lipschitz functions in a way of semi-norm. We also mention the quasi-Töplitz functions introduced in [10] for nonlinear Hamiltonian derivative wave equations, which is also an improved version of Eliasson–Kuksin's Töplitz–Lipschitz functions. Comparing to the quasi-Töplitz functions, our simplified form is more easy to handle. For further work on the reduction of linear operators involving weak dispersion relations, please refer to references [21–23].

To state our main results, we introduce some definitions and assumptions on the potentials V in the Eqs. (1.3) and (1.4).

Definition 1.1 (Approximation function, [40, 45]). A non-decreasing function

$$\Delta : [0, \infty) \to [1, \infty)$$

is called an approximation function, if

$$\frac{\log \Delta(t)}{t} \downarrow 0, \quad 0 \le t \to \infty \tag{1.5}$$

and

$$\int_1^\infty \frac{\log \Delta(t)}{t^2} dt < \infty. \tag{1.6}$$

in addition, the normalization $\Delta(0) = 1$ is imposed for definition.

Remark 1.1 Below we list three typical approximation functions: $\Delta_1 = \exp(t^\alpha/\alpha)$, $0 < \alpha < 1$, $\Delta_2 = \exp\left(\frac{t}{1+\log^\alpha(1+t)}\right)$, $\alpha > 1$ and $\Delta_3 = \exp\left(\frac{t}{\log^\alpha t}\right)$, $\alpha > 1$.

Definition 1.2 (*Brjuno–Rüssmann frequency*) Let Δ be an approximation function. A vector $\omega \in \mathbb{R}^n$ is called Brjuno–Rüssmann frequency vector if it satisfies

$$|k \cdot \omega| \ge \frac{\gamma}{\Delta(|k|)}, \quad k \in \mathbb{Z}^n \setminus \{0\} \tag{1.7}$$

for some constant $\gamma > 0$.

Assumption 1 Suppose the function $V : \mathbb{T}^n \times [0, \pi] \to \mathbb{R}$ is real analytic in (θ, x). For $\theta \in \mathbb{T}^n$, $V(\theta, \cdot)$ is a $2\pi-$periodic, even function $V(\theta, x) = V(\theta, -x)$. Then it can be written as

$$V(\theta, x) = \sum_{j \ge 0} \widetilde{V}_j(\theta) \cos jx. \tag{1.8}$$

Moreover, suppose for all θ, the function $V(\theta, \cdot)$ extends to a complex analytic function on a strip $|Imx| < 2a$ for some $a > 0$. For all x, the function $V(\cdot, x)$ extends to a complex analytic function on a strip on $|Im\theta| < 2r$ for some $r > 0$. Then there is a positive constant $C_V > 0$ such that for $p \ge 0$,

$$\|V\|_{D(2r),2a,p} := \|\widetilde{V}_0\|_{D(2r)} + \sum_{j \ge 1} j^p e^{2aj} \|\widetilde{V}_j\|_{D(2r)} \le C_V, \tag{1.9}$$

where the norm $\|\cdot\|_{D(2r)}$ is defined in Sect. 2.

Let $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin jx,\ j \geq 1$ be the normalized Dirichlet eigenfunctions of the operator $D_m^2 := -\partial_{xx} + m$ associated to the eigenvalues $\lambda_j^2 = j^2 + m,\ j \geq 1$. We consider the Eqs. (1.3) and (1.4) in the following function space

$$\mathcal{H}_0^{a,p} = \left\{ u = \sum_{j\geq 1} q_j \phi_j : \|u\|_{a,p} = \sum_{j \geq 1} j^p \mathrm{e}^{aj} |q_j| < \infty \right\}. \tag{1.10}$$

Our main result is stated as follows.

Theorem 1.1 *Let $m \geq 0$. Under the Assumption 1 on the potential functions V, there is ϵ_0 so that for all $0 < \epsilon < \epsilon_0$ there exists $\mathcal{O}_\epsilon \subseteq [0, 2\pi)^n$ of positive Lebesgue measure such that for all $\omega \in \mathcal{O}_\epsilon$ satisfying Brjuno–Rüssmann non-resonance conditions, the above linear quasi-periodic wave Eq. (1.3) and half-wave Eq. (1.4) reduce to the linear equations with constant coefficients with respect to the time variable.*

In Sect. 5, we prove this theorem by the reducibility Theorem 4.1.

As a corollary of Theorem 1.1, we have the following conclusion concerning the solutions of the Eqs. (1.3) and (1.4):

Corollary 1.1 *Let the initial data $u_0 \in \mathcal{H}_0^{a,p}$, $v_0 \in \mathcal{H}_0^{a,p-1}$. Under the Assumption 1, there is ϵ_0 so that for all $0 < \epsilon < \epsilon_0$ and $\omega \in \mathcal{O}_\epsilon$,*

(i) *there exists a unique solution $(u(t, x), u_t(t, x)) \in \mathcal{H}_0^{a,p} \times \mathcal{H}_0^{a,p-1}$ of the wave Eq. (1.3) with $(u(0, x), u_t(0, x)) = (u_0, v_0)$. Moreover, $u(t, x)$ is almost-periodic in time and stable, i.e.,*

$$\begin{aligned}(1 - \varepsilon C)(\|u_0\|_{a,p} + \|v_0\|_{a,p-1}) &\leq \|u(t,\cdot)\|_{a,p} + \|u_t(t,\cdot)\|_{a,p-1} \\ &\leq (1 + \varepsilon C)(\|u_0\|_{a,p} + \|v_0\|_{a,p-1}),\end{aligned}$$

$\forall t \in \mathbb{R}$, *for some constant $C = C(a, p, \omega) > 0$.*

(ii) *there exists a unique solution $u(t, x) \in \mathcal{H}_0^{a,p}$ of the half-wave Eq. (1.4) with $u(0, x) = u_0$. Moreover, $u(t, x)$ is almost-periodic in time and stable, i.e.,*

$$(1 - \varepsilon C)\|u_0\|_{a,p} \leq \|u(t,\cdot)\|_{a,p} \leq (1 + \varepsilon C)\|u_0\|_{a,p}, \quad \forall t \in \mathbb{R}$$

for some constant $C = C(a, p, \omega) > 0$.

Remark 1.2 More recently, using the Renormalization Group method under Brjuno-type conditions without employing an approximation function, Corsi et al. [16] have constructed almost-periodic solutions with Gevrey regularity for the NLS equation with a convolution potential of arbitrarily high regularity.

2 Functional Setting

Let $\mathcal{O} \subset \mathbb{R}^n$ be a parameter set of positive Lebesgue measure. Throughout the paper, for any real or complex valued function depending on parameters $\xi \in \mathcal{O}$, its derivatives with respect to ξ are understood in the sense of Whitney. We denote by $C_W^1(\mathcal{O})$ the class of C^1 Whitney differentiable functions on $\mathcal{O}$.

Suppose $f \in C_W^1(\mathcal{O})$, we define its norm as

$$|f|_{\mathcal{O}} := \sup_{\xi \in \mathcal{O}} \left(|f(\xi)| + |\frac{\partial f}{\partial \xi}(\xi)| \right),$$

where $|\cdot|$ denotes the sup-norm of complex vectors.

Given an n-torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ and its complex neighborhood

$$D(r) = \{\theta \in \mathbb{C}^n : |\mathrm{Im}\theta| < r,\ r > 0\}.$$

Consider a real analytic function $f(\theta;\xi)$ on $\theta \in D(r)$. It is also C_W^1 on $\xi \in \mathcal{O}$. Its Fourier expansion reads $f(\theta;\xi) = \sum_{k\in\mathbb{Z}^n} \widehat{f}(k;\xi)\mathrm{e}^{\mathrm{i}k\cdot\theta}$, then we define its norm as

$$\|f\|_{D(r)\times\mathcal{O}} := \sum_{k\in\mathbb{Z}^n} |\widehat{f}(k;\cdot)|_{\mathcal{O}}\mathrm{e}^{|k|r},$$

where $k\cdot\theta = \sum_{i=1}^n k_i\theta_i$ and $|k| = \sum_{i=1}^n |k_i|$.

Let $K > 0$. For $f(\theta;\xi)$ above, its $K-$order Fourier truncation $\mathcal{T}_K f$ is defined as

$$(\mathcal{T}_K f)(\theta) := \sum_{k\in\mathbb{Z}^n,\ |k|<K} \widehat{f}(k)\mathrm{e}^{\mathrm{i}k\cdot\theta}.$$

The remainder $\mathcal{R}_K f$ of f is defined by $(\mathcal{R}_K f)(\theta) := f(\theta) - \mathcal{T}_K f(\theta)$. Suppose $0 < 2\sigma < r$, we have the following estimate for $\mathcal{R}_K f$:

$$\|\mathcal{R}_K f\|_{D(r-2\sigma)\times\mathcal{O}} \le 32\sigma^{-2}\mathrm{e}^{-K\sigma}\|f\|_{D(r)\times\mathcal{O}}. \tag{2.1}$$

The average $[f]$ of f on $\mathbb{T}^n$ is defined as

$$[f] := \widehat{f}(0) = (2\pi)^{-n}\int_{\mathbb{T}^n} f(\theta)d\theta.$$

Let $a, p > 0$, we introduce the Banach space $\ell^{a,p}$ of all real or complex sequences $z = (z_j)_{j\in\mathbb{Z}}$ with

$$\|z\|_{a,p} = \sum_{j\in\mathbb{Z}} \mathrm{e}^{aj} j^p |z_j| < \infty.$$

Given $r, s > 0$, we define the phase space

$$\mathcal{P}^{a,p} := \mathbb{T}^n \times \mathbb{R}^n \times \ell^{a,p} \times \ell^{a,p} \ni w := (\theta, I, z, \bar{z})$$

and a complex neighborhood

$$D(r,s) = \{w : |\mathrm{Im}\theta| < r, |I| < s^2, \|z\|_{a,p} < s, \|\bar{z}\|_{a,p} < s\}$$

of $\mathcal{T}_0^n := \mathbb{T}^n \times \{I = 0\} \times \{z = 0\} \times \{\bar{z} = 0\}$ in $\mathcal{P}_{\mathbb{C}}^{a,p} := \mathbb{C}^n \times \mathbb{C}^n \times \ell^{a,p} \times \ell^{a,p}$.

Consider a real analytic function $f(\theta, I, z, \bar{z};\xi)$ on $D(r,s)$, which is also C_W^1 on $\xi \in \mathcal{O}$. Its Taylor–Fourier expansion reads

$$f(\theta, I, z, \bar{z};\xi) = \sum_{l,\alpha,\beta} f_{l\alpha\beta}(\theta;\xi) I^l z^\alpha \bar{z}^\beta = \sum_{k\in\mathbb{Z}^n, l,\alpha,\beta} \widehat{f_{l\alpha\beta}}(k;\xi)\mathrm{e}^{\mathrm{i}k\cdot\theta} I^l z^\alpha \bar{z}^\beta,$$

where we use the multi-index notations $l = (l_j)_{j=1}^n$, $\alpha = (\alpha_j)_{j\ge1}$, $\beta = (\beta_j)_{j\ge1}$ with $l_j, \alpha_j, \beta_j \in \mathbb{N}$. α and β have only finitely many nonzero components. $I^l z^\alpha \bar{z}^\beta = (\prod_{i=1}^n I_i^{l_i})(\prod_{j\in\mathbb{Z}} z_j^{\alpha_j} \bar{z}_j^{\beta_j})$.

We define the majorant of f as

$$\lfloor f \rceil_{D(r)\times\mathcal{O}} \equiv \lfloor f(\cdot, I, z, \bar{z};\cdot) \rceil_{D(r)\times\mathcal{O}} := \sum_{l,\alpha,\beta} \|f_{l\alpha\beta}\|_{D(r)\times\mathcal{O}} |I^l||z^\alpha||\bar{z}^\beta|$$

and the norm of f as

$$\begin{aligned}\|f\|_{D(r,s)\times\mathcal{O}} &:= \sup_{|I|<s^2, \|z\|_{a,p}<s, \|\bar{z}\|_{a,p}<s} \lfloor f\rceil_{D(r)\times\mathcal{O}} \\ &= \sup_{|I|<s^2, \|z\|_{a,p}<s, \|\bar{z}\|_{a,p}<s} \sum_{l,\alpha,\beta} \|f_{l\alpha\beta}\|_{D(r)\times\mathcal{O}} |I^l||z^\alpha||\bar{z}^\beta|.\end{aligned}$$

Consider an infinite dimensional dynamical system on $D(r,s)$:

$$\dot{w} = X(w), \quad w = (\theta, I, z, \bar{z}) \in D(r,s),$$

where the vector field

$$X(w) = (X^{(\theta)}(w), X^{(I)}(w), X^{(z)}(w), X^{(\bar{z})}(w)),$$

Suppose vector field $X(w;\xi)$ is real analytic on $D(r,s)$ and C^1_W smooth on $\mathcal{O}$, we define the weighted norm of X as follows

$$\begin{aligned}&\|X\|_{s;D(r,s)\times\mathcal{O}} \\ &= \sup_{|I|<s^2, \|z\|_{a,p}<s, \|\bar{z}\|_{a,p}<s} \left\{ \sum_{i=1}^n \lfloor X^{(\theta_i)}\rceil_{D(r)\times\mathcal{O}} + \frac{1}{s^2}\sum_{i=1}^n \lfloor X^{(I_i)}\rceil_{D(r)\times\mathcal{O}} \right. \\ &\quad \left. + \frac{1}{s}\sum_{j\in\mathbb{Z}} \mathrm{e}^{aj} j^p (\lfloor X^{(z_j)}\rceil_{D(r)\times\mathcal{O}} + \lfloor X^{(\bar{z}_j)}\rceil_{D(r)\times\mathcal{O}}) \right\}.\end{aligned}$$

3 Töplitz–Lipschitz Functions

3.1 Definitions

In this section, we introduce a class of real analytic functions with exponentially off-diagonal decay.

Definition 3.1 Let $r, s, \rho > 0$. Suppose $P(\theta, z, \bar{z}; \xi)$ is real analytic on $(\theta, z, \bar{z}) \in D(r,s)$ and C^1_W−smooth on parameters $\xi \in \mathcal{O}$. We say that P is Töplitz–Lipschitz and write $P \in \mathcal{T}^\rho_{D(r,s)\times\mathcal{O}}$ if

$$\langle P\rangle_{\rho,D(r,s)\times\mathcal{O}} < \infty, \tag{3.1}$$

where the semi-norm $\langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}$ is the smallest non-negative real number that satisfies the following conditions

(T1) Exponentially off-diagonal decay.

$$\left\|\frac{\partial^2 P}{\partial z_i \partial z_j}\right\|_{D(r,s)\times\mathcal{O}} \le \langle P\rangle_{\rho,D(r,s)\times\mathcal{O}} \mathrm{e}^{-\rho|i+j|}. \tag{3.2}$$

$$\left\|\frac{\partial^2 P}{\partial z_i \partial \bar{z}_j}\right\|_{D(r,s)\times\mathcal{O}} \le \langle P\rangle_{\rho,D(r,s)\times\mathcal{O}} \mathrm{e}^{-\rho|i-j|}. \tag{3.3}$$

$$\left\|\frac{\partial^2 P}{\partial \bar{z}_i \partial \bar{z}_j}\right\|_{D(r,s)\times\mathcal{O}} \le \langle P\rangle_{\rho,D(r,s)\times\mathcal{O}} \mathrm{e}^{-\rho|i+j|}. \tag{3.4}$$

(T2) Asymptotically Töplitz. The limits

$$\lim_{t\in\mathbb{Z},\,t\to\infty}\frac{\partial^2 P}{\partial z_{i+t}\partial z_{j-t}},\quad \lim_{t\in\mathbb{Z},\,t\to\infty}\frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}}\quad\text{and}\quad \lim_{t\in\mathbb{Z},\,t\to\infty}\frac{\partial^2 P}{\partial \bar{z}_{i+t}\partial \bar{z}_{j-t}}$$

exist and are finite for all $i, j \in \mathbb{Z}$.

(T3) Lipschitz at infinity. For sufficiently large $|t|$, $t \in \mathbb{Z}$, the following hold.

$$\left\|\frac{\partial^2 P}{\partial z_{i+t}\partial z_{j-t}} - \lim_{t\to\infty}\frac{\partial^2 P}{\partial z_{i+t}\partial z_{j-t}}\right\|_{D(r,s)\times\mathcal{O}} \le |t|^{-1}\langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i+j|},\tag{3.5}$$

$$\left\|\frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}} - \lim_{t\to\infty}\frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}}\right\|_{D(r,s)\times\mathcal{O}} \le |t|^{-1}\langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i-j|},\tag{3.6}$$

$$\left\|\frac{\partial^2 P}{\partial \bar{z}_{i+t}\partial \bar{z}_{j-t}} - \lim_{t\to\infty}\frac{\partial^2 P}{\partial \bar{z}_{i+t}\partial \bar{z}_{j-t}}\right\|_{D(r,s)\times\mathcal{O}} \le |t|^{-1}\langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i+j|},\tag{3.7}$$

Remark 3.1 By the definition of $\langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}$, it is not difficult to verify that

- $\langle P\rangle_{\rho,D(r,s)\times\mathcal{O}} \ge 0$;
- $\langle \lambda P\rangle_{\rho,D(r,s)\times\mathcal{O}} = |\lambda|\langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}$ for all $\lambda \in \mathbb{C}$;
- $\langle P+F\rangle_{\rho,D(r,s)\times\mathcal{O}} \le \langle P\rangle_{\rho,D(r,s)\times\mathcal{O}} + \langle F\rangle_{\rho,D(r,s)\times\mathcal{O}}$.

Note that $\langle P\rangle_{\rho,D(r,s)\times\mathcal{O}} = 0$ could not imply $P = 0$. This means $\langle\cdot\rangle_{\rho,D(r,s)\times\mathcal{O}}$ is only a semi-norm.

Remark 3.2 From (T1) and (T3), the limits in (T3) satisfy

$$\left\|\lim_{t\to\infty}\frac{\partial^2 P}{\partial z_{i+t}\partial z_{j-t}}\right\|_{D(r,s)\times\mathcal{O}} \le \langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i+j|};\tag{3.8}$$

$$\left\|\lim_{t\to\infty}\frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}}\right\|_{D(r,s)\times\mathcal{O}} \le \langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i-j|};\tag{3.9}$$

$$\left\|\lim_{t\to\infty}\frac{\partial^2 P}{\partial \bar{z}_{i+t}\partial \bar{z}_{j-t}}\right\|_{D(r,s)\times\mathcal{O}} \le \langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i+j|}.\tag{3.10}$$

Remark 3.3 By the definition of the semi-norm $\langle\cdot\rangle_{\rho,D(r,s)\times\mathcal{O}}$, it is not difficult to verify that the following conclusions hold:

(1) $\langle P\rangle_{\rho,D(r',s')\times\mathcal{O}} \le \langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}$ if $0 < r' \le r,\ 0 < s' \le s$;
(2) $\langle P\rangle_{\rho',D(r,s)\times\mathcal{O}} \le \langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}$ if $0 < \rho' \le \rho$;
(3) Let $K > 0$, then the Fourier truncation $\mathcal{T}_K P$ of P satisfies

$$\langle \mathcal{T}_K P\rangle_{\rho,D(r,s)\times\mathcal{O}} \le \langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}$$

and the remainder $\mathcal{R}_K P$ of P satisfies

$$\langle \mathcal{R}_K P\rangle_{\rho,D(r',s)\times\mathcal{O}} \le \mathrm{e}^{-K(r-r')}\langle P\rangle_{\rho,D(r,s)\times\mathcal{O}}$$

if $0 < r' \le r$.

Definition 3.2 Let $\ell_0^{a,p}$ be the unilateral infinite sequences space defined by

$$\ell_0^{a,p} = \left\{ z = (z_j)_{j\ge 1} : \|z\|_{a,p} = \sum_{j\ge 1}|z_j||j|^p\mathrm{e}^{a|j|} < \infty \right\}.\tag{3.11}$$

Given a real analytic function $P(\theta, z, \bar{z})$ with $(z, \bar{z}) \in \ell_0^{a,p} \times \ell_0^{a,p}$, we lift it from $\ell_0^{a,p} \times \ell_0^{a,p}$ to $\ell^{a,p} \times \ell^{a,p}$ by $\widetilde{P}(\theta, \tilde{z}, \bar{\tilde{z}}) = P(\theta, z, \bar{z})$, where $(\tilde{z}, \bar{\tilde{z}}) \in \ell^{a,p} \times \ell^{a,p}$ and $\tilde{z} = z_j, \bar{\tilde{z}} = \bar{z}_j$ for all $j \geq 1$.

We say that the function P is Töplitz–Lipschitz and write $P \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$ if $\widetilde{P}(\theta, \tilde{z}, \bar{\tilde{z}})$ is Töplitz–Lipschitz and define

$$\langle P \rangle_{\rho, D(r,s)\times\mathcal{O}} := \langle \widetilde{P} \rangle_{\rho, D(r,s)\times\mathcal{O}} < \infty. \tag{3.12}$$

Below we focus on a class of quadratic functions on $(z, \bar{z})$ of the form

$$P(\theta, z, \bar{z}; \xi) = \sum_{|\alpha|+|\beta|=2} P_{\alpha\beta}(\theta; \xi) z^{\alpha} \bar{z}^{\beta}.$$

We study the Töplitz–Lipschitz property for these functions under the action of the Poisson bracket, the flow of linear Hamiltonian system and the canonical transformation.

Proposition 3.1 (Poisson bracket). *Let* $0 < \delta < \min\{\rho, 1\}$. *Suppose the quadratic functions* $R,\ F \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$, *then* $\{R, F\} \in \mathcal{T}^{\rho-\delta}_{D(r,s)\times\mathcal{O}}$ *and there exists a constant* $C > 0$ *so that*

$$\langle \{R, F\} \rangle_{\rho-\delta, D(r,s)\times\mathcal{O}} \leq \frac{C}{\delta} \langle R \rangle_{\rho, D(r,s)\times\mathcal{O}} \langle F \rangle_{\rho, D(r,s)\times\mathcal{O}}. \tag{3.13}$$

Proof The Poisson bracket $\{R, F\}$ reads

$$\{R, F\} = \mathrm{i} \sum_{k\in\mathbb{Z}} \left(\frac{\partial R}{\partial z_k} \frac{\partial F}{\partial \bar{z}_k} - \frac{\partial R}{\partial \bar{z}_k} \frac{\partial F}{\partial z_k} \right).$$

In what follows, it remains to analyze the second derivative $\frac{\partial^2 \{R,F\}}{\partial z_i \partial \bar{z}_j}$ with respect to $z_i, \bar{z}_j$, and the other second derivatives could be similarly done.

Since the functions R and F are both quadratic on $(z, \bar{z})$, their third derivatives vanish. Then we have

$$\begin{aligned} &\frac{\partial^2 \{R, F\}}{\partial z_i \partial \bar{z}_j} \\ &\quad = \sum_{k\in\mathbb{Z}} \mathrm{i} \left(\frac{\partial^2 R}{\partial z_k \partial \bar{z}_j} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_k} + \frac{\partial^2 R}{\partial z_i \partial z_k} \frac{\partial^2 F}{\partial \bar{z}_k \partial \bar{z}_j} - \frac{\partial^2 R}{\partial \bar{z}_k \partial \bar{z}_j} \frac{\partial^2 F}{\partial z_i \partial z_k} - \frac{\partial^2 R}{\partial \bar{z}_k \partial z_i} \frac{\partial^2 F}{\partial z_k \partial \bar{z}_j} \right). \end{aligned} \tag{3.14}$$

- We first verify the property (T1) for $\frac{\partial^2 \{R,F\}}{\partial z_i \partial \bar{z}_j}$. It suffices to consider the sums $\sum_{k\geq 1} \frac{\partial^2 R}{\partial z_k \partial \bar{z}_j} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_k}$ and $\sum_{k\geq 1} \frac{\partial^2 R}{\partial z_i \partial z_k} \frac{\partial^2 F}{\partial \bar{z}_k \partial \bar{z}_j}$ in (3.14), and the others can be similarly done. Since the functions R and F satisfy the property (T1), then we have

$$\begin{aligned} \left\| \sum_k \frac{\partial^2 R}{\partial z_k \partial \bar{z}_j} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_k} \right\|_{D(r,s)\times\mathcal{O}} &\leq \sum_k \left\| \frac{\partial^2 R}{\partial z_k \partial \bar{z}_j} \right\|_{D(r,s)\times\mathcal{O}} \left\| \frac{\partial^2 F}{\partial z_i \partial \bar{z}_k} \right\|_{D(r,s)\times\mathcal{O}} \\ &\leq \langle R \rangle_{\rho, D(r,s)\times\mathcal{O}} \langle F \rangle_{\rho, D(r,s)\times\mathcal{O}} \sum_k \mathrm{e}^{-\rho(|i-k|+|k-j|)} \\ &\leq \langle R \rangle_{\rho, D(r,s)\times\mathcal{O}} \langle F \rangle_{\rho, D(r,s)\times\mathcal{O}} \mathrm{e}^{-(\rho-\delta)(|i-j|)} \sum_k \mathrm{e}^{-\delta(|i-k|+|k-j|)} \\ &\leq C\delta^{-1} \langle R \rangle_{\rho, D(r,s)\times\mathcal{O}} \langle F \rangle_{\rho, D(r,s)\times\mathcal{O}} \mathrm{e}^{-(\rho-\delta)(|i-j|)} \end{aligned}$$

and

$$\begin{aligned}\left\|\sum_k \frac{\partial^2 R}{\partial z_i \partial z_k}\frac{\partial^2 F}{\partial \bar{z}_j \partial \bar{z}_k}\right\|_{D(r,s)\times\mathcal{O}} &\leq \sum_k \left\|\frac{\partial^2 R}{\partial z_i \partial z_k}\right\|_{D(r,s)\times\mathcal{O}}\left\|\frac{\partial^2 F}{\partial \bar{z}_j \partial \bar{z}_k}\right\|_{D(r,s)\times\mathcal{O}}\\ &\leq \langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\langle F\rangle_{\rho,D(r,s)\times\mathcal{O}}\sum_k \mathrm{e}^{-\rho(|i+k|+|k+j|)}\\ &\leq C\delta^{-1}\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\langle F\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-(\rho-\delta)(|i-j|)}.\end{aligned}$$

here we use the inequality $\sum_k \mathrm{e}^{-\delta(|i-k|+|k-j|)} \leq C\delta^{-1}$ (see Lemma 7.5, Appendix).

• We then verify the property (T2) for $\frac{\partial^2\{R,F\}}{\partial z_i \partial \bar{z}_j}$. From the above analysis, we know that the functional series $\sum_{k\geq 1}\frac{\partial^2 R}{\partial z_k \partial \bar{z}_j}\frac{\partial^2 F}{\partial z_i \partial \bar{z}_k}$ and $\sum_{k\geq 1}\frac{\partial^2 R}{\partial z_i \partial z_k}\frac{\partial^2 F}{\partial \bar{z}_k \partial \bar{z}_j}$ converge uniformly on $D(r,s)\times\mathcal{O}$. Since the limits $\lim_{t\to\infty}\frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}}$, $\lim_{t\to\infty}\frac{\partial^2 P}{\partial z_{i+t}\partial z_{j-t}}$ and $\lim_{t\to\infty}\frac{\partial^2 P}{\partial \bar{z}_{i+t}\partial z_{j-t}}$ exist and are finite, then the limits

$$\lim_{t\to\infty}\sum_k \frac{\partial^2 R}{\partial z_{k+t}\partial \bar{z}_{j+t}}\frac{\partial^2 F}{\partial z_{i+t}\partial \bar{z}_{k+t}}$$

and

$$\lim_{t\to\infty}\sum_k \frac{\partial^2 R}{\partial z_{i+t}\partial z_{k-t}}\frac{\partial^2 F}{\partial \bar{z}_{j+t}\partial \bar{z}_{k-t}}$$

also exist and are finite. This implies the property (T2) holds for $\frac{\partial^2\{R,F\}}{\partial z_i \partial \bar{z}_j}$.

• Finally, we verify the property (T3) for $\frac{\partial^2\{R,F\}}{\partial z_i \partial \bar{z}_j}$. For the sake of convenience, we introduce the notations

$$P^{11}_{ij,\infty} := \lim_{t\to\infty}\frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}},$$
$$P^{20}_{ij,\infty} := \lim_{t\to\infty}\frac{\partial^2 P}{\partial z_{i+t}\partial z_{j-t}}$$

and

$$P^{02}_{ij,\infty} := \lim_{t\to\infty}\frac{\partial^2 P}{\partial \bar{z}_{i+t}\partial z_{j-t}}.$$

In view of $R,\ F \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$ and thanks to the difference equality

$$AB - ab = (A-a)b + a(B-b) + (A-a)(B-b), \tag{3.15}$$

and the inequality in Lemma 7.5, we have

$$\begin{aligned}&\left\|\sum_k \frac{\partial^2 R}{\partial z_{k+t}\partial \bar{z}_{j+t}}\frac{\partial^2 F}{\partial z_{i+t}\partial \bar{z}_{k+t}} - \lim_{t\to\infty}\sum_k \frac{\partial^2 R}{\partial z_{k+t}\partial \bar{z}_{j+t}}\frac{\partial^2 F}{\partial z_{i+t}\partial \bar{z}_{k+t}}\right\|_{D(r,s)\times\mathcal{O}}\\ &\leq \sum_k \left\|\frac{\partial^2 R}{\partial z_{k+t}\partial \bar{z}_{j+t}} - R^{11}_{kj,\infty}\right\|_{D(r,s)\times\mathcal{O}}\left\|F^{11}_{ik,\infty}\right\|_{D(r,s)\times\mathcal{O}}\\ &\quad + \sum_k \left\|R^{11}_{kj,\infty}\right\|_{D(r,s)\times\mathcal{O}}\left\|\frac{\partial^2 F}{\partial z_{i+t}\partial \bar{z}_{k+t}} - F^{11}_{ik,\infty}\right\|_{D(r,s)\times\mathcal{O}}\end{aligned}$$

$$
\begin{aligned}
&+\sum_k \left\| \frac{\partial^2 R}{\partial z_{k+t}\partial \bar{z}_{j+t}} - R^{11}_{kj,\infty} \right\|_{D(r,s)\times\mathcal{O}} \left\| \frac{\partial^2 F}{\partial z_{i+t}\partial \bar{z}_{k+t}} - F^{11}_{ik,\infty} \right\|_{D(r,s)\times\mathcal{O}} \\
&\le |t|^{-1}\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\langle F\rangle_{\rho,D(r,s)\times\mathcal{O}} \sum_k \mathrm{e}^{-\rho(|i-k|+|k-j|)} \\
&\quad + |t|^{-1}\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\langle F\rangle_{\rho,D(r,s)\times\mathcal{O}} \sum_k \mathrm{e}^{-\rho(|i-k|+|k-j|)} \\
&\quad + |t|^{-2}\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\langle F\rangle_{\rho,D(r,s)\times\mathcal{O}} \sum_k \mathrm{e}^{-\rho(|i-k|+|k-j|)} \\
&\le |t|^{-1}C\delta^{-1}\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\langle F\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-(\rho-\delta)(|i-j|)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \sum_k \frac{\partial^2 R}{\partial z_{i+t}\partial z_{k-t}} \frac{\partial^2 F}{\partial \bar{z}_{j+t}\partial \bar{z}_{k-t}} - \lim_{t\to\infty}\sum_k \frac{\partial^2 R}{\partial z_{i+t}\partial z_{k-t}} \frac{\partial^2 F}{\partial \bar{z}_{j+t}\partial \bar{z}_{k-t}} \right\|_{D(r,s)\times\mathcal{O}} \\
&\le \sum_k \left\| \frac{\partial^2 R}{\partial z_{i+t}\partial z_{k-t}} - R^{20}_{ik,\infty} \right\|_{D(r,s)\times\mathcal{O}} \left\| F^{02}_{jk,\infty} \right\|_{D(r,s)\times\mathcal{O}} \\
&\quad + \sum_k \left\| R^{20}_{ik,\infty} \right\|_{D(r,s)\times\mathcal{O}} \left\| \frac{\partial^2 F}{\partial \bar{z}_{j+t}\partial \bar{z}_{k-t}} - F^{02}_{jk,\infty} \right\|_{D(r,s)\times\mathcal{O}} \\
&\quad + \sum_k \left\| \frac{\partial^2 R}{\partial z_{i+t}\partial z_{k-t}} - R^{20}_{ik,\infty} \right\|_{D(r,s)\times\mathcal{O}} \left\| \frac{\partial^2 F}{\partial \bar{z}_{j+t}\partial \bar{z}_{k-t}} - F^{02}_{jk,\infty} \right\|_{D(r,s)\times\mathcal{O}} \\
&\le |t|^{-1}C\delta^{-1}\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\langle F\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-(\rho-\delta)(|i-j|)}.
\end{aligned}
$$

These imply that

$$
\begin{aligned}
&\left\| \frac{\partial^2\{R,F\}}{\partial z_{i+t}\partial \bar{z}_{j+t}} - \lim_{t\to\infty} \frac{\partial^2\{R,F\}}{\partial z_{i+t}\partial \bar{z}_{j+t}} \right\|_{D(r,s)\times\mathcal{O}} \\
&\quad \le |t|^{-1}C\delta^{-1}\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\langle F\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-(\rho-\delta)(|i-j|)}.
\end{aligned}
$$

□

3.2 Töplitz–Lipschitz Matrices

Denote by $\mathcal{M}_2(\mathbb{C})$ the space of 2×2 complex matrices. Let $\|\cdot\|$ be any sub-multiplicative norm on $\mathcal{M}_2(\mathbb{C})$. Consider a bilateral infinite dimensional $\mathcal{M}_2(\mathbb{C})-$valued matrix

$$
\begin{aligned}
A : \mathbb{Z}\times\mathbb{Z} &\to \mathcal{M}_2(\mathbb{C}) : \\
(i,j) &\mapsto A_{ij} = \begin{pmatrix} A^{11}_{ij} & A^{12}_{ij} \\ A^{21}_{ij} & A^{22}_{ij} \end{pmatrix}.
\end{aligned}
$$

The matrix multiplication is defined by $(AB)_{ij} = \sum_{k\in\mathbb{Z}} A_{ik}B_{kj}$.

Now we consider the matrices depend on $(\theta,\xi)\in D(r)\times\mathcal{O}$.

Definition 3.3 (*Matrices with Töplitz–Lipschitz property*) Let $r,\rho>0$. We say that a matrix $A = A(\theta,\xi)$ on $D(r)\times\mathcal{O}$ is Töplitz–Lipschitz and write $A\in\mathfrak{M}^{\rho}_{r,\mathcal{O}}$ if $\langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}}<\infty$, where the norm $\langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}}$ is defined by the following conditions:

(T1′) Exponentially off-diagonal decay

$$\|A_{ij}^{11}\|_{D(r)\times\mathcal{O}} \le \langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}} e^{-\rho|i-j|}, \tag{3.16}$$

$$\|A_{ij}^{12}\|_{D(r)\times\mathcal{O}} \le \langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}} e^{-\rho|i+j|}, \tag{3.17}$$

$$\|A_{ij}^{21}\|_{D(r)\times\mathcal{O}} \le \langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}} e^{-\rho|i+j|}, \tag{3.18}$$

$$\|A_{ij}^{22}\|_{D(r)\times\mathcal{O}} \le \langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}} e^{-\rho|i-j|}. \tag{3.19}$$

(T2′) Asymptotically Töplitz The limits

$$\lim_{t\in\mathbb{Z},\, t\to\infty} A_{i+t,j+t}^{11}, \quad \lim_{t\in\mathbb{Z},\, t\to\infty} A_{i+t,j-t}^{12}, \quad \lim_{t\in\mathbb{Z},\, t\to\infty} A_{i+t,j-t}^{21} \text{ and } \lim_{t\in\mathbb{Z},\, t\to\infty} A_{i+t,j+t}^{22}$$

exist and are finite for all $i, j \in \mathbb{Z}$.

(T3′) Lipschitz at infinity For sufficiently large $|t|$, $t \in \mathbb{Z}$, the following hold.

$$\|A_{i+t,j+t}^{11} - \lim_{t\to\infty} A_{i+t,j+t}^{11}\|_{D(r)\times\mathcal{O}} \le |t|^{-1}\langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}} e^{-\rho|i-j|}. \tag{3.20}$$

$$\|A_{i+t,j-t}^{12} - \lim_{t\to\infty} A_{i+t,j-t}^{12}\|_{D(r)\times\mathcal{O}} \le |t|^{-1}\langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}} e^{-\rho|i+j|}. \tag{3.21}$$

$$\|A_{i+t,j-t}^{21} - \lim_{t\to\infty} A_{i+t,j-t}^{21}\|_{D(r)\times\mathcal{O}} \le |t|^{-1}\langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}} e^{-\rho|i+j|}. \tag{3.22}$$

$$\|A_{i+t,j+t}^{22} - \lim_{t\to\infty} A_{i+t,j+t}^{22}\|_{D(r)\times\mathcal{O}} \le |t|^{-1}\langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}} e^{-\rho|i-j|}. \tag{3.23}$$

Definition 3.4 Given a unilateral infinite dimensional $\mathcal{M}_2(\mathbb{C})-$valued matrix

$$A : \mathbb{N}\times\mathbb{N} \to \mathcal{M}_2(\mathbb{C}),$$

we lift it from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{Z}\times\mathbb{Z}$ by

$$\tilde{A}_{ij} = \begin{cases} A_{ij}, & i \ge 1, \quad j \ge 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.24}$$

We say that A is Töplitz–Lipschitz and write $A \in \mathfrak{M}_{r,\mathcal{O}}^{\rho}$ if $\tilde{A}$ is Töplitz–Lipschitz and define

$$\langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}} := \langle\langle \tilde{A}\rangle\rangle_{\rho,r,\mathcal{O}} < \infty. \tag{3.25}$$

The following conclusion indicate that $\mathfrak{M}_{r,\mathcal{O}}^{\rho}$ is an algebra. This important property will be applied to Proposition 3.4.

Proposition 3.2 *Let* $0 < \delta < \rho$. *Suppose the matrices* $A, B \in \mathfrak{M}_{r,\mathcal{O}}^{\rho}$. *Then their product* $AB \in \mathfrak{M}_{r,\mathcal{O}}^{\rho-\delta}$ *and there exists a constant* $C > 0$ *so that*

$$\langle\langle AB\rangle\rangle_{\rho-\delta,r,\mathcal{O}} \le C\delta^{-1}\langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}}\langle\langle B\rangle\rangle_{\rho,r,\mathcal{O}}.$$

The proof is given in Sect. 7.2, Appendix.

3.3 Flow of Linear Hamiltonian System

In this section, we study the Hamiltonian flow generated by a quadratic Töplitz–Lipschitz function $F(\theta, z, \bar{z}; \xi) \in \mathcal{T}_{D(r,s)\times\mathcal{O}}^{\rho}$.

In the sequel, we use the notations $Z = (Z_j)_{j\in\mathbb{Z}}^T$ with $Z_j = (z_j, \bar{z}_j)^T$. The Hessian $\partial_Z^2 F$ of F with respect to Z reads

$$\partial_Z^2 F = \left(\nabla_{Z_j}\nabla_{Z_i} F\right)_{i,j\in\mathbb{Z}}$$

where

$$\nabla_{Z_j}\nabla_{Z_i}F = \begin{pmatrix} \frac{\partial^2 F}{\partial z_i \partial z_j} & \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} \\ \frac{\partial^2 F}{\partial \bar{z}_i \partial z_j} & \frac{\partial^2 F}{\partial \bar{z}_i \partial \bar{z}_j} \end{pmatrix}.$$

Denote $A = J\partial_Z^2 F$, where

$$J = diag\left\{J_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}_{j\in\mathbb{Z}},$$

then

$$A_{ij} = \begin{pmatrix} \frac{\partial^2 F}{\partial \bar{z}_i \partial z_j} & \frac{\partial^2 F}{\partial \bar{z}_i \partial \bar{z}_j} \\ -\frac{\partial^2 F}{\partial z_i \partial z_j} & -\frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} \end{pmatrix}. \tag{3.26}$$

By the definitions of Töplitz–Lipschitz function and Töplitz–Lipschitz matrix, they have the following relation.

Lemma 3.3 *Let $\rho > 0$. Suppose $F(\theta, z, \bar{z}, ; \xi)$ is a quadratic function on $D(r, s) \times \mathcal{O}$. Then $F \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$ if and only if $A = J\partial_Z^2 F \in \mathfrak{M}^{\rho}_{r,\mathcal{O}}$. Moreover,*

$$\langle\langle A\rangle\rangle_{\rho,r,\mathcal{O}} = \langle F\rangle_{\rho,D(r,s)\times\mathcal{O}}. \tag{3.27}$$

The Hamiltonian equation associated to the quadratic function F reads

$$\begin{cases} (\dot{\theta}(t), \dot{I}(t), \dot{z}(t), \dot{\bar{z}}(t)) = X_F(\theta(t), I(t), z(t), \bar{z}(t)), \\ (\theta(0), I(0), z(0), \bar{z}(0)) = (\theta^0, I^0, z^0, \bar{z}^0). \end{cases} \tag{3.28}$$

Under the new notation Z, the quadratic function $F \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$ can be rewritten as

$$F(\theta, Z) = \frac{1}{2}Z^T A(\theta)Z = \frac{1}{2}Z^T\partial_Z^2 F(\theta, 0)Z \tag{3.29}$$

and the Eq. (3.28) reads

$$\begin{cases} \dot{\theta}(t) = 0, \\ \dot{I}(t) = -\partial_\theta F(\theta(t), Z(t)), \\ \dot{Z}(t) = A(\theta(t))Z = J\partial_Z F(\theta(t), 0)Z(t), \\ (\theta(0), I(0), Z(0)) = (\theta^0, I^0, Z^0). \end{cases} \tag{3.30}$$

The Jacobian $\partial_{Z^0}Z$ (the derivative of $Z(t)$ with respect to Z^0) is

$$\partial_{Z^0}Z = \left(\partial_{Z_j^0}Z_i\right)_{i,j\in\mathbb{Z}} = \left(\begin{pmatrix} \frac{\partial z_i}{\partial z_j^0} & \frac{\partial z_i}{\partial \bar{z}_j^0} \\ \frac{\partial \bar{z}_i}{\partial z_j^0} & \frac{\partial \bar{z}_i}{\partial \bar{z}_j^0} \end{pmatrix}\right)_{i,j\in\mathbb{Z}}.$$

Proposition 3.4 *Let $0 < \delta < \min\{\rho, 1\}$ and $0 < \sigma < r/3$. Suppose $Cs\sigma \leq \ln 2$ and $Cs^2 \leq 2$ and the quadratic function $F \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$ and*

$$\|X_F\|_{s;D(r-\sigma,s)\times\mathcal{O}} + \langle F\rangle_{\rho,D(r-\sigma,s)\times\mathcal{O}} < C\sigma. \tag{3.31}$$

Then the solution $(\theta(t), I(t), Z(t))$ of the Eq. (3.30) with initial condition $(\theta^0, I^0, Z^0) \in D(r-\sigma, \frac{s}{4})$ satisfies $(\theta(t), I(t), Z(t)) \in D(r, \frac{s}{2})$ for all $0 \leq t \leq 1$. Moreover, the Jacobian $\partial_{Z^0}Z(t)$ satisfies

$$\langle\langle \partial_{Z^0}Z(t) - Id\rangle\rangle_{\rho-\delta,r-\sigma,\mathcal{O}} \leq C\langle F\rangle_{\rho,D(r-\sigma,s)\times\mathcal{O}}. \tag{3.32}$$

where the notation Id is the identity mapping.

Proof Since $\dot\theta(t)=0$, then $\theta(t)\equiv\theta^0\in D(r-\sigma)$ remains unchanged.

Consider the equation for Z :

$$\begin{cases}\dot Z=A(\theta^0)Z:=J\partial_Z F(\theta^0,0)Z,\\ Z(0)=Z^0.\end{cases}\tag{3.33}$$

It is a linear system with constant coefficients, thus its solution is

$$Z(t)=\mathrm{e}^{tA(\theta^0)}Z^0.\tag{3.34}$$

By (3.26) and (3.31),

$$\|A\|_{\ell^{a,p}\to\ell^{a,p}}\le s\|X_F\|_{s;D(r-\sigma,s)\times\mathcal{O}}\le Cs\sigma.$$

Thus thanks to $Cs\sigma\le\ln 2$, for all $0\le t\le 1$,

$$\|Z(t)\|_{\ell^{a,p}}\le \mathrm{e}^{\|A\|_{\ell^{a,p}\to\ell^{a,p}}}\|Z^0\|_{\ell^{a,p}}\le \mathrm{e}^{Cs\sigma}\frac{s}{4}\le\frac{s}{2}.$$

Consider the equation for I. By (3.30) and (3.34), we have

$$\begin{cases}\dot I(t)=-\frac12 Z^T\partial_\theta A(\theta)Z,\\ I(0)=I^0.\end{cases}\tag{3.35}$$

The integral form of the above Eq. (3.35) is

$$I(t)=I^0-\frac12\int_0^t Z^T(\tau)\partial_\theta A(\theta)Z(\tau)d\tau.\tag{3.36}$$

Then thanks to $Cs^2\le 2$, for all $0\le t\le 1$,

$$|I(t)|\le|I^0|+\frac{1}{2\sigma}\|A\|_{\ell^{a,p}\to\ell^{a,p}}\|Z(t)\|^2_{\ell^{a,p}}\le\frac{s}{4}+\frac{Cs^3}{8}\le\frac{s}{2}.$$

Thus the flow X_F^t exists for all $0\le t\le 1$ and it maps the domain $D(r-\sigma,\frac{s}{4})$ to $D(r,\frac{s}{2})$. Denote the solution $(\theta(t),I(t),z(t),\bar z(t))=X_F^t(\theta^0,I^0,z^0,\bar z^0)$, then for $0\le t\le 1$ and $(\theta^0,I^0,z^0,\bar z^0)\in D(r-\sigma,\frac{s}{4})$, the solution $(\theta(t),I(t),z(t),\bar z(t))\in D(r,\frac{s}{2})$.

Now we prove the estimate (3.32). Rewrite the solution $Z(t)$ in (3.34) as

$$Z(t)=(Id+B(t))Z^0,\tag{3.37}$$

where

$$B(t)=\mathrm{e}^{tA(\theta)}-Id=\sum_{k=1}^{\infty}\frac{t^k}{k!}A^k(\theta).$$

By Proposition 3.2 and Lemma 3.3, for all $0\le t\le 1$,

$$\begin{aligned}\langle\langle B\rangle\rangle_{\rho-\delta,r-\sigma,\mathcal{O}}&\le\sum_{k=1}^{\infty}\frac{(k-1)^{k-1}}{k!}\left(\frac{C}{\delta}\right)^{k-1}\langle\langle A\rangle\rangle^k_{\rho,r-\sigma}\\&\le\sum_{k=1}^{\infty}\frac{\mathrm{e}^{k-1}}{k}\left(\frac{C}{\delta}\right)^{k-1}\langle F\rangle^k_{\rho,D(r-\sigma,s)\times\mathcal{O}}\\&\le C\langle F\rangle_{\rho,D(r-\sigma,s)\times\mathcal{O}}.\end{aligned}\tag{3.38}$$

This completes the proof of the estimate (3.32). □

Proposition 3.5 (Canonical transformation). *Let* $0 < \delta < \min\{\rho/3, 1\}$, $0 < \sigma < r$ *and* $R, F \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$, *where the Hamiltonian* F *is a quadratic function. Assume that the Hamiltonian* F *satisfies* (3.31). *Then the composition* $R \circ X_F^1 \in \mathcal{T}^{\rho-3\delta}_{D(r-\sigma,s/4)\times\mathcal{O}}$ *and there exists a constant* $C > 0$ *so that*

$$\langle R \circ X_F^1 \rangle_{\rho-3\delta, D(r-\sigma,s/4)\times\mathcal{O}} \leq C\delta^{-2} \langle R \rangle_{\rho, D(r,s/2)\times\mathcal{O}}. \tag{3.39}$$

Proof By Proposition 3.4, the time-1 mapping X_F^1 maps $(\theta^0, I^0, Z^0) \in D(r-\sigma, \frac{s}{4})$ to $(\theta, I, Z) := X_F^1(\theta^0, I^0, Z^0) \in D(r, \frac{s}{2})$.

Since the mapping Z is linear in Z^0, the Hessian $\partial^2_{Z^0} Z = 0$. Then the Hessian $\partial^2_{Z^0}(R \circ X_F^1)$ of $R \circ X_F^1$ with respect to Z^0 becomes

$$\partial^2_{Z^0}(R \circ X_F^1) = (\partial_{Z^0} Z)^T \partial_Z^2 R(X_F^1) \partial_{Z^0} Z.$$

Note that $\langle\langle J^T (\partial_{Z^0} Z)^T J \rangle\rangle_{\rho,r} = \langle\langle \partial_{Z^0} Z \rangle\rangle_{\rho,r}$ and $J^T = J^{-1} = -J$, then by Lemma 3.3 and Proposition 3.4, we have

$$\begin{aligned}
&\langle R \circ X_F^1 \rangle_{\rho-3\delta, D(r-\sigma,s/4)\times\mathcal{O}} \\
&= \langle\langle J \partial^2_{Z^0}(R \circ X_F^1) \rangle\rangle_{\rho-3\delta, D(r-\sigma,s/4)\times\mathcal{O}} \\
&\quad \leq C\delta^{-2} \langle\langle J^T (\partial_{Z^0} Z)^T J \rangle\rangle_{\rho-\delta, r-\sigma} \langle\langle J \partial_Z^2 R \rangle\rangle_{\rho, D(r,s/2)\times\mathcal{O}} \langle\langle \partial_{Z^0} Z \rangle\rangle_{\rho-\delta, r-\sigma} \\
&\quad = C\delta^{-2} \langle R \rangle_{\rho, D(r,s/2)\times\mathcal{O}} \langle\langle \partial_{Z^0} Z \rangle\rangle^2_{\rho-\delta, r-\sigma} \\
&\quad \leq C\delta^{-2} \langle R \rangle_{\rho, D(r,s/2)\times\mathcal{O}}.
\end{aligned} \tag{3.40}$$

□

4 A Reducibility Theorem Under Brjuno Condition

Consider the following quadratic Hamiltonian with time quasi-periodic perturbation:

$$\begin{aligned}
H(\omega t, z, \bar{z}) &= \sum_{j\geq 1} \Omega_j z_j \bar{z}_j + P(\omega t, z, \bar{z}) \\
&= \sum_{j\geq 1} \Omega_j z_j \bar{z}_j + \sum_{|\alpha|+|\beta|=2} P_{\alpha\beta}(\omega t) z^\alpha \bar{z}^\beta,
\end{aligned} \tag{4.1}$$

where $(z, \bar{z}) \in \ell_0^{a,p} \times \ell_0^{a,p}$, the space $\ell_0^{a,p}$ is the unilateral infinite sequences space defined in (3.11). The forcing frequency vector $\omega \in [0, 2\pi)^n$ and the normal frequencies $\Omega_j \in \mathbb{R}$ for all $j \geq 1$. Then the associated linear Hamiltonian system reads

$$\begin{cases} \dot{z}_j = \mathrm{i}\Omega_j z_j + \mathrm{i}\frac{\partial}{\partial \bar{z}_j} P(\omega t, z, \bar{z}), \\ \dot{\bar{z}}_j = -\mathrm{i}\Omega_j \bar{z}_j - \mathrm{i}\frac{\partial}{\partial z_j} P(\omega t, z, \bar{z}), \quad j \geq 1. \end{cases} \tag{4.2}$$

Introducing the angle variables $\theta = \omega t \in \mathbb{T}^n$, and the auxiliary action variables $I \in \mathbb{R}^n$, then we obtain an autonomous Hamiltonian system

$$\begin{cases} \dot{z}_j = \mathrm{i}\Omega_j z_j + \mathrm{i}\frac{\partial}{\partial \bar{z}_j} P(\omega t, z, \bar{z}), \\ \dot{\bar{z}}_j = -\mathrm{i}\Omega_j \bar{z}_j - \mathrm{i}\frac{\partial}{\partial z_j} P(\omega t, z, \bar{z}), & j \geq 1, \\ \dot{\theta}_i = \omega_i, & i = 1 \cdots n, \\ \dot{I}_i = -\frac{\partial}{\partial \theta_i} P(\theta, z, \bar{z}), & i = 1 \cdots n. \end{cases} \tag{4.3}$$

on the phase space $\mathcal{P}_0^{a,p} := \mathbb{T}^n \times \mathbb{R}^n \times \ell_0^{a,p} \times \ell_0^{a,p}$ with respect to the symplectic form

$$\sum_{i=1}^{n} d\theta_i \wedge dI_i + \mathrm{i}\sum_{j\geq 1} dz_j \wedge d\bar{z}_j.$$

The new Hamiltonian is

$$\begin{aligned} H(\theta, I, z, \bar{z}; \omega) &= N + P(\theta, z, \bar{z}; \omega) \\ &= \sum_{i=1}^{n} \omega_i I_i + \sum_{j\geq 1} \Omega_j z_j \bar{z}_j + \sum_{|\alpha|+|\beta|=2} P_{\alpha\beta}(\theta; \omega) z^{\alpha} \bar{z}^{\beta}. \end{aligned} \tag{4.4}$$

Given $s, r > 0$, in the following, we investigate Hamiltonian (4.4) on the domain $D(r, s) \subseteq \mathcal{P}_{0,\mathbb{C}}^{a,p}$. The forcing frequency $\omega \in [0, 2\pi)^n$ will play the role of parameters. Suppose $H(\theta, I, z, \bar{z}; \omega)$ in (4.4) is real analytic on $(\theta, I, z, \bar{z}; \omega)$ and C_W^1-smooth in compact subset $\mathcal{O} \subseteq [0, 2\pi)^n$ with positive Lebesgue measure. Furthermore, suppose Hamiltonian (4.4) satisfies the following assumptions.

(A1) Asymptotics of normal frequencies:

$$\Omega_j = j + \breve{\Omega}_j(\omega), \quad j \geq 1, \tag{4.5}$$

where $\breve{\Omega}_j \in C_W^1(\mathcal{O})$ and there exist positive constants A_0, such that $\sup_{j\geq 1, \omega\in\mathcal{O}} |\breve{\Omega}_j| \leq A_0$. $\sup_{j\geq 1} \sup_{\omega\in\mathcal{O}} |\partial_\omega \breve{\Omega}_j| \leq \varepsilon_0$.

(A2) Non-resonance conditions: There exist a constant $0 < \gamma \leq 1$ and some fixed approximation function Δ such that uniformly on $\mathcal{O}$, for all $(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty \backslash \{0\}$,

$$\begin{aligned} |k \cdot \omega| &\geq \frac{\gamma}{\Delta(|k|)}, \; k \neq 0, \\ |k \cdot \omega + l \cdot \Omega(\omega)| &\geq \frac{\gamma}{\Delta(|k|)}, \quad |l| = 2 \end{aligned} \tag{4.6}$$

where $|k| = |k_1| + \cdots + |k_n|$, $|l| = \sum_j |l_j|$.

(A3) Regularity: The Hamiltonian vector field $X_P = (0, -P_\theta, \mathrm{i}P_{\bar{z}}, -\mathrm{i}P_z)^T$ of perturbation P defines a map

$$X_P : D(r, s) \times \mathcal{O} \to \mathcal{P}_{0,\mathbb{C}}^{a,p},$$

$X_P(\cdot; \omega)$ is real analytic in $D(r, s)$ for each $\omega \in \mathcal{O}$, and $P(\chi; \cdot)$ is C_W^1-smooth in $\mathcal{O}$ for each $\chi \in D(r, s)$.

(A4) Töplitz–Lipschitz property: $\breve{\Omega} := diag(\breve{\Omega}_j)_{j\geq 1} \in \mathfrak{M}_{r,\mathcal{O}}^{\rho}$ and $P \in \mathcal{T}_{D(r,s)\times\mathcal{O}}^{\rho}$ for some $\rho > 0$.

Denote

$$[P]_{s; D(r,s)\times\mathcal{O}}^{\rho} := \|X_P\|_{s; D(r,s)\times\mathcal{O}} + \langle P \rangle_{\rho, D(r,s)\times\mathcal{O}}. \tag{4.7}$$

Theorem 4.1 *Let Δ be an approximation function such that*

$$\sum_{k\in\mathbb{Z}^n} \frac{1}{\sqrt{\Delta(|k|)}} < +\infty. \tag{4.8}$$

If the Hamiltonian $H = N + P$ in (4.4) satisfies the above assumptions (A1)–(A4) and there exists $0 < \varepsilon_0 < \min\{\frac{\gamma}{4}(\sqrt{\Delta(1)} - 1), (C_*\gamma 2^5)^{\frac{3}{2}}, \frac{1}{12n}\}$ *so that*

$$\langle\langle\breve{\Omega}\rangle\rangle_{\rho,r,\mathcal{O}} < \varepsilon_0 \quad \text{and} \quad [P]^{\rho}_{s;D(r,s)\times\mathcal{O}} < \varepsilon_0.$$

Then there exist

(i) *a Cantor subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with Lebesgue measure $\mathrm{mes}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\sqrt{\gamma})$ as $\gamma \to 0$;*

(ii) *a C^1_W-smooth family of real analytic, symplectic coordinate transformations $\Phi = \Phi_\omega : \mathcal{P}^{a,0}_0 \times \mathcal{O}_\gamma \to \mathcal{P}^{a,0}_0$ of the form*

$$\Phi_\omega \begin{pmatrix} \theta \\ I \\ Z \end{pmatrix} = \begin{pmatrix} \theta \\ I + \frac{1}{2} Z^T M_\omega(\theta) Z \\ L_\omega(\theta) Z \end{pmatrix} \tag{4.9}$$

where $Z = (Z_j)^T_{j\geq 1}$ with $Z_j = (z_j, \bar{z}_j)^T$. $M_\omega(\theta)$ and $L_\omega(\theta)$ are linear bounded operators on $\ell^{a,p}_0 \times \ell^{a,p}_0$ for all $p \geq 0$, and $L_\omega(\theta)$ is also invertible;

(iii) *a C^1_W-smooth family of new normal forms*

$$N^\infty = \sum_{j=1}^{n} \omega_j I_j + \sum_{j\geq 1} \Omega^\infty_j z_j \bar{z}_j \tag{4.10}$$

such that on $\mathcal{P}^{a,0}_0 \times \mathcal{O}_\gamma$,

$$H \circ \Phi = N^\infty.$$

Moreover the new normal frequencies are close to the original ones

$$|\Omega^\infty - \Omega|_{\mathcal{O}_\gamma} \leq c\varepsilon,$$

and the the new frequencies satisfy a non-resonant condition: for all $\omega \in \mathcal{O}_\gamma$,

$$|k \cdot \omega| \geq \frac{\gamma}{2\Delta(|k|)}, \quad \forall k \neq 0,$$
$$|k \cdot \omega + l \cdot \Omega^\infty(\omega)| \geq \frac{\gamma}{2\Delta(|k|)}, \quad \forall k \in \mathbb{Z}^n, \quad |l| = 2.$$

5 Applications to Some Linear Hamiltonian PDEs

We give the proof of Theorem 1.1 by Theorem 4.1.

5.1 The Hamiltonian Derivative Wave Equations

We consider the wave Eq. (1.3). Let

$$\begin{cases} w = \frac{1}{\sqrt{2}}(\mathbf{D}_m u + \mathrm{i} u_t), \\ \bar{w} = \frac{1}{\sqrt{2}}(\mathbf{D}_m u - \mathrm{i} u_t). \end{cases} \tag{5.1}$$

Then the Eq. (1.3) is written as a non-autonomous Hamiltonian equation

$$\begin{cases} w_t = -\mathrm{i}\frac{\partial}{\partial \bar{w}} H(t, w, \bar{w}) = -\mathrm{i}\mathbf{D}_m w - \frac{\mathrm{i}\epsilon}{2} V(\omega t, x)(w + \bar{w}), \\ \bar{w}_t = \mathrm{i}\frac{\partial}{\partial w} H(t, w, \bar{w}) = \mathrm{i}\mathbf{D}_m w + \frac{\mathrm{i}\epsilon}{2} V(\omega t, x)(w + \bar{w}). \end{cases} \tag{5.2}$$

with the Hamiltonian

$$H(t,w,\bar{w}) = \int_0^\pi \left[\bar{w}\mathbf{D}_m w + \frac{\epsilon}{2}V(\omega t,x)(w+\bar{w})^2\right]dx.$$

Recall the function space $\mathcal{H}_0^{a,p}$ in (1.10). Through the inverse discrete Fourier transform $\mathcal{S}:\ell_0^{a,p}\to\mathcal{H}_0^{a,p}$, the space $\mathcal{H}_0^{a,p}$ can be identified with the space $\ell_0^{a,p}$.

We expand $w(t,x),\ \bar{w}(t,x)$ on the eigenfunctions

$$w(t,x) = \sum_{j\geq 1} q_j(t)\phi_j(x) \in \mathcal{H}_0^{a,p},\ \ \bar{w}(t,x) = \sum_{j\geq 1} \bar{q}_j(t)\phi_j(x) \in \mathcal{H}_0^{a,p}$$

with $q=(q_j)_{j\geq 1},\ \bar{q}=(\bar{q}_j)_{j\geq 1}\in\ell_0^{a,p}$. Then the Eq. (4.3) becomes

$$\begin{cases} \dot{q}_j = -\mathrm{i}\frac{\partial}{\partial \bar{q}_j}H(t,q,\bar{q}) = -\mathrm{i}\lambda_j q_j - \mathrm{i}\frac{\partial}{\partial \bar{q}_j}G, \\ \dot{\bar{q}}_j = \mathrm{i}\frac{\partial}{\partial q_j}H(t,q,p) = \mathrm{i}\lambda_j q_j + \mathrm{i}\frac{\partial}{\partial \bar{q}_j}G, \end{cases} \tag{5.3}$$

where

$$\begin{aligned} H(t,q,p) &= \Lambda + G, \\ \Lambda &= \sum_{j\geq 1}\lambda_j q_j \bar{q}_j,\ \ \lambda_j = \sqrt{j^2+m}, \\ G &= \frac{\epsilon}{2}\sum_{i,j\geq 1}(q_i+\bar{q}_i)(q_j+\bar{q}_j)\int_0^\pi V(t\omega,x)\phi_i(x)\phi_j(x)dx. \end{aligned}$$

Now we introduce the angle variables $\theta=\omega t\in\mathbb{T}^n$, the auxiliary action variables $I\in\mathbb{R}^n$ and the complex coordinates $z=(z_j)_{j\geq 1},\ \bar{z}=(\bar{z}_j)_{j\geq 1}$ via letting $z_j=-q_j,\ \bar{z}_j=-\bar{q}_j$. Then we obtain an autonomous Hamiltonian system

$$\begin{cases} \dot{z}_j = \mathrm{i}\lambda_j z_j + \mathrm{i}\frac{\partial}{\partial \bar{z}_j}P(\theta,z,\bar{z}) & j\geq 1, \\ \dot{\bar{z}}_j = -\mathrm{i}\lambda_j \bar{z}_j - \mathrm{i}\frac{\partial}{\partial z_j}P(\theta,z,\bar{z}) & j\geq 1, \\ \dot{\theta}_i = \omega_i & i=1\cdots n, \\ \dot{I}_i = -\frac{\partial}{\partial\theta_i}P(\theta,z,\bar{z}) & i=1\cdots n. \end{cases} \tag{5.4}$$

on the phase space $\mathcal{P}_0^{a,p}$ with respect to the symplectic form

$$\sum_{i=1}^n d\theta_i\wedge dI_i + \mathrm{i}\sum_{j\geq 1}dz_j\wedge d\bar{z}_j.$$

The Hamiltonian associated to the system (5.4) is

$$H = N + P \tag{5.5}$$

where

$$\begin{aligned} N &= \sum_{j=1}^n \omega_j I_j + \sum_{j\geq 1}\lambda_j z_j \bar{z}_j, \\ P &= \frac{\epsilon}{2}\sum_{i,j\geq 1}(z_i+\bar{z}_i)(z_j+\bar{z}_j)\int_0^\pi V(\theta,x)\phi_i(x)\phi_j(x)dx. \end{aligned} \tag{5.6}$$

In the following, we check that the Hamiltonian (5.5) satisfies the assumptions **(A1)–(A4)**. Let r be that in Assumption 1.1 and $s > 0$ be a suitable positive number. Take $\varepsilon_0 = (2^{p+1} + 2^4 + \frac{18n}{r})C_V\epsilon > 0$.

(1) *Verifying the assumption* **(A1)**.
Since $\lambda_j = \sqrt{j^2+m} = j + \frac{m}{2j} - \frac{m^2}{8j^3} + \cdots$, then we take $\Omega_j = j + \breve{\Omega}_j = j + O(\frac{1}{j})$. Note that $\breve{\Omega}_j$ does not depend on $\omega \in [0, 2\pi)^n$, thus $\partial_\omega \breve{\Omega}_j = 0$ and $\breve{\Omega}_j \in C^1_W([0, 2\pi)^n)$. Take $A_0 = 1+m$. Since $\breve{\Omega}_j = O(\frac{1}{j})$ and $\partial_\omega \breve{\Omega}_j = 0$, then for all $j \geq 1$ and $\omega \in [0, 2\pi)^n$, we have $|\breve{\Omega}_j| \leq A_0$ and $|\partial_\omega \breve{\Omega}_j| \leq \varepsilon_0$.

(2) *Verifying the assumption* **(A2)**.
Take the vector $v = (\text{sgn}(k_1), \ldots, \text{sgn}(k_n))$ then $k \cdot v = |k|$. Let $\omega = \omega_\mu = \mu v + w$ with $\mu \in \mathbb{R}$, $w \in v^\perp$. Consider the function $f(\mu) = k \cdot \omega_\mu + l \cdot \Omega = |k|\mu + k \cdot w + l \cdot \Omega$. Thanks to $\partial_\omega \Omega = 0$, we have

$$|f'(\mu)| = |k|.$$

By Lemma 7.6 in Appendix, we have

$$\text{mes}\{\mu : \mu v + w \in [0, 2\pi)^n, |f(\mu)| \leq \delta\} \leq \frac{4\delta}{|k|}.$$

It follows that the measure

$$\begin{aligned}
&\text{mes}\left\{\omega \in [0, 2\pi)^n : |k \cdot \omega + l \cdot \Omega| \leq \frac{\gamma}{\Delta(|k|)}, |l| = 0, 2\right\} \\
&\leq \text{diam}^{n-1}([0, 2\pi)^n)\, \text{mes}\left\{\mu : \mu v + w \in [0, 2\pi)^n, |f(\mu)| \leq \frac{\gamma}{\Delta(|k|)}\right\} \\
&\leq (2\pi)^{n(n-1)} \frac{4\gamma}{|k|\Delta(|k|)}.
\end{aligned} \tag{5.7}$$

Thus there is a subset $\mathcal{O} \subset [0, 2\pi)^n$ of positive Lebesgue measure with $\text{mes}\,\mathcal{O} \geq (2\pi)^n(1 - O(\gamma))$ such that the assumption**(A2)** holds on $\mathcal{O}$.

(3) *Verifying the assumption* (A3).
The perturbation P in (5.6) reads

$$P(\theta, z, \bar{z}) = \frac{\epsilon}{2} \sum_{ij \geq 1} p^{20}_{ij}(\theta) z_i z_j + \epsilon \sum_{ij \geq 1} p^{11}_{ij}(\theta) z_i \bar{z}_j + \frac{\epsilon}{2} \sum_{ij \geq 1} p^{02}_{ij}(\theta) \bar{z}_i \bar{z}_j,$$

where

$$\begin{aligned}
p^{20}_{ij}(\theta) = p^{11}_{ij}(\theta) = p^{02}_{ij}(\theta) &= \int_0^\pi V(\theta, x)\phi_i(x)\phi_j(x)dx \\
&= \begin{cases} \frac{1}{2}(\widetilde{V}_{i-j}(\theta) - \widetilde{V}_{i+j}(\theta)), & i > j, \\ \widetilde{V}_0(\theta) - \frac{1}{2}\widetilde{V}_{2j}(\theta), & i = j, \\ \frac{1}{2}(\widetilde{V}_{j-i}(\theta) - \widetilde{V}_{i+j}(\theta)), & i < j. \end{cases}
\end{aligned} \tag{5.8}$$

Now we investigate the regularity of the perturbation vector field $X_P = (0, -\frac{\partial P}{\partial \theta}, \mathrm{i}\frac{\partial P}{\partial \bar{z}}, -\mathrm{i}\frac{\partial P}{\partial z})$. Note that the vector field X_P does not depend on ω. For the above $r, s > 0$, we estimate the vector field norm

$$\|X_P\|_{s; D(r,s) \times \mathcal{O}}$$

$$= \frac{1}{s^2}\sum_{h=1}^{n}\left\|\frac{\partial P}{\partial\theta_h}\right\|_{D(r,s)\times\mathcal{O}} + \frac{1}{s}\sup_{\|z\|_{a,p}<s,\|\bar z\|_{a,p}<s}\sum_{i=1}^{\infty} i^p e^{ai}\left(\left\|\frac{\partial P}{\partial\bar z_i}\right\|_{D(r)\times\mathcal{O}} + \left\|\frac{\partial P}{\partial z_i}\right\|_{D(r)\times\mathcal{O}}\right).$$

- We first estimate the sum

$$\begin{aligned}
&\sum_{h=1}^{n}\left\|\frac{\partial P}{\partial\theta_h}\right\|_{D(r,s)\times\mathcal{O}}\\
&\quad= \epsilon\sup_{\|z\|_{a,p}<s,\|\bar z\|_{a,p}<s}\sum_{h=1}^{n}\sum_{i,j\geq 1}\left(\frac{1}{2}\|\frac{\partial p_{ij}^{20}}{\partial\theta_h}\|_{D(r)}|z_i||z_j|\right.\\
&\qquad\left.+\|\frac{\partial p_{ij}^{11}}{\partial\theta_h}\|_{D(r)}|z_i||\bar z_j| + \frac{1}{2}\|\frac{\partial p_{ij}^{02}}{\partial\theta_h}\|_{D(r)}|\bar z_i||\bar z_j|\right)\\
&\quad\leq \frac{n\epsilon}{r}\sup_{\|z\|_{a,p}<s,\|\bar z\|_{a,p}<s}\left(\sum_{i,j\geq 1}\|p_{ij}^{20}\|_{D(2r)}|z_i||z_j|\right.\\
&\qquad\left.+\sum_{i,j\geq 1}\|p_{ij}^{11}\|_{D(2r)}|z_i||\bar z_j| + \sum_{i,j\geq 1}\|p_{ij}^{02}\|_{D(2r)}|\bar z_i||\bar z_j|\right).
\end{aligned}$$

For this purpose, it suffices to estimate each of three sums on the last line:

$$\begin{aligned}
&\sum_{i,j\geq 1}\|p_{ij}^{11}\|_{D(2r)}|z_i||\bar z_j|\\
&\quad= \sum_{j\geq 1}\|p_{jj}^{11}\|_{D(2r)}|z_j||\bar z_j| + \sum_{j\geq 1}\sum_{1\leq i\leq j-1}\|p_{ij}^{11}\|_{D(2r)}|z_i||\bar z_j| + \sum_{j\geq 1}\sum_{i\geq j+1}\|p_{ij}^{11}\|_{D(2r)}|z_i||\bar z_j|\\
&\quad\leq \sum_{j\geq 1}(\|\widetilde V_0(\theta)\|_{D(2r)} + \|\widetilde V_{2j}(\theta)\|_{D(2r)})|z_j||\bar z_j|\\
&\qquad+\sum_{j\geq 1}\sum_{1\leq i\leq j-1}(\|\widetilde V_{j-i}\|_{D(2r)} + \|\widetilde V_{j+i}\|_{D(2r)})|z_i||\bar z_j|\\
&\qquad+\sum_{j\geq 1}\sum_{i\geq j+1}(\|\widetilde V_{i-j}\|_{D(2r)} + \|\widetilde V_{j+i}\|_{D(2r)})|z_i||\bar z_j|\\
&\quad\leq 6C_V\|z\|_{a,p}\|\bar z\|_{a,p}.
\end{aligned}$$

Similarly, we have

$$\sum_{i,j\geq 1}\|p_{ij}^{20}\|_{D(2r)}|z_i||z_j| \leq 6C_V\|z\|_{a,p}\|z\|_{a,p}$$

and

$$\sum_{i,j\geq 1}\|p_{ij}^{02}\|_{D(2r)}|\bar z_i||\bar z_j| \leq 6C_V\|\bar z\|_{a,p}\|\bar z\|_{a,p}.$$

This shows that

$$\frac{1}{s^2}\sum_{h=1}^{n}\|\frac{\partial P}{\partial\theta_h}\|_{D(r,s)\times\mathcal{O}} \leq \frac{18n}{r}C_V\epsilon. \tag{5.9}$$

- We turn to the estimate for

$$\frac{1}{s}\sup_{\|z\|_{a,p}<s,\|\bar z\|_{a,p}<s}\sum_{i=1}^{\infty} i^p e^{ai}\left(\left\|\frac{\partial P}{\partial\bar z_i}\right\|_{D(r)\times\mathcal{O}} + \left\|\frac{\partial P}{\partial z_i}\right\|_{D(r)\times\mathcal{O}}\right).$$

It suffices to consider

$$\sum_{i=1}^{\infty} i^p e^{ai} \left\| \frac{\partial P}{\partial z_i} \right\|_{D(r)\times\mathcal{O}} = \epsilon \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^p e^{ai} \|p_{ij}^{11}\|_{D(r)} |z_j| + \epsilon \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^p e^{ai} \|p_{ij}^{02}\|_{D(r)} |\bar{z}_j|.$$

By (5.8),

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^p e^{ai} \|p_{ij}^{11}\|_{D(r)} |z_j| \\
&\quad = \sum_{j\geq 1} j^p e^{aj} \|\widetilde{V}_0(\theta) - \frac{1}{2}\widetilde{V}_{2j}(\theta)\|_{D(2r)} |z_j| \quad \cdots (*1) \\
&\quad + \sum_{j\geq 1}\sum_{1\leq i\leq j-1} i^p e^{ai} \|\frac{1}{2}(\widetilde{V}_{j-i}(\theta) - \widetilde{V}_{i+j}(\theta))\|_{D(2r)} |z_j| \quad \cdots (*2) \\
&\quad + \sum_{j\geq 1}\sum_{i\geq j+1} i^p e^{ai} \|\frac{1}{2}(\widetilde{V}_{i-j}(\theta) - \widetilde{V}_{i+j}(\theta))\|_{D(2r)} |z_j| \quad \cdots (*3),
\end{aligned}$$

where

$$(*1) = \sum_{j\geq 1} j^p e^{aj} \|\widetilde{V}_0(\theta) - \frac{1}{2}\widetilde{V}_{2j}(\theta)\|_{D(2r)} |z_j| \leq 2\|V\|_{D(2r),b,p} \|z\|_{a,p}.$$

$$\begin{aligned}
(*2) &= \sum_{j\geq 1}\sum_{1\leq i\leq j-1} i^p e^{ai} \|\frac{1}{2}(\widetilde{V}_{j-i}(\theta) - \widetilde{V}_{i+j}(\theta))\|_{D(2r)} |z_j| \\
&\leq \sum_{j\geq 1}\sum_{1\leq i\leq j-1} i^p e^{ai} \frac{1}{2}\|V\|_{D(2r),b,p} (j-i)^{-p} e^{-b(j-i)} |z_j| \\
&\quad + \sum_{j\geq 1}\sum_{1\leq i\leq j-1} i^p e^{ai} \frac{1}{2}\|V\|_{D(2r),b,p} (j+i)^{-p} e^{-b(j+i)} |z_j| \\
&\leq \frac{1}{2}\|V\|_{D(2r),b,p} \sum_{j\geq 1}\sum_{1\leq i\leq j-1} \left(\frac{i}{j-i}\right)^p e^{ai} e^{-b(j-i)} |z_j| \\
&\quad + \frac{1}{2}\|V\|_{D(2r),b,p} \sum_{j\geq 1}\sum_{1\leq i\leq j-1} \left(\frac{i}{j+i}\right)^p e^{ai} e^{-b(j+i)} |z_j| \\
&\leq \frac{1}{2}\|V\|_{D(2r),b,p} \sum_{j\geq 1} |z_j| j^p 2e^{aj} + \frac{1}{2}\|V\|_{D(2r),b,p} \sum_{j\geq 1} 2|z_j| \\
&\leq 2C_V \|z\|_{a,p}.
\end{aligned}$$

$$\begin{aligned}
(*3) &= \sum_{j\geq 1}\sum_{i\geq j+1} i^p e^{ai} \|\frac{1}{2}(\widetilde{V}_{i-j}(\theta) - \widetilde{V}_{i+j}(\theta))\|_{D(2r)} |z_j| \\
&\leq \sum_{j\geq 1}\sum_{i\geq j+1} i^p e^{ai} \frac{1}{2}\|V\|_{D(2r),b,p} (i-j)^{-p} e^{-b(i-j)} |z_j| \\
&\quad + \sum_{j\geq 1}\sum_{i\geq j+1} i^p e^{ai} \frac{1}{2}\|V\|_{D(2r),b,p} (j+i)^{-p} e^{-b(j+i)} |z_j|
\end{aligned}$$

$$\leq \frac{1}{2}\|V\|_{D(2r),b,p}\sum_{j\geq 1}\sum_{i\geq j+1}\left(\frac{i}{i-j}\right)^p \mathrm{e}^{ai}\mathrm{e}^{-b(i-j)}|z_j|$$
$$+\frac{1}{2}\|V\|_{D(2r),b,p}\sum_{j\geq 1}\sum_{i\geq j+1}\left(\frac{i}{j+i}\right)^p \mathrm{e}^{ai}\mathrm{e}^{-b(j+i)}|z_j|$$
$$\leq (2^p+2)C_V\|z\|_{a,p}.$$

Then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^p \mathrm{e}^{ai}\|p_{ij}^{11}\|_{D(r)}|z_j| \leq (*1)+(*2)+(*3) \leq (2^p+6)C_V\|z\|_{a,p}.$$

By the similar argument, we get

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i^p \mathrm{e}^{ai}\|p_{ij}^{02}\|_{D(r)}|\bar{z}_j| \leq (2^p+6)C_V\|\bar{z}\|_{a,p}.$$

It follows that

$$\frac{1}{s}\sup_{\|z\|_{a,p}<s,\|\bar{z}\|_{a,p}<s}\sum_{i=1}^{\infty} i^p \mathrm{e}^{ai}\left(\|\frac{\partial P}{\partial \bar{z}_i}\|_{D(r)\times\mathcal{O}}+\|\frac{\partial P}{\partial z_i}\|_{D(r)\times\mathcal{O}}\right) \leq 2(2^p+6)C_V\epsilon. \tag{5.10}$$

We conclude from (5.9) and (5.10) that

$$\|X_P\|_{s;D(r,s)\times\mathcal{O}}$$
$$=\frac{1}{s^2}\sum_{h=1}^{\mathrm{n}}\|\frac{\partial P}{\partial \theta_h}\|_{D(r,s)\times\mathcal{O}}+\frac{1}{s}\sup_{\|z\|_{a,p}<s,\|\bar{z}\|_{a,p}<s}\sum_{i=1}^{\infty} i^p\mathrm{e}^{ai}\left(\|\frac{\partial P}{\partial \bar{z}_i}\|_{D(r)\times\mathcal{O}}+\|\frac{\partial P}{\partial z_i}\|_{D(r)\times\mathcal{O}}\right)$$
$$\leq (2^{p+1}+12+\frac{18n}{r})C_V\epsilon \leq \varepsilon_0.$$

Thus we complete the verification of the regularity for X_P.

(4) *Verifying the assumption* (A4).

• We verify $\breve{\Omega} := diag(\breve{\Omega}_j)_{j\geq 1}$ satisfies Töplitz–Lipschitz property. During the verification of the assumption (A1), we have obtained $|\breve{\Omega}_j| \leq \frac{C_0}{j}$, where C_0 is a constant depending on m. It is evident that $\lim_{t\to\infty}\breve{\Omega}_{j+t} = 0$ and

$$\left\|\lim_{t\to\infty}\breve{\Omega}_{j+t}\right\|_{\mathcal{O}} \leq C_0.$$
$$\left\|\breve{\Omega}_{j+t}-\lim_{t\to\infty}\breve{\Omega}_{j+t}\right\|_{\mathcal{O}} = \left\|\breve{\Omega}_{j+t}-\lim_{t\to\infty}\breve{\Omega}_{j+t}\right\|_{\mathcal{O}} \leq \frac{C_0}{|j+t|} \leq \frac{C_0}{|t|}.$$

• Taking $\rho = 2a$, we verify the perturbation $P \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$.

We first consider $\frac{\partial^2 P}{\partial z_i \partial \bar{z}_j}$. By (5.8), we have for $t \geq 1$,

$$\frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}} = \epsilon p^{11}_{i+t,j+t}(\theta) = \begin{cases} \frac{\epsilon}{2}(\widetilde{V}_{i-j}(\theta)-\widetilde{V}_{i+j+2t}(\theta)), & i>j, \\ \epsilon\widetilde{V}_0(\theta)-\frac{\epsilon}{2}\widetilde{V}_{2j+2t}(\theta), & i=j, \\ \frac{\epsilon}{2}(\widetilde{V}_{j-i}(\theta)-\widetilde{V}_{i+j+2t}(\theta)), & i<j. \end{cases}$$

Due to $\|\widetilde{V}_j\|_{D(2r)} \le C_V \mathrm{e}^{-2aj}$, $j \ge 1$, the limit $\lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}}$ exists and

$$\lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}} = \begin{cases} \frac{\epsilon}{2}\widetilde{V}_{i-j}(\theta), & i > j, \\ \epsilon \widetilde{V}_0(\theta), & i = j, \\ \frac{\epsilon}{2}\widetilde{V}_{j-i}(\theta), & i < j. \end{cases}$$

Moreover,

$$\left\| \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}} \right\|_{D(r,s)\times\mathcal{O}} \le \epsilon \left\| \widetilde{V}_{|i-j|}(\theta) \right\|_{D(r,s)\times\mathcal{O}} \le \varepsilon_0 \mathrm{e}^{-\rho|i-j|}.$$

Thanks to the exponentially decay of $\widetilde{V}_j$, we also have

$$\begin{aligned} &\left\| \frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}} - \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}} \right\|_{D(r,s)\times\mathcal{O}} \\ &= \begin{cases} \frac{\epsilon}{2}\left\|\widetilde{V}_{i+j+2t}(\theta))\right\|_{D(r,s)\times\mathcal{O}}, & i > j, \\ \frac{\epsilon}{2}\left\|\widetilde{V}_{2j+2t}(\theta)\right\|_{D(r,s)\times\mathcal{O}}, & i = j, \\ \frac{\epsilon}{2}\left\|\widetilde{V}_{i+j+2t}(\theta))\right\|_{D(r,s)\times\mathcal{O}}, & i < j, \end{cases} \\ &\le \frac{\epsilon}{2}\|V\|_{D(2r),b,p}\mathrm{e}^{-2a(i+j+2t)} \le \frac{\varepsilon_0}{t}\mathrm{e}^{-\rho|i-j|}. \end{aligned}$$

where we use the inequality $\mathrm{e}^{-2a(i+j+2t)} = \mathrm{e}^{-2a(i+j)}\mathrm{e}^{-2t} \le \frac{1}{t}\mathrm{e}^{-2a|i-j|}$.
As to the second derivative $\frac{\partial^2 P}{\partial z_i \partial z_j} = \frac{\epsilon}{2} p^{20}_{ij}(\theta)$, we consider the lift $\widetilde{P}(\theta, \tilde{z}, \bar{\tilde{z}}) = P(\theta, z, \bar{z})$, where $(\tilde{z}, \bar{\tilde{z}}) \in \ell^{a,p} \times \ell^{a,p}$ and $\tilde{z} = z_j, \bar{\tilde{z}} = \bar{z}_j$ when $j \ge 1$. (recall the Definition 3.2). Then

$$\frac{\partial^2 \widetilde{P}}{\partial \tilde{z}_i \partial \tilde{z}_j} = \begin{cases} \frac{\partial^2 P}{\partial z_i \partial z_j}, i \ge 1, j \ge 1, \\ 0, \text{ otherwise.} \end{cases} \tag{5.11}$$

When $|t|$ is sufficiently large, we have either $i + t < 0$ or $j - t < 0$, then $\frac{\partial^2 \widetilde{P}}{\partial \tilde{z}_{i+t}\partial \tilde{z}_{j-t}} = 0$ and thus the limit $\lim_{t\to\infty} \frac{\partial^2 \widetilde{P}}{\partial \tilde{z}_{i+t}\partial \tilde{z}_{j-t}} = 0$. It is obvious that

$$\left\| \lim_{t\to\infty} \frac{\partial^2 \widetilde{P}}{\partial \tilde{z}_{i+t}\partial \tilde{z}_{j-t}} \right\|_{D(r,s)\times\mathcal{O}} \le \varepsilon_0 \mathrm{e}^{-\rho|i+j|}$$

and

$$\left\| \frac{\partial^2 \widetilde{P}}{\partial \tilde{z}_{i+t}\partial \tilde{z}_{j-t}} - \lim_{t\to\infty} \frac{\partial^2 \widetilde{P}}{\partial \tilde{z}_{i+t}\partial \tilde{z}_{j-t}} \right\|_{D(r,s)\times\mathcal{O}} \le \frac{\varepsilon_0}{t}\mathrm{e}^{-\rho|i+j|}.$$

Similar argument also applies to the second derivative $\frac{\partial^2 P}{\partial \bar{z}_i \partial \bar{z}_j}$.
It follows that $P \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$ and $\langle P\rangle_{\rho, D(r,s)\times\mathcal{O}} \le \varepsilon_0$.

5.2 The Half-Wave Equations

Denote the inner product $\langle u, v\rangle = Re \int_0^{\pi} u(x)\overline{v(x)}dx$. The half-wave Eq. (1.4) can be written as

$$u_t = \mathrm{i}\nabla H(t, u) = \mathrm{i}\mathbf{D}_0 u + \mathrm{i}\varepsilon V(\omega t, x)u. \tag{5.12}$$

where the Hamiltonian

$$H(t,u) = \frac{1}{2}\langle \mathbf{D}_0 u, u\rangle + \frac{\varepsilon}{2}\int^{\pi} V(\omega t, x)|u|^2 dx.$$

We expand $u(t,x)$ on the eigenfunctions

$$u(t,x) = \sum_{j\geq 1} q_j(t)\phi_j(x) \in \mathcal{H}_0^{a,p},$$

(see (1.10) on the space $\mathcal{H}_0^{a,p}$) where $q = (q_j)_{j\geq 1} \in \ell_0^{a,p}$. Then the Eq. (5.12) becomes

$$\dot{q}_j = 2\mathrm{i}\frac{\partial}{\partial \bar{q}_j} H(t,q,\bar{q}) = \lambda_j q_j + 2\frac{\partial}{\partial \bar{q}_j} G, \tag{5.13}$$

where

$$H(t,q,\bar{q}) = \Lambda + G,$$
$$\Lambda = \sum_{j\geq 1} \frac{\lambda_j}{2} q_j \bar{q}_j,\ \ \lambda_j = j,$$
$$G = \frac{\varepsilon}{2}\sum_{j,k\geq 1} q_j \bar{q}_k \int_0^{\pi} V(t\omega, x)\phi_j(x)\phi_k(x)dx.$$

To rewrite the above equation as an autonomous Hamiltonian system, we introduce the angle variables $\theta = \omega t \in \mathbb{T}^n$, the action variables $I \in \mathbb{R}^n$ and the complex coordinates $z = (z_j)_{j\geq 1},\ \bar{z} = (\bar{z}_j)_{j\geq 1}$ through

$$z_j = \frac{1}{\sqrt{2}} q_j,\ \ \bar{z}_j = \frac{1}{\sqrt{2}} \bar{q}_j.$$

Then we obtain an autonomous Hamiltonian system

$$\begin{cases} \dot{z}_j = \mathrm{i}\lambda_j z_j + \mathrm{i}\frac{\partial}{\partial \bar{z}_j} P(\theta, z, \bar{z}) & j \geq 1, \\ \dot{\bar{z}}_j = -\mathrm{i}\lambda_j \bar{z}_j - \mathrm{i}\frac{\partial}{\partial z_j} P(\theta, z, \bar{z}) & j \geq 1, \\ \dot{\theta}_i = \omega_i & i = 1\cdots n, \\ \dot{I}_i = -\frac{\partial}{\partial \theta_i} P(\theta, z, \bar{z}) & i = 1\cdots n. \end{cases} \tag{5.14}$$

on the phase space $\mathcal{P}_0^{a,p}$ with respect to the symplectic form

$$\sum_{i=1}^{n} d\theta_i \wedge dI_i + \mathrm{i}\sum_{j\geq 1} dz_j \wedge d\bar{z}_j.$$

The new Hamiltonian associated to the system (5.14) is

$$H = N + P \tag{5.15}$$

where

$$N = \sum_{i=1}^{n} \omega_i I_i + \sum_{j\geq 1} \lambda_j z_j \bar{z}_j,$$
$$P = \varepsilon \sum_{l,k\geq 1} z_l \bar{z}_k \int_0^{\pi} V(t\omega, x)\phi_l(x)\phi_k(x)dx.$$

The next is the verification of the assumptions **(A1)–(A4)** for the Hamiltonian (5.15). Let r be that in Assumption 1.1 and $s > 0$ be a suitable positive number. Take $\varepsilon_0 = (2^{p+1} + 2^4 + \frac{n}{2r})C_V \epsilon > 0$.

(1) *Verifying the assumption* **(A1)**.
Since $\lambda_j = j$, then we take $\Omega_j = j + \breve{\Omega}_j$ with $\breve{\Omega}_j = 0$, thus $\breve{\Omega}_j \in C^1_W([0, 2\pi)^n)$. Let $A_0 = 1$. It is obvious that for all $j \geq 1$ and $\omega \in [0, 2\pi)^n$, $|\breve{\Omega}_j| \leq A_0$ and $|\partial_\omega \breve{\Omega}_j| \leq \varepsilon_0$.

(2) *Verifying the assumption* **(A2)**.
Following the verification of the assumption **(A2)**, we can also prove that there is a subset $\mathcal{O} \subset [0, 2\pi)^n$ of positive Lebesgue measure with mes $\mathcal{O} \geq (2\pi)^n(1 - O(\gamma))$ such that the assumption **(A2)** holds for (5.15) on $\mathcal{O}$.

(3) *Verifying the assumption* **(A3)**.
The perturbation P in (5.15) reads

$$P = \varepsilon \sum_{ij \geq 1} p_{ij}(\theta) z_i \bar{z}_j, \tag{5.16}$$

where

$$\begin{aligned} p_{ij}(\theta) &:= \int_0^\pi V(\theta, x)\phi_i(x)\phi_j(x)dx \\ &= \begin{cases} \frac{1}{2}(\widetilde{V}_{i-j}(\theta) - \widetilde{V}_{i+j}(\theta)), & i > j, \\ \widetilde{V}_0(\theta) - \frac{1}{2}\widetilde{V}_{2j}(\theta), & i = j, \\ \frac{1}{2}(\widetilde{V}_{j-i}(\theta) - \widetilde{V}_{i+j}(\theta)), & i < j. \end{cases} \end{aligned}$$

Following the arguments in the verification of the assumption **(A3)** for the wave Eq. (1.3), one can prove that

$$\|X_P\|_{s;D(r,s)\times\mathcal{O}} \leq (2^{p+1} + 12 + \frac{n}{2r})\|V\|_{D(2r),b,p}\epsilon \leq \varepsilon_0.$$

This shows the regularity of Hamiltonian vector field X_P.

(4) *Verifying the assumption* **(A4)**.
Let $\rho = 2a$. Thanks to $\breve{\Omega}_j \equiv 0$, it is obvious that $\breve{\Omega} := diag(\breve{\Omega}_j)_{j\geq 1} \in \mathfrak{M}^\rho_{r,\mathcal{O}}$.
Now we verify $P \in \mathcal{T}^\rho_{D(r,s)\times\mathcal{O}}$. By (5.16), we have

$$\frac{\partial^2 P}{\partial z_i \partial \bar{z}_j} = \epsilon p_{ij}(\theta) \text{ and } \frac{\partial^2 P}{\partial z_i \partial z_j} = 0 = \frac{\partial^2 P}{\partial \bar{z}_i \partial \bar{z}_j}.$$

Following the arguments in verifying the assumption **(A4)**, we have the limit $\lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}}$ exists. Moreover,

$$\left\| \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}} \right\|_{D(r,s)\times\mathcal{O}} \leq \varepsilon_0 \mathrm{e}^{-\rho|i-j|}$$

and

$$\left\| \frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}} - \lim_{t\to\infty} \frac{\partial^2 P}{\partial z_{i+t}\partial \bar{z}_{j+t}} \right\|_{D(r,s)\times\mathcal{O}} \leq \frac{\epsilon}{2}\|V\|_{D(2r),b,p}\mathrm{e}^{-b(i+j+2t)} \leq \frac{\varepsilon_0}{t}\mathrm{e}^{-\rho|i-j|}.$$

This together with $\frac{\partial^2 P}{\partial z_i \partial z_j} = 0 = \frac{\partial^2 P}{\partial \bar{z}_i \partial \bar{z}_j}$ shows that the perturbation $P \in \mathcal{T}^\rho_{D(r,s)\times\mathcal{O}}$ and $\langle P \rangle_{\rho,D(r,s)\times\mathcal{O}} \leq \varepsilon_0$.

5.3 Proof of Corollaries 1.1 and'1.2

Below, we provide the proof for Corollaries 1.1 and 1.2, focusing on the case of the half-wave equation. The same argument applies to the derivative wave equation.

From Theorem 4.1, in the new coordinates $(\theta^\infty, I^\infty, z^\infty, \bar{z}^\infty) = \Phi_\omega^{-1}(\theta, I, z, \bar{z})$, the dynamics are linear with I^∞ invariant:

$$\begin{cases} \dot{\theta}_j^\infty = \omega_j & j = 1, \dots, n, \\ \dot{I}_j^\infty = 0 & j = 1, \dots, n, \\ \dot{z}_j^\infty = i\Omega_j^\infty z_j^\infty & j \geq 1, \\ \dot{\bar{z}}_j^\infty = -i\Omega_j^\infty \bar{z}_j^\infty & j \geq 1. \end{cases}$$

As (1.4) is equivalent to system (5.14), the solutions $u(t, x)$ of (1.4) with initial data $u_0(x) = \sum_{j\geq 1} z_j(0)\phi_j(x)$ read

$$u(t, x) = \sum_{j\geq 1} z_j(t)\phi_j(x)$$

with

$$(z(t), \bar{z}(t))^T = L_\omega(\omega t)\left(z^\infty(0)e^{i\Omega^\infty t}, \bar{z}^\infty(0)e^{-i\Omega^\infty t}\right)^T$$

and

$$(z^\infty(0), \bar{z}^\infty(0))^T = L_\omega^{-1}(0)(z(0), \bar{z}(0))^T.$$

Thus,

$$u(t, x) = \sum_{j\geq 1} \psi_j(\omega t, x)e^{i\Omega_j^\infty t},$$

where

$$\psi_j(\theta, x) = \sum_{\ell\geq 1}[L_\omega(\theta)L_\omega^{-1}(0)(z(0), \bar{z}(0))^T]_\ell \phi_\ell(x).$$

Therefore, the solutions are almost-periodic in time with a non-resonant frequency vector $(\omega, \Omega_1^\infty, \Omega_2^\infty, \dots)$. Furthermore, we observe that $\psi_j(\omega t, x)e^{i\Omega_j^\infty t}$ solves (1.4) if and only if $k \cdot \omega + \Omega_j^\infty$ is an eigenvalue of the operator K_2 (above Corollary 1.2). This demonstrates that the spectrum of the Floquet operator K_2 equals $\{k \cdot \omega + \Omega_j^\infty : k \in \mathbb{Z}^n, j \geq 1\}$, thereby proving Corollary 1.2.

For Corollary 1.1, the key point is that when V is real analytic and satisfies (1.9), the perturbation P in (5.15) satisfies Assumption (A3) for all $p \geq 0$. That is, X_P maps smoothly from $\mathcal{P}^{a,p}$ into itself. Therefore, Theorem 4.1 applies for all $p \geq 2$, and by (4.9), the canonical transformation Φ is close to the identity in the $\mathcal{P}^{a,p}$-norm. Since in the new variables, $(\theta^\infty, I^\infty, z^\infty, \bar{z}^\infty) = \Phi_\omega^{-1}(\theta, I, z, \bar{z})$, the modulus of z_j^∞ is invariant. We deduce that there exists a constant C such that

$$(1 - \varepsilon C)\|z(0)\|_{a,p} \leq \|z(t)\|_{a,p} \leq (1 + \varepsilon C)\|z(0)\|_{a,p},$$

which in turn implies that

$$(1 - \varepsilon C)\|u_0\|_{a,p} \leq \|u(t, \cdot)\|_{a,p} \leq (1 + \varepsilon C)\|u_0\|_{a,p}, \quad \forall t \in \mathbb{R}.$$

6 Proof of the Reducibility Theorem 4.1

6.1 Basic Strategy

The reducibility Theorem 4.1 is proved by KAM method. We construct a sequence of Hamiltonian $H = N + P$ of the form (4.4). Suppose the perturbation $P = O(\varepsilon)$, then we construct a symplectic coordinate transformation Φ such that it transforms $H = N + P$ into a new Hamiltonian $H_+ = H \circ \Phi = N_+ + P_+$ with new normal form N_+ and a smaller perturbation $P_+ = O(\varepsilon^\kappa),\ 1 < \kappa < 2$, than the old perturbation P.

The above transformation Φ is constructed via the flow X_F^t generated by a quadratic Hamiltonian F. Taking $\Phi = X_F^1$ and denoting $R = \mathcal{T}_K P$, then

$$
\begin{aligned}
H \circ \Phi = H \circ X_F^1 = N \circ X_F^1 + R \circ X_F^1 + (P - R) \circ X_F^1 \\
= N + \{N, F\} + \int_0^1 (1-t)\{\{N, F\}, F\} \circ X_F^t dt \\
+ R + \int_0^1 \{R, F\} \circ X_F^t dt + (P - R) \circ X_F^1.
\end{aligned}
$$

The new normal form is defined as $N_+ = N + \hat{N}$. This leads to the following homological equation

$$\{N, F\} + R = \hat{N},$$

where the unknowns are F and $\hat{N}$. We solve this homological equation in the next section.

6.2 Solving the Homological Equation

Consider the homological equation

$$\{N, F\} + R = \hat{N} \tag{6.1}$$

on $D(r, s) \times \mathcal{O}$, where

$$N = \sum_{i=1}^{n} \omega_i I_i + \sum_{j\geq 1} \Omega_j(\xi) z_j \bar{z}_j$$

with the fixed tangential frequencies $\omega(\xi) \in \mathbb{R}^n$. The normal frequencies $\Omega_j(\xi) \in \mathbb{R},\ j \geq 1$ satisfy (4.5). The Hamiltonian R is a quadratic on $(z, \bar{z})$ of the form

$$
\begin{aligned}
R(\theta, z, \bar{z}; \xi) &= \langle R^{20}(\theta)z, z\rangle + \langle R^{11}(\theta)z, \bar{z}\rangle + \langle R^{02}(\theta)z, \bar{z}\rangle \\
&= \sum_{|k|\leq K} \sum_{i,j\geq 1} [R^{20}_{kij}(\xi) z_i z_j + R^{11}_{kij}(\xi) z_i \bar{z}_j + R^{02}_{kij}(\xi) \bar{z}_i \bar{z}_j] \mathrm{e}^{\mathrm{i}k\cdot\theta}.
\end{aligned} \tag{6.2}
$$

It does not depend on the action variables I and satisfies $R = \mathcal{T}_K R$. We define its mean value $[R]$ with respect to θ by

$$[R] = \sum_{j\geq 1} R^{11}_{0jj}(\xi) z_j \bar{z}_j.$$

In the following, we use the notations

$$\Gamma_{ab}(\sigma) = \sup_{t\geq 0} (1+t)^a \Delta^b(t) \mathrm{e}^{-t\sigma},\ \ a,\ b \in \mathbb{N}.$$

Proposition 6.1 *Let $\gamma > 0$ and $0 < 5\sigma < r$. Suppose N and R satisfy the above conditions (A1)–(A2), then the homological Eq. (6.1) has the unique solutions F and $\widehat{N}$ satisfying $[F] = 0$ and the estimates*

$$\|X_F\|_{s;D(r-\sigma,s)\times\mathcal{O}} \leq (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\|X_R\|_{s;D(r,s)\times\mathcal{O}}, \tag{6.3}$$

$$\|X_{\widehat{N}}\|_{s;D(r,s)\times\mathcal{O}} \leq \|X_R\|_{s;D(r,s)\times\mathcal{O}}, \tag{6.4}$$

where the constant $\mathcal{C}_0$ depends only on A_0.

Proof We look for a Hamiltonian F of the form

$$\begin{aligned} F(\theta, z, \bar{z}; \xi) &= \langle F^{20}(\theta)z, z\rangle + \langle F^{11}(\theta)z, \bar{z}\rangle + \langle F^{02}(\theta)z, \bar{z}\rangle \\ &= \sum_{|k|\leq K}\sum_{i,j\geq 1}[F^{20}_{kij}(\xi)z_i z_j + F^{11}_{kij}(\xi)z_i\bar{z}_j + F^{02}_{kij}(\xi)\bar{z}_i\bar{z}_j]\mathrm{e}^{\mathrm{i}k\cdot\theta}. \end{aligned} \tag{6.5}$$

Denote $\omega\cdot\nabla f(\theta) := \sum_{b=1}^{n}\omega_b\frac{\partial f}{\partial\theta_b}$. We take $\hat{N} = [R]$. By the comparison of coefficients, the homological Eq. (6.1) is equivalent to the following scalar form: For all $i, j \geq 1$,

$$\omega\cdot\nabla F^{20}_{ij} + \mathrm{i}(\Omega_i + \Omega_j)F^{20}_{ij} = R^{20}_{ij}, \tag{6.6}$$

$$\omega\cdot\nabla F^{11}_{ij} + \mathrm{i}(\Omega_i - \Omega_j)F^{11}_{ij} = R^{11}_{ij} - \delta_{ij}[R^{11}_{ij}], \tag{6.7}$$

and

$$\omega\cdot\nabla F^{02}_{ij} - \mathrm{i}(\Omega_i + \Omega_j)F^{02}_{ij} = R^{02}_{ij}, \tag{6.8}$$

here $\delta_{ij} = 1$, if $i = j$, and 0, otherwise.

Consider the Eq. (6.7). For $i = j$, the Eq. (6.7) becomes

$$\partial_\omega F^{11}_{jj} = R^{11}_{jj} - [R^{11}_{jj}], \tag{6.9}$$

then by Fourier expansion,

$$F^{11}_{kjj} = \begin{cases} 0, & k = 0, \\ \frac{R^{11}_{kjj}}{\mathrm{i}k\cdot\omega}, & 0 < |k| \leq K \end{cases}$$

and we obtain the form solution

$$F^{11}_{jj} = \sum_{0<|k|\leq K}\frac{R^{11}_{kjj}}{\mathrm{i}k\cdot\omega}\mathrm{e}^{\mathrm{i}k\cdot\theta}.$$

For $i \neq j$, by Fourier expansion, the Eq. (6.7) becomes

$$F^{11}_{kij} = \frac{R^{11}_{kij}}{\mathrm{i}(k\cdot\omega + \Omega_i - \Omega_j)}$$

and we obtain the form solution

$$F^{11}_{ij}(\theta) = \sum_{0\leq|k|\leq K}\frac{R^{11}_{kij}}{\mathrm{i}(k\cdot\omega + \Omega_i - \Omega_j)}\mathrm{e}^{\mathrm{i}k\cdot\theta}. \tag{6.10}$$

Now we give the estimate for F^{11}_{ij}. Denote $S_{ij} = k\cdot\omega + \Omega_i - \Omega_j$. For all $1 \leq a \leq n$,

$$|\partial_{\xi_a}S_{i,j}| = |k\cdot\partial_{\xi_a}\omega + \partial_{\xi_a}\breve{\Omega}_i - \partial_{\xi_a}\breve{\Omega}_j| \leq \mathcal{C}_0(1+|k|),$$

where the constant $\mathcal{C}_0 = \mathcal{C}_0(E, L)$ depends only on E and L.

Then

$$\begin{aligned}
&\|F_{ij}^{11}\|_{D(r-\sigma)\times\mathcal{O}}\\
&\quad\le \sum_{|k|\le K}\left(\frac{|R_{k,ij}^{11}|_{\mathcal{O}}}{|S_{ij}|}+\frac{|\partial_\xi S_{ij}||R_{k,ij}^{11}|_{\mathcal{O}}}{|S_{ij}^2|}\right)\mathrm{e}^{|k|(r-\sigma)}\\
&\quad\le \sum_{|k|\le K}(1+\mathcal{C}_0)(1+|k|)\gamma^{-2}\Delta^2(|k|)\mathrm{e}^{-|k|\sigma}|R_{k,ij}^{11}|_{\mathcal{O}}\mathrm{e}^{|k|r}\\
&\quad\le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\|R_{ij}^{11}\|_{D(r)\times\mathcal{O}}.
\end{aligned} \tag{6.11}$$

Similarly, we have

$$\|F_{ij}^{20}\|_{D(r-\sigma)\times\mathcal{O}}\le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\|R_{ij}^{20}\|_{D(r)\times\mathcal{O}},$$
$$\|F_{ij}^{02}\|_{D(r-\sigma)\times\mathcal{O}}\le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\|R_{ij}^{02}\|_{D(r)\times\mathcal{O}}.$$

Note that the derivative

$$\frac{\partial F}{\partial z_i}=\sum_{j\ge 1}F_{ji}^{20}z_j+F_{ij}^{20}z_j+F_{ij}^{11}\bar{z}_j, \tag{6.12}$$

then

$$\left\lfloor\frac{\partial F}{\partial z_i}\right\rfloor_{D(r-\sigma)\times\mathcal{O}}\le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\left\lfloor\frac{\partial R}{\partial z_i}\right\rfloor_{D(r-\sigma)\times\mathcal{O}}. \tag{6.13}$$

Similarly,

$$\left\lfloor\frac{\partial F}{\partial \bar{z}_i}\right\rfloor_{D(r-\sigma)\times\mathcal{O}}\le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\left\lfloor\frac{\partial R}{\partial \bar{z}_i}\right\rfloor_{D(r-\sigma)\times\mathcal{O}}. \tag{6.14}$$

For each $1\le b\le n$, by (6.10), the norm of the derivative $\frac{\partial F_{ij}^{11}}{\partial\theta_b}$ is

$$\left\|\frac{\partial F_{ij}^{11}}{\partial\theta_b}\right\|_{D(r-\sigma)\times\mathcal{O}}\le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\left\|\frac{\partial R_{ij}^{11}}{\partial\theta_b}\right\|_{D(r)\times\mathcal{O}}.$$

Similarly, we have

$$\left\|\frac{\partial F_{ij}^{20}}{\partial\theta_b}\right\|_{D(r-\sigma)\times\mathcal{O}}\le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\left\|\frac{\partial R_{ij}^{20}}{\partial\theta_b}\right\|_{D(r)\times\mathcal{O}},$$

and

$$\left\|\frac{\partial F_{ij}^{02}}{\partial\theta_b}\right\|_{D(r-\sigma)\times\mathcal{O}}\le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\left\|\frac{\partial R_{ij}^{02}}{\partial\theta_b}\right\|_{D(r)\times\mathcal{O}}.$$

It follows that

$$\left\|\frac{\partial F}{\partial\theta}\right\|_{D(r-\sigma,s)\times\mathcal{O}}\le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\left\|\frac{\partial R}{\partial\theta}\right\|_{D(r,s)\times\mathcal{O}} \tag{6.15}$$

From (6.13), (6.14) and (6.15), we obtain the estimate for the Hamiltonian vector field X_F :

$$\|X_F\|_{s;D(r-\sigma,s)\times\mathcal{O}}\le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\|X_R\|_{s;D(r,s)\times\mathcal{O}}.$$

The estimates of $X_{\hat{N}}$ follow from the observation that $\hat{N}_{z\bar{z}}$ is the diagonal of the mean value of $R_{z\bar{z}}$. □

The above lemma implies the estimate for the Jacobian DX_F :

$$\|DX_F\|_{s;D(r-2\sigma,s)\times\mathcal{O}} \le C\sigma^{-1}(1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\|X_R\|_{s;D(r,s)\times\mathcal{O}}. \tag{6.16}$$

Now we verify the Töplitz–Lipschitz property of the solutions of homological Eq. (6.1).

Proposition 6.2 *Suppose N and R satisfy the above conditions (A1)–(A2) and $R \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$, then there exists a constant $C := 5+4\mathcal{C}_0$ such that for any $0<\sigma<r$, the solutions F and $\hat{N}$ of homological Eq. (6.1) are Töplitz–Lipschitz on $D(r,s)\times\mathcal{O}$, i.e., $F \in \mathcal{T}^{\rho}_{D(r-\sigma,s)\times\mathcal{O}}$, $\hat{N} \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$, and*

$$\langle F\rangle_{\rho,D(r-\sigma,s)\times\mathcal{O}} \le C\gamma^{-3}\Gamma_{13}(\sigma)\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}, \tag{6.17}$$

$$\langle \hat{N}\rangle_{\rho,D(r,s)\times\mathcal{O}} \le \langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}. \tag{6.18}$$

Proof The estimation of $\hat{N}$ follows from the observation that $\hat{N}_{z\bar{z}}$ is the diagonal of the mean value of $R_{z\bar{z}}$. In the following, we prove the estimation (6.17).

From (6.10) in the proof of Lemma 6.1, the second derivative of F w.r.t. $z_i,\ \bar{z}_j$ is

$$\frac{\partial^2 F}{\partial z_i\partial\bar{z}_j} = F^{11}_{ij}(\theta) = \sum_{0\le|k|\le K}\frac{R^{11}_{kij}}{\mathrm{i}(k\cdot\omega+\Omega_i-\Omega_j)}\mathrm{e}^{\mathrm{i}k\cdot\theta}.$$

• We first verify the exponentially off-diagonal decay of $\frac{\partial^2 F}{\partial z_i\partial\bar{z}_j}$.

Since $R\in\mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$, we have

$$\left\|\frac{\partial^2 R}{\partial z_i\partial\bar{z}_j}\right\|_{D(r,s)\times\mathcal{O}} \le \langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i-j|}.$$

Then

$$\begin{aligned}\left\|\frac{\partial^2 F}{\partial z_i\partial\bar{z}_j}\right\|_{D(r-\sigma)\times\mathcal{O}} &\le \sum_{|k|\le K}\gamma^{-1}\Delta(|k|)\mathrm{e}^{-|k|\sigma}|R^{11}_{k,ij}|_{\mathcal{O}}\mathrm{e}^{|k|r}\\ &\le \gamma^{-1}\Gamma_{11}(\sigma)\left\|\frac{\partial^2 R}{\partial z_i\partial\bar{z}_j}\right\|_{D(r,s)\times\mathcal{O}}\\ &\le \gamma^{-1}\Gamma_{11}(\sigma)\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i-j|}.\end{aligned}$$

•We then check the asymptotically Töplitz property of $\frac{\partial^2 F}{\partial z_i\partial\bar{z}_j}$.

Since $\Omega_j = j+\breve{\Omega}_j,\ j\ge 1$, and $\langle\langle\breve{\Omega}\rangle\rangle_{\rho,r,\mathcal{O}}<\varepsilon_0$, the limits $\lim_{t\to\infty}\breve{\Omega}_{j+t}$ exist and satisfy

$$\left\|\lim_{t\to\infty}\breve{\Omega}_{j+t}\right\|_{\mathcal{O}} \le \varepsilon_0, \tag{6.19}$$

$$\left\|\breve{\Omega}_{j+t}-\lim_{t\to\infty}\breve{\Omega}_{j+t}\right\|_{\mathcal{O}} \le \frac{\varepsilon_0}{|t|}. \tag{6.20}$$

Note that

$$\Omega_{i+t}-\Omega_{j+t} = i-j+\breve{\Omega}_{i+t}-\breve{\Omega}_{j+t},$$

then for all i,j the limits $\Omega_{i,j,\infty} := \lim_{t\to\infty}(\Omega_{i+t}-\Omega_{j+t})$ exist and satisfy the non-resonance conditions

$$|k\cdot\omega+\Omega_{i,j,\infty}| \ge \frac{\gamma}{\Delta(|k|)}. \tag{6.21}$$

Denote $S_{ij,\infty} := k\cdot\omega + \Omega_{i,j,\infty}$. For $1 \le a \le n$,

$$\begin{aligned}|\partial_{\xi_a} S_{ij,\infty}| &= |k\cdot\partial_{\xi_a}\omega + \partial_{\xi_a}\breve{\Omega}_{i,\infty} - \partial_{\xi_a}\breve{\Omega}_{j,\infty}|\\ &\le |k||\omega|_{\mathcal{O}} + 2|\breve{\Omega}|_{\mathcal{O}} \le \mathcal{C}_0(1+|k|),\end{aligned}$$

where the constant $\mathcal{C}_0 = \mathcal{C}_0(E, L)$.

Since $R \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$, the limit $R^{11}_{ij,\infty} := \lim_{t\to\infty} R^{11}_{i+t,j+t}$. exists. Consider a similar equation to the Eq. (6.7):

$$\partial_\omega u + \mathrm{i}\Omega_{i,j,\infty} u = R^{11}_{ij,\infty}.$$

By the non-resonance conditions (6.21), the solution $F^{11}_{ij,\infty}$ of the above equation exists:

$$F^{11}_{ij,\infty} = \sum_{0\le|k|\le K} \frac{R^{11}_{k,ij,\infty}}{\mathrm{i}S_{ij,\infty}} \mathrm{e}^{\mathrm{i}k\cdot\theta}. \tag{6.22}$$

Moreover, similar to the estimate for $\|F^{11}_{ij}\|_{D(r-\sigma)\times\mathcal{O}}$ in (6.11), we obtain

$$\left\|F^{11}_{ij,\infty}\right\|_{D(r-\sigma)\times\mathcal{O}} \le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i-j|},$$

thus

$$\left\|\lim_{t\to\infty}\frac{\partial^2 F}{\partial z_{i+t}\partial\bar{z}_{j+t}}\right\|_{D(r-\sigma)\times\mathcal{O}} \le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i-j|}.$$

• Finally, we check that $\frac{\partial^2 F}{\partial z_i\partial\bar{z}_j}$ is Lipschitz at infinity.

By (6.10) and (6.22), we write the difference $F^{11}_{i+t,j+t} - F^{11}_{ij,\infty}$ as

$$F^{11}_{i+t,j+t} - F^{11}_{ij,\infty} = \sum_{|k|\le K} \mathcal{F}_{k,ij}(\xi)\mathrm{e}^{k\cdot\theta}.$$

where

$$\mathrm{i}\mathcal{F}_{k,ij}(\xi) = \frac{R^{11}_{k,i+t,j+t}}{S_{i+t,j+t}} - \frac{R^{11}_{k,ij,\infty}}{S_{ij,\infty}}.$$

For $a = 1,\ldots,n$, the Whitney derivatives of $\mathcal{F}_{k,ij}(\xi)$ with respect to ξ_a are

$$\begin{aligned}\mathrm{i}\partial_{\xi_a}\mathcal{F}_{k,ij}(\xi) =& \frac{\partial_{\xi_a}(R^{11}_{k,i+t,j+t} - R^{11}_{k,ij,\infty})}{S_{i+t,j+t}} - \frac{\partial_{\xi_a}S_{i+t,j+t}}{S^2_{i+t,j+t}}(R^{11}_{k,i+t,j+t} - R^{11}_{k,ij,\infty})\\ &+ \left(\frac{S_{ij,\infty} - S_{i+t,j+t}}{S_{i+t,j+t}S_{ij,\infty}}\right)\partial_{\xi_a}R^{11}_{k,ij,\infty} - \left(\frac{\partial_{\xi_a}S_{i+t,j+t} - \partial_{\xi_a}S_{ij,\infty}}{S^2_{i+t,j+t}}\right)R^{11}_{k,ij,\infty}\\ &- \partial_{\xi_a}S_{ij,\infty}(S_{ij,\infty} - S_{i+t,j+t})\left(\frac{1}{S^2_{i+t,j+t}S_{ij,\infty}} + \frac{1}{S_{i+t,j+t}S^2_{ij,\infty}}\right)R^{11}_{k,ij,\infty}.\end{aligned}$$

In view of $\langle\langle\breve{\Omega}\rangle\rangle_{\rho,r,\mathcal{O}} < \varepsilon_0$, we have

$$|S_{ij,\infty} - S_{i+t,j+t}| \le 2|t|^{-1}\varepsilon_0.$$

and for $a = 1,\ldots,n$,

$$|\partial_{\xi_a}S_{ij,\infty} - \partial_{\xi_a}S_{i+t,j+t}| \le 2|t|^{-1}\varepsilon_0.$$

It follows that

$$|\mathcal{F}_{k,ij}(\xi)| \le \gamma^{-1}\Delta(|k|)|R^{11}_{k,i+t,j+t} - R^{11}_{k,ij,\infty}| + \gamma^{-2}\Delta^2(|k|)2|t|^{-1}\varepsilon_0|R^{11}_{k,ij,\infty}|$$

and

$$\begin{aligned}&|\partial_{\xi_a}\mathcal{F}_{k,ij}(\xi)|\\ &\quad\le \Delta(|k|)\gamma^{-1}|\partial_{\xi_a}(R^{11}_{k,i+t,j+t} - R^{11}_{k,ij,\infty})| + \mathcal{C}_0(1+|k|)\Delta^2(|k|)\gamma^{-2}|R^{11}_{k,i+t,j+t} - R^{11}_{k,ij,\infty}|\\ &\qquad + 2|t|^{-1}\varepsilon_0\Delta^2(|k|)\gamma^{-2}(|\partial_{\xi_a}R^{11}_{k,ij,\infty}| + |R^{11}_{k,ij,\infty}|) + 4\mathcal{C}_0(1+|k|)|t|^{-1}\varepsilon_0\Delta^3(|k|)\gamma^{-3}|R^{11}_{k,ij,\infty}|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|F^{11}_{i+t,j+t} - F^{11}_{ij,\infty}\|_{D(r-\sigma)\times\mathcal{O}} &= \sum_{|k|\le K}|\mathcal{F}_{k,ij}(\xi)|_{\mathcal{O}}\mathrm{e}^{|k|(r-\sigma)}\\ &\le (2\gamma^{-1}\Gamma_{01} + \mathcal{C}_0\gamma^{-2}\Gamma_{12})\|R^{11}_{i+t,j+t} - R^{11}_{ij,\infty}\|_{D(r)\times\mathcal{O}}\\ &\quad + (5\gamma^{-2}\Gamma_{02} + 4\mathcal{C}_0\gamma^{-3}\Gamma_{13})|t|^{-1}\|R^{11}_{ij,\infty}\|_{D(r)\times\mathcal{O}}.\end{aligned}$$

This together with $R \in \mathcal{T}^{\rho}_{D(r,s)\times\mathcal{O}}$ shows that

$$\left\|\frac{\partial^2 F}{\partial z_{i+t}\partial\bar{z}_{j+t}} - \lim_{t\to\infty}\frac{\partial^2 F}{\partial z_{i+t}\partial\bar{z}_{j+t}}\right\|_{D(r-\sigma)\times\mathcal{O}} \le (5+4\mathcal{C}_0)\gamma^{-3}\Gamma_{13}|t|^{-1}\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i-j|}.$$

Similarly, we have

$$\begin{aligned}&\left\|\lim_{t\to\infty}\frac{\partial^2 F}{\partial z_{i+t}\partial z_{j-t}}\right\|_{D(r-\sigma)\times\mathcal{O}}, \left\|\lim_{t\to\infty}\frac{\partial^2 F}{\partial\bar{z}_{i+t}\partial\bar{z}_{j-t}}\right\|_{D(r-\sigma)\times\mathcal{O}}\\ &\quad\le (1+\mathcal{C}_0)\gamma^{-2}\Gamma_{12}(\sigma)\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i+j|}\end{aligned}$$

and

$$\begin{aligned}&\left\|\frac{\partial^2 F}{\partial\bar{z}_{i+t}\partial\bar{z}_{j-t}} - \lim_{t\to\infty}\frac{\partial^2 F}{\partial\bar{z}_{i+t}\partial\bar{z}_{j-t}}\right\|_{D(r-\sigma)\times\mathcal{O}}, \left\|\frac{\partial^2 F}{\partial z_{i+t}\partial z_{j-t}} - \lim_{t\to\infty}\frac{\partial^2 F}{\partial z_{i+t}\partial z_{j-t}}\right\|_{D(r-\sigma)\times\mathcal{O}}\\ &\quad\le (5+4\mathcal{C}_0)\gamma^{-3}\Gamma_{13}(\sigma)|t|^{-1}\langle R\rangle_{\rho,D(r,s)\times\mathcal{O}}\mathrm{e}^{-\rho|i+j|}.\end{aligned}$$

Thus we complete the proof of the estimation (6.17). □

6.3 KAM Iteration and Convergence

Let C_* be a constant that is twice the maximum of all implicit constants used during the KAM step, and it depends only on n, A_0 and ρ_0.

We take the Hamiltonian $H = N + P$ in (4.4) as the initial Hamiltonian $H_0 = N_0 + P_0$. Similarly, we set other initial quantities as those in Sect. 4. Namely, we set $r_0 = r,\ s_0 = s,\ \gamma_0 = \gamma,\ \rho_0 = \rho,\ K_0 = K,\ \mathcal{O}_0 = \mathcal{O}$.

For $\nu \ge 0$,

$$\begin{aligned}\gamma_\nu &= \frac{\gamma_0}{2}(1+2^{-\nu}),\\ \delta_\nu &= 2^{-(\nu+4)}\rho_0, \quad \rho_{\nu+1} = \rho_\nu - 4\delta_\nu.\end{aligned}$$

Denote

$$\Gamma(\sigma) = \Gamma_{23}(\sigma) = \sup_{t\ge 0}(1+t)^2\Delta^3(t)\mathrm{e}^{-\sigma t}.$$

$$\kappa_\nu = \kappa^{-(\nu+1)}, \quad \kappa = \frac{4}{3}.$$

Given $\sigma > 0$ with $6\sigma < r_0$. There exists a non-increasing positive sequence

$$\sigma_0 \geq \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_\nu \geq \sigma_{\nu+1} \geq \cdots > 0$$

such that

$$\sum_{\nu=0}^{\infty} \sigma_\nu = \sigma \tag{6.23}$$

and

$$\Xi(\sigma) = \inf_{\tilde{\sigma}_0 \geq \tilde{\sigma}_1 \geq \cdots > 0, \tilde{\sigma}_0 + \tilde{\sigma}_1 + \cdots \leq \sigma} \prod_{\mu=0}^{\infty} \Gamma^{\kappa_\mu}(\tilde{\sigma}_\mu) = \prod_{\mu=0}^{\infty} \Gamma^{\kappa_\mu}(\sigma_\mu) < \infty, \tag{6.24}$$

see Appendix for the proof. For such a fixed sequence $\{\sigma_\nu\}$, we define

$$\Gamma_\nu = 2C_* \Gamma(\sigma_\nu), \tag{6.25}$$

and

$$\varepsilon_{\nu+1} = \Gamma_\nu \varepsilon_\nu^\kappa, \tag{6.26}$$

then

$$\varepsilon_\nu = \left(\prod_{\mu=0}^{\nu-1} \Gamma_\mu^{\kappa_\mu} \varepsilon_0 \right)^{\kappa^\nu}, \quad \nu \geq 1. \tag{6.27}$$

The order K_ν of Fourier truncation is defined implicitly by

$$C_* \mathrm{e}^{-K_\nu \sigma_\nu} = \Gamma_\nu \varepsilon_\nu^{1/2}. \tag{6.28}$$

Finally, we set

$$r_\nu = r_0 - 3 \sum_{\mu=0}^{\nu-1} \sigma_\mu, \quad s_{\nu+1} = \frac{1}{4} s_\nu$$

and denote the domain $D_\nu = D(r_\nu, s_\nu)$.

Remark 6.1 Recall that the non-resonance conditions in our KAM iterative steps are of Brjuno-type and are given by a class of approximation functions $\Delta(t)$. This differs from the usual Diophantine non-resonance conditions, which are given by an explicit power function t^τ. Thus, some iterative parameters such as perturbation parameters ε_ν and K_ν cannot be constructed explicitly but rather implicitly.

Below we provide some heuristic considerations about the construction of ε_ν and K_ν. Now for some iterative sequences, we drop the index ν and write '+' for '$\nu + 1$' to simplify notation. Suppose a Hamiltonian $H = N + P$ on $D(r, s)$, where the perturbation P is of size ε under the norm "[·]" as defined in (3.7). From (6.46), after one iteration step, the new perturbation P_+ on $D(r_+, s_+)$ is of the form

$$P_+ = O\left(\Gamma(\sigma)\varepsilon^\kappa\right) + O\left(\delta^{-2}\mathrm{e}^{-K\sigma}\varepsilon\right),$$

where $\kappa = 4/3$ and $\Gamma(\sigma) = \Gamma_{23}(\sigma)$. To ensure the iterative scheme follows a Newton-like form, the size ε_+ of the new perturbation P_+ will be of the form $\varepsilon_+ \sim \Gamma\varepsilon^\kappa$ with $\Gamma \sim \Gamma(\sigma)$. Therefore, it is necessary to set up

$$\delta^{-2}\mathrm{e}^{-K\sigma}\varepsilon \leq \Gamma(\sigma)\varepsilon^\kappa,$$

i.e., $e^{-K\sigma} \leq \delta^2\Gamma(\sigma)\varepsilon^{1/3}$. Since $\varepsilon \ll \delta$, we let $e^{-K\sigma} \sim \Gamma\varepsilon^{1/2}$, which leads us to define the sequence K_ν implicitly as in (6.28). Note that $\varepsilon_+ \sim \Gamma\varepsilon^\kappa$ gives the sequence ε_ν in (6.26) and (6.27). From (6.27), the definition of the quantity $\Xi(\sigma)$ in (6.24) is natural.

Lemma 6.3 (Iterative Lemma). *Let* $0 < \varepsilon_0 < \min\{(C_*\gamma_0 2^5)^{\frac{3}{2}}, \delta_0^{12}, (\gamma_0\delta_0)^{9/2}, \frac{1}{12n}\}$. *Given a sequence of parameter domains*

$$\mathcal{O}_0 \supseteq \mathcal{O}_1 \supseteq \cdots \supseteq \mathcal{O}_\nu.$$

Suppose for $\ell = 0, 1, \ldots, \nu$, *the Hamiltonian* $H_\ell = N_\ell + P_\ell$ *are regular on* $D_\ell \times \mathcal{O}_\ell$, *where the normal forms*

$$N_\ell = \sum_{j=1}^{n} \omega_j I_j + \sum_{j\in\mathbb{Z}} \Omega_{\ell,j}(\omega) z_j \bar{z}_j \tag{6.29}$$

with $\Omega_{\ell,j}(\omega) = j + \widetilde{\Omega}_{\ell,j}(\omega)$ *satisfies*

$$|\widetilde{\Omega}_\ell|_{\mathcal{O}} \leq A_0 + \sum_{b=1}^{\ell-1} \varepsilon_b \quad \text{and} \quad \langle\langle \widetilde{\Omega}_\ell \rangle\rangle_{\rho_\ell, r_\ell, \mathcal{O}_\ell} \leq \varepsilon_0 + \sum_{b=1}^{\ell-1} \varepsilon_b, \tag{6.30}$$

$$\begin{aligned}
&|k\cdot\omega| \geq \frac{\gamma_\ell}{\Delta(|k|)}, \quad \forall 0 < |k| \leq K_\ell,\\
&|k\cdot\omega + \Omega_{\ell,i}(\omega) + \Omega_{\ell,j}(\omega)| \geq \frac{\gamma_\ell}{\Delta(|k|)}, \quad \forall |k| \leq K_\ell, \quad i, j \geq 1,\\
&|k\cdot\omega + \Omega_{\ell,i}(\omega) - \Omega_{\ell,j}(\omega)| \geq \frac{\gamma_\ell}{\Delta(|k|)}, \quad \forall |k| \leq K_\ell, \quad i \neq j,
\end{aligned} \tag{6.31}$$

on $\mathcal{O}_\ell$, *and the perturbation* P_ℓ *satisfies*

$$P_\ell \in \mathcal{T}^{\rho_\ell}_{D_\ell\times\mathcal{O}_\ell} \text{ and } [P_\ell]^{\rho_\ell}_{s_\ell; D_\ell\times\mathcal{O}_\ell} < \varepsilon_\ell. \tag{6.32}$$

Then there exists a Whitney smooth family of real analytic symplectic transformations $\Phi_{\nu+1} : D_{\nu+1}\times\mathcal{O}_\nu \to D_\nu$ *satisfying*

$$\|\Phi_{\nu+1} - id\|_{s_\nu; D_{\nu+1}\times\mathcal{O}_\nu}, \ \|D\Phi_{\nu+1} - I\|_{s_\nu; D_{\nu+1}\times\mathcal{O}_\nu} \leq \varepsilon_\nu^{5/12}, \tag{6.33}$$

and a closed subset of $\mathcal{O}_\nu$:

$$\mathcal{O}_{\nu+1} = \mathcal{O}_\nu \setminus \bigcup_{|k|>K_\nu} \left(\mathcal{R}_k^{\nu+1}(\gamma_{\nu+1}) \cup \bigcup_{i,j} \mathcal{R}_{kij}^{+,\nu+1}(\gamma_{\nu+1}) \cup \bigcup_{i\neq j} \mathcal{R}_{kij}^{-,\nu+1}(\gamma_{\nu+1}) \right), \tag{6.34}$$

where

$$\begin{aligned}
\mathcal{R}_k^{\nu+1}(\gamma_{\nu+1}) &= \left\{ \omega \in \mathcal{O}_\nu : |k\cdot\omega| < \frac{\gamma_{\nu+1}}{\Delta(|k|)} \right\},\\
\mathcal{R}_{kij}^{+,\nu+1}(\gamma_{\nu+1}) &= \left\{ \omega \in \mathcal{O}_\nu : |k\cdot\omega + \Omega_{\nu+1,i}(\omega) + \Omega_{\nu+1,j}(\omega)| < \frac{\gamma_{\nu+1}}{\Delta(|k|)} \right\},\\
\mathcal{R}_{kij}^{-,\nu+1}(\gamma_{\nu+1}) &= \left\{ \omega \in \mathcal{O}_\nu : |k\cdot\omega + \Omega_{\nu+1,i}(\omega) - \Omega_{\nu+1,j}(\omega)| < \frac{\gamma_{\nu+1}}{\Delta(|k|)} \right\},
\end{aligned}$$

such that $\Phi_{\nu+1}$ *transforms* H_ν *into*

$$H_{\nu+1} = H_\nu \circ \Phi_{\nu+1} = N_{\nu+1} + P_{\nu+1},$$

and on the domain $D_{\nu+1}\times\mathcal{O}_{\nu+1}$, $N_{\nu+1}$ *and* $P_{\nu+1}$ *satisfy the conditions* $(6.29)_{\nu+1}$, $(6.30)_{\nu+1}$, $(6.31)_{\nu+1}$ *and* $(6.32)_{\nu+1}$.

Proof ♦ *The construction of symplectic transformation* $\Phi_{\nu+1}$.

Let $R_\nu = T_{K_\nu} P_\nu$ be the Fourier truncation of order K_ν of P_ν. Using the inequalities

$$\|X_{R_\nu}\|_{s_\nu;D_\nu\times\mathcal{O}_\nu} \le \|X_{P_\nu}\|_{s_\nu;D_\nu\times\mathcal{O}_\nu} \le \varepsilon_\nu,$$
$$\langle R_\nu\rangle_{\rho_\nu,D_\nu\times\mathcal{O}_\nu} \le \langle P_\nu\rangle_{\rho_\nu,D_\nu\times\mathcal{O}_\nu} \le \varepsilon_\nu,$$

and by Propositions 6.1 and 6.2, under the non-resonance conditions $(6.31)_\nu$, the homological equation

$$\{N_\nu, F\} + R_\nu = \hat{N} \tag{6.35}$$

has a set of unique solutions $F = F_\nu$ and $\hat{N} = \hat{N}_\nu$ satisfying the estimates

$$\|X_{F_\nu}\|_{s_\nu;D(r_\nu-\sigma_\nu,s_\nu)\times\mathcal{O}_\nu} \le C\gamma_\nu^{-2}\Gamma_{12}(\sigma_\nu)\|X_{R_\nu}\|_{s_\nu;D_\nu\times\mathcal{O}_\nu} \le C\gamma_\nu^{-2}\Gamma_{12}(\sigma_\nu)\varepsilon_\nu, \tag{6.36}$$
$$\|X_{\widehat{N}_\nu}\|_{s_\nu;D_\nu\times\mathcal{O}_\nu} \le \|X_{R_\nu}\|_{s_\nu;D_\nu\times\mathcal{O}_\nu} \le \varepsilon_\nu, \tag{6.37}$$
$$\langle F_\nu\rangle_{\rho_\nu,D(r_\nu-\sigma_\nu,s_\nu)\times\mathcal{O}_\nu} \le C\gamma_\nu^{-3}\Gamma_{13}(\sigma_\nu)\langle R_\nu\rangle_{\rho_\nu,D(r_\nu,s_\nu)\times\mathcal{O}_\nu} \le C\gamma_\nu^{-3}\Gamma_{13}(\sigma_\nu)\varepsilon_\nu, \tag{6.38}$$

and

$$\langle \hat{N}_\nu\rangle_{\rho_\nu,D_\nu\times\mathcal{O}_\nu} \le \langle R_\nu\rangle_{\rho_\nu,D_\nu\times\mathcal{O}_\nu} \le \varepsilon_\nu. \tag{6.39}$$

Since $\Gamma_{12} \le \Gamma_{13}$ by the definition and $\Gamma_{13} \le \sigma\Gamma_{23}$ by Lemma 7.1 in Appendix, we have

$$\begin{aligned}[F_\nu]^{\rho_\nu}_{s_\nu;D(r_\nu-\sigma_\nu,s_\nu)\times\mathcal{O}_\nu} &\overset{(4.7)}{=} \|X_{F_\nu}\|_{s_\nu;D(r_\nu-\sigma_\nu,s_\nu)\times\mathcal{O}_\nu} + \langle F_\nu\rangle_{\rho_\nu,D(r_\nu-\sigma_\nu,s_\nu)\times\mathcal{O}_\nu}\\ &\le C\gamma_\nu^{-2}\Gamma_{12}(\sigma_\nu)\varepsilon_\nu + C\gamma_\nu^{-3}\Gamma_{13}(\sigma_\nu)\varepsilon_\nu\\ &\le C\sigma_\nu.\end{aligned} \tag{6.40}$$

Then by Lemma 3.4, the flow $X^t_{F_\nu}$ generated by the Hamiltonian vector field X_{F_ν} exists on $D(r_\nu-\sigma_\nu, \frac{s_\nu}{4})$ for all $0 \le t \le 1$. Taking $\Phi_{\nu+1} = X^1_{F_\nu}$, it maps $D(r_\nu-\sigma_\nu, \frac{s_\nu}{4})$ into $D(r_\nu, \frac{s_\nu}{2})$.

Now we prove the estimate (6.33). Since $\varepsilon_\nu \ll 1$ and γ_ν and σ_ν are both bounded sequences, it follows from (6.4), (6.25) and (6.28) that

$$\begin{aligned}\|\Phi_{\nu+1} - id\|_{s_\nu;D_{\nu+1}\times\mathcal{O}_\nu} &\le 2\|X_{F_\nu}\|_{s_\nu;D(r_\nu-\sigma_\nu,s_\nu)\times\mathcal{O}_\nu}\\ &\overset{(6.4)}{\le} 2C\gamma_\nu^{-2}\Gamma_{12}(\sigma_\nu)\varepsilon_\nu\\ &\le 2C\gamma_\nu^{-2}\sigma_\nu\Gamma_\nu(\sigma_\nu)\varepsilon_\nu\\ &\overset{(6.25)}{\le} 2C\gamma_\nu^{-2}\sigma_\nu\Gamma_\nu(2C_*)^{-1}\varepsilon_\nu\\ &\overset{(6.28)}{=} C\gamma_\nu^{-2}\sigma_\nu e^{-K_\nu\sigma_\nu}\varepsilon_\nu^{1/2}\\ &\le \varepsilon_\nu^{5/12}.\end{aligned} \tag{6.41}$$

By the Cauchy estimate and (6.16), using the same approach as for (6.41), we obtain the estimate

$$\begin{aligned}\|D\Phi_{\nu+1} - I\|_{s_\nu;D_{\nu+1}\times\mathcal{O}_\nu} &\le 2\|DX_{F_\nu}\|_{s_\nu;D(r_\nu-\sigma_\nu,s_\nu)\times\mathcal{O}_\nu}\\ &\le \sigma_\nu^{-1}2C\gamma_\nu^{-2}\Gamma_{12}(\sigma_\nu)\varepsilon_\nu.\end{aligned} \tag{6.42}$$

♦ *The new Hamiltonian* $H_{\nu+1}$.

Using the Taylor formula together with the homological Eq. (6.35), we define the new Hamiltonian

$$H_{\nu+1} = H_\nu\circ\Phi_{\nu+1} = N_\nu\circ\Phi_{\nu+1} + R_\nu\circ\Phi_{\nu+1} + (P_\nu - R_\nu)\circ\Phi_{\nu+1}$$

$$
\begin{aligned}
&= N_\nu + \{N_\nu, F_\nu\} + \int_0^1 (1-t)\{\{N_\nu, F_\nu\}, F_\nu\} \circ X^t_{F_\nu} dt \\
&\quad + R_\nu + \int_0^1 \{R_\nu, F_\nu\} \circ X^t_{F_\nu} dt + (P_\nu - R_\nu) \circ X^1_{F_\nu} \\
&= N_{\nu+1} + P_{\nu+1},
\end{aligned} \tag{6.43}
$$

where

$$
\begin{aligned}
N_{\nu+1} &= N_\nu + \hat{N}_\nu, \\
P_{\nu+1} &= \int_0^1 \{\widehat{R}_\nu(t), F_\nu\} \circ X^t_{F_\nu} dt + (P_\nu - R_\nu) \circ X^1_{F_\nu}
\end{aligned}
$$

with $\widehat{R}_\nu(t) = (1-t)\hat{N}_\nu + tR_\nu$.

- *The estimation for* $P_{\nu+1}$.

We first consider the estimation for $\|X_{P_{\nu+1}}\|_{s_{\nu+1}; D_{\nu+1}\times\mathcal{O}_{\nu+1}}$. Note that

$$
X_{P_{\nu+1}} = \int_0^1 (X^t_{F_\nu})^*[X_{\widehat{R}_\nu(t)}, X_{F_\nu}]dt + (X^1_{F_\nu})^*(X_{P_\nu} - X_{R_\nu}).
$$

Then using the classical estimates for the pull-back of a vector field and the Lie bracket of two vector fields (see Sect. 3 in [42]), and by (2.1) and (6.3), we obtain the estimate

$$
\begin{aligned}
\|X_{P_{\nu+1}}\|_{s_{\nu+1}; D_{\nu+1}\times\mathcal{O}_{\nu+1}} &\le \int_0^1 \|(X^t_{F_\nu})^*[X_{\widehat{R}_\nu(t)}, X_{F_\nu}]\|_{s_{\nu+1}; D_{\nu+1}\times\mathcal{O}_{\nu+1}} dt \\
&\quad + \|(X^1_{F_\nu})^*(X_{P_\nu} - X_{R_\nu})\|_{s_{\nu+1}; D_{\nu+1}\times\mathcal{O}_{\nu+1}} \\
&\le 2\|[X_{R_\nu}, X_{F_\nu}]\|_{s_{\nu+1}; D(r_\nu - 2\sigma_\nu, s_{\nu+1})\times\mathcal{O}_{\nu+1}} \\
&\quad + 2\|X_{P_\nu} - X_{R_\nu}\|_{s_{\nu+1}; D(r_\nu - \sigma_\nu, s_{\nu+1})\times\mathcal{O}_{\nu+1}} \\
&\overset{(2.1)}{\le} 2C\sigma_\nu^{-1}\|X_{R_\nu}\|_{s_\nu; D_\nu\times\mathcal{O}_\nu}\|X_{F_\nu}\|_{s_\nu; D_\nu\times\mathcal{O}_\nu} \\
&\quad + 2\mathrm{e}^{-K_\nu\sigma_\nu}\|X_{P_\nu}\|_{s_\nu; D_\nu\times\mathcal{O}_\nu}.
\end{aligned} \tag{6.44}
$$

We then consider the estimation for $\langle P_{\nu+1}\rangle^{\rho_{\nu+1}}_{D_{\nu+1}\times\mathcal{O}_{\nu+1}}$. Using Remarks 3.1 and 3.3(3), Propositions 3.1 and 3.5, we have:

$$
\begin{aligned}
\langle P_{\nu+1}\rangle_{\rho_{\nu+1}, D_{\nu+1}\times\mathcal{O}_{\nu+1}} &\overset{Rem.\ 3.1}{\le} \int_0^1 \langle\{\widehat{R}_\nu(t), F_\nu\} \circ X^t_{F_\nu}\rangle_{\rho_{\nu+1}, D_{\nu+1}\times\mathcal{O}_{\nu+1}} dt \\
&\quad + \langle(P_\nu - R_\nu) \circ X^1_{F_\nu}\rangle_{\rho_{\nu+1}, D_{\nu+1}\times\mathcal{O}_{\nu+1}} \\
&\overset{Pro.\ 3.5}{\le} C\delta_\nu^{-2}\int_0^1 \langle\{\widehat{R}_\nu(t), F_\nu\}\rangle_{\rho_\nu - \delta_\nu, D(r_\nu - 2\sigma_\nu, s_\nu)\times\mathcal{O}_{\nu+1}} dt \\
&\quad + C\delta_\nu^{-2}\langle P_\nu - R_\nu\rangle_{\rho_\nu - \delta_\nu, D(r_\nu - 2\sigma_\nu, s_\nu)\times\mathcal{O}_{\nu+1}} \\
&\overset{Pro.\ 3.1 + Rem.\ 3.3(3)}{\le} C\delta_\nu^{-3}\gamma_\nu^{-3}\Gamma_{13}(\sigma_\nu)\varepsilon_\nu^2 + C\delta_\nu^{-2}\mathrm{e}^{-K_\nu\sigma_\nu}\varepsilon_\nu.
\end{aligned} \tag{6.45}
$$

It follows from (4.7), (6.44), (6.45), (6.26) and (6.28) that

$$
\begin{aligned}
[P_{\nu+1}]^{\rho_{\nu+1}}_{s_{\nu+1}; D_{\nu+1}\times\mathcal{O}_{\nu+1}} &\overset{(4.7)}{=} \|X_{P_{\nu+1}}\|_{s_{\nu+1}; D_{\nu+1}\times\mathcal{O}_{\nu+1}} + \langle P_{\nu+1}\rangle_{\rho_{\nu+1}, D_{\nu+1}\times\mathcal{O}_{\nu+1}} \\
&\overset{(6.44)+(6.45)}{\le} C\delta_\nu^{-3}\sigma_\nu^{-1}\gamma_\nu^{-3}\Gamma_{13}(\sigma_\nu)\varepsilon_\nu^2 + C\delta_\nu^{-2}\mathrm{e}^{-K_\nu\sigma_\nu}\varepsilon_\nu \\
&\le C\Gamma_{23}(\sigma_\nu)\varepsilon_\nu^\kappa + C\delta_\nu^{-2}\mathrm{e}^{-K_\nu\sigma_\nu}\varepsilon_\nu
\end{aligned}
$$

$$\overset{(6.26)+(6.28)}{\leq} \Gamma_\nu \varepsilon_\nu^\kappa = \varepsilon_{\nu+1}. \tag{6.46}$$

• *The new frequency and non-resonance condition.*

In the new normal form $N_{\nu+1}$, the frequencies $\Omega_{\nu+1,j} = j + \breve{\Omega}_{\nu+1,j} = \Omega_{\nu,j} + \widehat{\Omega}_{\nu,j}$, where $\widehat{\Omega}_{\nu,j} = \frac{\partial^2 \widehat{N}_\nu}{\partial z_j \partial \bar{z}_j}$. Thus

$$|\widehat{\Omega}_{\nu,j}|_{\mathcal{O}} \leq \|X_{\widehat{N}_\nu}\|_{s_\nu; D_\nu \times \mathcal{O}_\nu} \leq \|X_{R_\nu}\|_{s_\nu; D_\nu \times \mathcal{O}_\nu} \leq \varepsilon_\nu.$$

Recall the proof of Proposition 6.1, $\hat{N}_\nu$ is the average of $R_\nu = \mathcal{T}_K P_\nu$ from the perturbation P_ν with respect to θ, i.e., $\hat{N}_\nu = [R_\nu]$. So, by Remark 3.3 (3),

$$\langle \hat{N}_\nu \rangle_{\rho_\nu, D_\nu \times \mathcal{O}_\nu} \leq \langle P_\nu \rangle_{\rho_\nu, D_\nu \times \mathcal{O}_\nu} \leq \varepsilon_\nu.$$

Then following the definition of the semi-norm $\langle \hat{N}_\nu \rangle_{\rho_\nu, D_\nu \times \mathcal{O}_\nu}$, we have

$$|\lim_{t\to\infty} \widehat{\Omega}_{\nu,j+t}|_{\mathcal{O}_\nu} \leq \|\lim_{t\to\infty} \frac{\partial^2 \widehat{N}_\nu}{\partial z_{j+t} \partial \bar{z}_{j+t}}\|_{s_\nu; D_\nu \times \mathcal{O}_\nu} \leq \langle \hat{N}_\nu \rangle_{\rho_\nu, D_\nu \times \mathcal{O}_\nu} \leq \varepsilon_\nu. \tag{6.47}$$

$$\begin{aligned} |\widehat{\Omega}_{\nu,j+t} - \lim_{t\to\infty} \widehat{\Omega}_{\nu,j+t}|_{\mathcal{O}_\nu} &\leq \|\frac{\partial^2 \widehat{N}_\nu}{\partial z_{j+t} \partial \bar{z}_{j+t}} - \lim_{t\to\infty} \frac{\partial^2 \widehat{N}_\nu}{\partial z_{j+t} \partial \bar{z}_{j+t}}\|_{s_\nu; D_\nu \times \mathcal{O}_\nu} \\ &\leq |t|^{-1} \langle \hat{N}_\nu \rangle_{\rho_\nu, D_\nu \times \mathcal{O}_\nu} \leq |t|^{-1} \varepsilon_\nu. \end{aligned} \tag{6.48}$$

These imply

$$|\widehat{\Omega}_\nu|_{\mathcal{O}} \leq \varepsilon_\nu, \ \ \langle\langle \widehat{\Omega}_\nu \rangle\rangle_{\rho_\nu, r_\nu, \mathcal{O}_\nu} \leq \varepsilon_\nu.$$

Therefore,

$$|\breve{\Omega}_{\nu+1}|_{\mathcal{O}} \leq A_0 + \sum_{b=1}^{\nu} \varepsilon_b \text{ and } \langle\langle \breve{\Omega}_{\nu+1} \rangle\rangle_{\rho_\nu, r_\nu, \mathcal{O}_\nu} \leq \varepsilon_0 + \sum_{b=1}^{\nu} \varepsilon_b,$$

♦Finally, we consider the construction of $\mathcal{O}_{\nu+1}$. It suffices to verify

$$|k \cdot \omega + \Omega_{\nu+1,i}(\omega) - \Omega_{\nu+1,j}(\omega)| \geq \frac{\gamma_\ell}{\Delta(|k|)}, \ \forall |k| \leq K_\nu, \ i \neq j.$$

By the definition of γ_ν, Γ_ν and K_ν, we have

$$\begin{aligned} \frac{\gamma_0}{2^{\nu+3} \varepsilon_\nu \Delta(K_\nu)} &= \frac{\gamma_0 e^{-K_\nu \sigma_\nu}}{2^{\nu+3} \varepsilon_\nu \Delta(K_\nu) e^{-K_\nu \sigma_\nu}} \\ &\geq \frac{\gamma_0 e^{-K_\nu \sigma_\nu}}{2^{\nu+3} \varepsilon_\nu \Gamma(\sigma_\nu)} \\ &= \frac{2\gamma_0 \varepsilon_\nu^{\kappa - 5/6}}{2^{\nu+3} \varepsilon_\nu} \\ &= \frac{\gamma_0}{2^{\nu+2} \varepsilon_\nu^{1/2}} \geq 1. \end{aligned}$$

This implies $\gamma_\nu - \gamma_{\nu+1} \geq 2\varepsilon_\nu \Delta(|k|)$ for all $0 < |k| \leq K_\nu$, thus

$$\begin{aligned} |k \cdot \omega + \Omega_{\nu+1,i}(\omega) - \Omega_{\nu+1,j}(\omega)| &\geq |k \cdot \omega + \Omega_{\nu,i}(\omega) - \Omega_{\nu,j}(\omega)| - |\widehat{\Omega}_{\nu,i}(\omega)| - |\widehat{\Omega}_{\nu,j}(\omega)| \\ &\geq \frac{\gamma_\nu}{\Delta(|k|)} - 2\varepsilon_\nu \geq \frac{\gamma_{\nu+1}}{\Delta(|k|)}. \end{aligned}$$

Then after removing the resonance zones for $K_\nu < |k| \le K_{\nu+1}$, we get a closed set $\mathcal{O}_{\nu+1} \subseteq \mathcal{O}_\nu$ with the desired properties. □

The Convergence Proof.

By the iterative Lemma 6.3, we obtain a sequence of decreasing domains $D_\nu \times \mathcal{O}_\nu$ and symplectic transformations $\Phi^\nu = \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_\nu : D_\nu \times \mathcal{O}_{\nu-1} \to D_{\nu-1},\ \nu \ge 1$. Then by (6.33) and following the arguments in [42], the sequence Φ^ν of symplectic transformations converge uniformly on $D(r/2) \times \mathcal{O}_\gamma$ to a real analytic torus embedding $\Phi : \mathbb{T}^n \to \mathcal{P}^{a,p}$, for which we also need to verify

(a) the symplectic coordinate transformation Φ is of the form given in (4.9);
(b) the new Hamiltonian eventually reduces to the new normal form, i.e., $P^\infty = 0$;
(c) the symplectic coordinate transformation Φ, which is defined by Theorem 4.1 on each $\mathcal{P}^{a,p}$, extends to $\mathcal{P}^{a,0}$.

In fact, by (3.34) and (3.36) in Section 3.36, the the symplectic coordinate transformation Φ_ν at the νth-step has the form the form

$$\Phi_\nu \begin{pmatrix} \theta \\ I \\ Z \end{pmatrix} = \begin{pmatrix} \theta \\ \Phi_\nu^{(I)} \\ \Phi_\nu^{(Z)} \end{pmatrix} = \begin{pmatrix} \theta \\ I + \frac{1}{2} Z^T M_\nu(\theta) Z \\ L_\nu(\theta) Z \end{pmatrix}. \tag{6.49}$$

In particular, the linear operator $L_\nu(\theta) = \mathrm{e}^{J A_\nu(\theta)}$ is invertible. Then property (**a**) is satisfied at each step, and thus we can iterate the process. It follows that the limiting transformation $\Phi = \Phi_1 \circ \Phi_2 \circ \cdots$ also satisfies the property (**a**). Similar to the initial Hamiltonian, the transformed Hamiltonian is linear in I and quadratic in Z, which implies that the new Hamiltonian eventually reduces to the new normal form, i.e., $P^\infty = 0$.

Since $\Phi^{(Z)}$ is a linear symplectomorphism, then following Prop.1.3 [34] by duality, it extends on $\ell^{a,p} \times \ell^{a,p}$ for all $p \in [-2, 2]$ and thus the conclusion (**c**) holds if we take $p = 0$.

The sequence of closed subset $\mathcal{O}_\nu$ converges to a closed set

$$\mathcal{O}_\gamma = \bigcap_{\nu \ge 0} \mathcal{O}_\nu.$$

By the construction of γ_ν and $|\Omega_{\nu+1} - \Omega_\nu|_{\mathcal{O}} = |\widehat{\Omega}_\nu|_{\mathcal{O}} \le \varepsilon_\nu$, we have $|\Omega^\infty - \Omega|_{\mathcal{O}} \le \varepsilon_0^{1/2}$, and thus for all $\omega \in \mathcal{O}_\gamma$,

$$|\langle k, \omega\rangle| \ge \frac{\gamma}{2\Delta(|k|)}, \quad \forall k \ne 0,$$
$$|\langle k, \omega\rangle + l \cdot \Omega^\infty(\omega)| \ge \frac{\gamma}{2\Delta(|k|)}, \quad \forall k \in \mathbb{Z}^n, \quad |l| = 2.$$

The measure estimate of $\mathcal{O} \setminus \mathcal{O}_\gamma$ of bad frequencies is given in the next section.

6.4 Measure Estimate

In this subsection, we complete the Lebesgue measure estimate of the parameter set $\mathcal{O} \setminus \mathcal{O}_\gamma$. In the process of constructing iterative sequences, we obtain a decreasing sequence of closed sets $\mathcal{O}_0 \supset \mathcal{O}_1 \supset \cdots$ such that $\mathcal{O}_\gamma = \bigcap_{\nu \ge 0} \mathcal{O}_\nu$ and

$$\mathcal{O} \setminus \mathcal{O}_\gamma = \bigcup_{\nu \ge 0} \bigcup_{K_{\nu-1} < |k| \le K_\nu, i, j} \left(\mathcal{R}_k^\nu(\gamma_\nu) \cup \bigcup_{i,j} \mathcal{R}_{kij}^{+,\nu}(\gamma_\nu) \cup \bigcup_{i \ne j} \mathcal{R}_{kij}^{-,\nu}(\gamma_\nu) \right), \tag{6.50}$$

where $\mathcal{R}_k^\nu, \mathcal{R}_{kij}^{+,\nu}, \mathcal{R}_{kij}^{-,\nu}$ are defined in (6.34).

Below we only consider the most difficult resonance set $\mathcal{R}_{kij}^{-,\nu}(\gamma_\nu)$. The proof for other resonance sets $\mathcal{R}_k^\nu,\ \mathcal{R}_{kij}^{+,\nu}$ are more simple, and thus omitted.

Since $\Omega_{\nu,j} = j+\breve{\Omega}_{\nu,j}$, then by (6.30), there is a constant $A_1 > 0$ such that $|\Omega_{\nu,i}-\Omega_{\nu,j}| \geq A_1|i-j|$. Denote $A_2 = (1+2A_1+2A_0)/A_1$. Note that when $|i-j| > A_2|k|$,

$$|k\cdot\omega + \Omega_{\nu,i} - \Omega_{\nu,j}| \geq (1+A_0+A_1)|k|,$$

thus in this case there is no small divisor, and in the following it remains to consider the case of $|i-j| \leq A_2|k|$.

Denote

$$\begin{aligned} S_{k,i,j}^\nu &= k\cdot\omega + \Omega_{\nu,i} - \Omega_{\nu,j}, \\ S_{k,i,j,\infty}^\nu &= k\cdot\omega + \lim_{t\to\infty}(\Omega_{\nu,i+t} - \Omega_{\nu,j+t}) \end{aligned}$$

and introduce the following resonant sets

$$\mathcal{R}_{k,i+t,j+t}^{-,\nu}(\gamma_\nu) = \left\{\omega\in\mathcal{O}_{\nu-1} : |S_{k,i+t,j+t}^\nu| < \frac{\gamma_\nu}{\Delta(|k|)}\right\},$$

Lemma 6.4 *For $i, j \geq 1$ with $|i-j| \leq A_2|k|$, there exist $i', \ j' \geq 1$ satisfying $i' \leq 2A_2|k|,\ \ j' \leq 2A_2|k|$ and $t \geq 1$ such that $i = i'+t,\ \ j = j'+t$. Consequently,*

$$\bigcup_{i,j,|i-j|\leq A_2|k|} \mathcal{R}_{kij}^\nu \subset \bigcup_{i',j'\leq 2A_2|k|}\bigcup_{t\geq 1}\mathcal{R}_{k,i'+t,j'+t}^\nu. \tag{6.51}$$

Proof Without loss of generalization, we assume $j > i$. For given $i,\ j$, choosing a $t_0 \geq 1$ such that $0 \leq i - t_0 \leq A_2|k|$. Let $i' = i - t_0$ and $j' = i' + j - i = j - t_0$, then

$$j' \leq i' + |j-i| \leq 2A_2|k|.$$

It follows that (6.51) holds. □

Lemma 6.5 *For fixed k, i', j',*

$$\text{mes}\left(\bigcup_{t\geq 1}\mathcal{R}_{k,i'+t,j'+t}^\nu\right) \leq (20+8B_0)(2\pi)^{n(n-1)}\frac{\sqrt{\gamma}}{|k|^2\sqrt{\Delta(|k|)}}.$$

Proof For $\omega \in \bigcup_{t>\sqrt{\frac{\Delta(|k|)}{\gamma}}}\mathcal{R}_{k,i'+t,j'+t}^{-,\nu}(\gamma_\nu)$, suppose $\omega\in\mathcal{R}_{k,i'+t_0,j'+t_0}^{-,\nu}(\gamma_\nu)$ for some $t_0 > \sqrt{\frac{\Delta(|k|)}{\gamma}}$.

From the Töplitz–Lipschitz property of P_ν and $\breve{\Omega}_\nu$, we conclude that

$$|S_{k,i'+t,j'+t}^\nu - S_{k,i,j,\infty}^\nu| < \frac{2(1+B_0)}{|t|}.$$

Then

$$\begin{aligned} |S_{k,i',j',\infty}^\nu| &\leq |S_{k,i'+t_0,j'+t_0}^\nu| + |S_{k,i'+t_0,j'+t_0}^\nu - S_{k,i',j',\infty}^\nu| \\ &\leq \frac{\gamma_\nu}{\Delta(|k|)} + \frac{2(1+B_0)}{|t_0|} \leq (3+2B_0)\frac{\sqrt{\gamma}}{\sqrt{\Delta(|k|)}}. \end{aligned}$$

Thus

$$\bigcup_{t>\sqrt{\frac{\Delta(|k|)}{\gamma}}} \mathcal{R}^{-,\nu}_{k,i+t,j+t}(\gamma_\nu) \subseteq \left\{\omega \in \mathcal{O}_{\nu-1} : |S^{\nu}_{k,i',j',\infty}| < (3+2B_0)\frac{\sqrt{\gamma}}{\sqrt{\Delta(|k|)}}\right\}$$
$$=: \mathcal{Q}^{\nu}_{k,i',j',\infty}.$$

We give the estimate of $\mathcal{Q}^{\nu}_{k,i',j',\infty}$. Taking the vector $v = |k|(\mathrm{sgn}(k_1), \ldots, \mathrm{sgn}(k_n))$, then $k \cdot v = |k|^2$. Let $\omega = \omega_\mu = \mu v + w$ with $\mu \in \mathbb{R}$, $w \in v^{\perp}$. Let

$$f(\mu) = S^{\nu}_{k,i,j,\infty} = k \cdot \omega_\mu + \lim_{t\to\infty}(\Omega_{\nu,i+t}(\omega_\mu) - \Omega_{\nu,j+t}(\omega_\mu)).$$

Due to $\sup_{\omega\in\mathcal{O}} |\lim_{t\to\infty} \partial_\omega \widetilde{\Omega}_{\nu,i+t}| \le 3\varepsilon_0$ and $\varepsilon_0 \le \frac{1}{12n}$, the derivative

$$\begin{aligned} |f'(\mu)| &= ||k|^2 + \lim_{t\to\infty} v \cdot (\partial_\omega \widetilde{\Omega}_{\nu,i+t}(\omega_\mu) - \partial_\omega \widetilde{\Omega}_{\nu,j+t}(\omega_\mu))| \\ &\ge |k|^2 - 6n|k|\varepsilon_0 \\ &\ge \frac{1}{2}|k|^2. \end{aligned} \tag{6.52}$$

Then by Lemma 7.6, one has

$$\mathrm{mes}\{\mu : \mu v + w \in \mathcal{O}_{\nu-1}, |f(\mu)| \le \delta\} \le \frac{4\delta}{|k|^2}.$$

It follows that, by Fubini's theorem,

$$\begin{aligned} &\mathrm{mes}\left(\mathcal{Q}^{\nu}_{k,i',j',\infty}\right) \\ &\quad\le \mathrm{diam}^{n-1}(\mathcal{O}_{\nu-1})\, \mathrm{mes}\{\mu : \mu v + w \in \mathcal{O}_{\nu-1}, |f(\mu)| \le (3+2B_0)\frac{\sqrt{\gamma}}{\sqrt{\Delta(|k|)}}\} \\ &\quad\le 4(2\pi)^{n(n-1)}(3+2B_0)\frac{\sqrt{\gamma}}{|k|^2\sqrt{\Delta(|k|)}}. \end{aligned} \tag{6.53}$$

Similarly, for the resonant set $\mathcal{R}^{-,\nu}_{k,i'+t,j'+t}$, following the argument of estimating $\mathrm{mes}\left(\mathcal{Q}^{\nu}_{k,i',j',\infty}\right)$, we have

$$\mathrm{mes}\left(\mathcal{R}^{-,\nu}_{k,i'+t,j'+t}\right) \le (2\pi)^{n(n-1)}\frac{4\gamma_\nu}{|k|^2\Delta(|k|)}. \tag{6.54}$$

Then

$$\begin{aligned} \mathrm{mes}\left(\bigcup_{t\le\sqrt{\frac{\Delta(|k|)}{\gamma}}} \mathcal{R}^{-,\nu}_{k,i'+t,j'+t}\right) &\le 2\sqrt{\frac{\Delta(|k|)}{\gamma}}(2\pi)^{n(n-1)}\frac{4\gamma}{|k|^2\Delta(|k|)} \\ &\le 8(2\pi)^{n(n-1)}\frac{\sqrt{\gamma}}{|k|^2\sqrt{\Delta(|k|)}}. \end{aligned} \tag{6.55}$$

Using (6.53) and (6.55), we complete the proof. □

Finally, we give the estimate of $\mathrm{mes}\left(\mathcal{O}\backslash\mathcal{O}_\gamma\right)$.

Lemma 6.6 *Let Δ be an approximation function satisfying (4.8),i.e., $\sum_{k\in\mathbb{Z}^n}\frac{1}{\sqrt{\Delta(|k|)}}<\infty$. Then the total measure of resonant set should be excluded during the KAM iteration is*

$$\operatorname{mes}\left(\mathcal{O}\setminus\mathcal{O}_\gamma\right)=O(\sqrt{\gamma}),$$

where the implicit constants in "O" depend only on n, A_2, B_0, Δ and are made explicit in the proof.

Proof By Lemma 6.5,

$$\begin{aligned}\operatorname{mes}&\left(\bigcup_{1\le i',j'\le 2A_2|k|}\bigcup_{t\ge1}\mathcal{R}^\nu_{k,i'+t,j'+t}\right)\\ &\le(2A_2|k|)^2(20+8B_0)(2\pi)^{n(n-1)}\frac{\sqrt{\gamma}}{|k|^2\sqrt{\Delta(|k|)}}\\ &\le A_2^2(80+32B_0)(2\pi)^{n(n-1)}\frac{\sqrt{\gamma}}{\sqrt{\Delta(|k|)}}.\end{aligned}$$

Then

$$\begin{aligned}\operatorname{mes}\left(\bigcup_{\nu\ge0}\bigcup_{K_{\nu-1}<|k|\le K_\nu}\bigcup_{i,j}\mathcal{R}^\nu_{kij}\right)&\le\sum_{\nu\ge0}\sum_{K_{\nu-1}<|k|\le K_\nu}\operatorname{mes}\left(\bigcup_{|i'|,|j'|\le 2A_2|k|}\bigcup_{|t|\ge1}\mathcal{R}^\nu_{k,i'+t,j'+t}\right)\\ &\le A_2^2(80+32B_0)(2\pi)^{n(n-1)}\sum_{\nu\ge0}\sum_{K_{\nu-1}<|k|\le K_\nu}\frac{\sqrt{\gamma}}{\sqrt{\Delta(|k|)}}\\ &\le A_2^2(80+32B_0)(2\pi)^{n(n-1)}\sqrt{\gamma}\sum_k\frac{1}{\sqrt{\Delta(|k|)}}.\end{aligned}$$

Consequently, the measure of the set $\mathcal{O}\backslash\mathcal{O}_\gamma$ is

$$\operatorname{mes}(\mathcal{O}\setminus\mathcal{O}_\gamma)=O(\sqrt{\gamma}).$$

□

7 Appendix

7.1 Some Properties of Approximation Functions

Lemma 7.1 *For all integers $k\ge1$, $l\ge0$,*

$$\Gamma_k(\sigma)\le\sigma^l\Gamma_{k+l}(\sigma),$$

where $\Gamma_k(\sigma)=\Gamma_{k3}(\sigma)=\sup_{t\ge0}(1+t)^k\Delta^3(t)\mathrm{e}^{-t\sigma}$.

Proof Let

$$f(t)=k\log(1+t)+\log\Delta^3(t)-t\sigma.$$

Its derivative

$$f'(t)=\frac{k}{1+t}+\frac{d}{dt}\log\Delta^3(t)-\sigma.$$

If $\sigma(1+t) \le 1$, then

$$f'(t) \ge \frac{k-1}{t+1} + \frac{d}{dt}\log \Delta^3(t) \ge 0.$$

It follows that $(1+t)^k\Delta^3(t)e^{-t\sigma}$ arrive at its supremum at some point t_* with $\sigma(1+t_*) \ge 1$. Therefore, for all $l \ge 0$,

$$\begin{aligned}\Gamma_k(\sigma) &= (1+t_*)^k\Delta^3(t_*)e^{-t_*\sigma}\\ &\le \sigma^l(1+t_*)^{k+l}\Delta^3(t_*)e^{-t_*\sigma} \le \sigma^l\Gamma_{k+l}(\sigma).\end{aligned}$$

□

Recall Ξ defined in (6.24):

$$\Xi(\sigma) = \inf_{\tilde\sigma_0\ge\tilde\sigma_1\ge\cdots>0,\tilde\sigma_0+\tilde\sigma_1+\cdots\le\sigma} \prod_{\mu=0}^{\infty}\Gamma^{\kappa_\mu}(\tilde\sigma_\mu) = \prod_{\mu=0}^{\infty}\Gamma^{\kappa_\mu}(\sigma_\mu) < \infty,$$

where $\Gamma(\sigma) = \Gamma_2(\sigma) = \Gamma_{23}(\sigma) = \sup_{t\ge0}(1+t)^2\Delta^3(t)e^{-t\sigma}$.

Lemma 7.2 *The* Ξ *defined in* (6.24) *is finite for all* $\sigma > 0$. *In particular, let* $T > 0$, *if*

$$\frac{1}{\log\kappa}\int_T^{\infty}\frac{\log\Delta(t)}{t^2}dt < \sigma,$$

then

$$\Xi(\sigma) \le e^{\sigma T}.$$

Proof Let $\delta(t) = \log(1+t)^2\Delta^3(t)$ and

$$t_\nu = \kappa^{\nu+1}T,\ \ \sigma_\nu = \frac{\delta(t_\nu)}{t_\nu}$$

for $\nu \ge 0$. By condition (1.5) and the hypotheses, $\sigma_0 \ge \sigma_1 \ge \cdots > 0$ and

$$\sum_{\nu=0}^{\infty}\sigma_\nu \le \int_{-1}^{\infty}\frac{\delta(t_\nu)}{t_\nu}d\nu \le \frac{1}{\log\kappa}\int_T^{\infty}\frac{\delta(t)}{t^2}dt \le \sigma.$$

Since $\delta(t) - \sigma_\nu t \le 0$ for $t \ge t_\nu$, then by condition (1.5) again the supremum of $\delta(t) - \sigma_\nu t$ is obtained on the interval $[0, t_\nu]$ and thus smaller than $\delta(t_\nu)$. It follows that

$$\Gamma(\sigma_\nu) = \sup_{t\ge0} e^{\delta(t)-\sigma_\nu t} \le e^{\delta(t_\nu)} = e^{\sigma_\nu t_\nu}$$

in view of the definition of σ_ν and hence by $\kappa_\nu t_\nu = T$,

$$\Xi(\sigma) \le \prod_{\mu=0}^{\infty} e^{\kappa_\nu\sigma_\nu t_\nu} \le e^{\sigma T}.$$

□

Lemma 7.3

$$\sum_{k\in\mathbb{Z}^n} \frac{1}{\sqrt{\Delta(|k|)}} \le 2^n \int_0^\infty \binom{n+t}{n} \frac{d\log\sqrt{\Delta(t)}}{\sqrt{\Delta(t)}} dt$$

provided that $t^n/\Delta(t)$ *as* $t \to 0$.

Proof Note that

$$\sum_{k\in\mathbb{Z}^n} \frac{1}{\sqrt{\Delta(|k|)}} \le 2^n \sum_{k\in\mathbb{N}^n} \frac{1}{\sqrt{\Delta(|k|)}}.$$

Let $V_n(t) = \mathrm{card}\{k \in \mathbb{N}^n : |k| \le t\}$. Then By the monotonicity of approximation functions the sum above may be written as a Stieltjes integral

$$\sum_{k\in\mathbb{N}^n} \frac{1}{\sqrt{\Delta(|k|)}} \le \inf_{0=t_0<t_1<t_2<\cdots} \{1 + \sum_{\nu=0}^\infty \frac{V_n(t_{\nu+1}) - V_n(t_\nu)}{\sqrt{\Delta(t_\nu)}}\}$$
$$\le 1 + \int_0^\infty \frac{dV_n(t)}{\sqrt{\Delta(t)}} = \int_0^\infty V_n(t) \frac{d\log\sqrt{\Delta(t)}}{\sqrt{\Delta(t)}}$$

by partial integration. From the proof of Lemma 8.3 in [40],

$$V_n(t) \le \binom{n+t}{n},$$

this prove the lemma. □

Lemma 7.4 *There are approximation functions* Δ *such that*

$$\sum_{k\in\mathbb{Z}^n} \frac{1}{\sqrt{\Delta(|k|)}} \le K^{n\log\log n}$$

for all sufficiently large n *with some constant* K.

Proof For $t \le n$,

$$\binom{n+t}{n} \le \binom{2n}{n} \le \frac{(2n)!}{(n!)^2} \le 4^n$$

for all $n \ge 1$. Hence

$$\int_0^n \binom{n+t}{n} \frac{d\log\sqrt{\Delta(t)}}{\sqrt{\Delta(t)}} dt \le 4^n \int_0^\infty \frac{d\log\sqrt{\Delta(t)}}{\sqrt{\Delta(t)}} dt = 4^n$$

for every approximation function Δ.

For $t \ge n$,

$$\binom{n+t}{n} = \frac{1}{n!}(t+1)\cdots(t+n) \le \frac{2^n}{n!} t^n.$$

Let φ be given by $\varphi(s) = \log^2 s$, and define Δ by stipulating that $t \mapsto s = \log\sqrt{\Delta(t)}$ is the inverse function of $s \mapsto t = s\varphi(s)$, at least for large t and s respectively. Let $s_n = n/\varphi(n)$ Since

$$s\varphi(s)|_{s_n} = \frac{n}{\varphi(n)}\varphi(\frac{n}{\varphi(n)}) \leq n$$

by the monotonicity of φ, then

$$\begin{aligned}\int_0^n \binom{n+t}{n} \frac{d\log\sqrt{\Delta(t)}}{\sqrt{\Delta(t)}}dt &\leq \frac{2^n}{n!}\int_0^\infty t^n \frac{d\log\sqrt{\Delta(t)}}{\sqrt{\Delta(t)}}dt \\ &\leq \frac{2^n}{n!}\int_{s_n}^\infty s^n\varphi^n(s)\mathrm{e}^{-s}ds. \end{aligned} \tag{7.1}$$

For all large n and $s \geq s_n$,

$$\varphi(s) = \log^2 s \leq s^{h_n}, \;\; h_n = \frac{\log\varphi(s_n)}{\log s_n} \leq \frac{4\log\log n}{\log n}.$$

Thus, for all large n,

$$\begin{aligned}\int_0^n \binom{n+t}{n} \frac{d\log\sqrt{\Delta(t)}}{\sqrt{\Delta(t)}}dt &\leq \frac{2^n}{n!}\int_0^\infty s^{n+nh_n}\mathrm{e}^{-s}ds \\ &\leq \frac{2^n}{n^n}(n+nh_n)^{n+nh_n+1} = 2^n A_n^n n^{nh_n+1}\end{aligned}$$

here $A_n = (1+h_n)^{1+h_n+1/n}$. The final estimate follows, since $A_n \to 1$ as $n \to \infty$ and $nh_n\log n = 4n\log\log n$. □

7.2 Proof of Proposition 3.2

Proof We only give the proof for the estimate of $(AB)_{ij}^{(11)}$ and $(AB)_{ij}^{(12)}$, the proofs for the estimates of $(AB)_{ij}^{(21)}$ and $(AB)_{ij}^{(22)}$ are similar.

By the matrix multiplication, we have

$$(AB)_{ij}^{(11)} = \sum_{k\in\mathbb{Z}}\left(A_{ik}^{11}B_{kj}^{11} + A_{ik}^{12}B_{kj}^{21}\right)$$

and

$$(AB)_{ij}^{(12)} = \sum_{k\in\mathbb{Z}}\left(A_{ik}^{11}B_{kj}^{12} + A_{ik}^{12}B_{kj}^{22}\right).$$

•Verifying the property (**T1**$'$). In view of $A, B \in \mathfrak{M}_r^\rho$ and the inequality in Lemma 7.5, we have

$$\begin{aligned}\|(AB)_{i,j}^{(11)}\|_{D(r)\times\mathcal{O}} &\leq \sum_{k\in\mathbb{Z}}\|A_{i,k}^{11}\|_{D(r)\times\mathcal{O}}\|B_{k,j}^{11}\|_{D(r)\times\mathcal{O}} + \sum_{k}\|A_{i,k}^{12}\|_{D(r)\times\mathcal{O}}\|B_{k,j}^{21}\|_{D(r)\times\mathcal{O}} \\ &\leq \langle\langle A\rangle\rangle_{\rho,r}\langle\langle B\rangle\rangle_{\rho,r}\left(\sum_k \mathrm{e}^{-\rho(|i-k|+|k-j|)} + \sum_{k\in\mathbb{Z}}\mathrm{e}^{-\rho(|i+k|+|k+j|)}\right)\end{aligned}$$

$$\leq C\delta^{-1}\langle\langle A\rangle\rangle_{\rho,r}\langle\langle B\rangle\rangle_{\rho,r}\mathrm{e}^{-(\rho-\delta)(|i-j|)}$$

and

$$\begin{aligned}\|(AB)^{(12)}_{i,j}\|_{D(r)\times\mathcal{O}} &\leq \sum_{k\in\mathbb{Z}}\|A^{11}_{i,k}\|_{D(r)\times\mathcal{O}}\|B^{12}_{k,j}\|_{D(r)\times\mathcal{O}}+\sum_{k}\|A^{12}_{i,k}\|_{D(r)\times\mathcal{O}}\|B^{22}_{k,j}\|_{D(r)\times\mathcal{O}}\\ &\leq \langle\langle A\rangle\rangle_{\rho,r}\langle\langle B\rangle\rangle_{\rho,r}\left(\sum_{k}\mathrm{e}^{-\rho(|i-k|+|k+j|)}+\sum_{k\in\mathbb{Z}}\mathrm{e}^{-\rho(|i+k|+|k-j|)}\right)\\ &\leq C\delta^{-1}\langle\langle A\rangle\rangle_{\rho,r}\langle\langle B\rangle\rangle_{\rho,r}\mathrm{e}^{-(\rho-\delta)(|i+j|)}.\end{aligned}$$

•Verifying the property (**T2**$'$). In view of $A, B\in\mathfrak{M}^{\rho}_{r}$, then following the verification of Property (**T1**$'$), we have

$$\begin{aligned}\|\lim_{t\to\infty}(AB)^{(11)}_{i+t,j+t}\|_{D(r)\times\mathcal{O}} &\leq \sum_{k}\|\lim_{t\to\infty}A^{11}_{i+t,k+t}\|_{D(r)\times\mathcal{O}}\|\lim_{t\to\infty}B^{11}_{k+t,j+t}\|_{D(r)\times\mathcal{O}}\\ &\quad+\sum_{k}\|\lim_{t\to\infty}A^{12}_{i+t,k-t}\|_{D(r)\times\mathcal{O}}\|\lim_{t\to\infty}B^{21}_{k-t,j+t}\|_{D(r)\times\mathcal{O}}\\ &\leq \langle\langle A\rangle\rangle_{\rho,r}\langle\langle B\rangle\rangle_{\rho,r}\left(\sum_{k}\mathrm{e}^{-\rho(|i-k|+|k-j|)}+\sum_{k}\mathrm{e}^{-\rho(|i+k|+|k+j|)}\right)\\ &\leq C\delta^{-1}\langle\langle A\rangle\rangle_{\rho,r}\langle\langle B\rangle\rangle_{\rho,r}\mathrm{e}^{-(\rho-\delta)(|i-j|)}\end{aligned}$$

and

$$\begin{aligned}\|\lim_{t\to\infty}(AB)^{(12)}_{i+t,j-t}\|_{D(r)\times\mathcal{O}} &\leq \sum_{k}\|\lim_{t\to\infty}A^{11}_{i+t,k+t}\|_{D(r)\times\mathcal{O}}\|\lim_{t\to\infty}B^{12}_{k+t,j-t}\|_{D(r)\times\mathcal{O}}\\ &\quad+\sum_{k}\|\lim_{t\to\infty}A^{12}_{i+t,k-t}\|_{D(r)\times\mathcal{O}}\|\lim_{t\to\infty}B^{22}_{k-t,j-t}\|_{D(r)\times\mathcal{O}}\\ &\leq \langle\langle A\rangle\rangle_{\rho,r}\langle\langle B\rangle\rangle_{\rho,r}\left(\sum_{k}\mathrm{e}^{-\rho(|i-k|+|k+j|)}+\sum_{k}\mathrm{e}^{-\rho(|i+k|+|k-j|)}\right)\\ &\leq C\delta^{-1}\langle\langle A\rangle\rangle_{\rho,r}\langle\langle B\rangle\rangle_{\rho,r}\mathrm{e}^{-(\rho-\delta)(|i-j|)}.\end{aligned}$$

These imply the property (T2) holds.

•Verifying the property (**T3**$'$). Denote $A^{11}_{i,j,\infty}:=\lim_{t\to\infty}A^{11}_{i+t,j+t}$ and $A^{12}_{i,j,\infty}:=\lim_{t\to\infty}A^{12}_{i+t,j-t}$. Similarly for other terms.

Then by the difference equality (3.15) and the inequality in Lemma 7.5, we have

$$\begin{aligned}&\|(AB)^{(11)}_{i+t,j+t}-\lim_{t\to\infty}(AB)^{(11)}_{i+t,j+t}\|_{D(r)\times\mathcal{O}}\\ &\leq \sum_{k}\|A^{11}_{i+t,k+t}-A^{11}_{i,k,\infty}\|_{D(r)\times\mathcal{O}}\|B^{11}_{k,j,\infty}\|_{D(r)\times\mathcal{O}}\\ &\quad+\sum_{k}\|A^{11}_{i,k,\infty}\|_{D(r)\times\mathcal{O}}\|B^{11}_{k+t,j+t}-B^{11}_{k,j,\infty}\|_{D(r)\times\mathcal{O}}\\ &\quad+\sum_{k}\|A^{11}_{i+t,k+t}-A^{11}_{i,k,\infty}\|_{D(r)\times\mathcal{O}}\|B^{11}_{k+t,j+t}-B^{11}_{k,j,\infty}\|_{D(r)\times\mathcal{O}}\\ &\quad+\sum_{k}\|A^{12}_{i+t,k-t}-A^{12}_{i,k,\infty}\|_{D(r)\times\mathcal{O}}\|B^{21}_{k,j,\infty}\|_{D(r)\times\mathcal{O}}\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_k \|A^{12}_{i,k,\infty}\|_{D(r)\times\mathcal{O}} \|B^{21}_{k-t,j+t} - B^{21}_{k,j,\infty}\|_{D(r)\times\mathcal{O}} \\
&\quad + \sum_k \|A^{12}_{i+t,k-t} - A^{12}_{i,k,\infty}\|_{D(r)\times\mathcal{O}} \|B^{21}_{k-t,j+t} - B^{21}_{k,j,\infty}\|_{D(r)\times\mathcal{O}} \\
&\le |t|^{-1} \langle\langle A\rangle\rangle_{\rho,r} \langle\langle B\rangle\rangle_{\rho,r} \left(\sum_k \mathrm{e}^{-\rho(|i-k|+|k-j|)} + \sum_k \mathrm{e}^{-\rho(|i+k|+|k+j|)} \right) \\
&\le |t|^{-1} C\delta^{-1} \langle\langle A\rangle\rangle_{\rho,r} \langle\langle B\rangle\rangle_{\rho,r} \mathrm{e}^{-(\rho-\delta)(|i-j|)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(AB)^{(12)}_{i+t,j-t} - \lim_{t\to\infty} (AB)^{(12)}_{i+t,j-t}\|_{D(r)\times\mathcal{O}} \\
&\le \sum_k \|A^{11}_{i+t,k+t} - A^{11}_{i,k,\infty}\|_{D(r)\times\mathcal{O}} \|B^{12}_{k,j,\infty}\|_{D(r)\times\mathcal{O}} \\
&\quad + \sum_k \|A^{11}_{i,k,\infty}\|_{D(r)\times\mathcal{O}} \|B^{12}_{k+t,j-t} - B^{12}_{k,j,\infty}\|_{D(r)\times\mathcal{O}} \\
&\quad + \sum_k \|A^{11}_{i+t,k+t} - A^{11}_{i,k,\infty}\|_{D(r)\times\mathcal{O}} \|B^{12}_{k+t,j-t} - B^{12}_{k,j,\infty}\|_{D(r)\times\mathcal{O}} \\
&\quad + \sum_k \|A^{12}_{i+t,k-t} - A^{12}_{i,k,\infty}\|_{D(r)\times\mathcal{O}} \|B^{22}_{k,j,\infty}\|_{D(r)\times\mathcal{O}} \\
&\quad + \sum_k \|A^{12}_{i,k,\infty}\|_{D(r)\times\mathcal{O}} \|B^{22}_{k-t,j-t} - B^{22}_{k,j,\infty}\|_{D(r)\times\mathcal{O}} \\
&\quad + \sum_k \|A^{12}_{i+t,k-t} - A^{12}_{i,k,\infty}\|_{D(r)\times\mathcal{O}} \|B^{22}_{k-t,j-t} - B^{22}_{k,j,\infty}\|_{D(r)\times\mathcal{O}} \\
&\le |t|^{-1} \langle\langle A\rangle\rangle_{\rho,r} \langle\langle B\rangle\rangle_{\rho,r} \left(\sum_k \mathrm{e}^{-\rho(|i-k|+|k+j|)} + \sum_k \mathrm{e}^{-\rho(|i+k|+|k-j|)} \right) \\
&\le |t|^{-1} C\delta^{-1} \langle\langle A\rangle\rangle_{\rho,r} \langle\langle B\rangle\rangle_{\rho,r} \mathrm{e}^{-(\rho-\delta)(|i+j|)}.
\end{aligned}
$$

□

7.3 Some Technical Lemmas

Lemma 7.5 *Let* $0 < \delta \le 1$.

$$\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\delta(|i-k|+|k-j|)} \le C\delta^{-1}, \tag{7.2}$$

where C *is a positive constant that does not depend on* δ *and* i, j.

Proof Without loss of generality, we assume $i \ge j$.

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\delta(|i-k|+|k-j|)} &= \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\delta(|k|+|i-j-k|)} \\
&= \left(\sum_{k<0} + \sum_{0\le k\le i-j} + \sum_{k>i-j} \right) \mathrm{e}^{-\delta(|k|+|i-j-k|)} \\
&= \sum_{k<0} \mathrm{e}^{-\delta(-2k+i-j)} + \sum_{0\le k\le i-j} \mathrm{e}^{-\delta(i-j)} + \sum_{k>i-j} \mathrm{e}^{-\delta(2k-(i-j))}
\end{aligned}
$$

$$= \frac{2e^{-\delta(i-j)}}{e^{2\delta}-1} + (i-j+1)e^{-\delta(i-j)}.$$

Since the function $f(x) = (x+1)e^{-\delta x},\ (x \geq 0)$ reaches its maximum at $x = \frac{1}{\delta} - 1$, then if $0 < \delta \leq 1$,

$$f(x) \leq \frac{1}{\delta}e^{-1+\delta} \leq \frac{1}{\delta}.$$

It follows that the inequality holds. □

Lemma 7.6 [45] *Let* $f : [a,b] \to \mathbb{R}$ *be a* $q-$*times continuously differentiable function satisfying*

$$|f^{(q)}(t)| \geq \beta,\ \forall\ t \in [a,b]$$

for some $q \in \mathbb{N}$ *and* $\beta > 0$. *Then we have the estimate*

$$\text{mes}\{t \in [a,b] : |f(t)| \leq \varepsilon\} \leq 4\left(\frac{q!}{2\beta}\varepsilon\right)^{\frac{1}{q}},\ \forall\ \varepsilon > 0.$$

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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