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On Traveling Fronts of Combustion Equations in Spatially Periodic Media

Yasheng Lyu[1,2] · Hongjun Guo[3] · Zhi-Cheng Wang[1]

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Abstract
This paper is concerned with traveling fronts of spatially periodic reaction–diffusion equations with combustion nonlinearity in $\mathbb{R}^N$. It is known that for any given propagation direction $e \in \mathbb{S}^{N-1}$, the equation admits a pulsating front connecting two equilibria 0 and 1. In this paper we firstly give exact asymptotic behaviors of the pulsating front and its derivatives at infinity, and establish uniform decay estimates of the pulsating fronts at infinity on the propagation direction $e \in \mathbb{S}^{N-1}$. Following the uniform estimates, we then show continuous Fréchet differentiability of the pulsating fronts with respect to the propagation direction. Lastly, using the differentiability, we establish the existence, uniqueness and stability of curved fronts with V-shape in $\mathbb{R}^2$ by constructing suitable super- and subsolutions.

Keywords Reaction–diffusion equations · Spatial periodicity · Combustion nonlinearity · Pulsating front · Differentiability · Curved front

Mathematics Subject Classification 35K57 · 35C07 · 35B30 · 35B40

1 Introduction

In this paper, we investigate spatially periodic reaction–diffusion equations of the type

$$u_t - \Delta_z u = f(z, u) \ \text{ in } (t, z) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.1}$$

where $N \in \mathbb{N}$, $u = u(t, z)$, $u_t = \frac{\partial u}{\partial t}$, Δ_z denotes the Laplace operator with respect to the spatial variables $z \in \mathbb{R}^N$, and the reaction term $f(z, u)$ satisfies the following assumptions:

✉ Zhi-Cheng Wang
wangzhch@lzu.edu.cn

1 School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, Gansu, People's Republic of China

2 School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, Shaanxi, People's Republic of China

3 School of Mathematical Sciences, and Institute for Advanced Study, Tongji University, Shanghai 200092, People's Republic of China

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(F1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is of class $C^\infty(\mathbb{R}^{N+1})$ and satisfies

$$\|f\|_{C^k(\mathbb{R}^{N+1})} = \sum_{i=0}^{k} \left\| D^i f \right\|_{L^\infty(\mathbb{R}^{N+1})} < +\infty \quad \text{for all } k \in \mathbb{N}. \tag{1.2}$$

(F2) For each $u \in \mathbb{R}$, the function $f(\cdot, u) : \mathbb{R}^N \to \mathbb{R}$ is $L-$periodic. Here a function $h : \mathbb{R}^N \to \mathbb{R}$ is said to be $L-$periodic if $h(z_1, \ldots, z_k+L_k, \ldots, z_N) = h(z_1, \ldots, z_N)$ for all $1 \le k \le N$ and all $(z_1, \ldots, z_N) \in \mathbb{R}^N$, where $L_1, \ldots, L_N$ are given positive constants. In such case, $\mathbb{L}^N := (0, L_1) \times \cdots \times (0, L_N)$ is called the cell of periodicity.

(F3) There exists $p \in (0, 1)$ such that

$$\begin{cases} \forall\, (z, u) \in \mathbb{R}^N \times [0, p] \cup \{1\}, \;\; f(z, u) = 0; \\ \forall\, (z, u) \in \mathbb{R}^N \times (p, 1), \;\; f(z, u) \ge 0; \\ \forall\, u \in (p, 1), \;\; \exists\, z \in \mathbb{R}^N, \;\; s.t. \; f(z, u) > 0. \end{cases} \tag{1.3}$$

(F4) There holds $\sup_{z\in\mathbb{R}^N} f_u(z, 1) < 0$.

Assumption (F1) can be relaxed to $f \in C^m\left(\mathbb{R}^{N+1}\right)$ for sufficiently large $m \in \mathbb{N}$, but for the sake of convenience, we suppose $f \in C^\infty\left(\mathbb{R}^{N+1}\right)$. Clearly, assumptions (F3) and (F4) imply that the reaction term f is of combustion type. Denote

$$-K_1 := \inf_{z\in\mathbb{R}^N} f_u(z, 1) \;\; \text{and} \;\; -\kappa_1 := \sup_{z\in\mathbb{R}^N} f_u(z, 1). \tag{1.4}$$

By assumptions (F1) and (F4), one has $0 < \kappa_1 \le K_1 < +\infty$. It follows from (1.2) and (1.4) that there exists a positive constant $\gamma_\star < 1$ such that

$$f_u(z, u) \le -\frac{\kappa_1}{2}, \;\; \forall (z, u) \in \mathbb{R}^N \times [1 - \gamma_\star, 1 + \gamma_\star]. \tag{1.5}$$

For reaction–diffusion equations in spatially periodic media, important advances have recently been made in its propagation dynamics. To describe the propagation dynamics of spatially periodic equations, it is necessary to introduce an important notion–pulsating front, which is a natural extension of the classical notion–planar traveling wave solution in homogeneous media.

Definition 1.1 (Pulsating Front) A pair (U_e, c_e) with $U_e : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $c_e \in \mathbb{R}$ is said to be a pulsating front of (1.1) with effective speed c_e in the direction $e \in \mathbb{S}^{N-1}$ connecting two equilibria 0 and 1, if the following three properties are satisfied:

(i) the function $u(t, z) := U_e(z \cdot e - c_e t, z)$ is an entire (classical) solution of the parabolic Eq. (1.1).

(ii) the profile U_e satisfies $U_e(s, z) = U_e(s, z + y)$ for all $(s, z) \in \mathbb{R} \times \mathbb{R}^N$ and $y \in \prod_{i=1}^{N} L_i\mathbb{Z}$.

(iii) the profile U_e satisfies

$$\lim_{s\to+\infty} U_e(s, z) = 0 \;\; \text{and} \;\; \lim_{s\to-\infty} U_e(s, z) = 1 \quad \text{uniformly for } \; z \in \mathbb{R}^N.$$

The notion of pulsating front was introduced first by Shigesada et al. [46] and Xin [58–60]. Now we recall the existing results on pulsating fronts of a general reaction–diffusion equation in spatially periodic media

$$u_t = \nabla \cdot (A(z)\nabla u) + q(z) \cdot \nabla u + f(z, u) \;\; \text{in } (t, z) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.6}$$

where $A(z) = \left(A_{ij}(z)\right)_{1\le i,j\le N}$ is a matrix field and $q(z) = (q_1(z), \cdots, q_N(z))$ is a vector field. For the monostable nonlinearity f, it was shown that for any propagation direction $e \in \mathbb{S}^{N-1}$, there exists a minimal wave speed c_e^* such that Eq. (1.6) admits a pulsating front if and only if the wave speed $c \ge c_e^*$, see Berestycki and Hamel [3], Liang and Zhao [37], Weinberger [57], etc. Furthermore, the uniqueness and stability of pulsating fronts were studied in [29, 35]. For the combustion nonlinearity, it follows from Berestycki and Hamel [3] and Xin [58, 60] that for any $e \in \mathbb{S}^{N-1}$, there exists a unique (up to time shift) pulsating front U_e with wave speed c_e. In particular, the speed c_e is also unique. For the bistable nonlinearity f, the existence and nonexistence of pulsating traveling fronts were studied intensively. For one dimensional case, see Ding, Hamel and Zhao [15], Ducrot, Giletti and Matano [18], Fang and Zhao [22], Giletti and Matano [23], and Nolen and Ryzhik [42]. For higher dimensional case, see Ducrot [17], Giletti and Rossi [24], Xin [59]. More recently, Ding and Giletti [14] showed in any spatial dimension that for an arbitrary large number of directions, there exists a spatially periodic bistable type equation to achieve any combination of speeds in those given directions, provided that those speeds have the same sign. In particular, even if in one dimensional space, any pair of rightward and leftward wave speeds is admissible, which is completely different from the Fisher-KPP case. They also showed that these variations in the speeds of bistable pulsating fronts lead to strongly asymmetrical situations in the multistable equations. Besides these existence results for pulsating fronts of bistable equations, it was also shown that there may not exist pulsating fronts for bistable equations in spatially periodic media. Zlatoš [64] constructed a periodic pure bistable reaction such that there is no pulsating fronts of (1.1). We also refer to [15, 61, 62] for some nonexistence results. For the unique and stability of pulsating fronts for bistable equations in spatially periodic media, we refer to Ding, Hamel and Zhao [15].

As reported above, in spatially periodic media, pulsating traveling fronts may not exist for bistable equations. In fact, unlike in the homogeneous case, the equation in spatially periodic media is no longer invariant by rotation, and hence the wave profile and the wave speed may be different depending on its direction even if the pulsating fronts exist. Therefore, many researchers paid attention to the dependence of propagation phenomena on the direction in spatially periodic media. In [2], Alfaro and Giletti considered a spatially periodic reaction–diffusion equation with either combustion or monostable nonlinearity in high-dimensional space. They showed that the (minimal) wave speed of pulsating fronts of the equation depends continuously on the direction of propagation, and so does its associated profile (up to time shifts). They also showed that the spreading properties in [57] are uniform with respect to the direction. Guo [25] studied a spatially periodic reaction–diffusion equation with bistable nonlinearity in high-dimensional space. Under the a priori assumption that there exist pulsating fronts with nonzero speeds for every direction of propagation, they showed the continuity and differentiability of wave speeds and profiles of the underlying pulsating fronts with respect to the direction of propagation. They also proved that the propagating speed of any transition front is larger than the infimum of speeds of pulsating fronts and less than the supremum of speeds of pulsating fronts. More recently, Ding et al. [16] revisited the continuity and further proved the continuity of wave speeds on the direction without the extra assumption that the wave speeds are nonzero in all directions.

In high-dimensional space, even if for homogeneous reaction–diffusion equation, there exist various traveling fronts whose level sets admit different shapes, such as V-shaped traveling fronts, pyramidal traveling fronts, and conical traveling fronts. These fronts have been found in experimental observations and numerical calculations for the Bunsen burners and Belousov–Zhabotinskii chemical reaction, see [27, 43, 47] for flames of various kinds of smooth shapes, and [6, 44] for V-shaped chemical waves. In the past thirty years, there were many important studies concentrating on the rigorous mathematical analysis to these fronts. See [32, 33, 40, 48–52] for bistable equations, [11, 12, 34, 55] for monostable equations, [5, 9, 11, 12, 30, 31, 55] for combustion equations, and [41, 45, 53] for reaction–diffusion systems. For inhomogeneous (heterogeneous) reaction–diffusion equations, there also were some literatures concerning curved fronts of the equations. See [54, 56, 63] for time periodic bistable and combustion equations, [19, 20] for monostable and combustion equations with periodic shear flow, and [10] for space-time periodic monostable equations. In particular, Guo et al. [26] studied Eq. (1.1) under bistable assumption and gave some sufficient conditions to the existence of curved fronts in $\mathbb{R}^2$. They further showed that the curved front is unique and asymptotically stable.

In this paper we continuously investigate the propagation phenomena of periodic Eq. (1.1) under assumptions (F1)–(F4). Namely, we consider Eq. (1.1) with combustion nonlinearity. By (F3), we have that there exists $p \in (0, 1)$ such that $f(z, u) = 0$ for any $(z, u) \in \mathbb{R}^N \times [0, p] \cup \{1\}$, which is different from the bistable case studied in [26]. In this case the equilibrium 0 is degenerate, which in turn raises some difficulties in our study. In fact, to overcome the difficulty due to the degenerate, we have to work under some weighted sense. Under assumptions (F1)–(F4), it follows from [3] that for any given propagation direction $e \in \mathbb{S}^{N-1}$, (1.1) admits a unique pulsating front connecting two equilibria 0 and 1. Based on the existence of pulsating fronts, in this paper we turn to investigate the properties of the pulsating fronts and establish curved fronts for (1.1). Firstly, we give exponentially asymptotic behaviors of the pulsating front and its derivatives at infinity, and establish uniform decay estimates of the pulsating fronts at infinity on the propagation direction $e \in \mathbb{S}^{N-1}$. According to the uniform estimates, we then show continuous Fréchet differentiability of the pulsating fronts with respect to the propagation direction. Lastly, using the differentiability, we establish the existence, uniqueness and stability of curved fronts with V-shape in $\mathbb{R}^2$.

The following sections are devoted to stating and proving the results of this paper. In Sect. 2 we introduce some known results as preliminaries and state our main results. Section 3 is concerned with the asymptotic behaviors of pulsating fronts, including Theorems 2.5 and 2.7. In Sect. 4, we mainly investigate the Fréchet differentiability of pulsating fronts with respect to the direction of propagation, namely Theorems 2.8 and 2.10. Lastly, Sect. 5 is devoted to the proof of the existence, uniqueness and stability of curved fronts in $\mathbb{R}^2$, that is, we prove Theorems 2.12, 2.15 and 2.16.

2 Preliminaries and Main Results

In this section, we firstly introduce some known results on the pulsating fronts of Eq. (1.1) as preliminaries, and then state our main results. Here we emphasize that (F1)–(F4) always hold throughout this paper.

2.1 Preliminaries

As reported in Sect. 1, the existence and uniqueness of pulsating fronts of (1.1) with combustion nonlinearity has been established by Berestycki and Hamel [3], namely, for any given propagation direction $e \in \mathbb{S}^{N-1}$, (1.1) admits a unique pulsating front connecting two equilibria 0 and 1.

Theorem 2.1 ([3], Theorem 1.13) *Assume that* (F1)–(F4) *hold. Let* $e \in \mathbb{S}^{N-1}$. *Then*

(i) *there exists* (U_e, c_e) *such that* $u(t,z) := U_e(z\cdot e - c_e t, z)$ *is a classical solution of* (1.1).
(ii) *the speed* c_e *is unique and positive. The profile* $U_e(s,z)$ *is unique up to transition in the variable* s.
(iii) *the profile* $U_e(s,z) : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ *is strictly decreasing in* s.

From the above definition, pulsating front U_e satisfies a semilinear degenerate elliptic equation of the type

$$c_e\partial_s U_e + \partial_{ss}U_e + 2\nabla_z\partial_s U_e\cdot e + \Delta_z U_e + f(z,U_e) = 0 \ \text{ in } \mathbb{R}\times\mathbb{R}^N, \tag{2.1}$$

where $e\in\mathbb{S}^{N-1}$. We now recall some results of the asymptotic behaviors of pulsating front U_e at infinite. Due to assumption (F4), the asymptotic behaviors of pulsating front as tending to the equilibrium state 1 can directly follows from Bu, Wang and Liu [13]. Recently, Bu and He [8] also gave the asymptotic behaviors of pulsating front as tending to the equilibrium state 0. We say that $U_e(s,z)\sim C_1e^{-c_es}$ as $s\to+\infty$ uniformly in $z\in\mathbb{R}^N$, if

$$\liminf_{s\to+\infty}\min_{z\in\mathbb{R}^N}\frac{U_e(s,z)}{C_1e^{-c_es}} = \limsup_{s\to+\infty}\max_{z\in\mathbb{R}^N}\frac{U_e(s,z)}{C_1e^{-c_es}} = 1.$$

Theorem 2.2 ([8, 13]) *Assume that* (F1)–(F4) *hold. Let* $e\in\mathbb{S}^{N-1}$. *Assume that* (U_e, c_e) *is the unique pulsating front of* (1.1). *Then*

(i) *there exist two nonzero constants* C_1 *and* C_2 *such that*

$$U_e(s,z)\sim C_1e^{-c_es} \ \textit{ and } \ \partial_sU_e(s,z)\sim C_2e^{-c_es} \ \textit{ as } s\to+\infty$$

uniformly in $z\in\mathbb{R}^N$.
(ii) *there exist two nonzero constants* C_1', C_2' *and a positive constant* τ_e *dependent on* e *such that*

$$1-U_e(s,z)\sim C_1'e^{\tau_es}\varphi_{\tau_e}(z) \ \textit{ and } \ \partial_sU_e(s,z)\sim C_2'e^{\tau_es}\varphi_{\tau_e}(z) \ \textit{ as } s\to-\infty$$

uniformly in $z\in\mathbb{R}^N$, *where* $0<\varphi_{\tau_e}(z)\in C^2(\mathbb{R}^N)$ *is L-periodic and* $\|\varphi_{\tau_e}\|_{L^\infty(\mathbb{R}^N)} = 1$.

For spatially periodic media, pulsating fronts in different propagation directions are different in general. Thus the dependency of pulsating front with respect to the propagation direction is vital to investigate problems involving more than one pulsating front. Alfaro and Giletti [2] got the continuity of pulsating fronts with respect to the propagation direction for spatially periodic reaction–diffusion equations with combustion nonlinearity.

Theorem 2.3 ([2], Theorems 2.4 and 2.5). *Assume that assumptions* (F1)–(F4) *hold. Let* $e\in\mathbb{S}^{N-1}$. *Assume that* (U_e, c_e) *is the unique pulsating front of* (1.1). *Then*

(i) *the mappig* $e\in\mathbb{S}^{N-1}\mapsto c_e$ *is continuous.*
(ii) *there exist two positive constants* κ *and* K *such that*

$$0<\kappa := \inf_{e\in\mathbb{S}^{N-1}} c_e \le \sup_{e\in\mathbb{S}^{N-1}} c_e =: K < +\infty.$$

(iii) *the mapping* $e\in\mathbb{S}^{N-1}\mapsto U_e$ *is continuous under the topology* $\|\cdot\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}$, *by normalization as* $\min_{z\in\mathbb{R}^N}U_e(0,z) = (1+p)/2$, *where* p *is defined in* (1.3).

Remark 2.4 Here we emphasize that the continuity of pulsating fronts in Theorem 2.3 (iii) is proved under the normalization $\min_{z\in\mathbb{R}^N}U_e(0,z) = (1+p)/2$.

2.2 Main Results

In this subsection we list our main results in this paper. The first part is concerned with the properties of pulsating fronts, including the exponentially asymptotic behaviors of pulsating front and its derivatives at infinite, and the continuity and Fréchet differentiability of pulsating fronts and wave speeds with respect to the propagation direction. The second part is concerned with curved fronts of (1.1) in $\mathbb{R}^2$.

• *Pulsating fronts*

In this part we mainly focus on the continuity and Fréchet differentiability of pulsating fronts and wave speeds with respect to the propagation direction. To do that, we firstly establish the exponentially asymptotic behaviors of pulsating front and its derivatives at infinite.

Theorem 2.5 *Assume that assumptions* (F1)–(F4) *hold. Let* $e \in \mathbb{S}^{N-1}$. *Assume that* (U_e, c_e) *is the unique pulsating front of* (1.1). *Then for any nonnegative integers* k *and* l, *there exists a constant* C_{kl} *dependent on* k *and* l, *such that*

$$\lim_{s\to+\infty} \frac{D_z^k D_s^l U_e}{U_e} = C_{kl}, \tag{2.2}$$

$$\lim_{s\to+\infty} \frac{\partial_s U_e}{U_e} = -c_e, \tag{2.3}$$

$$\lim_{s\to+\infty} \frac{\partial_{ss} U_e}{U_e} = c_e^2, \tag{2.4}$$

$$\lim_{s\to+\infty} \frac{|\nabla_z U_e|,\ |\nabla_z \partial_s U_e|}{U_e} = 0, \tag{2.5}$$

$$\lim_{s\to+\infty} \frac{\Delta_z U_e}{U_e} = 0 \tag{2.6}$$

uniformly in $z \in \mathbb{R}^N$, *where* ∇_z *denotes the gradient operator with respect to* $z \in \mathbb{R}^N$.

Remark 2.6 Here we point out that the asymptotic behaviors in Theorem 2.5 may rely on the propagation direction e.

Furthermore, we have the following uniform estimates.

Theorem 2.7 *Assume that* (F1)–(F4) *hold. Let* $e \in \mathbb{S}^{N-1}$. *Assume that* (U_e, c_e) *is the unique pulsating front of* (1.1). *Normalize* U_e *as* $\min_{z\in\mathbb{R}^N} U_e(0, z) = (1 + p)/2$, *where* p *is defined in* (1.3). *Then there exist positive constants* $\overline{K}$ *and* κ_2, *both independent of* $e \in \mathbb{S}^{N-1}$, *such that*

$$|U_e(s,z)|,\ |DU_e(s,z)|,\ |D^2U_e(s,z)|,\ |D^3U_e(s,z)| \le \overline{K}e^{-\frac{3\kappa}{4}s} \ \ in\ [0,+\infty)\times\mathbb{R}^N,$$
$$|1-U_e(s,z)|,\ |DU_e(s,z)|,\ |D^2U_e(s,z)|,\ |D^3U_e(s,z)| \le \overline{K}e^{\kappa_2 s} \ \ in\ (-\infty,0]\times\mathbb{R}^N,$$

where κ *is defined in Theorem 2.3;* D, D^2 *and* D^3 *denote any first-order, second-order and third-order derivative with respect to* $(s, z) \in \mathbb{R} \times \mathbb{R}^N$ *respectively.*

Now we consider the continuity and Fréchet differentiability of pulsating fronts and wave speeds with respect to the propagation direction. Denote

$$\rho = \rho(s) \overset{\text{def}}{=} 1 + e^{2\varepsilon s}, \quad \forall s \in \mathbb{R}, \tag{2.7}$$

where ε is a constant satisfying $0 < \varepsilon \ll \kappa$. Let us now define a weighted L^2 space

$$L^2_\rho(\mathbb{R}\times\mathbb{L}^N) \stackrel{\text{def}}{=} \Bigg\{u : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R} \,\Bigg|\, \|u\|_{L^2_\rho} \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}\times\mathbb{L}^N} u^2\rho\, dsdz\right)^{1/2} < \infty \\ \text{and } u(s,z) = u(s,z+y) \text{ a.e. in } \mathbb{R}\times\mathbb{R}^N \text{ for any } y \in \prod_{i=1}^N L_i\mathbb{Z}\Bigg\} \tag{2.8}$$

and a weighted H^1 space

$$H^1_\rho(\mathbb{R}\times\mathbb{L}^N) \stackrel{\text{def}}{=} \left\{u \in L^2_\rho :\ \|u\|^2_{H^1_\rho} \stackrel{\text{def}}{=} \|u\|^2_{L^2_\rho} + \int_{\mathbb{R}\times\mathbb{L}^N} |\nabla u|^2\rho\, dsdz < \infty\right\}, \tag{2.9}$$

where ∇ denotes the gradient operator with respect to $(s, z) \in \mathbb{R}\times\mathbb{R}^N$. Similarly, one can obtain the definition of weighted space $H^n_\rho,\ n \geq 2$.

In the sequel of this paper, the profile of pulsating fronts U_e are always normalized as

$$\int_{\mathbb{R}^+\times\mathbb{L}^N} U_e^2\rho\, dsdz = 1 \ \text{ for all } e \in \mathbb{S}^{N-1}, \tag{2.10}$$

for the sake of considering the Fréchet differentiability of pulsating front. Under normalization (2.10), the continuity of pulsating fronts with respect to the propagation direction is given in below theorem.

Theorem 2.8 *Assume that assumptions* (F1)–(F4) *hold. Let $e \in \mathbb{S}^{N-1}$. Assume that (U_e, c_e) is the unique pulsating front of* (1.1). *Then*

(i) the mappig $e \in \mathbb{S}^{N-1} \mapsto c_e$ is continuous.

(ii)

$$0 < \kappa = \inf_{e\in\mathbb{S}^{N-1}} c_e \leq \sup_{e\in\mathbb{S}^{N-1}} c_e = K < +\infty,$$

where κ and K coincide with those in Theorem 2.3.

(iii) the mapping $e \in \mathbb{S}^{N-1} \mapsto U_e$ is continuous under the topology $\|\cdot\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}$, by normalization (2.10).

Remark 2.9 Clearly, the uniform estimates in Theorem 2.7 is established under the normalization U_e as $\min_{z\in\mathbb{R}^N} U_e(0,z) = (1+p)/2$. In fact, the conclusions in Theorem 2.7 still hold for the normalization (2.10).

In the following theorem, we give the continuous differentiability of pulsating fronts with respect to the propagation direction $e \in \mathbb{S}^{N-1}$ for the case of combustion nonlinearity. For the case of bistable nonlinearity, continuity and differentiability properties of the pulsating fronts U_e and speeds c_e with respect to the direction $e \in \mathbb{S}^{N-1}$ under topology $\|\cdot\|_{L^2(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}}$ has been studied by Guo [25]. For any $b \in \mathbb{R}^N \setminus \{0\}$, define

$$U_b := U_{\frac{b}{|b|}} \ \text{ and } \ c_b := c_{\frac{b}{|b|}}. \tag{2.11}$$

It is clear that U_b and c_b are well defined.

Theorem 2.10 *Assume that assumptions* (F1)–(F4) *hold. Let $e \in \mathbb{S}^{N-1}$. Assume that (U_e, c_e) is the unique pulsating front of* (1.1). *Normalize U_e as* (2.10). *Then U_b and c_b are doubly continuously Fréchet differentiable in $b \in \mathbb{R}^N$ everywhere at $\mathbb{R}^N \setminus \{0\}$ under the topology $\|\cdot\|_{C^2(\mathbb{R}\times\mathbb{R}^N)\times\mathbb{R}}$.*

Remark 2.11 According to the proof of Theorem 2.10, U_b and c_b are also second-order continuously Fréchet differentiable in $b \in \mathbb{R}^N$ everywhere at $\mathbb{R}^N \setminus \{0\}$ under the topology $\|\cdot\|_{H^2_\beta(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}}$.

• *Curved fronts*

In this part we only consider the case $N = 2$. Namely, we establish the existence, uniqueness and stability of curved fronts with V-shape for Eq. (1.1) in $\mathbb{R}^2$ on the base of the asymptotic behavior results (Theorem 2.5) and Fréchet differentiability results (Theorem 2.10).

Let $N = 2$ and $z := (x, y) \in \mathbb{R}^2$. Let θ be an arbitrary angle, then it follows from Theorem 2.1 that there exists a unique pulsating front in the sense of Definition 1.1 with the propagation direction $(\cos\theta, \sin\theta)$, denoted by (U_θ, c_θ). Under the normalization (2.10), define

$$U^-_{\alpha\beta}(t,x,y) \overset{\text{def}}{=} \max\left\{U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y),\ U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y)\right\},$$

which is evidently a subsolution of (1.1). The following theorem shows the existence of curved fronts, which converge to pulsating fronts along its asymptotic lines under some conditions on angles α and β. The curved front is actually a transition front connecting 0 and 1 (see [4, 7, 28]), whose interface can be chosen as a V-shaped curve. For convenience, denote

$$\xi_\alpha := x\cos\alpha + y\sin\alpha - c_\alpha t \ \text{ and } \ \xi_\beta := x\cos\beta + y\sin\beta - c_\beta t.$$

Define

$$g(\theta) := \frac{c_\theta}{\sin\theta} \ \text{ for all } \theta \in (0, \pi).$$

Theorem 2.12 *Assume that assumptions* (F1)–(F4) *hold. Let α and β be two angles satisfying $0 < \alpha < \beta < \pi$, such that $g'(\alpha) < 0$, $g'(\beta) > 0$, and*

$$\frac{c_\alpha}{\sin\alpha} = \frac{c_\beta}{\sin\beta} =: c_{\alpha\beta} > \frac{c_\theta}{\sin\theta}, \quad \forall\theta \in (\alpha, \beta). \tag{2.12}$$

Then there exists an entire solution $V(t,x,y)$ of (1.1)*, which satisfies $0 < V < 1$,*

$$\lim_{R\to+\infty} \sup_{x^2+(y-c_{\alpha\beta}t)^2>R^2} \frac{\left|V(t,x,y) - U^-_{\alpha\beta}(t,x,y)\right|}{\min\left\{1, e^{-\nu_\star \min\{\xi_\alpha/\sin\alpha,\, \xi_\beta/\sin\beta\}}\right\}} = 0 \tag{2.13}$$

and

$$\partial_t V(t,x,y) > 0 \tag{2.14}$$

in $\mathbb{R}\times\mathbb{R}^2$, where $\nu_\star$ is a positive constant.

Remark 2.13 The condition of Theorem 2.12 is not empty. In fact, there are infinite pairs (α, β) satisfying the condition.

Proof of Remark 2.13 It follows from Theorem 2.10 that the function c_θ is continuously differentiable in $[0,\pi]$, which implies that $\max_{\theta\in[0,\pi]} c'_\theta$ is bounded. By virtue of Theorem 2.8, one gets

$$g'(\theta) = \frac{1}{\sin\theta}\left(c'_\theta - c_\theta\cot\theta\right) \le \frac{1}{\sin\theta}\left(\max_{\theta\in[0,\pi]} c'_\theta - \kappa\cot\theta\right)$$

for all $0 < \theta < \pi/2$. Thus there is a constant $\theta_1 \in (0, \pi/2)$ such that $g'(\theta) < 0$ for all $\theta \in (0, \theta_1)$. Similarly, there is a constant $\theta_2 \in (\pi/2, \pi)$ such that $g'(\theta) > 0$ for all $\theta \in (\theta_2, \pi)$. Since

$$g(\theta) = \frac{c_\theta}{\sin\theta} \geq \frac{\kappa}{\sin\theta} \to +\infty \ \text{ as } \theta \to 0 \text{ or } \pi$$

and

$$g(\theta) = \frac{c_\theta}{\sin\theta} \leq \frac{K}{\sin\theta} \leq \frac{K}{\min\{\sin\theta_1, \sin\theta_2\}} \ \text{ in } [\theta_1, \theta_2],$$

where κ and K are given in Theorem 2.8, it is clear that Remark 2.13 is valid. □

Remark 2.14 Based on below uniqueness result (Theorem 2.15), we can get a fact that

$$V(t, x, y) = V(t + L_2 k/c_{\alpha\beta}, x, y + L_2 k) \ \text{ in } \mathbb{R} \times \mathbb{R}^2, \ \forall k \in \mathbb{Z},$$

where L_2 given in the definition of $\mathbb{L}^N$ is the period of y. As Remark 1.4 of Guo et al. [26], we can show that the curved front $V(t, x, y)$ established in Theorem 2.12 is a transition front of Eq. (1.1) connecting two equilibria 0 and 1, see [4, 7, 28] for the definition of transition fronts. According to Remark 2.13, there exist α_1 and β_1 with $0 < \alpha_1 < \beta_1 < \pi$ such that for any $\alpha \in (0, \alpha_1)$, there exists $\beta \in (\beta_1, \pi)$ such that (2.12) is satisfied and there is a curved front $V(t, x, y)$ of (1.1) satisfying (2.13) and (2.14). This gives a sufficient condition to the existence of curved fronts in $\mathbb{R}^2$. That is, condition (2.12) holds when angle α close to 0 and angle β close to π. See also Corollary 1.5 of Guo et al. [26]. In addition, as mentioned by Guo et al. [26], one can rotate the coordinate such that the y-axis points to any direction. Though the periodicity can not be preserved by rotation, the same proofs of Theorem 2.12 can be applied to obtaining the existence of a curved front by using any two pulsating fronts whose propagation directions are closed to reversed with each other.

The following two theorems give the uniqueness and stability of the curved front $V(t, x, y)$ in Theorem 2.12 respectively.

Theorem 2.15 *Assume that assumptions* (F1)–(F4) *hold. Let* α, β, $V(t, x, y)$ *be given in Theorem 2.12. If there is an entire solution* $V_1(t, x, y)$ *of* (1.1) *satisfying* $0 \leq V_1 \leq 1$ *and*

$$\lim_{R\to+\infty} \sup_{x^2+(y-c_{\alpha\beta}t)^2>R^2} \left| V_1(t, x, y) - U^-_{\alpha\beta}(t, x, y) \right| = 0, \tag{2.15}$$

then $V_1(t, x, y) \equiv V(t, x, y)$ *in* $\mathbb{R}^3$.

Theorem 2.16 *Assume that assumptions* (F1)–(F4) *hold. Let* α, β, $V(t, x, y)$ *be given in Theorem 2.12. Assume that* $u_0 \in C(\mathbb{R}^2, [0, 1])$ *satisfies*

$$U^-_{\alpha\beta}(0, x, y) \leq u_0(x, y) \tag{2.16}$$

for all $(x, y) \in \mathbb{R}^2$, *and*

$$\lim_{R\to+\infty} \sup_{x^2+y^2>R^2} \frac{\left| u_0(x, y) - U^-_{\alpha\beta}(0, x, y) \right|}{\min\left\{1, e^{-\nu \min\{\xi_\alpha/\sin\alpha,\, \xi_\beta/\sin\beta\}}\right\}} = 0 \tag{2.17}$$

for some $\nu > 0$, *where* ξ_α *and* ξ_β *are evaluated at* $(0, x, y)$. *Then the solution* $u(t, x, y)$ *of Cauchy problem* (1.1) *for* $t \geq 0$ *with initial condition* $u(0, x, y) = u_0(x, y)$, *satisfies*

$$\lim_{t\to+\infty} \|u(t, \cdot, \cdot) - V(t, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^2)} = 0.$$

Theorems 2.12, 2.15 and 2.16 investigate the existence, uniqueness and stability of curved fronts of (1.1) in $\mathbb{R}^2$. In particular, Theorem 2.12 implies that condition (2.12) together with $g'(\alpha) < 0$ and $g'(\beta) < 0$ is sufficient to the existence of the curved front $V(t, x, y)$. The following theorem shows that condition (2.12) is necessary.

Theorem 2.17 *Assume that assumptions* (F1)–(F4) *hold. Suppose that there exist a constant* $c_{\alpha\beta} > 0$ *and two angles* α *and* β *with* $0 < \alpha < \beta < \pi$ *such that* (1.1) *admits an entire solution* $V(t, x, y)$ *satisfying* $0 < V < 1$ *and* (2.13) *for some positive constant* $\nu_\star$*, then the constant* $c_{\alpha\beta}$ *and the angles* α *and* β *satisfy* (2.12).

Finally, we give some comments with respect to the results in this part.

Remark 2.18 Here we would like to give some comments with respect to the results on curved fronts, and list some interesting issues which should be considered in the future.

(1) Clearly, the stability of curved fronts in Theorem 2.16 is only established for the case that $u_0(x, y) \geq U^-_{\alpha\beta}(0, x, y)$ for all $(x, y) \in \mathbb{R}^2$. A natural question is whether the solution of Eq. (1.1) with initial value u_0 satisfying $0 \leq u_0(x, y) \leq U^-_{\alpha\beta}(0, x, y)$ for all $(x, y) \in \mathbb{R}^2$ still converge to the curved front? For homogeneous equations, the answer is positive, and hence, we conjecture that the answer is also positive for periodic Eq. (1.1). But to confirm the conclusion for (1.1), it is needed to construct some new super- and subsolutions, which seems not easy.
(2) In this paper we only consider the existence of curved fronts in $\mathbb{R}^2$. As reported in Sect. 1, for homogeneous equation in $\mathbb{R}^3$, it has been found that there exist various curved fronts with nonplanar level sets. Therefore, it is valuable to investigate possible curved fronts of periodic Eq. (1.1) in $\mathbb{R}^3$. In addition, it was shown that there exist nonplanar traveling fronts in homogeneous equation with degenerate monostable nonlinearity. Therefore, it is also interesting to consider the existence of curved fronts of Eq. (1.1) with degenerate monostable nonlinearity. Besides, as done in [7, 26, 28, 45], curved fronts with varying interfaces should also be considered.
(3) As mentioned in Sect. 1, El Smaily [19] has considered the existence, uniqueness and qualitative properties of curved traveling fronts to the reaction–advection–diffusion problem (which is precisely a periodic shear flow)

$$\partial_t u = \Delta u + q(x)\partial_y u + f(u), \quad (x, y) \in \mathbb{R}^2, \tag{2.18}$$

where $f \in C^{1+\delta}$ satisfies

$$\exists p \in (0, 1) \text{ such that } f(u) = 0 \text{ for } u \in [0, p) \cup \{1\}, \; f(u) > 0 \text{ for } u \in [p, 1), \; f'(1) < 0.$$

Under the assumption $q(x) = q(-x)$ for all $x \in \mathbb{R}$, by constructing a pair of sub- and supersolutions which consist of the right and left moving fronts, El Smaily [19] established the existence of curved fronts of Eq. (2.18) satisfying conical conditions (see (1.5) in [19]). Furthermore, he showed the uniqueness and monotonicity of curved fronts by using the comparison principle and the sliding method. In our opinion, the main feature of El Smaily [19] is to consider the influence of spatially periodic advection term on the curved front. As a counterpart, this paper concentrates on the influence of spatially periodic reaction term on the curved front. Technically, the essential difference is due to that the reaction term f depends on spatial variables z, which brings a lot of different difficulties. In this paper, by using the continuous Fréchet differentiability of the pulsating fronts with respect to the propagation direction done in Sect. 4, we first construct a supersolution via pulsating front with varying propagation direction (different from that in El Smaily [19]) and then establish the existence of curved fronts of Eq. (1.1) in $\mathbb{R}^2$. Furthermore, we obtain the uniqueness and stability by using the sliding technique and investigating the Omega-limit set of the solution of Eq. (1.1) with initial values satisfying (2.16) and (2.17). Clearly, there is no results on the stability of curved fronts in [19]. Of course, it will be more difficult and challenging to consider curved fronts in general Eq. (1.6) with advection term.

3 Properties of Pulsating Front

In this section we introduce some versions of the maximum principle in unbounded domains and give some properties of pulsating front.

Lemma 3.1 *Assume that $g(s, z, u)$ is a function defined in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, and $g(s, z, \cdot)$ is globally Lipschitz-continuous uniformly for (s, z). Let $c_e \neq 0$ and $\Sigma_h^+ := (h, +\infty) \times \mathbb{R}^N$. Assume that g is nonincreasing with respect to u in $\overline{\Sigma_h^+} \times (-\infty, \varrho]$ for some $\varrho \in \mathbb{R}$. Assume that $\phi^1(s, z)$ and $\phi^2(s, z)$ are two functions of $C^2(\Sigma_h^+)$, and $\|\phi^1\|_{C^0(\Sigma_h^+)}, \ \|\phi^2\|_{C^0(\Sigma_h^+)} < +\infty$. Assume that $g, \ \phi^1, \ \phi^2$ are periodic with respect to $z \in \mathbb{R}^N$, and $\mathbb{L}^N$ is the cell of periodicity independent of $s, \ u$. Let*

$$\begin{cases} \mathcal{N}_e\phi^1(s, z) + g\big(s, z, \phi^1(s, z)\big) \geq 0 \ \text{ in } \Sigma_h^+ \\ \mathcal{N}_e\phi^2(s, z) + g\big(s, z, \phi^2(s, z)\big) \leq 0 \ \text{ in } \Sigma_h^+ \\ \lim\limits_{s_0 \to +\infty} \sup\limits_{s \geq s_0, z \in \mathbb{R}^N} \left(\phi^1 - \phi^2\right)(s, z) \leq 0 \end{cases}$$

where

$$\mathcal{N}_e\phi := c_e\partial_s\phi + \partial_{ss}\phi + 2\nabla_z\partial_s\phi \cdot e + \Delta_z\phi.$$

Assume that $\phi^1 \leq \varrho$ in $\overline{\Sigma_h^+}$ and $\phi^1(h, z) \leq \phi^2(h, z)$ for all $z \in \mathbb{R}^N$. Then $\phi^1 \leq \phi^2$ in $\overline{\Sigma_h^+}$.

Lemma 3.2 *Assume that $g(s, z, u)$ is a function defined in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, and $g(s, z, \cdot)$ is globally Lipschitz-continuous uniformly for (s, z). Let $c_e \neq 0$ and $\Sigma_h^- := (-\infty, h) \times \mathbb{R}^N$. Assume that g is nonincreasing with respect to u in $\overline{\Sigma_h^-} \times (-\infty, \varrho]$ for some $\varrho \in \mathbb{R}$. Assume that $\phi^1(s, z)$ and $\phi^2(s, z)$ are two functions of $C^2(\Sigma_h^-)$, and $\|\phi^1\|_{C^0(\Sigma_h^-)}, \ \|\phi^2\|_{C^0(\Sigma_h^-)} < +\infty$. Assume that $g, \ \phi^1, \ \phi^2$ are periodic with respect to $z \in \mathbb{R}^N$, and $\mathbb{L}^N$ is the cell of periodicity independent of $s, \ u$. Let*

$$\begin{cases} \mathcal{N}_e\phi^1(s, z) + g\big(s, z, \phi^1(s, z)\big) \geq 0 \ \text{ in } \Sigma_h^-, \\ \mathcal{N}_e\phi^2(s, z) + g\big(s, z, \phi^2(s, z)\big) \leq 0 \ \text{ in } \Sigma_h^-, \\ \lim\limits_{s_0 \to -\infty} \sup\limits_{s \leq s_0, z \in \mathbb{R}^N} \left(\phi^1 - \phi^2\right)(s, z) \leq 0. \end{cases}$$

Assume that $\phi^1 \leq \varrho$ in $\overline{\Sigma_h^-}$ and $\phi^1(h, z) \leq \phi^2(h, z)$ for all $z \in \mathbb{R}^N$. Then $\phi^1 \leq \phi^2$ in $\overline{\Sigma_h^-}$.

Lemmas 3.1 and 3.2 can be proved similarly to Lemma 3.2 of [3]. In the following we firstly consider the asymptotic behavior of pulsating front tending to the equilibrium 0, that is Theorem 2.5. Then we give some estimates of the profile U_e and its derivatives.

Proof of Theorem 2.5 Fix an arbitrary propagation direction $e \in \mathbb{S}^{N-1}$.

Step 1: *we prove* (2.2).

It follows from Definition 1.1 that there exists a positive constant q_1 such that

$$U_e(s,z) \le p \ \text{ for all } (s,z) \in [q_1 - 1, +\infty) \times \mathbb{R}^N,$$

where p is defined in (1.3). Denote $u(t,z) := U_e(z \cdot e - c_e t, z)$ and define

$$\Omega := \{(t,z) : \ z \cdot e - c_e t > q_1\} \ \text{ and } \ \Omega_1 := \{(t,z) : \ z \cdot e - c_e t > q_1 - 1\}.$$

Clearly, one has $0 \le u \le p$ in Ω_1. By (1.3) and (2.1), $u(t,z)$ satisfies

$$\partial_t u - \Delta_z u = 0 \ \text{ in } \Omega_1.$$

It follows from Theorem 9 in Section 3 of Chapter 2 of [21] and Corollary 7.42 of [38] that, for any $k, l \in \mathbb{N}$ there exists a constant $\widetilde{C}_{kl} > 0$ such that

$$\left| D_z^k D_t^l u(t,z) \right| \le \widetilde{C}_{kl} u(t,z), \ \ \forall (t,z) \in \Omega, \tag{3.1}$$

which implies that there is a constant $\widehat{C}_{kl} > 1$ such that

$$\left| D_z^k D_s^l U_e \right| < \left(\widehat{C}_{kl} - 1 \right) U_e, \ \ \forall (s,z) \in [q_1, +\infty) \times \mathbb{R}^N. \tag{3.2}$$

For convenience, denote $\widetilde{U}_e := D_z^k D_s^l U_e + \widehat{C}_{kl} U_e$. Then $\widetilde{U}_e$ solves an equation of the type

$$c_e \partial_s \widetilde{U}_e + \partial_{ss} \widetilde{U}_e + 2\nabla_z \partial_s \widetilde{U}_e \cdot e + \Delta_z \widetilde{U}_e = 0 \ \text{ in } (q_1, +\infty) \times \mathbb{R}^N.$$

Since $\widetilde{U}_e = \left(D_z^k D_s^l U_e + \left(\widehat{C}_{kl} - 1 \right) U_e \right) + U_e$, it follows from (3.1), (3.2) and Theorem 2.2 that

$$\liminf_{s \to +\infty} \min_{z \in \mathbb{R}^N} \frac{\widetilde{U}_e}{e^{-c_e s}} \ge \liminf_{s \to +\infty} \min_{z \in \mathbb{R}^N} \frac{U_e}{e^{-c_e s}} = C_1 > 0$$

and

$$\limsup_{s \to +\infty} \max_{z \in \mathbb{R}^N} \frac{\widetilde{U}_e}{e^{-c_e s}} \le \limsup_{s \to +\infty} \max_{z \in \mathbb{R}^N} \frac{\left(2\widehat{C}_{kl} - 1 \right) U_e}{e^{-c_e s}} = \left(2\widehat{C}_{kl} - 1 \right) C_1 < +\infty.$$

With similar arguments as those in Theorem 2.2 of [8], by replacing ϕ with $\widetilde{U}_e$, one can prove that there exists a positive constant $\overline{C}_{kl}$ dependent on k and l such that

$$\widetilde{U}_e(s,z) \sim \overline{C}_{kl} e^{-c_e s} \ \text{ as } s \to +\infty$$

uniformly in $z \in \mathbb{R}^N$, and hence (2.2) holds.

Step 2: *we prove* (2.3).

According to Theorem 2.2, there exist two nonzero constants C_1 and C_2 such that

$$U_e(s,z) \sim C_1 e^{-c_e s} \ \text{ and } \ \partial_s U_e(s,z) \sim C_2 e^{-c_e s} \ \text{ as } s \to +\infty \tag{3.3}$$

uniformly in $z \in \mathbb{R}^N$. Thus in order to prove (2.3), one needs only to prove $\frac{C_2}{C_1} = -c_e$.

For any $n \in \mathbb{N}$, we define

$$w_n(\xi) := \frac{U_e(\xi + n, 0)}{U_e(n, 0)}, \ \ w(\xi) := e^{-c_e \xi}, \ \ w^*(\xi) := \frac{C_2}{C_1} e^{-c_e \xi}$$

where $\xi \in [-1,1]$. It is easy to verify that the sequence of functions $\{w_n'\}_{n \in \mathbb{N}}$ is convergent to the function w^* uniformly in $\xi \in [-1,1]$ as $n \to \infty$. Apparently, one has by (3.3) that the sequence of functions $\{w_n\}_{n \in \mathbb{N}}$ is convergent to w in $\xi \in [-1,1]$ as $n \to \infty$. Thus

$$w'(\xi) = \frac{d}{d\xi} \left(\lim_{n \to \infty} w_n \right) = \lim_{n \to \infty} w_n'(\xi) = w^*(\xi), \ \ \forall \xi \in [-1,1].$$

It is clear that $\frac{C_2}{C_1} = -c_e$.

Step 3: *we prove* (2.4) *and* (2.5).

From (2.2) and (3.3), one knows

$$U_e(s,z) \sim C_1 e^{-c_e s} \text{ and } \partial_{z_j} U_e(s,z) \sim C_{3j} e^{-c_e s} \text{ as } s \to +\infty$$

uniformly in $z \in \mathbb{R}^N$, where $j \in \{1, 2, \cdots, N\}$, C_{3j} is a constant, and C_1 is nonzero. Fix arbitrary $j \in \{1, 2, \cdots, N\}$, denote

$$\tilde{w}_n(z_j) := \frac{U_e(n, 0, \cdots, 0, z_j, 0, \cdots, 0)}{U_e(n, 0, \cdots, 0)}, \quad \tilde{w}(z_j) \equiv 1, \quad \tilde{w}^*(z_j) \equiv \frac{C_{3j}}{C_1}$$

for all $n \in \mathbb{N}$, where $z_j \in [-1, 1]$. With similar arguments as those in *Step 2*, one gets that

$$\frac{d}{dz_j}\left(\lim_{n\to\infty} \tilde{w}_n\right) = \lim_{n\to\infty} \tilde{w}_n' \quad \text{in } [-1, 1],$$

which implies $\tilde{w}' = \tilde{w}^*$. Consequently $C_{3j}/C_1 = 0$ for any $j \in \{1, 2, \cdots, N\}$. Therefore

$$\lim_{s\to+\infty} \frac{|\nabla_z U_e|}{U_e} = 0 \text{ uniformly in } z \in \mathbb{R}^N.$$

From (2.2) and (3.3), one knows

$$\partial_s U_e(s,z) \sim C_2 e^{-c_e s}, \ \partial_s(\partial_s U_e)(s,z) \sim C_4 e^{-c_e s}, \ \partial_{z_j}(\partial_s U_e)(s,z) \sim C_{5j} e^{-c_e s}$$

uniformly in $z \in \mathbb{R}^N$ as $s \to +\infty$, where $j \in \{1, 2, \cdots, N\}$, C_4 and C_{5j} are constants, and C_2 is nonzero. Then with similar arguments as above, one obtains that $C_4/C_2 = -c_e$ and $C_{5j}/C_2 = 0$ for all $1 \le j \le N$. It follows from (2.3) that

$$\lim_{s\to+\infty} \frac{\partial_{ss} U_e}{U_e} = c_e^2 \text{ and } \lim_{s\to+\infty} \frac{|\nabla_z \partial_s U_e|}{U_e} = 0$$

uniformly in $z \in \mathbb{R}^N$.

Step 4: *we prove* (2.6).

It follows from (2.1) that

$$\lim_{s\to+\infty} \frac{\Delta_z U_e}{U_e} = -\lim_{s\to+\infty} \frac{c_e \partial_s U_e + \partial_{ss} U_e + 2\nabla_z \partial_s U_e \cdot e}{U_e} = 0.$$

Hence (2.6) holds. The proof of Theorem 2.5 is thereby complete. □

Below we establish three propositions which provide some estimates of pulsating fronts U_e. Here we emphasize that these estimates are independent of the propagation direction $e \in \mathbb{S}^{N-1}$.

Proposition 3.3 *Assume that assumptions* (F1)–(F4) *hold, and that* (U_e, c_e) *is a pulsating front of* (1.1), *where* $e \in \mathbb{S}^{N-1}$. *Normalize* U_e *as* $\min_{z\in\mathbb{R}^N} U_e(0, z) = (1 + p)/2$, *where* p *is defined in* (1.3). *Then there exist two positive constants* K_2 *and* κ_2, *both independent of* $e \in \mathbb{S}^{N-1}$, *such that*

$$0 < U_e(s,z) \le K_2 e^{-\frac{3\kappa}{4}s} \text{ for all } (s,z) \in [0, +\infty) \times \mathbb{R}^N,$$
$$0 < 1 - U_e(s,z) \le K_2 e^{\kappa_2 s} \text{ for all } (s,z) \in (-\infty, 0] \times \mathbb{R}^N,$$

where κ *is defined in Theorem* 2.3.

Proof By virtue of the continuity of U_e in e with respect to the topology $\|\cdot\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}$ (from Theorem 2.3) and the monotonicity of $U_e(s,z)$ in s (from Theorem 2.1), there exists a constant q_2 such that

$$U_e(s,z) \le p \ \text{ for all } (s,z) \in [q_2,+\infty)\times\mathbb{R}^N \text{ and } e \in \mathbb{S}^{N-1},$$

where p is defined in (1.3). Thus $f(z,U_e) \equiv 0$ in $[q_2,+\infty)\times\mathbb{R}^N$ for all $e \in \mathbb{S}^{N-1}$. It then follows from (2.1) that

$$\mathcal{N}_e U_e = 0 \ \text{ in } [q_2,+\infty)\times\mathbb{R}^N$$

for all $e \in \mathbb{S}^{N-1}$, where the operator $\mathcal{N}_e$ is defined in Lemma 3.1. Denote $K_{2a} := e^{\frac{3\kappa}{4}q_2}$, then $U_e(q_2,z) \le 1 = K_{2a}e^{-\frac{3\kappa}{4}q_2}$ for all $z \in \mathbb{R}^N$ and $e \in \mathbb{S}^{N-1}$. By calculations, it holds from the conclusion (ii) of Theorem 2.3 that

$$\mathcal{N}_e\left(K_{2a}e^{-\frac{3\kappa}{4}s}\right) = K_{2a}e^{-\frac{3\kappa}{4}s}\left(-\frac{3\kappa}{4}c_e + \frac{9\kappa^2}{16}\right) \le -K_{2a}\frac{3\kappa^2}{16}e^{-\frac{3\kappa}{4}s} < 0$$

for all $e \in \mathbb{S}^{N-1}$. Furthermore, the asymptotic behavior of U_e (from Theorem 2.2, together with $\kappa \le c_e$) yields

$$\lim_{s_0\to+\infty}\ \sup_{\{s\ge s_0,\, z\in\mathbb{R}^N\}}\left(U_e - K_{2a}e^{-\frac{3\kappa}{4}s}\right) \le 0, \quad \forall e \in \mathbb{S}^{N-1}.$$

Then for any $e \in \mathbb{S}^{N-1}$, applying Lemma 3.1 to $g = 0$, $\varrho = p$, $h = q_2$, $\phi^1 = U_e$ and $\phi^2 = K_{2a}e^{-\frac{3\kappa}{4}s}$, we obtain

$$U_e(s,z) \le K_{2a}e^{-\frac{3\kappa}{4}s} \ \text{ in } [q_2,+\infty)\times\mathbb{R}^N.$$

Apparently, there exists a positive constant κ_2 satisfying

$$K\kappa_2 + \kappa_2^2 - \frac{\kappa_1}{2} \le 0, \tag{3.4}$$

where $K > 0$ is defined in Theorem 2.3 and $\kappa_1 > 0$ is defined in (1.4). Set $V_e := 1 - U_e$ in $\mathbb{R}\times\mathbb{R}^N$. It follows from (2.1) that

$$\mathcal{N}_e V_e + \tilde{f}(z,V_e) = 0 \ \text{ in } \mathbb{R}\times\mathbb{R}^N$$

for all $e \in \mathbb{S}^{N-1}$, where $\tilde{f}(z,u) := -f(z,1-u)$ in $\mathbb{R}^N\times\mathbb{R}$. It follows from (1.5) that

$$\tilde{f}_u(z,u) = f_u(z,1-u) \le -\frac{\kappa_1}{2}, \quad \forall (z,u) \in \mathbb{R}^N\times[0,\gamma_\star], \tag{3.5}$$

where $\gamma_\star$ is given in (1.5). By virtue of the continuity of U_e in e with respect to the topology $\|\cdot\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}$ (from Theorem 2.3) and the monotonicity of $U_e(s,z)$ in s (from Theorem 2.1), there exists a constant q_3 such that $V_e(s,z) = 1 - U_e(s,z) \le \gamma_\star$ for all $(s,z) \in (-\infty,q_3]\times\mathbb{R}^N$ and $e \in \mathbb{S}^{N-1}$. Set $K_{2b} := \gamma_\star e^{-\kappa_2 q_3}$ and then

$$V_e(q_3,z) \le \gamma_\star = K_{2b}e^{\kappa_2 q_3}, \quad \forall z \in \mathbb{R}^N.$$

Furthermore $K_{2b}e^{\kappa_2 s} \le \gamma_\star$ for all $(s,z) \in (-\infty,q_3]\times\mathbb{R}^N$. Thus, by calculations, one gets from (3.4), (3.5) and Theorem 2.3 that

$$\begin{aligned}\mathcal{N}_e\left(K_{2b}e^{\kappa_2 s}\right) + \tilde{f}\left(z,K_{2b}e^{\kappa_2 s}\right) &= \mathcal{N}_e\left(K_{2b}e^{\kappa_2 s}\right) + \tilde{f}\left(z,K_{2b}e^{\kappa_2 s}\right) - \tilde{f}(z,0)\\ &= \left(c_e\kappa_2 + \kappa_2^2 + \tilde{f}_u\left(z,\theta K_{2b}e^{\kappa_2 s}\right)\right)K_{2b}e^{\kappa_2 s}\end{aligned}$$

$$\leq \left(K\kappa_2 + \kappa_2^2 - \frac{\kappa_1}{2}\right) K_{2b} e^{\kappa_2 s}$$
$$\leq 0$$

in $(-\infty, q_3) \times \mathbb{R}^N$ for all $e \in \mathbb{S}^{N-1}$, where $\theta \in (0, 1)$. Lastly, using Theorem 2.2 yields

$$\lim_{s_0 \to -\infty} \sup_{s \leq s_0, z \in \mathbb{R}^N} \left(V_e - K_{2b} e^{\kappa_2 s}\right) = 0, \quad \forall e \in \mathbb{S}^{N-1}.$$

Therefore, for any $e \in \mathbb{S}^{N-1}$, applying Lemma 3.2 to $g = \tilde{f}$, $\varrho = \gamma_1$, $h = q_3$, $\phi^1 = V_e$ and $\phi^2 = K_{2b} e^{\kappa_2 s}$, we get

$$V_e(s, z) \leq K_{2b} e^{\kappa_2 s} \quad \text{in } (-\infty, q_3] \times \mathbb{R}^N.$$

Now set $K_2 := \max\{K_{2a}, K_{2b}/\gamma_\star\}$. The proof of Proposition 3.3 is thereby complete. □

Remark 3.4 By virtue of Theorem 2.8 which will be proved in Section 4, Proposition 3.3 also holds with another normalization as (2.10).

Corollary 3.5 *For any $e_1, e_2 \in \mathbb{S}^{N-1}$, there holds that $U_{e_1} - U_{e_2} \in L^2_\rho(\mathbb{R} \times \mathbb{L}^N)$.*

Proposition 3.6 *Assume that assumptions* (F1)–(F4) *hold, and that (U_e, c_e) is a pulsating front of* (1.1). *Normalize U_e as $\min_{z \in \mathbb{R}^N} U_e(0, z) = (1 + p)/2$, where p is defined in* (1.3). *Then there exists a positive constant K_3 independent of e, such that*

$$|DU_e(s, z)|,\ |D^2 U_e(s, z)|,\ |D^3 U_e(s, z)| \leq K_3 e^{-\frac{3\kappa}{4} s} \quad \text{in } [0, +\infty) \times \mathbb{R}^N, \tag{3.6}$$
$$|DU_e(s, z)|,\ |D^2 U_e(s, z)|,\ |D^3 U_e(s, z)| \leq K_3 e^{\kappa_2 s} \quad \text{in } (-\infty, 0] \times \mathbb{R}^N, \tag{3.7}$$

where κ is defined in Theorem 2.3, κ_2 is given in Proposition 3.3; D, D^2 and D^3 denote any first-order, second-order and third-order derivative with respect to $(s, z) \in \mathbb{R} \times \mathbb{R}^N$ respectively.

Proof *Step 1*: *we prove* (3.6).

For any $e \in \mathbb{S}^{N-1}$, denote $u(t, z; e) := U_e(z \cdot e - c_e t, z)$ for any $(t, z) \in \mathbb{R} \times \mathbb{R}^N$. By Definition 1.1, the function $u(t, z; e)$ is an entire (classical) solution of Eq. (1.1) for all $e \in \mathbb{S}^{N-1}$. It follows from [3] that there exists a constant $M > 0$ such that

$$\|u(\cdot, \cdot; e)\|_{C^2(\mathbb{R} \times \mathbb{R}^N)} \leq M, \quad \forall e \in \mathbb{S}^{N-1}. \tag{3.8}$$

It is clear that $u(t, z; e)$ solves a linear parabolic equation of the type

$$\partial_t u - \Delta_z u - \tilde{f}_1(t, z; e) u = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \tag{3.9}$$

where $\tilde{f}_1(t, z; e) := f(z, u(t, z; e))/u(t, z; e)$ in $\mathbb{R} \times \mathbb{R}^N$. We claim that

$$\left\| \tilde{f}_1(\cdot, \cdot; e) \right\|_{C^1(\mathbb{R} \times \mathbb{R}^N)} \leq p^{-2}(N + 1)(2M + 1) \|f\|_{C^1(\mathbb{R}^{N+1})}, \quad \forall e \in \mathbb{S}^{N-1}. \tag{3.10}$$

Since $f(z, 0) = 0$ in $\mathbb{R}^N$ from (1.3), one has from (1.2) that

$$\left\| \tilde{f}_1(\cdot, \cdot; e) \right\|_{C^0(\mathbb{R} \times \mathbb{R}^N)} \leq \|f\|_{C^1(\mathbb{R}^{N+1})}. \tag{3.11}$$

Since $\tilde{f}_1(t, z; e) \equiv 0$ for $(t, z) \in \mathbb{R} \times \mathbb{R}^N$ with $u(t, z; e) \leq p$, and

$$\partial_{z_i} \tilde{f}_1(t, z; e) = \frac{1}{u(t, z; e)} \left(f_{z_i}(z, u(t, z; e)) + f_u(z, u(t, z; e)) \partial_{z_i} u(t, z; e) \right) - \frac{f(z, u(t, z; e))}{u^2(t, z; e)} \partial_{z_i} u(t, z; e),$$

one gets

$$\left\| \partial_{z_i} \tilde{f}_1(\cdot, \cdot; e) \right\|_{C^0(\mathbb{R} \times \mathbb{R}^N)} \le p^{-2} \|f\|_{C^1(\mathbb{R}^{N+1})} \left(1 + 2 \|u(\cdot, \cdot; e)\|_{C^1(\mathbb{R} \times \mathbb{R}^N)}\right) \tag{3.12}$$

where $i \in \{1, 2, \cdots, N\}$. Similarly, one also has

$$\left\| \partial_t \tilde{f}_1(\cdot, \cdot; e) \right\|_{C^0(\mathbb{R} \times \mathbb{R}^N)} \le 2p^{-2} \|f\|_{C^1(\mathbb{R}^{N+1})} \|u(\cdot, \cdot; e)\|_{C^1(\mathbb{R} \times \mathbb{R}^N)} . \tag{3.13}$$

Following from (3.8) and (3.11)-(3.13), we conclude that (3.10) holds.

Using (3.10) and applying the Schauder interior estimates (see also Theorem 4.9 of [38]) to (3.9), we get

$$\|u(\cdot, \cdot; e)\|_{C^1(Q(\tilde{t}, \tilde{z}; 1))} \le C^* \|u(\cdot, \cdot; e)\|_{C^0(Q(\tilde{t}, \tilde{z}; 2))} , \quad \forall (\tilde{t}, \tilde{z}) \in \mathbb{R} \times \mathbb{R}^N, \tag{3.14}$$

where the constant $C^* > 0$ is independent of $(\tilde{t}, \tilde{z}) \in \mathbb{R} \times \mathbb{R}^N$ and $e \in \mathbb{S}^{N-1}$, and $Q(\tilde{t}, \tilde{z}; r) := \{(t, z) : \tilde{t} - r^2 \le t \le \tilde{t}, |z - \tilde{z}| \le r\}$. Since $U_e(s, z) = u(\frac{z \cdot e - s}{c_e}, z; e)$ for all $e \in \mathbb{S}^{N-1}$, together with $\kappa \le c_e \le K$ from Theorem 2.3, by virtue of (3.14) and Proposition 3.3, we get (3.6) for DU_e. By differentiating Eq. (1.1), (3.6) follows from the similar arguments as above.

Step 2. Now let us consider the function $v(t, z; e) := 1 - u(t, z; e)$ in $\mathbb{R} \times \mathbb{R}^N$, which solves a linear parabolic equation of the type

$$(\partial_t - \Delta_z)v + f_*(t, z; e)v = 0 \text{ in } \mathbb{R} \times \mathbb{R}^N,$$

where $f_*(t, z; e) := f(z, u(t, z; e))/(1 - u(t, z; e))$ in $\mathbb{R} \times \mathbb{R}^N$. In view of $f(z, 1) = 0$ in $\mathbb{R}^N$, applying similar arguments as those in *Step 1* to the function $v(t, z; e)$, we can get (3.7). □

Remark 3.7 Theorem 2.7 directly follows from Propositions 3.3 and 3.6. By virtue of Theorem 2.8 which will be proved in Section 4, Proposition 3.6 also holds with another normalization as (2.10).

Corollary 3.8 *Assume that assumptions* (F1)–(F4) *hold, and that* (U_e, c_e) *is a pulsating front of* (1.1). *Then* $DU_e(s, z)$, $D^2U_e(s, z)$, $D^3U_e(s, z) \in L^2_\rho(\mathbb{R} \times \mathbb{L}^N)$, *where* D, D^2 *and* D^3 *denote any first-order, second-order and third-order derivative with respect to* $(s, z) \in \mathbb{R} \times \mathbb{R}^N$ *respectively.*

Proposition 3.9 ([26], Lemma 2.5) *Normalize the profile* U_e *as* (2.10). *Then for any* $q > 0$, *there are two small positive constants* γ *and* r *independent of* $e \in \mathbb{S}^{N-1}$, *such that*

$$\gamma \le U_e(s, z) \le 1 - \gamma \ \ \textit{and} \ \ -\partial_s U_e(s, z) \ge r, \quad \forall (s, z) \in [-q, q] \times \mathbb{R}^N. \tag{3.15}$$

Proof From the continuity of U_e in e with respect to the topology $\|\cdot\|_{L^\infty(\mathbb{R} \times \mathbb{R}^N)}$ (see Theorem 2.8, which will be proved in Sect. 4) and the monotonicity of $U_e(s, z)$ in s (from Theorem 2.1), one has that the first inequality of (3.15) holds. The second inequality of (3.15) can be proved by similar arguments as those in the proof of Lemma 2.5 of [26]. This completes the proof. □

4 Fréchet Differentiability of Pulsating Fronts

This section is devoted to proving the Fréchet differentiability of pulsating fronts with respect to the propagation direction. At first, we prove the continuity of pulsating fronts with respect to the propagation direction under topology $\| \cdot \|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}$, that is Theorem 2.8. Then we establish several lemmas and prove the Fréchet differentiability of pulsating fronts with respect to the propagation direction, that is Theorem 2.10. Lastly, some estimates of Fréchet derivatives of pulsating fronts with respect to the propagation direction are given. Here we would like to roughly state the strategy of the proof of the Fréchet differentiability of pulsating fronts. Based on the results of [60] (see Lemmas 4.1 and 4.2), we can obtain a priori estimates and the spectral structure of the linearized operator $\mathcal{H}_e$ of Eq. (2.1) at the pulsating front U_e in the weighted Sobolev space $H^1_\rho(\mathbb{R}\times\mathbb{L}^N)$. Consequently, by studying two nonlinear operators $\mathcal{K}_e$ and $\mathcal{G}_e$, we can introduce a key linear operator $\mathcal{Q}_e$ and show that $\mathcal{Q}_e$ is invertible and the inverse operator $\mathcal{Q}_e^{-1}$ is bounded, see Lemma 4.6. This step is inspired by [15] and [25]. In order to fall into the scheme of the weighted Sobolev space, the continuity of pulsating fronts with respect to the propagation direction under the topology $\| \cdot \|_{H^2_\rho(\mathbb{R}\times\mathbb{L}^N)}$ is given by using Theorem 2.8, see Lemma 4.7. Finally after studying the continuity of the inverse operator $\mathcal{Q}_e^{-1}$ with respect to $e \in \mathbb{S}^{N-1}$ (see Lemma 4.8), we are ready to prove the Fréchet differentiability of pulsating fronts.

4.1 Continuity

In this subsection, we modify the proof of Theorem 2.3 to get Theorem 2.8. For simplification, we only give the modified part in the proof of Theorem 2.3, thus one needs to read Theorem 2.3 (actually read Theorems 2.4 and 2.5 of [2]) for a start. For convenience, we write here the stated normalization of U_e, that is (2.10), which is

$$\int_{\mathbb{R}^+\times\mathbb{L}^N} U_e^2(s,z)\rho(s)\,dsdz = 1, \quad \forall e \in \mathbb{S}^{N-1},$$

where the function $\rho(s) = 1 + e^{2\varepsilon s}$ is given in (2.7) for $0 < \varepsilon \ll \kappa$.

Proof of Theorem 2.8 Since conclusions (i) and (ii) do not rely on the normalization of U_e, we get them from Theorem 2.3. Below we prove conclusion (iii), that is to prove

$$\left\| U_{e_k} - U_e \right\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)} \to 0 \text{ as } k \to \infty, \tag{4.1}$$

if $e_k,\ e \in \mathbb{S}^{N-1}$ satisfy that $|e_k - e| \to 0$ as $k \to \infty$, and U_e and U_{e_k} are normalized as (2.10).

Let $S,\ s_k \in \mathbb{R}$ satisfy

$$\min_{z\in\mathbb{R}^N} U_e(S,z) = \frac{1+p}{2} \text{ and } \min_{z\in\mathbb{R}^N} U_{e_k}(s_k,z) = \frac{1+p}{2}, \quad \forall k \in \mathbb{N}, \tag{4.2}$$

where p is defined in (1.3). Since $\partial_s U_e < 0$ for all $e \in \mathbb{S}^{N-1}$, S and s_k are unique and well-defined. To prove (4.1), we need only to prove that the sequence $\{s_k\}_{k\in\mathbb{N}}$ is uniformly bounded by the proof of Theorem 2.3. We prove it by contradiction, and if not, two cases may occur.

Case 1: *up to extraction of a subsequence*, $s_k \to -\infty$ as $k \to \infty$.

By virtue of (4.2), it follows from Proposition 3.3 that

$$0 < U_{e_k}(s_k+s,z) \le K_2 e^{-\frac{3\kappa}{4}s}, \quad \forall (s,z) \in \mathbb{R}^+\times\mathbb{L}^N,\ k \in \mathbb{N},$$

where K_2 and κ are given in Proposition 3.3. Note $\varepsilon \ll \kappa$. By virtue of the normalization (2.10), one gets a contradiction

$$\begin{aligned}
1 &= \int_{\mathbb{R}^+\times\mathbb{L}^N} U_{e_k}^2(s,z)\rho(s)\,dsdz \\
&= \int_{[-s_k,+\infty)\times\mathbb{L}^N} U_{e_k}^2(s_k+s,z)\rho(s_k+s)\,dsdz \\
&\le \int_{[-s_k,+\infty)\times\mathbb{L}^N} K_2^2 e^{-\frac{3\kappa}{2}s}\left(1+e^{2\varepsilon(s_k+s)}\right)\,dsdz \\
&\to 0 \quad \text{as } k\to\infty.
\end{aligned}$$

Case 2: *up to extraction of a subsequence,* $s_k \to +\infty$ as $k \to \infty$.
It follows from Definition 1.1 that

$$\lim_{s\to-\infty} U_e(S+s,z) = 1 \;\text{ uniformly in } z\in\mathbb{L}^N.$$

Consequently, we can choose a large integer K_e such that

$$\int_{[-K_e,K_e]\times\mathbb{L}^N} U_e^2(S+s,z)\,dsdz \gg 1. \tag{4.3}$$

In addition, it follows from the proof of Theorem 2.3 that the sequence $U_{e_k}(s_k+\cdot,\cdot)$ converges to $U_e(S+\cdot,\cdot)$ in $L^\infty(\mathbb{R}\times\mathbb{L}^N)$. Thus together with (4.3), we get

$$\int_{[-K_e,K_e]\times\mathbb{L}^N} U_{e_k}^2(s_k+s,z)\,dsdz \gg 1 \;\text{ for all sufficiently large integer } k. \tag{4.4}$$

Since $s_k \to +\infty$ as $k\to\infty$, one has $s_k > K_e$ for all sufficiently large integer k. Therefore, one reaches a contradiction from (4.4), that is, for all sufficiently large integer k

$$\begin{aligned}
1 &= \int_{\mathbb{R}^+\times\mathbb{L}^N} U_{e_k}^2(s,z)\rho(s)\,dsdz \\
&= \int_{[-s_k,+\infty)\times\mathbb{L}^N} U_{e_k}^2(s_k+s,z)\rho(s_k+s)\,dsdz \\
&\ge \int_{[-K_e,K_e]\times\mathbb{L}^N} U_{e_k}^2(s_k+s,z)\rho(s_k+s)\,dsdz \\
&\gg 1.
\end{aligned}$$

Therefore $\{s_k\}_{k\in\mathbb{N}}$ is uniformly bounded, and the proof of Theorem 2.8 is complete. □

4.2 Fréchet Differentiability

In this subsection, we prove Theorem 2.10. Now define two linear operators $\mathcal{M}_e$ and $\mathcal{H}_e$

$$\begin{aligned}
\mathcal{M}_e(v) &:= c_e\partial_s v + \partial_{ss}v + 2\nabla_z\partial_s v\cdot e + \Delta_z v - \beta v \\
&= (e\partial_s+\nabla_z)^T(e\partial_s+\nabla_z)v + c_e\partial_s v - \beta v, \quad \forall e\in\mathbb{S}^{N-1}, \\
\mathcal{H}_e(v) &:= c_e\partial_s v + \partial_{ss}v + 2\nabla_z\partial_s v\cdot e + \Delta_z v + f_u(z,U_e)v \\
&= (e\partial_s+\nabla_z)^T(e\partial_s+\nabla_z)v + c_e\partial_s v + f_u(z,U_e)v, \quad \forall e\in\mathbb{S}^{N-1},
\end{aligned}$$

where $\beta > 0$ is a given constant. In above definitions, ∇_z denotes the gradient operator with respect to $z \in \mathbb{R}^N$. For any $e \in \mathbb{S}^{N-1}$, the domains of two linear operators $\mathcal{M}_e$ and $\mathcal{H}_e$ are defined by

$$\mathcal{D}(\mathcal{M}_e) = \mathcal{D}(\mathcal{H}_e) \stackrel{\text{def}}{=} \left\{ u(s,z) \in H^1_\rho(\mathbb{R} \times \mathbb{L}^N) : \ (e\partial_s + \nabla_z)^T (e\partial_s + \nabla_z) u \in L^2_\rho \right\}$$

endowed with the norm

$$\|u\|_{\mathcal{D}(\mathcal{M}_e)} := \|u\|_{H^1_\rho(\mathbb{R}\times\mathbb{L}^N)} + \left\| (e\partial_s + \nabla_z)^T (e\partial_s + \nabla_z) u \right\|_{L^2_\rho(\mathbb{R}\times\mathbb{L}^N)},$$

where the space $L^2_\rho(\mathbb{R} \times \mathbb{L}^N)$ and $H^1_\rho(\mathbb{R} \times \mathbb{L}^N)$ are given by (2.8) and (2.9), respectively. The following two lemmas coming from [60], are related to the linear operator $\mathcal{M}_e$.

Lemma 4.1 ([60], Lemmas 2.1−2.4) *Let $v \in \mathcal{D}(\mathcal{M}_e)$ solve an equation of the type $\mathcal{M}_e v = g$, where $g \in L^2_\rho(\mathbb{R} \times \mathbb{L}^N)$. Then there exists a positive constant M independent of $e \in \mathbb{S}^{N-1}$, such that*

$$\|v\|_{\mathcal{D}(\mathcal{M}_e)} \leq M \, \|g\|_{L^2_\rho(\mathbb{R}\times\mathbb{L}^N)}.$$

Proof With similar arguments as those in Lemmas 2.1−2.4 of [60], by replacing k, c and $A(s)$ with e, c_e and $-\beta$, respectively, since $\kappa \leq c_e \leq K$ from Theorem 2.8, we get Lemma 4.1. □

Lemma 4.2 ([60], Lemmas 2.5 and 2.6) *For all $e \in \mathbb{S}^{N-1}$, the linear operator*

$$\mathcal{M}_e : \ \mathcal{D}(\mathcal{M}_e) \to L^2_\rho(\mathbb{R} \times \mathbb{L}^N)$$

is invertible. Moreover, the inverse operator $\mathcal{M}_e^{-1} : \ L^2_\rho(\mathbb{R} \times \mathbb{L}^N) \to \mathcal{D}(\mathcal{M}_e)$ is uniformly bounded, that is

$$\left\| \mathcal{M}_e^{-1} \right\| \leq M, \quad \forall e \in \mathbb{S}^{N-1},$$

where the constant M independent of e is given in Lemma 4.1.

Proof With similar arguments as those in Lemmas 2.5 and 2.6 of [60], by replacing k, c and $A(s)$ with e, c_e and $-\beta$, respectively, one gets that

$$\mathcal{M}_e : \ \mathcal{D}(\mathcal{M}_e) \to L^2_\rho(\mathbb{R} \times \mathbb{L}^N) \quad \text{is invertible.}$$

Then it follows from Lemma 4.1 that $\|\mathcal{M}_e^{-1}\| \leq M$, where the constant M is given in Lemma 4.1, which is independent of $e \in \mathbb{S}^{N-1}$. This completes the proof. □

In the following lemma, we show the continuity of $\mathcal{M}_e^{-1}$ with respect to $e \in \mathbb{S}^{N-1}$ in some sense.

Lemma 4.3 *Let $e \in \mathbb{S}^{N-1}$. For any unit vector sequence $\{e_n\}_{n\in\mathbb{N}}$, if $|e_n - e| \to 0$ as $n \to \infty$, then*

$$\left\| \mathcal{M}_{e_n}^{-1}(g) - \mathcal{M}_e^{-1}(g) \right\|_{H^1_\rho(\mathbb{R}\times\mathbb{L}^N)} \to 0 \ \ as\ n \to \infty$$

uniformly with respect to $g \in B_A \stackrel{def}{=} \left\{ g \in H^1_\rho(\mathbb{R} \times \mathbb{L}^N) : \ \|g\|_{H^1_\rho} \leq A \right\}$ for every $A > 0$.

Proof For $g \in H^1_\rho(\mathbb{R} \times \mathbb{L}^N)$, denote

$$w_n := \mathcal{M}_{e_n}^{-1}(g) \quad \text{and} \quad w := \mathcal{M}_e^{-1}(g), \quad \forall n \in \mathbb{N}.$$

Then Lemma 4.2 implies

$$w_n \in \mathcal{D}(\mathcal{M}_{e_n}),\ w \in \mathcal{D}(\mathcal{M}_e),\ \|w_n\|_{H^1_\rho},\ \|w\|_{H^1_\rho} \leq M \|g\|_{L^2_\rho}, \quad \forall n \in \mathbb{N},$$

where $M > 0$ is defined by Lemma 4.2. Since $\mathcal{M}_e w = g$ and $g \in H^1_\rho$, using difference quotients in $(-L_1, 2L_1) \times \cdots \times (-L_N, 2L_N)$, by virtue of Lemma 4.1 and periodicity, we get that $w \in H^2_\rho$ and

$$\|w\|_{H^2_\rho} \leq 3^N M \|g\|_{H^1_\rho}. \tag{4.5}$$

By calculation,

$$\begin{aligned}\mathcal{M}_{e_n}(w_n - w) &= \mathcal{M}_{e_n} w_n - \mathcal{M}_{e_n} w + \mathcal{M}_e w - \mathcal{M}_e w = \mathcal{M}_e w - \mathcal{M}_{e_n} w \\ &= (c_e - c_{e_n})\partial_s w + 2\nabla_z \partial_s w \cdot (e - e_n)\end{aligned}$$

for all $n \in \mathbb{N}$. Thus it follows from Lemma 4.1 that

$$\|w_n - w\|_{H^1_\rho} \leq M \left\|(c_e - c_{e_n})\partial_s w + 2\nabla_z \partial_s w \cdot (e - e_n)\right\|_{L^2_\rho}, \quad \forall n \in \mathbb{N}. \tag{4.6}$$

Moreover, (4.5) and (4.6) yield

$$\|w_n - w\|_{H^1_\rho} \leq C \|g\|_{H^1_\rho} \left(|c_{e_n} - c_e| + |e_n - e|\right), \quad \forall n \in \mathbb{N}, \tag{4.7}$$

where C is a positive constant independent of n and g. Since the mapping $e \in \mathbb{S}^{N-1} \mapsto c_e$ is continuous by Theorem 2.8, the proof is complete by using (4.7). □

We emphasize here that $L^2_\rho(\mathbb{R} \times \mathbb{L}^N)$ and $H^1_\rho(\mathbb{R} \times \mathbb{L}^N)$ are Hilbert spaces with inner products $(\cdot, \cdot)_{L^2_\rho}$ and $(\cdot, \cdot)_{H^1_\rho}$, respectively, where

$$\begin{aligned}(v, u)_{L^2_\rho} &:= \int_{\mathbb{R} \times \mathbb{L}^N} vu\rho \, ds dz, \quad \forall v,\ u \in L^2_\rho(\mathbb{R} \times \mathbb{L}^N), \\ (v, u)_{H^1_\rho} &:= (v, u)_{L^2_\rho} + \sum_{|\alpha|=1} \left(D^\alpha v, D^\alpha u\right)_{L^2_\rho}, \quad \forall v,\ u \in H^1_\rho(\mathbb{R} \times \mathbb{L}^N).\end{aligned}$$

The following lemma, which comes from [60], gives some properties of the linear operator $\mathcal{H}_e$. The L^2_ρ adjoint operator of $\mathcal{H}_e$, denoted by $\mathcal{H}_e^*$, is given by

$$\left(\mathcal{H}_e^*(v), u\right)_{L^2_\rho(\mathbb{R} \times \mathbb{L}^N)} = (v, \mathcal{H}_e(u))_{L^2_\rho(\mathbb{R} \times \mathbb{L}^N)} \quad \text{for all } v,\ u \in H^2_\rho(\mathbb{R} \times \mathbb{L}^N).$$

Lemma 4.4 ([60], Propositions 2.1 and 2.2, Lemma 2.8). *Assume that assumptions* (F1)–(F4) *hold. Then*

(i) the linear operator $\mathcal{H}_e$ has algebraically simple eigenvalue 0 *and the kernel of $\mathcal{H}_e$ is generated by $\partial_s U_e$.*
(ii) the adjoint operator $\mathcal{H}_e^*$ has geometrically simple eigenvalue 0.
(iii) the range of the linear operator $\mathcal{H}_e$, denoted by $R(\mathcal{H}_e)$, is closed in $L^2_\rho(\mathbb{R} \times \mathbb{L}^N)$, and

$$L^2_\rho(\mathbb{R} \times \mathbb{L}^N) = R(\mathcal{H}_e) \oplus \ker(\mathcal{H}_e^*).$$

Proof Define

$$A(s,z) := \begin{cases} f_u(z,1) & \text{for } s>0 \text{ and } z\in\mathbb{R}^N, \\ 0 & \text{for } s\le 0 \text{ and } z\in\mathbb{R}^N. \end{cases}$$

Replacing k, c, $A(s)$ and $g'(U)$ with e, c_e, $A(s,z)$ and $f_u(z,U_e)$ in [60], respectively, we get the conclusion (i) from Proposition 2.1 of [60], and get the conclusion (ii) from Proposition 2.2 of [60], and get the conclusion (iii) from Lemma 2.8 of [60] and Chapter 6 of [36]. □

Now we define two nonlinear operators. For any $e\in\mathbb{S}^{N-1}$, the nonlinear operator $\mathcal{K}_e : H^2_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}\times\mathbb{R}^N \to L^2_\rho(\mathbb{R}\times\mathbb{L}^N)$ is defined by

$$\mathcal{K}_e(v,\gamma,\eta) := \gamma\partial_s(U_e+v) + 2\nabla_z\partial_s(U_e+v)\cdot\eta + f(z,U_e+v) - f(z,U_e) + \beta v$$

for all $(v,\gamma,\eta)\in H^2_\rho\times\mathbb{R}\times\mathbb{R}^N$. For any $e\in\mathbb{S}^{N-1}$, the nonlinear operator $\mathcal{G}_e:\ H^2_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}\times\mathbb{R}^N \to H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}$ is defined by

$$\mathcal{G}_e(v,\gamma,\eta) := \left(\mathcal{G}^1_e(v,\gamma,\eta),\ \mathcal{G}^2_e(v,\gamma,\eta)\right),$$

where

$$\mathcal{G}^1_e(v,\gamma,\eta) := v + \mathcal{M}_e^{-1}(\mathcal{K}_e(v,\gamma,\eta)),\ \ \mathcal{G}^2_e(v,\gamma,\eta) := \int_{\mathbb{R}^+\times\mathbb{L}^N}\left[(U_e+v)^2 - U_e^2\right]\rho\, dsdz$$

for all $(v,\gamma,\eta)\in H^2_\rho\times\mathbb{R}\times\mathbb{R}^N$. In particular, we emphasize here that the domain of the nonlinear operator $\mathcal{G}_e(\cdot,\cdot,0)$ is

$$\mathcal{D}_0 := \left\{(v,\gamma)\,\middle|\, v\in L^2_\rho(\mathbb{R}\times\mathbb{L}^N),\ \partial_s v\in L^2_\rho(\mathbb{R}\times\mathbb{L}^N),\ \gamma\in\mathbb{R}\right\}.$$

But here we only consider the restriction of the operator $\mathcal{G}_e(\cdot,\cdot,0)$ on $(v,\gamma)\in H^1_\rho\times\mathbb{R}$ and show its Fréchet differentiability in the following lemma.

Lemma 4.5 *Assume that assumptions* (F1)–(F4) *hold. Then for every* $e\in\mathbb{S}^{N-1}$, *the operator* $\mathcal{G}_e(\cdot,\cdot,0):\ H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}\to H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}$ *is continuously Fréchet differentiable.*

Proof Since $\mathcal{K}_e:\ H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}\times\{0\}\to L^2_\rho(\mathbb{R}\times\mathbb{L}^N)$ and $\mathcal{M}_e^{-1}:\ L^2_\rho(\mathbb{R}\times\mathbb{L}^N)\to\mathcal{D}(\mathcal{M}_e)$, it follows that $\mathcal{G}_e(\cdot,\cdot,0):\ H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}\to H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}$.

Step 1. One has

$$\begin{aligned} A_1|_{(v,\gamma)}(\tilde v,\tilde\gamma) &:= \lim_{t\to 0}\frac{1}{t}\left[\mathcal{G}^1_e(v+t\tilde v,\gamma+t\tilde\gamma,0) - \mathcal{G}^1_e(v,\gamma,0)\right] \\ &= \tilde v + \mathcal{M}_e^{-1}(\gamma\partial_s\tilde v + \tilde\gamma\partial_s(U_e+v) + f_u(z,U_e+v)\tilde v + \beta\tilde v). \end{aligned}$$

It holds from Lemma 4.2 that $A_1|_{(v,\gamma)}:\ H^1_\rho\times\mathbb{R}\to H^1_\rho$ is well defined. In addition, one has

$$\begin{aligned} \left\|A_1|_{(v,\gamma)}(\tilde v,\tilde\gamma)\right\|_{H^1_\rho} &\le \|\tilde v\|_{H^1_\rho} + \left\|\mathcal{M}_e^{-1}\right\|\,\|\gamma\partial_s\tilde v + \tilde\gamma\partial_s(U_e+v) + f_u(z,U_e+v)\tilde v + \beta\tilde v\|_{L^2_\rho} \\ &\le C\,\|(\tilde v,\tilde\gamma)\|_{H^1_\rho\times\mathbb{R}}, \end{aligned}$$

which implies that $A_1|_{(v,\gamma)}$ is also bounded, and then is the Gâteaux differentiable operator of $\mathcal{G}^1_e(\cdot,\cdot,0)$ at the point (v,γ). For any $\tilde v\in H^1_\rho(\mathbb{R}\times\mathbb{L}^N)$, we have

$$\left\|D(\tilde v^2\rho)\right\|_{L^1(\mathbb{R}\times\mathbb{L}^N)} = \left\|\tilde v^2 D\rho + 2\rho\tilde v D\tilde v\right\|_{L^1} \le (1+2\varepsilon)\left\|\tilde v^2\rho\right\|_{L^1} + \left\|(D\tilde v)^2\rho\right\|_{L^1},$$

where D denotes any first-order derivative with respect to $(s, z) \in \mathbb{R} \times \mathbb{R}^N$. Thus

$$\left\|\tilde{v}^2\rho\right\|_{W^{1,1}(\mathbb{R}\times\mathbb{L}^N)} \leq (N+2)(1+2\varepsilon)\left\|\tilde{v}\right\|_{H^1_\rho(\mathbb{R}\times\mathbb{L}^N)}, \quad \forall \tilde{v} \in H^1_\rho(\mathbb{R}\times\mathbb{L}^N). \tag{4.8}$$

Then it follows that

$$\begin{aligned}
&\left\|A_1|_{(v_1,\gamma_1)} - A_1|_{(v_2,\gamma_2)}\right\| \\
&= \sup_{\|(\tilde{v},\tilde{\gamma})\|_{H^1_\rho\times\mathbb{R}}\leq 1} \left\|A_1|_{(v_1,\gamma_1)}(\tilde{v},\tilde{\gamma}) - A_1|_{(v_2,\gamma_2)}(\tilde{v},\tilde{\gamma})\right\|_{H^1_\rho} \\
&\leq \sup_{\|(\tilde{v},\tilde{\gamma})\|_{H^1_\rho\times\mathbb{R}}\leq 1} \left\|\mathcal{M}_e^{-1}\right\| \|\partial_s\tilde{v}(\gamma_1-\gamma_2) + \tilde{\gamma}\partial_s(v_1-v_2) \\
&\quad + \tilde{v}\left(f_u(z,U_e+v_1) - f_u(z,U_e+v_2)\right)\|_{L^2_\rho} \\
&\leq M\left(|\gamma_1-\gamma_2| + \|v_1-v_2\|_{H^1_\rho}\right) + M \sup_{\|\tilde{v}\|_{H^1_\rho}\leq 1}\left\|\left(f_u(z,U_e+v_1) - f_u(z,U_e+v_2)\right)^2\tilde{v}^2\rho\right\|_{L^1} \\
&\leq M\left(|\gamma_1-\gamma_2| + \|v_1-v_2\|_{H^1_\rho}\right) \\
&\quad + M \sup_{\|\tilde{v}\|_{H^1_\rho}\leq 1} \left(2\|f_u\|_{L^\infty}\right)^{\frac{2N}{N+1}} \left\|\tilde{v}^2\rho\right\|_{L^{\frac{N+1}{N}}} \left\|\left(f_u(z,U_e+v_1) - f_u(z,U_e+v_2)\right)^{\frac{2}{N+1}}\right\|_{L^{N+1}} \\
&\leq M\left(|\gamma_1-\gamma_2| + \|v_1-v_2\|_{H^1_\rho}\right) + C\|f_u\|_{L^\infty}^{\frac{2N}{N+1}} \|f_{uu}\|_{L^\infty}^{\frac{2}{N+1}} \|v_1-v_2\|_{L^2}^{\frac{2}{N+1}},
\end{aligned}$$

where we have used the Sobolev imbedding theorem (see Theorem 4.12 of [1]). Therefore, the Gâteaux differentiable operator $A_1|_{(v,\gamma)}$ is continuous with respect to (v,γ). As a conclusion, $\mathcal{G}_e^1(\cdot,\cdot,0)$ is continuously Fréchet differentiable, and its Fréchet differentiable operator is $\partial_{(v,\gamma)}\mathcal{G}_e^1(\cdot,\cdot,0)|_{(v,\gamma)} = A_1|_{(v,\gamma)}$.

Step 2. It is clear that

$$\begin{aligned}
A_2|_{(v,\gamma)}(\tilde{v},\tilde{\gamma}) &:= \lim_{t\to 0}\frac{1}{t}\left[\mathcal{G}_e^2(v+t\tilde{v},\gamma+t\tilde{\gamma},0) - \mathcal{G}_e^2(v,\gamma,0)\right] \\
&= 2\int_{\mathbb{R}^+\times\mathbb{L}^N}(U_e+v)\,\tilde{v}\rho\,dsdz.
\end{aligned}$$

With similar arguments as those in *Step 1*, we get that $\mathcal{G}_e^2(\cdot,\cdot,0)$ is continuously Fréchet differentiable, and its Fréchet differentiable operator is $\partial_{(v,\gamma)}\mathcal{G}_e^2(\cdot,\cdot,0)|_{(v,\gamma)} = A_2|_{(v,\gamma)}$. □

Denote $\mathcal{Q}_e := \partial_{(v,\gamma)}\mathcal{G}_e(\cdot,\cdot,0)|_{(0,0)}$ for all $e \in \mathbb{S}^{N-1}$. Then it follows from Lemma 4.5 that the linear operator $\mathcal{Q}_e : H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R} \to H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}$ is defined by

$$\mathcal{Q}_e(\tilde{v},\tilde{\gamma}) := \left(\tilde{v} + \mathcal{M}_e^{-1}(\tilde{\gamma}\partial_s U_e + f_u(z,U_e)\tilde{v} + \beta\tilde{v})\,,\; 2\int_{\mathbb{R}^+\times\mathbb{L}^N} U_e\tilde{v}\rho\,dsdz\right). \tag{4.9}$$

Apparently, $H^1_\rho\times\mathbb{R}$ is a Hilbert space with the inner product $(\cdot,\cdot)_{H^1_\rho\times\mathbb{R}}$, which is defined by

$$((w,\mu),\ (v,\gamma))_{H^1_\rho\times\mathbb{R}} := (w,v)_{H^1_\rho} + \mu\gamma \ \text{ for all } (w,\mu),\ (v,\gamma) \in H^1_\rho\times\mathbb{R}.$$

The inverse operator of $\mathcal{Q}_e$ is studied in the following lemma and we postpone its proof in the appendix. For bistable equations, the corresponding result for one-dimensional case $z \in \mathbb{R}$ is given in Lemma 3.4 of [15], and then the general case $z \in \mathbb{R}^N$ is considered in [25] but without specific proof details. In the following lemma, we give the rigorous proof for combustion equations, which is also valid for bistable equations with slight modification.

Lemma 4.6 *Assume that assumptions* (F1)–(F4) *hold. Then for every* $e \in \mathbb{S}^{N-1}$*, the linear operator* $\mathcal{Q}_e : H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R} \to H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}$ *is invertible, and the inverse operator* $\mathcal{Q}_e^{-1}$ *is bounded.*

Theorem 2.8 shows that the mapping $e \in \mathbb{S}^{N-1} \mapsto U_e$ is continuous under the topology $\|\cdot\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}$, but indeed, the mapping is also continuous under the topology $\|\cdot\|_{H^2_\rho(\mathbb{R}\times\mathbb{L}^N)}$, that is the following lemma.

Lemma 4.7 *Assume that assumptions* (F1)–(F4) *hold, and that* (U_e, c_e) *is a pulsating front of* (1.1)*. Normalize* U_e *as* (2.10)*. Then the mapping* $e \in \mathbb{S}^{N-1} \mapsto U_e$ *is continuous under the topology* $\|\cdot\|_{H^2_\rho(\mathbb{R}\times\mathbb{L}^N)}$*, that is,*

$$\left\|U_{e_n} - U_e\right\|_{H^2_\rho(\mathbb{R}\times\mathbb{L}^N)} \to 0 \ \text{ as } |e_n - e| \to 0.$$

Proof Let $\{e_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^{N-1}$ satisfy $\lim_{n\to\infty} e_n = e$.

Step 1. It follows from Proposition 3.3 that

$$|U_{e_n} - U_e| \le \max\{U_{e_n}, U_e\} \le K_2 e^{-\frac{3\kappa}{4}s} \ \text{ in } [0,+\infty)\times\mathbb{L}^N, \tag{4.10}$$

$$|U_{e_n} - U_e| \le |1 - U_{e_n}| + |1 - U_e| \le 2K_2 e^{\kappa_2 s} \ \text{ in } (-\infty,0]\times\mathbb{L}^N, \tag{4.11}$$

where $K_2,\ \kappa,\ \kappa_2$ are given in Proposition 3.3. Using the Lebesgue dominated convergence theorem, together with (4.10), (4.11) and Theorem 2.8, one gets

$$\left\|U_{e_n} - U_e\right\|_{L^2_\rho} \to 0 \ \text{ as } n \to \infty. \tag{4.12}$$

Step 2. Note that U_{e_n} and U_e satisfy the following equations

$$\mathcal{M}_{e_n} U_{e_n} = -f(z, U_{e_n}) - \beta U_{e_n} \ \text{ and } \ \mathcal{M}_e U_e = -f(z, U_e) - \beta U_e,$$

respectively. Then it holds that

$$\begin{aligned}
&\mathcal{M}_{e_n}(U_{e_n} - U_e)\\
&\quad= (c_e - c_{e_n})\partial_s U_e + 2\nabla_z\partial_s U_e\cdot(e - e_n) + \beta(U_e - U_{e_n}) + f(z,U_e) - f(z,U_{e_n})\\
&\quad= (c_e - c_{e_n})\partial_s U_e + 2\nabla_z\partial_s U_e\cdot(e - e_n) + \left[\beta + f_u(z, \tau U_e + (1-\tau)U_{e_n})\right](U_e - U_{e_n})
\end{aligned} \tag{4.13}$$

where $\tau \in (0,1)$. It follows from Corollaries 3.5 and 3.8 that

$$\partial_s U_e,\ \nabla_z\partial_s U_e \in L^2_\rho(\mathbb{R}\times\mathbb{L}^N) \ \text{ and } \ (U_{e_n} - U_e) \in H^2_\rho(\mathbb{R}\times\mathbb{L}^N) \subset \mathcal{D}(\mathcal{M}_{e_n}). \tag{4.14}$$

Thus by virtue of (4.13), (4.14) and Lemma 4.1, since $f_u(\cdot,\cdot)$ is bounded from (1.2), we get that

$$\left\|U_{e_n} - U_e\right\|_{H^1_\rho} \le C\left(|c_{e_n} - c_e| + |e_n - e| + \left\|U_{e_n} - U_e\right\|_{L^2_\rho}\right), \tag{4.15}$$

where C is a constant independent of n. Then it follows from (4.12), (4.15) and Theorem 2.8 that

$$\left\|U_{e_n} - U_e\right\|_{H^1_\rho} \to 0 \ \text{ as } n \to \infty.$$

With similar arguments as above, one obtains

$$\left\|U_{e_n} - U_e\right\|_{H^2_\rho} \to 0 \ \text{ as } n \to \infty.$$

The proof of Lemma 4.7 is thereby complete. □

We focus on the continuity of the inverse operator $\mathcal{Q}_e^{-1}$ with respect to $e \in \mathbb{S}^{N-1}$ in the following lemma, whose proof is postponed to the appendix.

Lemma 4.8 *Assume that assumptions* (F1)–(F4) *hold. Then the linear operator $\mathcal{Q}_e^{-1}$ is uniformly bounded with respect to $e \in \mathbb{S}^{N-1}$. Furthermore, the linear operator $\mathcal{Q}_e^{-1}$ is continuous with respect to $e \in \mathbb{S}^{N-1}$, that is,*

$$\left\| \mathcal{Q}_{e_n}^{-1} - \mathcal{Q}_e^{-1} \right\| \to 0 \ \text{ as } |e_n - e| \to 0.$$

Now we are ready to prove the Fréchet differentiability of pulsating fronts with respect to the propagation direction.

Proof of Theorem 2.10 It follows immediately from Theorem 2.8 and Lemma 4.7 that (U_b, c_b) is continuous in $b \in \mathbb{R}^N$ everywhere at $\mathbb{R}^N \setminus \{0\}$ under the topology $\|\cdot\|_{H_\rho^2 \times \mathbb{R}}$.

Step 1: we prove that (U_b, c_b) is first-order continuously Fréchet differentiable in $b \in \mathbb{R}^N$ under the topology $\|\cdot\|_{H_\rho^1 \times \mathbb{R}}$.

It follows from (2.1) and (2.11) that for any $b \in \mathbb{R}^N \setminus \{0\}$, (U_b, c_b) solves the equation

$$c_b \partial_s U_b + \partial_{ss} U_b + 2\nabla_z \partial_s U_b \cdot \frac{b}{|b|} + \Delta_z U_b + f(z, U_b) = 0 \ \text{ in } \mathbb{R} \times \mathbb{R}^N. \tag{4.16}$$

Now fix arbitrary $e \in \mathbb{S}^{N-1}$, and let $h \in \mathbb{R}^N$ be small such that $e + h \in \mathbb{R}^N \setminus \{0\}$. Set

$$\widetilde{U}_h := U_{e+h} - U_e, \ \ \tilde{c}_h := c_{e+h} - c_e \in \mathbb{R}, \ \ \tilde{h} := \tfrac{e+h}{|e+h|} - e \in \mathbb{R}^N.$$

One obtains from Corollaries 3.5 and 3.8 that $\widetilde{U}_h \in H_\rho^2$ for all h. Furthermore, Lemma 4.7 yields

$$\left\| \widetilde{U}_h \right\|_{H_\rho^2} = \|U_{e+h} - U_e\|_{H_\rho^2} \to 0 \ \text{ as } |h| \to 0. \tag{4.17}$$

From Theorem 2.8, one knows

$$|\tilde{c}_h| = |c_{e+h} - c_e| \to 0 \ \text{ as } |h| \to 0.$$

It is trivial to get

$$\tilde{h} = h - (e \cdot h)e + o(|h|) \ \text{ as } |h| \to 0. \tag{4.18}$$

By virtue of (2.10) and (4.16), it holds that

$$\mathcal{G}_e\big(\widetilde{U}_h, \tilde{c}_h, \tilde{h}\big) = (0, 0).$$

Note $\mathcal{G}_e(0, 0, 0) = (0, 0)$. By Lemma 4.5, one gets that

$$\begin{aligned}
(0, 0) &= \mathcal{G}_e\big(\widetilde{U}_h, \tilde{c}_h, \tilde{h}\big) - \mathcal{G}_e(0, 0, 0) \\
&= \mathcal{G}_e\big(\widetilde{U}_h, \tilde{c}_h, \tilde{h}\big) - \mathcal{G}_e\big(\widetilde{U}_h, \tilde{c}_h, 0\big) + \mathcal{Q}_e\big(\widetilde{U}_h, \tilde{c}_h\big) + \omega_2\big(\widetilde{U}_h, \tilde{c}_h\big) \\
&= \Big(\mathcal{M}_e^{-1}\big(2\nabla_z \partial_s U_e \cdot \tilde{h}\big), 0\Big) + \omega_1\big(\widetilde{U}_h, \tilde{h}\big) + \mathcal{Q}_e\big(\widetilde{U}_h, \tilde{c}_h\big) + \omega_2\big(\widetilde{U}_h, \tilde{c}_h\big),
\end{aligned} \tag{4.19}$$

where $\omega_1(\widetilde{U}_h, \tilde{h}) := \big(\mathcal{M}_e^{-1}(2\nabla_z \partial_s \widetilde{U}_h \cdot \tilde{h}), 0\big)$ and

$$\omega_2\big(\widetilde{U}_h, \tilde{c}_h\big) = o\big(\big\|\big(\widetilde{U}_h, \tilde{c}_h\big)\big\|_{H_\rho^1 \times \mathbb{R}}\big) \ \text{ as } |h| \to 0. \tag{4.20}$$

Then it follows from (4.19) and Lemma 4.6 that

$$\big(\widetilde{U}_h, \tilde{c}_h\big) = -\mathcal{Q}_e^{-1}\Big(\mathcal{M}_e^{-1}\big(2\nabla_z \partial_s U_e \cdot \tilde{h}\big), 0\Big) - \mathcal{Q}_e^{-1}\big(\omega_1\big(\widetilde{U}_h, \tilde{h}\big)\big) - \mathcal{Q}_e^{-1}\big(\omega_2\big(\widetilde{U}_h, \tilde{c}_h\big)\big). \tag{4.21}$$

One computes

$$\left\| \mathcal{Q}_e^{-1}\left(\omega_1\left(\widetilde{U}_h, \tilde{h}\right)\right) \right\|_{H_\rho^1 \times \mathbb{R}} \leq 2 \left\| \mathcal{Q}_e^{-1} \right\| \left\| \mathcal{M}_e^{-1} \right\| \left\| \widetilde{U}_h \right\|_{H_\rho^2} |\tilde{h}|.$$

Thus one obtains from (4.17) and (4.18) that

$$\left\| \mathcal{Q}_e^{-1}\left(\omega_1\left(\widetilde{U}_h, \tilde{h}\right)\right) \right\|_{H_\rho^1 \times \mathbb{R}} = o(|h|) \text{ as } |h| \to 0. \tag{4.22}$$

We claim that

$$\left\| \mathcal{Q}_e^{-1}\left(\omega_2\left(\widetilde{U}_h, \tilde{c}_h\right)\right) \right\|_{H_\rho^1 \times \mathbb{R}} = o(|h|) \text{ as } |h| \to 0. \tag{4.23}$$

Since $\|(\widetilde{U}_h, \tilde{c}_h)\|_{H_\rho^2 \times \mathbb{R}} \to 0$ as $|h| \to 0$, together with (4.20), one has

$$\begin{aligned} \left\| \left(\widetilde{U}_h, \tilde{c}_h\right) + \mathcal{Q}_e^{-1}\left(\omega_2\left(\widetilde{U}_h, \tilde{c}_h\right)\right) \right\|_{H_\rho^1 \times \mathbb{R}} &\geq \left\| \left(\widetilde{U}_h, \tilde{c}_h\right) \right\|_{H_\rho^1 \times \mathbb{R}} - \left\| \mathcal{Q}_e^{-1} \right\| \left\| \omega_2\left(\widetilde{U}_h, \tilde{c}_h\right) \right\|_{H_\rho^1 \times \mathbb{R}} \\ &\geq \frac{1}{2} \left\| \left(\widetilde{U}_h, \tilde{c}_h\right) \right\|_{H_\rho^1 \times \mathbb{R}} \end{aligned}$$

as $|h| \to 0$. Then it holds from (4.18), (4.21) and (4.22) that

$$\begin{aligned} \frac{1}{2|h|} \left\| \left(\widetilde{U}_h, \tilde{c}_h\right) \right\|_{H_\rho^1 \times \mathbb{R}} &\leq \frac{1}{|h|} \left\| \left(\widetilde{U}_h, \tilde{c}_h\right) + \mathcal{Q}_e^{-1}\left(\omega_2\left(\widetilde{U}_h, \tilde{c}_h\right)\right) \right\|_{H_\rho^1 \times \mathbb{R}} \\ &= \frac{1}{|h|} \left\| \mathcal{Q}_e^{-1}\left(\mathcal{M}_e^{-1}\left(2\nabla_z \partial_s U_e \cdot \tilde{h}\right), 0\right) + \mathcal{Q}_e^{-1}\left(\omega_1\left(\widetilde{U}_h, \tilde{h}\right)\right) \right\|_{H_\rho^1 \times \mathbb{R}} \\ &\leq \left\| \mathcal{Q}_e^{-1} \right\| \left\| \mathcal{M}_e^{-1} \right\| \left\| 2\nabla_z \partial_s U_e \cdot \tfrac{\tilde{h}}{|h|} \right\|_{L_\rho^2} + \frac{o(|h|)}{|h|} < +\infty \end{aligned}$$

as $|h| \to 0$, which implies

$$\left\| \left(\widetilde{U}_h, \tilde{c}_h\right) \right\|_{H_\rho^1 \times \mathbb{R}} = O(|h|) \text{ as } |h| \to 0. \tag{4.24}$$

Consequently, claim (4.23) is valid from (4.20), (4.24) and the fact that $\mathcal{Q}_e^{-1}$ is bounded.

It follows from (4.18) and (4.21)–(4.23) that

$$(U_{e+h} - U_e, c_{e+h} - c_e) = -\mathcal{Q}_e^{-1}\left(\mathcal{M}_e^{-1}\left(2\nabla_z \partial_s U_e \cdot [h - (e \cdot h)e]\right), 0\right) + o(|h|)$$

as $|h| \to 0$ under the topology $\|\cdot\|_{H_\rho^1 \times \mathbb{R}}$, which means that (U_b, c_b) is Fréchet differentiable in $b \in \mathbb{R}^N$ everywhere at $e \in \mathbb{S}^{N-1}$ under the topology $\|\cdot\|_{H_\rho^1 \times \mathbb{R}}$. Denote the Fréchet derivative of (U_b, c_b) in b at e by (U_e', c_e'). Then its form is

$$\left(U_e'(h), c_e'(h)\right) = \mathcal{Q}_e^{-1}\left(\mathcal{M}_e^{-1}\left(2\nabla_z \partial_s U_e \cdot [(e \cdot h)e - h]\right), 0\right), \quad \forall h \in \mathbb{R}^N. \tag{4.25}$$

Furthermore, for any $b \in \mathbb{R}^N \setminus \{0\}$ one can get by (2.11) that

$$(U_{b+h}, c_{b+h}) - (U_b, c_b) = \left(U'_{\frac{b}{|b|}}\left(\tfrac{h}{|b|} - \tfrac{b \cdot h}{|b|^3} b\right), c'_{\frac{b}{|b|}}\left(\tfrac{h}{|b|} - \tfrac{b \cdot h}{|b|^3} b\right)\right) + o(|h|) \tag{4.26}$$

as $|h| \to 0$. Thus, (U_b, c_b) is first-order Fréchet differentiable in $b \in \mathbb{R}^N$ everywhere at $\mathbb{R}^N \setminus \{0\}$ under the topology $\|\cdot\|_{H_\rho^1(\mathbb{R} \times \mathbb{L}^N) \times \mathbb{R}}$.

By virtue of Lemmas 4.2, 4.3, 4.7 and 4.8, it follows from (4.25) that

$$\begin{aligned} &\left\| \left(U_{e_n}', c_{e_n}'\right) - \left(U_e', c_e'\right) \right\| \\ &= \sup_{h \in \mathbb{S}^{N-1}} \left\| \left(U_{e_n}'(h), c_{e_n}'(h)\right) - \left(U_e'(h), c_e'(h)\right) \right\|_{H_\rho^1 \times \mathbb{R}} \to 0 \text{ as } e_n \to e. \end{aligned}$$

Therefore by (4.26), (U_b, c_b) is first-order continuously Fréchet differentiable in $b \in \mathbb{R}^N$ everywhere at $\mathbb{R}^N \setminus \{0\}$ under the topology $\|\cdot\|_{H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}}$. For convenience, without of ambiguity, we sometimes use notations $U'_b \cdot h$ and $c'_b \cdot h$ as $U'_b(h)$ and $c'_b(h)$, where $b,\ h \in \mathbb{R}^N$.

Step 2. Since $f \in C^\infty(\mathbb{R}^{N+1})$, we can get that Proposition 3.6 holds for k_{th}-order derivatives and Lemmas 4.1-4.8 hold for $\|\cdot\|_{H^k_\rho}$, for all $k \in \mathbb{N}$. Hence with similar arguments as those in *Step 1*, one can prove that (U_b, c_b) is first-order continuously Fréchet differentiable in $b \in \mathbb{R}^N$ everywhere at $\mathbb{R}^N \setminus \{0\}$ under the topology $\|\cdot\|_{H^k_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}}$ for any $k \in \mathbb{N}$. Applying the Sobolev imbedding theorem (see Theorem 4.12 of [1]), by virtue of the periodicity, we also obtain that (U_b, c_b) is first-order continuously Fréchet differentiable under the topology $\|\cdot\|_{C^2(\mathbb{R}\times\mathbb{R}^N)\times\mathbb{R}}$.

Step 3: *we prove that* (U_b, c_b) *is second-order continuously Fréchet differentiable in* $b \in \mathbb{R}^N$ *under the topology* $\|\cdot\|_{H^1_\rho\times\mathbb{R}}$.

Now fix arbitrary $e \in \mathbb{S}^{N-1}$. We define two nonlinear operators $\mathcal{K}^*_e$ and $\mathcal{G}^*_e$. The operator

$$\mathcal{K}^*_e:\ H^2_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}\times H^2_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to L^2_\rho(\mathbb{R}\times\mathbb{L}^N)$$

is defined by

$$\begin{aligned}
\mathcal{K}^*_e(v_1,\vartheta_1,v_2,\vartheta_2,h_1,h_2) := {} & \vartheta_2\partial_s(U_e+v_1) + c'_e(h_1)\partial_s v_1 + \vartheta_1\partial_s\big(U'_e(h_1)+v_2\big) \\
& + 2\nabla_z\partial_s v_1\cdot\left[\frac{h_1}{|e+h_2|} - \frac{(e+h_2)\cdot h_1}{|e+h_2|^3}(e+h_2)\right] \\
& + 2\nabla_z\partial_s U_e\cdot\left[\frac{h_1}{|e+h_2|} - \frac{(e+h_2)\cdot h_1}{|e+h_2|^3}(e+h_2) - h_1 + (e\cdot h_1)e\right] \\
& + \big(2\nabla_z\partial_s v_2 + 2\nabla_z\partial_s U'_e(h_1)\big)\cdot\left[\frac{e+h_2}{|e+h_2|} - e\right] \\
& + f_u(z,U_e+v_1)\big(U'_e(h_1)+v_2\big) - f_u(z,U_e)U'_e(h_1) + \beta v_2
\end{aligned}$$

for all $(v_1,\vartheta_1,v_2,\vartheta_2,h_1,h_2) \in H^2_\rho\times\mathbb{R}\times H^2_\rho\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$. The operator

$$\mathcal{G}^*_e:\ H^2_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}\times H^2_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}$$

is defined by

$$\begin{aligned}
&\mathcal{G}^*_e(v_1,\vartheta_1,v_2,\vartheta_2,h_1,h_2) \\
&:= \left(v_2 + \mathcal{M}_e^{-1}\big(\mathcal{K}^*_e(v_1,\vartheta_1,v_2,\vartheta_2,h_1,h_2)\big),\ 2\int_{\mathbb{R}^+\times\mathbb{L}^N}\big[U'_e(h_1)v_1 + v_2(U_e+v_1)\big]\rho\, dsdz\right)
\end{aligned}$$

for all $(v_1,\vartheta_1,v_2,\vartheta_2,h_1,h_2) \in H^2_\rho\times\mathbb{R}\times H^2_\rho\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$.

For any $h_2 \in \mathbb{R}^N$ satisfying $e + h_2 \in \mathbb{R}^N \setminus \{0\}$, denote $\widetilde{U}_{h_2} := U_{e+h_2} - U_e$, $\tilde{c}_{h_2} := c_{e+h_2} - c_e$, and $\widetilde{U}'_{h_2}(h_1) := U'_{e+h_2}(h_1) - U'_e(h_1)$, $\tilde{c}'_{h_2}(h_1) := c'_{e+h_2}(h_1) - c'_e(h_1)$ for all $h_1 \in \mathbb{S}^{N-1}$. Then it holds from *Step* 2 that

$$\big(\widetilde{U}_{h_2}, \tilde{c}_{h_2}, \widetilde{U}'_{h_2}(h_1), \tilde{c}'_{h_2}(h_1), h_1, h_2\big) \in H^2_\rho\times\mathbb{R}\times H^2_\rho\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N.$$

Differentiating Eq. (4.16) at b on the direction $h \in \mathbb{R}^N$ yields

$$\begin{aligned}
&\left(\tfrac{b}{|b|}\partial_s + \nabla_z\right)^T\left(\tfrac{b}{|b|}\partial_s + \nabla_z\right)\big(U'_b\cdot h\big) + \big(c'_b\cdot h\big)\partial_s U_b + c_b\partial_s\big(U'_b\cdot h\big) \\
&\quad + 2\nabla_z\partial_s U_b\cdot\left(\tfrac{h}{|b|} - \tfrac{b\cdot h}{|b|^3}b\right) + f_u(z,U_b)\big(U'_b\cdot h\big) = 0 \qquad (4.27)
\end{aligned}$$

in $\mathbb{R} \times \mathbb{R}^N$. By calculation, one can obtain from (4.27) and (2.10) that

$$\begin{aligned}
&\mathcal{G}_e^*\left(\widetilde{U}_{h_2}, \tilde{c}_{h_2}, \widetilde{U}_{h_2}'(h_1), \tilde{c}_{h_2}'(h_1), h_1, h_2\right) \\
&\quad = \left(0,\ 2\int_{\mathbb{R}^+\times\mathbb{L}^N} \left[U_{e+h_2}'(h_1)U_{e+h_2} - U_e'(h_1)U_e\right] \rho \, dsdz\right) \\
&\quad = \left(0,\ \int_{\mathbb{R}^+\times\mathbb{L}^N} \left(U_{e+h_2}^2\rho\right)'(h_1) - \left(U_e^2\rho\right)'(h_1) \, dsdz\right) \\
&\quad = (0, 0)
\end{aligned} \tag{4.28}$$

for all $|h_2| \le \frac{1}{2}$ and $h_1 \in \mathbb{S}^{N-1}$. Then by virtue of (4.28) one computes

$$\begin{aligned}
(0,0) = {} & \mathcal{G}_e^*\left(\widetilde{U}_{h_2}, \tilde{c}_{h_2}, \widetilde{U}_{h_2}'(h_1), \tilde{c}_{h_2}'(h_1), h_1, h_2\right) - \mathcal{G}_e^*\left(0, 0, \widetilde{U}_{h_2}'(h_1), \tilde{c}_{h_2}'(h_1), h_1, 0\right) \\
& + \mathcal{Q}_e\left(\widetilde{U}_{h_2}'(h_1), \tilde{c}_{h_2}'(h_1)\right)
\end{aligned}$$

for all $|h_2| \le \frac{1}{2}$ and $h_1 \in \mathbb{S}^{N-1}$, where $\mathcal{Q}_e$ is given in Lemma 4.6. Thus we have

$$\begin{aligned}
&\left(\widetilde{U}_{h_2}'(h_1), \tilde{c}_{h_2}'(h_1)\right) \\
&\quad = \mathcal{Q}_e^{-1}\left[\mathcal{G}_e^*\left(\widetilde{U}_{h_2}, \tilde{c}_{h_2}, \widetilde{U}_{h_2}'(h_1), \tilde{c}_{h_2}'(h_1), h_1, h_2\right) - \mathcal{G}_e^*\left(0, 0, \widetilde{U}_{h_2}'(h_1), \tilde{c}_{h_2}'(h_1), h_1, 0\right)\right]
\end{aligned} \tag{4.29}$$

for all $|h_2| \le \frac{1}{2}$ and $h_1 \in \mathbb{S}^{N-1}$. It is trivial that

$$\frac{e+h_2}{|e+h_2|} - e = h_2 - (e\cdot h_2)e + o(|h_2|) \text{ as } |h_2| \to 0 \tag{4.30}$$

and

$$\begin{aligned}
&\frac{h_1}{|e+h_2|} - \frac{(e+h_2)\cdot h_1}{|e+h_2|^3}(e+h_2) - h_1 + (e\cdot h_1)e \\
&\quad = [3(e\cdot h_1)(e\cdot h_2) - (h_1\cdot h_2)]\, e - (e\cdot h_2)h_1 - (e\cdot h_1)h_2 + o(|h_2|)
\end{aligned} \tag{4.31}$$

as $|h_2| \to 0$ uniformly for $h_1 \in \mathbb{S}^{N-1}$. Denote

$$\begin{aligned}
\widetilde{\Psi}_{h_1} &:= h_1 - (e\cdot h_1)e, \quad \widetilde{\Psi}_{h_2} := h_2 - (e\cdot h_2)e, \\
\widehat{\Psi}_{h_1h_2} &:= [3(e\cdot h_1)(e\cdot h_2) - (h_1\cdot h_2)]\, e - (e\cdot h_2)h_1 - (e\cdot h_1)h_2
\end{aligned}$$

for all $h_1,\ h_2 \in \mathbb{R}^N$. Now we define a bilinear operator $A^* :\ \mathbb{R}^N \times \mathbb{R}^N \to H_\rho^1(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}$, whose form is

$$\begin{aligned}
&A^*(h_1, h_2) \\
&\quad := \Big(\mathcal{M}_e^{-1}\left[c_e'(h_1)\partial_s U_e'(h_2) + c_e'(h_2)\partial_s U_e'(h_1) + 2\nabla_z\partial_s U_e'(h_2)\cdot\widetilde{\Psi}_{h_1} + 2\nabla_z\partial_s U_e\cdot\widehat{\Psi}_{h_1h_2}\right. \\
&\qquad \left. + 2\nabla_z\partial_s U_e'(h_1)\cdot\widetilde{\Psi}_{h_2} + f_{uu}(z, U_e)U_e'(h_1)U_e'(h_2)\right],\ 2\int_{\mathbb{R}^+\times\mathbb{L}^N} U_e'(h_1)U_e'(h_2)\rho\, dsdz\Big)
\end{aligned}$$

for all $h_1,\ h_2 \in \mathbb{R}^N$. By calculation, we obtain from *Step 2* that

$$\begin{aligned}
&\left\|A^*(h_1, h_2)\right\|_{H_\rho^1\times\mathbb{R}} \\
&\quad \le \left\|\mathcal{M}_e^{-1}\right\| \left\|c_e'(h_1)\partial_s U_e'(h_2) + c_e'(h_2)\partial_s U_e'(h_1) + 2\nabla_z\partial_s U_e\cdot\widehat{\Psi}_{h_1h_2}\right\|_{L_\rho^2} \\
&\qquad + \left\|\mathcal{M}_e^{-1}\right\| \left\|2\nabla_z\partial_s U_e'(h_2)\cdot\widetilde{\Psi}_{h_1} + 2\nabla_z\partial_s U_e'(h_1)\cdot\widetilde{\Psi}_{h_2}\right\|_{L_\rho^2}
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\mathcal{M}_e^{-1}\right\|\left\|f_{uu}(z,U_e)U_e'(h_1)U_e'(h_2)\right\|_{L^2_\rho}+2\left\|U_e'(h_1)\right\|_{L^2_\rho}\left\|U_e'(h_2)\right\|_{L^2_\rho}\\
\le&\left\|\mathcal{M}_e^{-1}\right\|\left(2\left\|c_e'\right\|\left\|U_e'\right\|+12\left\|U_e\right\|_{H^2_\rho}\right)|h_1||h_2|+2\left\|U_e'\right\|^2|h_1||h_2|\\
&+4\left\|\mathcal{M}_e^{-1}\right\|\left(\left\|U_e'(h_2)\right\|_{H^2_\rho}|h_1|+\left\|U_e'(h_1)\right\|_{H^2_\rho}|h_2|\right)\\
&+\left\|\mathcal{M}_e^{-1}\right\|\left\|f_{uu}\right\|_{L^\infty}\left\|U_e'(h_1)\right\|_{L^\infty}\left\|U_e'(h_2)\right\|_{L^2_\rho}\\
\le& C|h_1||h_2|,
\end{aligned}\tag{4.32}
$$

where C is a constant independent of h_1 and h_2. For convenience, define an operator $B^*:\mathbb{R}^N\times\mathbb{R}^N\to H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}$, whose form is

$$
\begin{aligned}
&B^*(h_1,h_2)\\
&:=\mathcal{G}_e^*\left(\widetilde{U}_{h_2},\tilde{c}_{h_2},\widetilde{U}_{h_2}'(h_1),\tilde{c}_{h_2}'(h_1),h_1,h_2\right)\\
&\quad-\mathcal{G}_e^*\left(0,0,\widetilde{U}_{h_2}'(h_1),\tilde{c}_{h_2}'(h_1),h_1,0\right)-A^*(h_1,h_2)
\end{aligned}
$$

for all $h_1,\ h_2\in\mathbb{R}^N$. Since (U_b,c_b) is first-order continuously Fréchet differentiable in b under both topologies $\|\cdot\|_{H^2_\rho\times\mathbb{R}}$ and $\|\cdot\|_{C^0\times\mathbb{R}}$ from *Step 2*, together with (4.30) and (4.31), one can get that

$$
\left\|B^*(h_1,h_2)\right\|_{H^1_\rho\times\mathbb{R}}=o(|h_2|)\ \text{ as } |h_2|\to 0 \tag{4.33}
$$

uniformly for all $h_1\in\mathbb{S}^{N-1}$. The proof of (4.33) is long and tedious, but not difficult, thus we omit it.

Since $\mathcal{Q}_e^{-1}$ is bounded from Lemma 4.6, by virtue of (4.29), (4.32) and (4.33), we conclude that (U_b,c_b) is second-order Fréchet differentiable in $b\in\mathbb{R}^N$ everywhere at $e\in\mathbb{S}^{N-1}$ under the topology $\|\cdot\|_{H^1_\rho\times\mathbb{R}}$. Denote the second-order Fréchet derivative of (U_b,c_b) in b at e by (U_e'',c_e''). Then its form is

$$
\left(U_e'',c_e''\right)(h_2)(h_1)=\mathcal{Q}_e^{-1}(A^*(h_1,h_2))\ \text{ for all } h_1,\ h_2\in\mathbb{R}^N.
$$

Furthermore, by virtue of Lemmas 4.2, 4.3, 4.7, 4.8 and *Step 2*, one obtains

$$
\left\|\left(U_{e_n}'',c_{e_n}''\right)-\left(U_e'',c_e''\right)\right\|\to 0\ \text{ as } e_n\to e.
$$

Then it follows from (2.11) that (U_b,c_b) is second-order continuously Fréchet differentiable in $b\in\mathbb{R}^N$ everywhere at $\mathbb{R}^N\setminus\{0\}$ under the topology $\|\cdot\|_{H^1_\rho(\mathbb{R}\times\mathbb{L}^N)\times\mathbb{R}}$. For convenience, without of ambiguity, we sometimes use notations $U_b''\cdot h_2\cdot h_1$ and $c_b''\cdot h_2\cdot h_1$ as $U_b''(h_2)(h_1)$ and $c_b''(h_2)(h_1)$, where $b,\ h_1,\ h_2\in\mathbb{R}^N$.

Step 4. With the same arguments as those in *Step 2*, we can get that (U_b,c_b) is second-order continuously Fréchet differentiable in $b\in\mathbb{R}^N$ everywhere at $\mathbb{R}^N\setminus\{0\}$ under the topology $\|\cdot\|_{C^2(\mathbb{R}\times\mathbb{R}^N)\times\mathbb{R}}$. The proof of Theorem 2.10 is thereby complete. □

In the following proposition, we establish some estimates of Fréchet derivatives of U_e with respect to $e\in\mathbb{S}^{N-1}$, which are especially independent of $e\in\mathbb{S}^{N-1}$.

Proposition 4.9 *Assume that assumptions* (F1)-(F4) *hold. Normalize* U_e *as* (2.10). *Then* ∇U_b *is Fréchet differentiable in b, and*

$$
(\nabla U_b)'\cdot h_1=\nabla\left(U_b'\cdot h_1\right), \tag{4.34}
$$

where ∇ denotes the gradient operator with respect to $(s,z)\in\mathbb{R}\times\mathbb{R}^N$. In addition, for any $e,\ h_1,\ h_2\in\mathbb{S}^{N-1}$, there exists a positive constant K_4 independent of e, h_1 and h_2, such that

$$\left|\left(U_e'\cdot h_1\right)(s,z)\right|,\ \left|\left(U_e''\cdot h_2\cdot h_1\right)(s,z)\right|,\ \left|\nabla\left(U_b'\cdot h_1\right)(s,z)\right|\le K_4e^{-\frac{\kappa}{2}s} \tag{4.35}$$

for all $(s,z)\in[0,+\infty)\times\mathbb{R}^N$, and

$$\left|\left(U_e'\cdot h_1\right)(s,z)\right|,\ \left|\left(U_e''\cdot h_2\cdot h_1\right)(s,z)\right|,\ \left|\nabla\left(U_b'\cdot h_1\right)(s,z)\right|\le K_4e^{\frac{\kappa_2}{2}s} \tag{4.36}$$

for all $(s,z)\in(-\infty,0]\times\mathbb{R}^N$, where κ and κ_2 are given in Proposition 3.3.

Proof *Step 1*: *we prove* (4.34).

Since $f\in C^\infty(\mathbb{R}^{N+1})$, it follows from *Step 2* of the proof of Theorem 2.10 that

$$\left\|U_{b+h_1}-U_b-U_b'\cdot h_1\right\|_{C^3(\mathbb{R}\times\mathbb{R}^N)}=o(|h_1|)\ \text{ as } |h_1|\to 0,$$

which implies

$$\left\|\nabla U_{b+h_1}-\nabla U_b-\nabla\left(U_b'\cdot h_1\right)\right\|_{C^2(\mathbb{R}\times\mathbb{R}^N)}=o(|h_1|)\ \text{ as } |h_1|\to 0. \tag{4.37}$$

Consequently, ∇U_b is Fréchet differentiable in $b\in\mathbb{R}^N$ under the topology $\|\cdot\|_{C^2(\mathbb{R}\times\mathbb{R}^N)}$. In the meantime, (4.34) is valid from (4.37).

Step 2: *we prove* (4.35).

By virtue of the continuity of U_e in e with respect to the topology $\|\cdot\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}$ and the monotonicity of $U_e(s,z)$ in s, there exists a constant $q_2>0$ such that

$$U_e(s,z)\le p\ \text{ for all } (s,z)\in[q_2,+\infty)\times\mathbb{R}^N \text{ and } e\in\mathbb{S}^{N-1},$$

where p is defined in (1.3). Thus $f_u(z,U_e)\equiv 0$ in $[q_2,+\infty)\times\mathbb{R}^N$ for all $e\in\mathbb{S}^{N-1}$. Recalling (4.27), the function $U_e'\cdot h_1$ satisfies an equation of the type

$$0=\mathcal{N}_e\left(U_e'\cdot h_1\right)+\left(c_e'\cdot h_1\right)\partial_sU_e+2\nabla_z\partial_sU_e\cdot(h_1-(e\cdot h_1)e)=:\mathcal{N}_e\left(U_e'\cdot h_1\right)+\tilde{g} \tag{4.38}$$

in $[q_2,+\infty)\times\mathbb{R}^N$ for all $e,\ h_1\in\mathbb{S}^{N-1}$, where $\mathcal{N}_e$ is defined in Lemma 3.1.

By the continuity of U_e' in e under the topology $\|\cdot\|_{C^2(\mathbb{R}\times\mathbb{R}^N)}$ from Theorem 2.10, there exists a constant $\widetilde{K}$ such that $|(U_e'\cdot h_1)(q_2,z)|\le\widetilde{K}$ for all $z\in\mathbb{R}^N$ and $e,\ h_1\in\mathbb{S}^{N-1}$. Denote

$$K_4:=\max\left\{\widetilde{K}e^{\frac{\kappa}{2}q_2},4K_3\left(\sup_{e\in\mathbb{S}^{N-1}}\left\|c_e'\right\|+4N\right)/\kappa^2\right\},$$

then

$$\left(U_e'\cdot h_1\right)(q_2,z)\le\widetilde{K}\le K_4e^{-\frac{\kappa}{2}q_2}\ \text{ for all } z\in\mathbb{R}^N \text{ and } e,\ h_1\in\mathbb{S}^{N-1}.$$

By calculation, it holds from Theorem 2.8 and Proposition 3.6 that

$$\begin{aligned}\mathcal{N}_e\left(K_4e^{-\frac{\kappa}{2}s}\right)+\tilde{g}&=K_4e^{-\frac{\kappa}{2}s}\left(-\frac{\kappa}{2}c_e+\frac{\kappa^2}{4}\right)+\left(c_e'\cdot h_1\right)\partial_sU_e+2\nabla_z\partial_sU_e\cdot(h_1-(e\cdot h_1)e)\\&\le\left(-K_4\frac{\kappa^2}{4}+\left(\left\|c_e'\right\|+4N\right)K_3\right)e^{-\frac{\kappa}{2}s}\\&\le 0\end{aligned}$$

in $[q_2,+\infty)\times\mathbb{R}^N$ for all $e,\ h_1\in\mathbb{S}^{N-1}$.

Noting that $U_e' \cdot h_1 \in C^2(\mathbb{R}\times\mathbb{R}^N)\cap L^2_\rho(\mathbb{R}\times\mathbb{L}^N)$ from Theorem 2.10, it is easy to verify that

$$\lim_{s_0\to+\infty}\ \sup_{\{s\geq s_0,\, z\in\mathbb{R}^N\}} \left(U_e'\cdot h_1\right)(s,z)=0 \ \text{ for each } e,\ h_1\in\mathbb{S}^{N-1}, \tag{4.39}$$

which implies

$$\lim_{s_0\to+\infty}\ \sup_{\{s\geq s_0,\, z\in\mathbb{R}^N\}} \left(U_e'\cdot h_1 - K_4 e^{-\frac{\kappa}{2}s}\right)=0 \ \text{ for all } e,\ h_1\in\mathbb{S}^{N-1}.$$

Then applying Lemma 3.1 to $g=\tilde{g}$, $\varrho=+\infty$, $h=q_2$, $\phi^1=U_e'\cdot h_1$ and $\phi^2=K_4e^{-\frac{\kappa}{2}s}$, one infers

$$U_e'\cdot h_1(s,z)\leq K_4e^{-\frac{\kappa}{2}s} \ \text{ in } \Sigma_h^+$$

for all $e,\ h_1\in\mathbb{S}^{N-1}$. Note that $-U_e'\cdot h_1=U_e'(-h_1)$ for all $e,\ h_1\in\mathbb{S}^{N-1}$. Consequently

$$\left|\left(U_e'\cdot h_1\right)(s,z)\right|\leq K_4e^{-\frac{\kappa}{2}s} \ \text{ in } [q_2,+\infty)\times\mathbb{R}^N \tag{4.40}$$

for all $e,\ h_1\in\mathbb{S}^{N-1}$. Furthermore with similar arguments as those in Proposition 3.6, even if it means increasing K_4, one can get that

$$\left|D\left(U_e'\cdot h_1\right)(s,z)\right|,\ \left|D^2\left(U_e'\cdot h_1\right)(s,z)\right|\leq K_4e^{-\frac{\kappa}{2}s} \ \text{ in } [q_2,+\infty)\times\mathbb{R}^N \tag{4.41}$$

for all $e,\ h_1\in\mathbb{S}^{N-1}$.

Differentiating Eq. (4.27), it holds that $U_e''\cdot h_2\cdot h_1$ solves an equation of the type

$$\begin{aligned}
-\mathcal{N}_e\left(U_e''\cdot h_2\cdot h_1\right) &= \left(c_e'\cdot h_2\right)\partial_s\left(U_e'\cdot h_1\right)+2\nabla_z\partial_s\left(U_e'\cdot h_1\right)\cdot\left(h_2-(e\cdot h_2)e\right)\\
&\quad+\left(c_e'\cdot h_1\right)\partial_s\left(U_e'\cdot h_2\right)+\left(c_e''\cdot h_2\cdot h_1\right)\partial_sU_e\\
&\quad+2\nabla_z\partial_s\left(U_e'\cdot h_2\right)\cdot\left(h_1-(e\cdot h_1)e\right)\\
&\quad+2\nabla_z\partial_sU_e\cdot\left(-(e\cdot h_1)h_2-(h_2\cdot h_1)e\right)
\end{aligned}$$

in $[q_2,+\infty)\times\mathbb{R}^N$ for all $e,\ h_1,\ h_2\in\mathbb{S}^{N-1}$. Then by virtue of (4.41) and Proposition 3.6, with similar arguments as above, even if it means increasing K_4, we obtain

$$\left|\left(U_e''\cdot h_2\cdot h_1\right)(s,z)\right|\leq K_4e^{-\frac{\kappa}{2}s} \ \text{ in } [q_2,+\infty)\times\mathbb{R}^N \tag{4.42}$$

for all $e,\ h_1,\ h_2\in\mathbb{S}^{N-1}$. Finally (4.35) follows by (4.40)–(4.42).

Step 3: *we prove* (4.36).

Clearly, there exists a constant $q_3<0$ such that

$$1-\gamma_\star\leq U_e(s,z)\leq 1 \ \text{ for all } (s,z)\in(-\infty,q_3]\times\mathbb{R}^N \text{ and } e\in\mathbb{S}^{N-1}, \tag{4.43}$$

where $\gamma_\star$ is given in (1.5). It follows from (4.27) that the function $U_e'\cdot h_1$ satisfies

$$\mathcal{N}_e\left(U_e'\cdot h_1\right)+\hat{g}\left(s,z,U_e'\cdot h_1\right)=0 \ \text{ in } (-\infty,q_3]\times\mathbb{R}^N \tag{4.44}$$

for all $e,\ h_1\in\mathbb{S}^{N-1}$, where $\mathcal{N}_e$ is defined in Lemma 3.1 and

$$\hat{g}(s,z,u):=f_u(z,U_e)u+\left(c_e'\cdot h_1\right)\partial_sU_e+2\nabla_z\partial_sU_e\cdot\left(h_1-(e\cdot h_1)e\right)$$

for all $(s,z,u)\in(-\infty,q_3]\times\mathbb{R}^N\times\mathbb{R}$. Increasing K_4, it follows from (1.5), (3.4), (4.43), Theorem 2.8 and Proposition 3.6 that

$$\mathcal{N}_e\left(K_4e^{\frac{\kappa_2}{2}s}\right)+\hat{g}\left(s,z,K_4e^{\frac{\kappa_2}{2}s}\right)\leq\left[\left(-K\frac{\kappa_2}{2}-\frac{3}{4}\kappa_2^2\right)K_4+\left(\left\|c_e'\right\|+4N\right)K_3\right]e^{\frac{\kappa_2}{2}s}\leq 0$$

in $(-\infty, q_3] \times \mathbb{R}^N$ for all $e,\ h_1 \in \mathbb{S}^{N-1}$. By Theorem 2.10, there is a constant $\widehat{K}$ such that $|(U_e' \cdot h_1)(q_3, z)| \leq \widehat{K}$ for all $z \in \mathbb{R}^N$ and $e,\ h_1 \in \mathbb{S}^{N-1}$. Even if it means increasing K_4, one has

$$\left|\left(U_e' \cdot h_1\right)(q_3, z)\right| \leq \widehat{K} \leq K_4 e^{\frac{\kappa_2}{2} q_3} \text{ for all } z \in \mathbb{R}^N \text{ and } e,\ h_1 \in \mathbb{S}^{N-1}.$$

Noting that $U_e' \cdot h_1 \in C^2(\mathbb{R} \times \mathbb{R}^N) \cap L^2_\rho(\mathbb{R} \times \mathbb{L}^N)$ from Theorem 2.10, it is easy to verify that

$$\lim_{s_0 \to -\infty} \sup_{\{s \leq s_0,\, z \in \mathbb{R}^N\}} \left(U_e' \cdot h_1\right)(s, z) = 0 \text{ for each } e,\ h_1 \in \mathbb{S}^{N-1}.$$

Lastly, applying Lemma 3.2 to $g = \hat{g}$, $h = q_3$, $\varrho = +\infty$, $\phi^1 = U_e' \cdot h_1$ and $\phi^2 = K_4 e^{\frac{\kappa_2}{2} s}$, we obtain

$$\left|\left(U_e' \cdot h_1\right)(s, z)\right| \leq K_4 e^{\frac{\kappa_2}{2} s} \text{ in } (-\infty, q_3] \times \mathbb{R}^N \tag{4.45}$$

for all $e,\ h_1 \in \mathbb{S}^{N-1}$. Then with similar arguments as above and those in *Step 2*, we can get (4.36). The proof of Proposition 4.9 is thereby complete. □

5 Curved Fronts in $\mathbb{R}^2$

This section is devoted to the existence, uniqueness and stability of curved fronts admitting a shape similar to a V-shaped curve, namely Theorems 2.12, 2.15 and 2.16. Here the asymptotic behaviors and the Fréchet differentiability of pulsating fronts with respect to the propagation direction play key roles in our proof. Here we would like to notice that the main strategy in this section are similar to those for bistable equations, see [26]. However, as a counterpart to bistable equation [26], the degeneracy of f at $u = 0$ gives rise to main difficulties in this section. To overcome the difficulties, we have to apply sophisticated asymptotic behaviors of pulsating fronts near the state 0 to construct more complicated supersolutions, see also [55]. Throughout this section, we investigate the problem in two space dimensions, that is $N = 2$ in Eq. (1.1). Set $z := (x, y) \in \mathbb{R}^2$.

5.1 Existence

At first, we introduce some properties of the hyperbolic function $\mathrm{sech}(x)$, which can be checked easily.

Lemma 5.1 *One has*

(i) $|sech'(x)|,\ |sech''(x)| \leq sech(x)$ *for all* $x \in \mathbb{R}$,
(ii) $sech'(x) > 0$ *for all* $x < 0$; $sech'(x) < 0$ *for all* $x > 0$,
(iii) *there is a positive constant* q *such that* $sech''(x) > 0$ *for all* $|x| \geq q$.

The following lemma comes from [26], which gives a smooth function with two asymptotic lines.

Lemma 5.2 ([26], Lemma 2.2) *For any angles* $0 < \alpha < \beta < \pi$, *there is a smooth function* $\psi(x)$ *for* $x \in \mathbb{R}$ *with* $y = -x \cot \alpha$ *and* $y = -x \cot \beta$ *being its asymptotic lines and there* *are two positive constants* k_1 *and* K_5 *such that*

$$\begin{cases} \psi''(x) > 0 & \text{for all } x \in \mathbb{R}, \\ -\cot\alpha < \psi'(x) < -\cot\beta & \text{for all } x \in \mathbb{R}, \\ k_1 sech(x) \le \psi'(x) + \cot\alpha \le K_5 sech(x) & \text{for all } x < 0, \\ \left| \frac{1}{\sqrt{\psi'^2(x)+1}} - \sin\alpha \right| \le K_5 sech(x) & \text{for all } x < 0, \\ k_1 sech(x) \le -\cot\beta - \psi'(x) \le K_5 sech(x) & \text{for all } x \ge 0, \\ \left| \frac{1}{\sqrt{\psi'^2(x)+1}} - \sin\beta \right| \le K_5 sech(x) & \text{for all } x \ge 0, \\ \max\left\{ \left|\psi''(x)\right|, \left|\psi'''(x)\right| \right\} \le K_5 sech(x) & \text{for all } x \in \mathbb{R}. \end{cases} \tag{5.1}$$

Remark 5.3 In fact, by the proof of Lemma 2.2 of [26], the function $\psi(x)$ has the form

$$\psi(x) = \begin{cases} -x\cot\alpha + \varsigma \operatorname{sech}(x), & \text{when } x \le -a, \\ -x\cot\beta + \varsigma \operatorname{sech}(x), & \text{when } x \ge b, \end{cases}$$

where positive constants ς, a and b are given in its proof.

Now, we construct a vector-valued function of the form

$$e(x) := (e_1(x),\ e_2(x)) = \left(-\frac{\psi'(\lambda x)}{\sqrt{\psi'^2(\lambda x)+1}},\ \frac{1}{\sqrt{\psi'^2(\lambda x)+1}} \right), \quad \forall x \in \mathbb{R},$$

where λ is a number to be determined. By Lemma 5.2, every component of $e(x)$ is smooth and

$$e(x) \to (\cos\alpha, \sin\alpha) \text{ as } x \to -\infty \ \text{ and } \ e(x) \to (\cos\beta, \sin\beta) \text{ as } x \to +\infty.$$

The derivatives of $e(x)$ can be denoted by

$$e'(x) = \left(e_1'(x),\ e_2'(x)\right) = \left(-\frac{\lambda\psi''(\lambda x)}{(\psi'^2(\lambda x)+1)^{\frac{3}{2}}},\ -\frac{\lambda\psi'(\lambda x)\psi''(\lambda x)}{(\psi'^2(\lambda x)+1)^{\frac{3}{2}}} \right)$$

and $e''(x) = \left(e_1''(x),\ e_2''(x)\right)$, where

$$e_1''(x) = -\frac{\lambda^2\psi'''(\lambda x)}{(\psi'^2(\lambda x)+1)^{\frac{3}{2}}} + \frac{3\lambda^2\psi'(\lambda x)\psi''^2(\lambda x)}{(\psi'^2(\lambda x)+1)^{\frac{5}{2}}},$$
$$e_2''(x) = -\frac{\lambda^2\psi''^2(\lambda x)}{(\psi'^2(\lambda x)+1)^{\frac{3}{2}}} - \frac{\lambda^2\psi'(\lambda x)\psi'''(\lambda x)}{(\psi'^2(\lambda x)+1)^{\frac{3}{2}}} + \frac{3\lambda^2\psi'^2(\lambda x)\psi''^2(\lambda x)}{(\psi'^2(\lambda x)+1)^{\frac{5}{2}}}.$$

Furthermore, it follows from Lemma 5.2 that there exists a positive constant K_6 such that

$$\left|e'(x)\right| \le \lambda K_6 \operatorname{sech}(\lambda x), \quad \left|e''(x)\right| \le \lambda^2 K_6 \operatorname{sech}(\lambda x), \quad \forall x \in \mathbb{R}. \tag{5.2}$$

Let $\omega(s)$ be a smooth function satisfying $\omega'(s) \ge 0$ and

$$\omega(s) = \begin{cases} 0, & \text{when } s \le -1, \\ \omega(s), & \text{when } s \in (-1, 1), \\ 1, & \text{when } s \ge 1. \end{cases} \tag{5.3}$$

For ease of reading, we write below some stated assumptions related to Theorem 2.12. Firstly, α and β are two given angles satisfying $0 < \alpha < \beta < \pi$. Secondly, let $c_{\alpha\beta}$ be a constant satisfying

$$c_{\alpha\beta} = \frac{c_\alpha}{\sin\alpha} = \frac{c_\beta}{\sin\beta}.$$

Thirdly, $(U_\alpha(s,x,y), c_\alpha)$ is the unique pulsating front with propagation direction $(\cos\alpha, \sin\alpha)$ and $\big(U_\beta(s,x,y), c_\beta\big)$ is the unique pulsating front with propagation direction $(\cos\beta, \sin\beta)$ in the sense of Definition 1.1. Fourthly, the function $U^-_{\alpha\beta}$ is given by

$$U^-_{\alpha\beta}(t,x,y) \overset{\text{def}}{=} \max\left\{U_\alpha(x\cos\alpha + y\sin\alpha - c_\alpha t, x, y),\ U_\beta(x\cos\beta + y\sin\beta - c_\beta t, x, y)\right\}, \tag{5.4}$$

which is a subsolution of (1.1). Lastly, denote ξ_α and ξ_β by

$$\xi_\alpha := x\cos\alpha + y\sin\alpha - c_\alpha t = \sin\alpha\left((y - c_{\alpha\beta}t) + x\cot\alpha\right), \tag{5.5}$$

$$\xi_\beta := x\cos\beta + y\sin\beta - c_\beta t = \sin\beta\left((y - c_{\alpha\beta}t) + x\cot\beta\right). \tag{5.6}$$

Now, we construct two functions ξ and η, where

$$\xi = \xi(t,x,y) := \frac{y - c_{\alpha\beta}t - \psi(\lambda x)/\lambda}{\sqrt{\psi'^2(\lambda x)+1}}, \tag{5.7}$$

$$\eta = \eta(t,x,y) := y - c_{\alpha\beta}t - \psi(\lambda x)/\lambda, \tag{5.8}$$

where the real number λ is to be determined.

Lemma 5.4 *Assume that assumptions* (F1)-(F4) *hold. Then there is a constant* $\delta^* > 0$ *such that the below statement is valid: for each* $\delta \in (0, \delta^*]$ *there exists a positive constant* $\varepsilon_0^+(\delta)$ *such that, for any* $0 < \varepsilon < \varepsilon_0^+(\delta)$ *there is a positive constant* $\lambda_0^+(\delta,\varepsilon)$ *such that for arbitrary* $0 < \lambda < \lambda_0^+(\delta,\varepsilon)$*, the function*

$$U^+(t,x,y) := U_{e(x)}(\xi,x,y) + \varepsilon sech(\lambda x) \times \left[U_\alpha^\delta(\eta,x,y)\omega(\xi) + (1-\omega(\xi))\right] \tag{5.9}$$

is a supersolution of Eq. (1.1)*. Futhermore*

$$\lim_{R\to+\infty}\ \sup_{x^2+(y-c_{\alpha\beta}t)^2>R^2}\left|U^+(t,x,y) - U^-_{\alpha\beta}(t,x,y)\right| \le \varepsilon, \tag{5.10}$$

$$U^+(t,x,y) \ge U^-_{\alpha\beta}(t,x,y)\ \ in\ \mathbb{R}^3, \tag{5.11}$$

$$\frac{\partial}{\partial t}U^+(t,x,y) > 0\ \ in\ \mathbb{R}^3. \tag{5.12}$$

Proof *Step 1*: *we prove that* U^+ *is a supersolution.*

The strategy is to find two numbers $X' > 1$ and $X'' > 1$ and show the inequality

$$\mathcal{L}U^+ := \partial_t U^+ - \Delta_{x,y}U^+ - f\big(x,y,U^+\big) \ge 0,\quad \forall (t,x,y) \in \mathbb{R}^3,$$

by considering three cases $\xi > X'$, $\xi < -X''$, and $\xi \in [-X'', X']$, respectively. Denote

$$I_1 := (\partial_t - \Delta_{x,y})\big(U_{e(x)}(\xi,x,y)\big)\ \text{ and }\ I_2 := (\partial_t - \Delta_{x,y})\Big(\varepsilon \operatorname{sech}(\lambda x)U_\alpha^\delta(\eta,x,y)\Big).$$

By virtue of Theorem 2.10 and Proposition 4.9, it follows from calculation that

$$\begin{aligned} I_1 = {} & \partial_s U_{e(x)}\xi_t - \Delta_{x,y}U_{e(x)} - 2\partial_{sx}U_{e(x)}\xi_x - 2\partial_{sy}U_{e(x)}\xi_y \\ & - \partial_s U_{e(x)}(\xi_{xx}+\xi_{yy}) - \partial_{ss}U_{e(x)}\big(\xi_x^2+\xi_y^2\big) - U''_{e(x)}\cdot e'(x)\cdot e'(x) \\ & - U'_{e(x)}\cdot e''(x) - 2\partial_x U'_{e(x)}\cdot e'(x) - 2\partial_s U'_{e(x)}\cdot e'(x)\xi_x, \end{aligned} \tag{5.13}$$

where $U_{e(x)}$ and all of its derivatives are evaluated at $(\xi(t,x,y),x,y)$. Now, we compute

$$\begin{cases} \xi_t = -\frac{c_{\alpha\beta}}{\sqrt{\psi'^2+1}}, \\ \xi_x = -\frac{\lambda\psi'\psi''}{\psi'^2+1}\xi - \frac{\psi'}{\sqrt{\psi'^2+1}}, \\ \xi_y = \frac{1}{\sqrt{\psi'^2+1}}, \\ \xi_{xx} = -\frac{\lambda^2\psi''^2+\lambda^2\psi'\psi'''}{\psi'^2+1}\xi + \frac{3\lambda^2\psi'^2\psi''^2}{(\psi'^2+1)^2}\xi + \frac{\lambda(\psi'^2-1)\psi''}{(\psi'^2+1)^{\frac{3}{2}}}, \\ \xi_{yy} = 0, \\ \xi_x - e_1(x) = -\frac{\lambda\psi'\psi''}{\psi'^2+1}\xi, \\ \xi_x^2+\xi_y^2-1 = \frac{\lambda^2\psi'^2\psi''^2}{(\psi'^2+1)^2}\xi^2 + \frac{2\lambda\psi'^2\psi''}{(\psi'^2+1)^{\frac{3}{2}}}\xi, \end{cases} \tag{5.14}$$

where functions ψ', ψ'', and ψ''' are evaluated at λx. Note that $(U_{e(x)}, c_{e(x)})$ solves

$$c_{e(x)}\partial_s U_{e(x)} + \partial_{ss}U_{e(x)} + 2\nabla_{x,y}\partial_s U_{e(x)}\cdot e(x) + \Delta_{x,y}U_{e(x)} + f\big(x,y,U_{e(x)}\big) = 0.$$

Since $e_2(x) = \xi_y$, by virtue of the above equation and (5.13), one gets that

$$\begin{aligned} I_1 = &\big(\xi_t + c_{e(x)}\big)\partial_s U_{e(x)} - 2\partial_{sx}U_{e(x)}(\xi_x - e_1(x)) - \partial_s U_{e(x)}(\xi_{xx}+\xi_{yy}) \\ &- \partial_{ss}U_{e(x)}\big(\xi_x^2+\xi_y^2-1\big) - U''_{e(x)}\cdot e'(x)\cdot e'(x) - U'_{e(x)}\cdot e''(x) \\ &- 2\partial_x U'_{e(x)}\cdot e'(x) - 2\partial_s U'_{e(x)}\cdot e'(x)\xi_x + f\big(x,y,U_{e(x)}\big), \end{aligned} \tag{5.15}$$

where $U_{e(x)}$ and all of its derivatives are evaluated at $(\xi(t,x,y),x,y)$. From Claim 2.9 of [26], there is a positive constant K_7 such that

$$\xi_t + c_{e(x)} = -\frac{c_{\alpha\beta}}{\sqrt{\psi'^2(\lambda x)+1}} + c_{e(x)} \le -K_7\operatorname{sech}(\lambda x) < 0 \text{ for all } x\in\mathbb{R}. \tag{5.16}$$

Now, it turns to compute I_2 and one has that

$$\begin{aligned} I_2 = &\varepsilon\delta\operatorname{sech}(\lambda x)U_\alpha^{\delta-1}\partial_s U_\alpha \eta_t - \varepsilon\lambda^2\operatorname{sech}''(\lambda x)U_\alpha^\delta \\ &- 2\varepsilon\delta\lambda\operatorname{sech}'(\lambda x)U_\alpha^{\delta-1}\times[\partial_x U_\alpha + \partial_s U_\alpha\eta_x] \\ &- \varepsilon\delta(\delta-1)\operatorname{sech}(\lambda x)U_\alpha^{\delta-2}\times\Big[(\partial_x U_\alpha + \partial_s U_\alpha\eta_x)^2 + \big(\partial_y U_\alpha + \partial_s U_\alpha\eta_y\big)^2\Big] \\ &- \varepsilon\delta\operatorname{sech}(\lambda x)U_\alpha^{\delta-1}\times\Big[\Delta_{x,y}U_\alpha + 2\nabla_{x,y}\partial_s U_\alpha\cdot(\eta_x,\eta_y) + \partial_{ss}U_\alpha\big(\eta_x^2+\eta_y^2\big) \\ &+ \partial_s U_\alpha(\eta_{xx}+\eta_{yy})\Big], \end{aligned} \tag{5.17}$$

where U_α and all of its derivatives are evaluated at $(\eta(t,x,y),x,y)$.

Case 1: $\xi(t,x,y) > X'$, *where* $X' > 1$ *is to be chosen.*

In this case, it holds that $U^+(t,x,y) = U_{e(x)}(\xi,x,y) + \varepsilon\operatorname{sech}(\lambda x)\times U_\alpha^\delta(\eta,x,y)$. Thus

$$\mathcal{L}U^+ = I_1 + I_2 - f\big(x,y,U^+\big).$$

Computing the derivatives of $\eta(t, x, y)$ yields

$$\begin{cases} \eta_t = -c_{\alpha\beta}, \\ \eta_x = -\psi'(\lambda x), \\ \eta_y = 1, \\ \eta_{xx} = -\lambda\psi''(\lambda x), \\ \eta_{yy} = 0, \\ \eta_x^2 + \eta_y^2 = \psi'^2(\lambda x) + 1. \end{cases} \tag{5.18}$$

By virtue of (5.17), (5.18) and Lemma 5.1, one gets that

$$\begin{aligned} I_2 = & -\varepsilon U_\alpha^\delta \times \lambda \left[\lambda \mathrm{sech}''(\lambda x) + 2\delta \mathrm{sech}'(\lambda x) \frac{\partial_x U_\alpha - \psi'(\lambda x)\partial_s U_\alpha}{U_\alpha} - \delta \mathrm{sech}(\lambda x)\psi''(\lambda x) \frac{\partial_s U_\alpha}{U_\alpha} \right] \\ & - \varepsilon U_\alpha^\delta \times \delta \mathrm{sech}(\lambda x) \left[\frac{\Delta_{x,y} U_\alpha + 2\nabla_{x,y}\partial_s U_\alpha \cdot \left(-\psi'(\lambda x), 1\right)}{U_\alpha} \right. \\ & + \delta \frac{\left(\partial_x U_\alpha - \psi'(\lambda x)\partial_s U_\alpha\right)^2 + \left(\partial_y U_\alpha + \partial_s U_\alpha\right)^2}{U_\alpha^2} \\ & - \frac{\left(\partial_x U_\alpha - \psi'(\lambda x)\partial_s U_\alpha\right)^2 + \left(\partial_y U_\alpha + \partial_s U_\alpha\right)^2}{U_\alpha^2} \\ & \left. + \frac{\partial_{ss} U_\alpha}{U_\alpha}\left(\psi'^2(\lambda x) + 1\right) + c_{\alpha\beta} \frac{\partial_s U_\alpha}{U_\alpha} \right] \\ \geq & -\varepsilon \mathrm{sech}(\lambda x) U_\alpha^\delta \times \lambda \left[\lambda + 2\delta \frac{\left|\partial_x U_\alpha - \psi'(\lambda x)\partial_s U_\alpha\right|}{U_\alpha} - \delta\psi''(\lambda x) \frac{\partial_s U_\alpha}{U_\alpha} \right] \\ & - \varepsilon \mathrm{sech}(\lambda x) U_\alpha^\delta \times \delta \left[\frac{\Delta_{x,y} U_\alpha + 2\nabla_{x,y}\partial_s U_\alpha \cdot (-\psi'(\lambda x), 1)}{U_\alpha} \right. \\ & + \delta \frac{\left(\partial_x U_\alpha - \psi'(\lambda x)\partial_s U_\alpha\right)^2 + \left(\partial_y U_\alpha + \partial_s U_\alpha\right)^2}{U_\alpha^2} \\ & - \frac{\left(\partial_x U_\alpha - \psi'(\lambda x)\partial_s U_\alpha\right)^2 + \left(\partial_y U_\alpha + \partial_s U_\alpha\right)^2}{U_\alpha^2} \\ & \left. + \frac{\partial_{ss} U_\alpha}{U_\alpha}\left(\psi'^2(\lambda x) + 1\right) + c_{\alpha\beta} \frac{\partial_s U_\alpha}{U_\alpha} \right] \\ =: & \; J_1 + J_2, \end{aligned} \tag{5.19}$$

where U_α and all of its derivatives are evaluated at $(\eta(t, x, y), x, y)$. Recalling (5.7) and (5.8), it follows from Lemma 5.2 that

$$\xi \leq \eta = \xi\sqrt{\psi'^2(\lambda x) + 1} \leq \xi\sqrt{\max\{\cot\alpha, -\cot\beta\}^2 + 1}. \tag{5.20}$$

By virtue of Theorem 2.5, it holds that

$$\begin{cases} \dfrac{\Delta_{x,y}U_\alpha + 2\nabla_{x,y}\partial_s U_\alpha \cdot (-\psi'(\lambda x), 1)}{U_\alpha} \longrightarrow 0, \\ \dfrac{\left(\partial_x U_\alpha - \psi'(\lambda x)\partial_s U_\alpha\right)^2 + \left(\partial_y U_\alpha + \partial_s U_\alpha\right)^2}{U_\alpha^2} \longrightarrow c_\alpha^2\left(\psi'^2(\lambda x)+1\right), \\ \dfrac{\partial_{ss}U_\alpha}{U_\alpha}\left(\psi'^2(\lambda x)+1\right) \longrightarrow c_\alpha^2\left(\psi'^2(\lambda x)+1\right), \\ c_{\alpha\beta}\dfrac{\partial_s U_\alpha}{U_\alpha} \longrightarrow -c_{\alpha\beta}c_\alpha \end{cases} \tag{5.21}$$

as $\eta \to +\infty$ uniformly in $(x, y) \in \mathbb{R}^2$. Set

$$\delta_1^* := \frac{c_{\alpha\beta}}{2c_\alpha\left(\max\{\cot\alpha, -\cot\beta\}^2 + 1\right)}.$$

Then for any $\delta \in (0, \delta_1^*]$, there exists a sufficiently large number $X_1' > 1$ such that

$$J_2 > \varepsilon \mathrm{sech}(\lambda x) U_\alpha^\delta \times \delta \frac{c_{\alpha\beta}c_\alpha}{4} \tag{5.22}$$

for all $(\eta, x, y) \in (X_1', +\infty) \times \mathbb{R}^2$. Thus by (5.19) and (5.22), there is a small positive constant $\lambda_1^+(\delta)$ such that, for arbitrary $\lambda \le \lambda_1^+(\delta)$, one has

$$I_2 > \varepsilon \mathrm{sech}(\lambda x) U_\alpha^\delta \times \delta \frac{c_{\alpha\beta}c_\alpha}{8} \tag{5.23}$$

for all $(\eta, x, y) \in (X_1', +\infty) \times \mathbb{R}^2$. Recalling $\partial_s U_{e(x)} < 0$, by virtue of (5.2), (5.14)–(5.16), Propositions 3.3, 3.6, 4.9 and Lemma 5.2, there exists a constant $\Lambda_1 > 0$ such that

$$I_1 \ge -\Lambda_1 \mathrm{sech}(\lambda x) e^{-\frac{\kappa}{4}\xi} + f\left(x, y, U_{e(x)}\right) \tag{5.24}$$

for all $(\xi, x, y) \in [0, +\infty) \times \mathbb{R}^2$, where κ is given in Theorem 2.8. Following from Theorem 2.2, we have that there exists a sufficiently large number $X_2' > 1$ such that

$$U_\alpha(\eta, x, y) \ge \frac{C_1}{2} e^{-c_\alpha \eta} \ge \frac{C_1}{2} e^{-K\eta}, \quad \forall (\eta, x, y) \in (X_2', +\infty) \times \mathbb{R}^2, \tag{5.25}$$

where C_1 is given in Theorem 2.2 and K is given in Theorem 2.8. Denote

$$\delta_2^* := \frac{\kappa}{8K\sqrt{\max\{\cot\alpha, -\cot\beta\}^2 + 1}}.$$

By virtue of (5.20) and (5.25), for each $\delta \in (0, \delta_2^*]$ one has

$$U_\alpha^\delta(\eta, x, y) \ge \left(\frac{C_1}{2}\right)^\delta e^{-\frac{\kappa}{8}\xi}, \quad \forall (\eta, x, y) \in (X_2', +\infty) \times \mathbb{R}^2. \tag{5.26}$$

Since (1.3) and Proposition 3.3, by virtue of (5.20), (5.23), (5.24) and (5.26), there exists a sufficiently large number $X_3' > 1$ such that

$$\begin{aligned} \mathcal{L}U^+ &= I_1 + I_2 - f\left(x, y, U^+\right) \\ &\ge -\Lambda_1 \mathrm{sech}(\lambda x) e^{-\frac{\kappa}{4}\xi} + \varepsilon \mathrm{sech}(\lambda x) U_\alpha^\delta \times \delta \frac{c_{\alpha\beta}c_\alpha}{8} \\ &\ge -\Lambda_1 \times \mathrm{sech}(\lambda x) e^{-\frac{\kappa}{4}\xi} + \left(\frac{C_1}{2}\right)^\delta \varepsilon\delta \frac{c_{\alpha\beta}c_\alpha}{8} \times \mathrm{sech}(\lambda x) e^{-\frac{\kappa}{8}\xi} \\ &> 0 \end{aligned} \tag{5.27}$$

for all $(\eta, x, y) \in (X_3', +\infty) \times \mathbb{R}^2$. Note that $\eta(t, x, y) \geq \xi(t, x, y)$ from (5.20), thus setting $\delta^* = \min\{\delta_1^*, \delta_2^*\}$, $X' = \max\{X_1', X_2', X_3'\}$ and $\lambda_0^+ \leq \lambda_1^+(\delta)$, we have $\mathcal{L}U^+ > 0$ in *Case 1*.

Case 2: $\xi(t, x, y) < -X''$, *where* $X'' > 1$ *is to be chosen.*

In this case, it holds that $U^+(t, x, y) = U_{e(x)}(\xi, x, y) + \varepsilon \operatorname{sech}(\lambda x)$. Recalling (5.15), (5.16) and $\partial_s U_{e(x)} < 0$, one has

$$
\begin{aligned}
\mathcal{L}U^+ &= I_1 - \varepsilon\lambda^2 \operatorname{sech}''(\lambda x) - f\big(x, y, U^+\big) \\
&> -2\partial_{sx} U_{e(x)}(\xi_x - e_1(x)) - \partial_s U_{e(x)}(\xi_{xx} + \xi_{yy}) \\
&\quad - \partial_{ss} U_{e(x)}\big(\xi_x^2 + \xi_y^2 - 1\big) - U''_{e(x)} \cdot e'(x) \cdot e'(x) - U'_{e(x)} \cdot e''(x) \\
&\quad - 2\partial_x U'_{e(x)} \cdot e'(x) - 2\partial_s U'_{e(x)} \cdot e'(x)\xi_x \\
&\quad + f\big(x, y, U_{e(x)}\big) - f\big(x, y, U^+\big) - \varepsilon\lambda^2 \operatorname{sech}''(\lambda x),
\end{aligned}
\tag{5.28}
$$

where $U_{e(x)}$ and all of its derivatives are evaluated at $(\xi(t, x, y), x, y)$. Set $\varepsilon_1^+ := \gamma_\star/2$, where $\gamma_\star$ is defined in (1.5). Then for any $\varepsilon < \varepsilon_1^+$, by virtue of (1.5) and Proposition 3.3, there is a sufficiently large number $X'' > 1$ such that

$$
f\big(x, y, U_{e(x)}\big) - f\big(x, y, U^+\big) > \frac{\kappa_1}{2} \varepsilon \operatorname{sech}(\lambda x) \tag{5.29}
$$

for all $(\xi, x, y) \in (-\infty, -X'') \times \mathbb{R}^2$, where κ_1 is given in (1.4). By (5.2), (5.14), (5.28), (5.29), Propositions 3.6, 4.9 and Lemma 5.2, one concludes that there is a constant $\Lambda_2 > 0$ such that

$$
\mathcal{L}U^+ > \operatorname{sech}(\lambda x) \times \left[-\varepsilon\lambda^2 - \Lambda_2\lambda + \frac{\kappa_1}{2}\varepsilon\right] \tag{5.30}
$$

for all $(\xi, x, y) \in (-\infty, -X'') \times \mathbb{R}^2$. It is trivial that there exists a constant $\lambda_2^+(\varepsilon) > 0$ such that

$$
-\varepsilon\lambda^2 - \Lambda_2\lambda + \frac{\kappa_1}{2}\varepsilon \geq 0 \quad \text{for all } \lambda \in \big(0, \lambda_2^+(\varepsilon)\big). \tag{5.31}
$$

Therefore setting $\varepsilon_0^+ \leq \varepsilon_1^+$ and $\lambda_0^+ \leq \lambda_2^+(\varepsilon)$, (5.30) and (5.31) yield that $\mathcal{L}U^+ > 0$ in *Case 2.*

Case 3: $-X'' \leq \xi(t, x, y) \leq X'$.

Recalling (5.14) and (5.18), there is a number $\Lambda_3 > 0$ such that

$$
\Big|(\partial_t - \Delta_{x,y})\Big(\varepsilon \operatorname{sech}(\lambda x)\Big[U_\alpha^\delta(\eta, x, y)\omega(\xi) + (1 - \omega(\xi))\Big]\Big)\Big| < \Lambda_3 \varepsilon \operatorname{sech}(\lambda x) \tag{5.32}
$$

for all $(\xi, x, y) \in [-X'', X'] \times \mathbb{R}^2$. It follows from (5.15) and (5.32) that

$$
\begin{aligned}
\mathcal{L}U^+ &> I_1 - f\big(x, y, U^+\big) - \Lambda_3 \varepsilon \operatorname{sech}(\lambda x) \\
&= \big(\xi_t + c_{e(x)}\big)\partial_s U_{e(x)} - 2\partial_{sx} U_{e(x)}(\xi_x - e_1(x)) - \partial_s U_{e(x)}(\xi_{xx} + \xi_{yy}) \\
&\quad - \partial_{ss} U_{e(x)}\big(\xi_x^2 + \xi_y^2 - 1\big) - U''_{e(x)} \cdot e'(x) \cdot e'(x) - U'_{e(x)} \cdot e''(x) \\
&\quad - 2\partial_x U'_{e(x)} \cdot e'(x) - 2\partial_s U'_{e(x)} \cdot e'(x)\xi_x \\
&\quad + f\big(x, y, U_{e(x)}\big) - f\big(x, y, U^+\big) - \Lambda_3 \varepsilon \operatorname{sech}(\lambda x)
\end{aligned}
\tag{5.33}
$$

for all $(\xi, x, y) \in [-X'', X'] \times \mathbb{R}^2$, where $U_{e(x)}$ and all of its derivatives are evaluated at $(\xi(t, x, y), x, y)$. From Proposition 3.9 and (5.16), there exists a number $r > 0$ such that

$$
\big(\xi_t + c_{e(x)}\big)\partial_s U_{e(x)} \geq K_7 r \operatorname{sech}(\lambda x) \tag{5.34}
$$

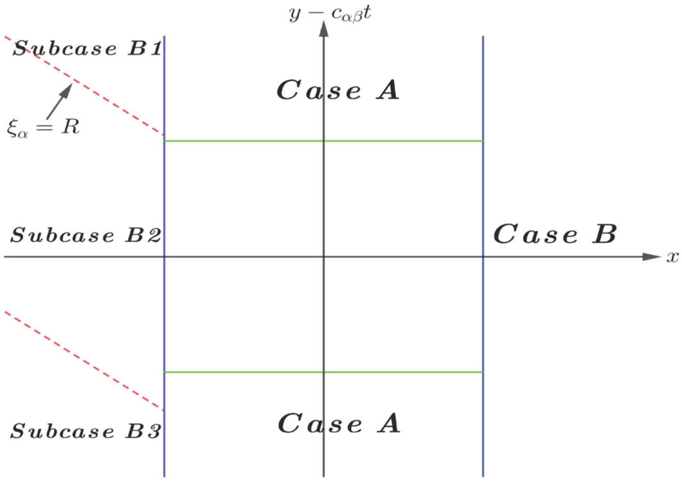

Fig. 1 Several cases to prove (5.38)

for all $(\xi, x, y) \in [-X'', X'] \times \mathbb{R}^2$. By virtue of (5.2), (5.14), Propositions 3.6, 4.9 and Lemma 5.2, since ξ is bounded, there exists a constant $\Lambda_4 > 0$ such that

$$\begin{aligned}&-2\partial_{sx}U_{e(x)}(\xi_x - e_1(x)) - \partial_s U_{e(x)}(\xi_{xx} + \xi_{yy}) - \partial_{ss}U_{e(x)}\left(\xi_x^2 + \xi_y^2 - 1\right)\\ &- U''_{e(x)} \cdot e'(x) \cdot e'(x) - U'_{e(x)} \cdot e''(x) - 2\partial_x U'_{e(x)} \cdot e'(x) - 2\partial_s U'_{e(x)} \cdot e'(x)\xi_x\\ &\ge -\Lambda_4 \lambda \operatorname{sech}(\lambda x)\end{aligned} \tag{5.35}$$

for all $(\xi, x, y) \in [-X'', X'] \times \mathbb{R}^2$. By (1.2) there is a number $\Lambda_5 > 0$ such that

$$f\left(x, y, U_{e(x)}\right) - f\left(x, y, U^+\right) > -\Lambda_5 \varepsilon \operatorname{sech}(\lambda x). \tag{5.36}$$

It follows immediately from (5.33)–(5.36) that

$$\mathcal{L}U^+ > \operatorname{sech}(\lambda x) \times [K_7 r - \Lambda_4 \lambda - \Lambda_5 \varepsilon - \Lambda_3 \varepsilon], \quad \forall (\xi, x, y) \in [-X'', X'] \times \mathbb{R}^2. \tag{5.37}$$

Denote

$$\lambda_3^+(\varepsilon) := \varepsilon \quad \text{and} \quad \varepsilon_2^+(\delta) := \frac{K_7 r}{\Lambda_3 + \Lambda_4 + \Lambda_5}.$$

Then setting $\varepsilon_0^+ \le \varepsilon_2^+(\delta)$ and $\lambda_0^+ \le \lambda_3^+(\varepsilon)$, $\mathcal{L}U^+ > 0$ holds in *Case 3* by (5.37). All in all, *Step 1* is complete by setting $\delta^* = \min\{\delta_1^*, \delta_2^*\}$, $\varepsilon_0^+ \le \min\{\varepsilon_1^+, \varepsilon_2^+(\delta), \gamma_\star/3\}$ and $\lambda_0^+ \le \min\{\lambda_1^+(\delta), \lambda_2^+(\varepsilon), \lambda_3^+(\varepsilon)\}$, where $\gamma_\star$ is defined in (1.5).

Step 2: *we prove* (5.10).

We claim that

$$\lim_{R \to +\infty} \sup_{x^2 + (y - c_{\alpha\beta}t)^2 > R^2} \left| U_{e(x)}(\xi, x, y) - U_{\alpha\beta}^-(t, x, y) \right| = 0, \tag{5.38}$$

which yields (5.10) immediately. Denote $T := x^2 + (y - c_{\alpha\beta}t)^2$. The proof of (5.38) is divided into several cases (see Fig. 1).

Case A: $|x|$ *is bounded.* In this case, $|y - c_{\alpha\beta}t|$ is unbounded. By (5.5)–(5.7), one has

$$\xi, \ \xi_\alpha, \ \xi_\beta \to +\infty \quad \text{or} \quad \xi, \ \xi_\alpha, \ \xi_\beta \to -\infty.$$

Then it follows from Proposition 3.3 that

$$\left|U_{e(x)}(\xi, x, y) - U^{-}_{\alpha\beta}(t, x, y)\right| \to 0 \text{ as } T \to +\infty \text{ in } \textit{Case A}.$$

Case B: $|x|$ *is unbounded.* We only discuss the case $x \to -\infty$, and the case $x \to +\infty$ is similar. It follows from (5.5), (5.6) and Remark 5.3 that

$$\begin{cases} \xi_\beta = \frac{\sin\beta}{\sin\alpha}\xi_\alpha - \frac{\sin(\beta-\alpha)}{\sin\alpha}x & \text{in } \mathbb{R}^3, \\ \psi(\lambda x)/\lambda = -x\cot\alpha + \varsigma \operatorname{sech}(\lambda x)/\lambda & \text{for all } x \le -a/\lambda. \end{cases} \tag{5.39}$$

Subcase B1: $x \to -\infty$ *and* $\xi_\alpha \to +\infty$. By (5.39), one gets

$$\xi_\beta \ge \frac{\sin\beta}{\sin\alpha}\xi_\alpha \text{ and } \eta \ge \frac{\xi_\alpha}{\sin\alpha} - \varsigma/\lambda,$$

which implies $\xi_\beta,\ \xi \to +\infty$ in *Subcase B1*. This yields by Proposition 3.3 that

$$\left|U_{e(x)}(\xi, x, y) - U^{-}_{\alpha\beta}(t, x, y)\right| \to 0 \text{ as } T \to +\infty \text{ in } \textit{Subcase B1}.$$

Subcase B2: $x \to -\infty$ *and* ξ_α *is bounded.* It holds from (5.39) that $\xi_\beta \to +\infty$, which yields from Propositions 3.3 and 3.9 that $U^{-}_{\alpha\beta} = U_\alpha(\xi_\alpha, x, y)$ in *Subcase B2*. By virtue of (5.39), Lemma 5.2, Remark 5.3, Propositions 3.6 and 4.9, one gets

$$\begin{aligned} \left|U_{e(x)}(\xi, x, y) - U_\alpha(\xi_\alpha, x, y)\right| &\le \left|U_{e(x)}(\xi, x, y) - U_\alpha(\xi, x, y)\right| + \left|U_\alpha(\xi, x, y) - U_\alpha(\xi_\alpha, x, y)\right| \\ &\le K_4\,|e(x) - (\cos\alpha, \sin\alpha)| + K_3\,|\xi - \xi_\alpha| \\ &\le K_4\,|e(x) - (\cos\alpha, \sin\alpha)| + \left(\frac{K_3\,|\xi_\alpha|}{\sin\alpha}K_5 + \frac{K_3\varsigma}{\lambda}\right)\operatorname{sech}(\lambda x) \end{aligned}$$

for all $x \le -a/\lambda$. Therefore,

$$\left|U_{e(x)}(\xi, x, y) - U^{-}_{\alpha\beta}(t, x, y)\right| \to 0 \text{ as } T \to +\infty \text{ in } \textit{Subcase B2}.$$

Subcase B3: $x \to -\infty$ and $\xi_\alpha \to -\infty$. It follows from (5.39) that $\eta \le \xi_\alpha/\sin\alpha$, which implies $\xi \to -\infty$ in *Subcase B3*. Thus we obtain from Proposition 3.3 that

$$\left|U_{e(x)}(\xi, x, y) - U^{-}_{\alpha\beta}(t, x, y)\right| \le \left|1 - U_{e(x)}(\xi, x, y)\right| + \left|1 - U_\alpha(\xi_\alpha, x, y)\right| \to 0$$

as $T \to +\infty$ in *Subcase B3*. In conclusion, the equality (5.38) is valid.

We claim that there exists positive constants $\nu_\star$ and $C_\star$ such that

$$\frac{\left|U^{-}_{\alpha\beta}(t, x, y)\right| + \left|U_{e(x)}(\xi, x, y)\right| + \left|U^{\delta}_{\alpha}(\eta, x, y)\right|}{\min\left\{1, e^{-2\nu_\star \min\{\xi_\alpha/\sin\alpha,\ \xi_\beta/\sin\beta\}}\right\}} \le C_\star \tag{5.40}$$

in $\mathbb{R}\times\mathbb{R}^2$. It is sufficient to consider $\min\{\xi_\alpha/\sin\alpha,\ \xi_\beta/\sin\beta\} > 0$. We only discuss the case $x \le 0$, and the case $x \ge 0$ is similar. Since $\xi_\alpha/\sin\alpha \le \xi_\beta/\sin\beta$, Proposition 3.3 yields that

$$\left|U_\alpha(\xi_\alpha, x, y)\right| + \left|U_\beta(\xi_\beta, x, y)\right| \le K_2\left(e^{-\frac{3\kappa}{4}\xi_\alpha} + e^{-\frac{3\kappa}{4}\xi_\beta}\right) \le K_2 e^{-2\nu\xi_\alpha/\sin\alpha}$$

for all $\nu \le 3\kappa\min\{\sin\alpha, \sin\beta\}/8$. It holds from Remark 5.3 that

$$|\eta - \xi_\alpha/\sin\alpha| = |\psi(\lambda x)/\lambda + x\cot\alpha| = |\varsigma \operatorname{sech}(\lambda x)/\lambda| \le \varsigma/\lambda$$

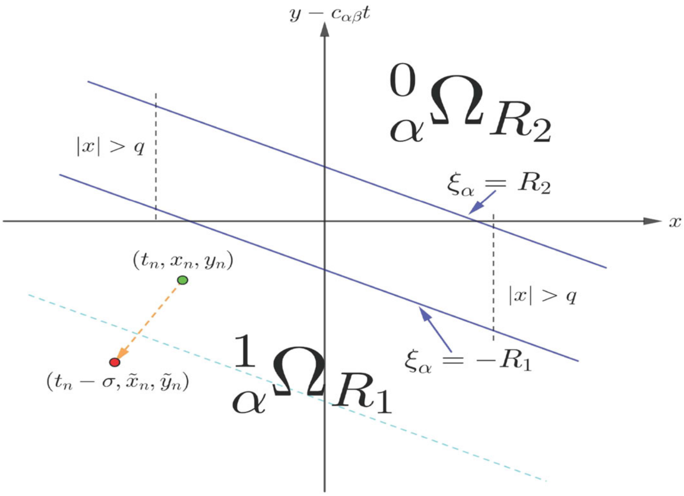

Fig. 2 The sequence (t_n, x_n, y_n)

for all $x \le -a/\lambda$. Then by virtue of Proposition 3.3, (5.7) and (5.8), we obtain that there is a positive constant $\nu_\star = \nu_\star(\delta, \lambda) < 3\kappa \min\{\sin\alpha, \sin\beta\}/8$ such that

$$\left|U_{e(x)}(\xi, x, y)\right| + \left|U_\alpha^\delta(\eta, x, y)\right| \le K_2 e^{-2\nu_\star \xi_\alpha / \sin\alpha}.$$

Therefore, the inequality (5.40) is valid.

Step 3: *we prove* (5.11) *by using the sliding method, inspired by Theorem 1.11 of* [4]. Denote

$${}^{0}_{\alpha}\Omega_R := \left\{(t, x, y) \in \mathbb{R}^3 : \xi_\alpha > R\right\} \text{ and } {}^{1}_{\alpha}\Omega_R := \left\{(t, x, y) \in \mathbb{R}^3 : \xi_\alpha < -R\right\}.$$

By (5.38) and Proposition 3.3, there are two numbers $R_1 > 1$ and $R_2 > 1$ such that

$$U_\alpha(\xi_\alpha, x, y) > 1 - \frac{\gamma_\star}{3}, \left|U_{e(x)}(\xi, x, y) - U_{\alpha\beta}^-(t, x, y)\right| < \frac{\gamma_\star}{3} \text{ in } {}^{1}_{\alpha}\Omega_{R_1 - 1}, \tag{5.41}$$

and $U_\alpha(\xi_\alpha, x, y) < p$ in ${}^{0}_{\alpha}\Omega_{R_2 - 1}$, where p and $\gamma_\star$ are defined in (1.3) and (1.5).

Denote ${}_{\alpha}\Omega_{R_{12}} := \mathbb{R}^3 \backslash ({}^{1}_{\alpha}\Omega_{R_1} \cup {}^{0}_{\alpha}\Omega_{R_2})$. Since ξ_α is bounded in ${}_{\alpha}\Omega_{R_{12}}$, there exists a number $\gamma_1 > 0$ such that

$$U_\alpha(\xi_\alpha, x, y) > \gamma_1 \text{ in } {}_{\alpha}\Omega_{R_{12}}. \tag{5.42}$$

By virtue of (5.38), there is a large positive number q such that

$$U^+(t, x, y) - U_\alpha(\xi_\alpha, x, y) \ge U_{e(x)}(\xi, x, y) - U_{\alpha\beta}^-(t, x, y) > -\frac{\gamma_1}{2} \tag{5.43}$$

for all $(t, x, y) \in {}_{\alpha}\Omega_{R_{12}} \cap \{|x| > q\}$. Since ξ_α is bounded in ${}_{\alpha}\Omega_{R_{12}}$, $\xi(t, x, y)$ is also bounded in ${}_{\alpha}\Omega_{R_{12}} \cap \{|x| \le q\}$. Then from Proposition 3.9, there exists a number $\gamma_2 > 0$ such that

$$U^+(t, x, y) \ge U_{e(x)}(\xi, x, y) > \gamma_2 \text{ in } {}_{\alpha}\Omega_{R_{12}} \cap \{|x| \le q\}. \tag{5.44}$$

Since $\xi_\alpha(t, x, y)$ is bounded in ${}_{\alpha}\Omega_{R_{12}}$, there is a number $\tau' > 0$ such that

$$U_\alpha\left(\xi_\alpha\left(t - \tau', x, y\right), x, y\right) < \min\left\{\frac{\gamma_1}{2}, \gamma_2\right\} \text{ in } {}_{\alpha}\Omega_{R_{12}}. \tag{5.45}$$

By (5.42)–(5.45), one gets

$$U^+(t,x,y) > U_\alpha\big(\xi_\alpha\big(t-\tau',x,y\big),x,y\big) \ \text{ in } {}_\alpha\Omega_{R_{12}}. \tag{5.46}$$

Define

$$\varepsilon^* := \inf\left\{\varepsilon > 0 : \ U^+(t,x,y) \geq U_\alpha\big(\xi_\alpha\big(t-\tau',x,y\big),x,y\big) - \varepsilon, \ \forall (t,x,y) \in {}^1_\alpha\Omega_{R_1}\right\}. \tag{5.47}$$

It follows from (5.41) that

$$U^+(t,x,y) + \frac{2\gamma_\star}{3} \geq U_{e(x)}(\xi,x,y) + \frac{2\gamma_\star}{3} \geq U^-_{\alpha\beta}(t,x,y) + \frac{\gamma_\star}{3} > 1 \ \text{ in } {}^1_\alpha\Omega_{R_1},$$

which implies $0 \leq \varepsilon^* \leq 2\gamma_\star/3$. We will prove that $\varepsilon^* = 0$, which means

$$U^+(t,x,y) \geq U_\alpha\big(\xi_\alpha\big(t-\tau',x,y\big),x,y\big) \ \text{ in } {}^1_\alpha\Omega_{R_1}. \tag{5.48}$$

Assume by contradiction that $\varepsilon^* > 0$. Denote

$$v(t,x,y) := U^+(t,x,y) - U_\alpha\big(\xi_\alpha\big(t-\tau',x,y\big),x,y\big) + \varepsilon^* \ \text{ in } {}^1_\alpha\Omega_{R_1} \cup {}_\alpha\Omega_{R_{12}},$$

then $v \geq 0$ in ${}^1_\alpha\Omega_{R_1} \cup {}_\alpha\Omega_{R_{12}}$ by (5.46). From the definition of ε^*, there exists a sequence of points $\{(t_n,x_n,y_n)\}_{n\in\mathbb{N}} \subset {}^1_\alpha\Omega_{R_1}$ such that

$$v(t_n,x_n,y_n) \to 0 \ \text{ as } n \to \infty. \tag{5.49}$$

By virtue of (5.38), there is a large positive number $\overline{R}_1 > R_1$ such that

$$\left|U_{e(x)}(\xi,x,y) - U^-_{\alpha\beta}(t,x,y)\right| < \frac{\varepsilon^*}{2} \quad \text{in } {}^1_\alpha\Omega_{\overline{R}_1},$$

which yields that $v(t,x,y) > \varepsilon^*/2$ in ${}^1_\alpha\Omega_{\overline{R}_1}$. Thus there exist a small number $\sigma > 0$ and a sequence of points (see Fig. 2)

$$\{(t_n-\sigma,\tilde{x}_n,\tilde{y}_n)\}_{n\in\mathbb{N}} \subset {}^1_\alpha\Omega_{\overline{R}_1}$$

satisfying

$$\text{dist}((\tilde{x}_n,\tilde{y}_n),\ (x_n,y_n)) \ \text{ is uniformly bounded w.r.t. } n. \tag{5.50}$$

Notice $\varepsilon^* \leq 2\gamma_\star/3$. By virtue of (1.5), (5.41) and U^+ is a supersolution of (1.1), it follows that

$$(\partial_t - \Delta_{x,y})\big(U^+ + \varepsilon^*\big) - f\big(x,y,U^+ + \varepsilon^*\big) \geq (\partial_t - \Delta_{x,y})U^+ - f\big(x,y,U^+\big) \geq 0 \tag{5.51}$$

in ${}^1_\alpha\Omega_{R_1-1}$. Since $U_\alpha(\xi_\alpha(t-\tau',x,y),x,y)$ is a solution, one gets that

$$(\partial_t - \Delta_{x,y})v(t,x,y) + b_1(t,x,y)v(t,x,y) \geq 0 \ \text{ in } {}^1_\alpha\Omega_{R_1-1}, \tag{5.52}$$

where $b_1(t,x,y) := \big(f(x,y,U_\alpha) - f(x,y,U^+ + \varepsilon^*)\big)/v$, and $\|b_1\|_{L^\infty}$ is bounded by (1.2).

Below there is a claim, which contradicts the fact that $v(t_n-\sigma,\tilde{x}_n,\tilde{y}_n) > \varepsilon^*/2$, thus we have proved $\varepsilon^* = 0$.

Claim 5.5 *There holds*

$$v(t_n-\sigma,\tilde{x}_n,\tilde{y}_n) \to 0 \ \text{ as } n \to \infty.$$

Proof Assume that by contradiction, up to a subsequence, there is a number $\gamma_3 > 0$ such that

$$v(t_n - \sigma, \tilde{x}_n, \tilde{y}_n) \geq \gamma_3, \quad \forall n \in \mathbb{N}.$$

Since the first-order derivatives of v are bounded, one has

$$v(t_n - \sigma, x, y) > \frac{\gamma_3}{2} \tag{5.53}$$

for all (x, y) in a ball $B_r(\tilde{x}_n, \tilde{y}_n)$ centered at $(\tilde{x}_n, \tilde{y}_n)$ with a small radius r independent of n, for all $n \in \mathbb{N}$. Let w_n be the solution of an equation of the type

$$(\partial_t - \Delta_{x,y})w(t, x, y) + b_1(t, x, y)w(t, x, y) = 0 \tag{5.54}$$

in a suitable domain $E_n \subset {}^{1}_{\alpha}\Omega_{R_1-1}$, which contains $(t_n - \sigma, t_n) \times B_r(\tilde{x}_n, \tilde{y}_n)$ and the point (t_n, x_n, y_n) away from ∂E_n independently of n, with initial condition

$$w(t_n - \sigma, \cdot, \cdot) \begin{cases} \equiv \frac{\gamma_2}{2} & \text{in } B_{\frac{r}{2}}(\tilde{x}_n, \tilde{y}_n), \\ \text{decays to } 0 & \text{in } B_r(\tilde{x}_n, \tilde{y}_n) \setminus B_{\frac{r}{2}}(\tilde{x}_n, \tilde{y}_n), \\ \equiv 0 & \text{outside } B_r(\tilde{x}_n, \tilde{y}_n), \end{cases}$$

and with lateral boundary condition $w = 0$. Furthermore, since (5.50), from the Harnack inequality, there exists a positive constant C independent of n such that

$$w_n(t_n, x_n, y_n) \geq C\gamma_2, \quad \forall n \in \mathbb{N}.$$

By virtue of (5.52), (5.53) and the fact that $v \geq 0$ in ${}^{1}_{\alpha}\Omega_{R_1-1}$, function v is just a supersolution of the problem (5.54), thus

$$v(t_n, x_n, y_n) \geq w_n(t_n, x_n, y_n) \geq C\gamma_2, \quad \forall n \in \mathbb{N},$$

which is a contradiction with (5.49). Therefore Claim 5.5 is valid. □

Similarly, we have

$$U^+(t, x, y) \geq U_\alpha\big(\xi_\alpha\big(t - \tau', x, y\big), x, y\big) \;\text{ in } {}^{0}_{\alpha}\Omega_{R_2}. \tag{5.55}$$

It follows immediately from (5.46), (5.48) and (5.55) that

$$U^+(t, x, y) \geq U_\alpha\big(\xi_\alpha\big(t - \tau', x, y\big), x, y\big) \;\text{ in } \mathbb{R}^3. \tag{5.56}$$

Set

$$\tau^* := \inf\left\{\tau > 0 : \; U^+(t, x, y) \geq U_\alpha(\xi_\alpha(t - \tau, x, y), x, y) \;\text{ in } \mathbb{R}^3\right\}. \tag{5.57}$$

Apparently τ^* is a well-defined nonnegative number by (5.56). In fact, $\tau^* = 0$ and we will prove it by contradiction. Assume $\tau^* > 0$ and define

$$\vartheta(t, x, y) := U^+(t, x, y) - U_\alpha(\xi_\alpha(t - \tau^*, x, y), x, y) \;\text{ in } \mathbb{R}^3,$$

then $\vartheta \geq 0$ in $\mathbb{R}^3$. Two cases may occur.

Case i: $\inf\{\vartheta(t, x, y) : \; (t, x, y) \in {}_{\alpha}\Omega_{R_{12}}\} > 0$.

Since the first-order derivatives of $U_\alpha(\xi_\alpha(t, x, y), x, y)$ are bounded, there is a number $\tau_0 \in (0, \tau^*)$ such that

$$U^+(t, x, y) > U_\alpha(\xi_\alpha(t - \tau_0, x, y), x, y) \;\text{ in } {}_{\alpha}\Omega_{R_{12}}.$$

Applying the above arguments again, one gets that

$$U^{+}(t,x,y) \geq U_{\alpha}(\xi_{\alpha}(t-\tau_0,x,y),x,y) \ \text{ in } \mathbb{R}^3.$$

This contradicts the minimality of τ^* and *Case i* is thus ruled out.

Case ii: $\inf\{\vartheta(t,x,y):\ (t,x,y)\in {}_{\alpha}\Omega_{R_{12}}\}=0$.

There exists a sequence of points $\{(\bar{t}_n,\bar{x}_n,\bar{y}_n)\}_{n\in\mathbb{N}}\subset {}_{\alpha}\Omega_{R_{12}}$ such that

$$\vartheta(\bar{t}_n,\bar{x}_n,\bar{y}_n)\to 0 \ \text{ as } n\to\infty. \tag{5.58}$$

Since $\xi_{\alpha}(t-\tau^*,x,y)$ is bounded in ${}_{\alpha}\Omega_{R_{12}}$, by Proposition 3.9 there are two numbers $\gamma_4>0$ and $r>0$ such that

$$U_{\alpha}(\xi_{\alpha}(t-\tau^*,x,y),x,y)<1-\gamma_4$$

and

$$U_{\alpha}(\xi_{\alpha}(t,x,y),x,y)-U_{\alpha}(\xi_{\alpha}(t-\tau^*,x,y),x,y)\geq r\tau^* \tag{5.59}$$

for all $(t,x,y)\in {}_{\alpha}\Omega_{R_{12}}$. Note that

$$\begin{cases}\xi_{\beta}=\frac{\sin\beta}{\sin\alpha}\xi_{\alpha}-\frac{\sin(\beta-\alpha)}{\sin\alpha}x & \text{in } \mathbb{R}^3,\\ \psi(\lambda x)/\lambda=-x\cot\beta+\varsigma\operatorname{sech}(\lambda x)/\lambda & \text{for all } x\geq b/\lambda.\end{cases} \tag{5.60}$$

It follows from (5.6), (5.7), (5.60) and Proposition 3.3 that $U_{e(x)}(\xi,x,y)\to 1$ as $x\to+\infty$ uniformly in ${}_{\alpha}\Omega_{R_{12}}$. Thus there exist two numbers $\bar{q}$ and N_0 such that

$$(\bar{t}_n,\bar{x}_n,\bar{y}_n)\in {}_{\alpha}\Omega_{R_{12}}\cap\{x<\bar{q}\},\quad \forall n\geq N_0. \tag{5.61}$$

By virtue of (5.38) and (5.59), there is a number $\tilde{q}<0$ such that

$$\begin{aligned}\vartheta(t,x,y)&\geq U_{e(x)}(\xi,x,y)-U_{\alpha}(\xi_{\alpha}(t-\tau^*,x,y),x,y)\\ &\geq U_{\alpha}(\xi_{\alpha}(t,x,y),x,y)-\frac{r\tau^*}{2}-U_{\alpha}(\xi_{\alpha}(t-\tau^*,x,y),x,y)\\ &\geq\frac{r\tau^*}{2}\end{aligned} \tag{5.62}$$

in ${}_{\alpha}\Omega_{R_{12}}\cap\{x<\tilde{q}\}$.

Since U^+ is a supersolution and $U_{\alpha}(\xi_{\alpha}(t-\tau^*,x,y),x,y)$ is a solution of (1.1), one has

$$(\partial_t-\Delta_{x,y})\vartheta(t,x,y)+b_2(t,x,y)\vartheta(t,x,y)\geq 0 \ \text{ in } \mathbb{R}^3,$$

where

$$b_2(t,x,y):=\left[f\left(x,y,U_{\alpha}\left(\xi_{\alpha}(t-\tau^*,x,y),x,y\right)\right)-f(x,y,U^+)\right]/\vartheta,$$

and $\|b_2\|_{L^\infty}$ is bounded by (1.2). Then by virtue of (5.58), (5.61) and the fact that $\vartheta\geq 0$ in $\mathbb{R}^3$, the same arguments as those in Claim 5.5 yield that there exist (see Fig. 3) a number $\hat{\sigma}>0$ and a sequence of points $\{(\hat{x}_n,\hat{y}_n)\}_{n\in\mathbb{N}}$ such that

$$\left\{(\bar{t}_n-\hat{\sigma},\hat{x}_n,\hat{y}_n)\right\}_{n\geq N_0}\subset {}_{\alpha}\Omega_{R_{12}}\cap\{x<\tilde{q}\} \ \text{ and } \ \vartheta(\bar{t}_n-\hat{\sigma},\hat{x}_n,\hat{y}_n)\to 0 \text{ as } n\to\infty,$$

which is impossible because of (5.62). Therefore *Case ii* is ruled out.

As a consequence, $\tau^*=0$ which implies that $U^+(t,x,y)\geq U_{\alpha}(\xi_{\alpha},x,y)$ in $\mathbb{R}^3$. Similarly, we also have $U^+(t,x,y)\geq U_{\beta}(\xi_{\beta},x,y)$ in $\mathbb{R}^3$, thus (5.11) is proved. Finally, one can get immediately (5.12) by calculation. Therefore, we complete the proof of Lemma 5.4. □

Here we note that the construction of the supersolution in Lemma 5.4 is motivated by Wang and Bu [55] and Guo et al. [26]. Below we prove the existence of curved fronts.

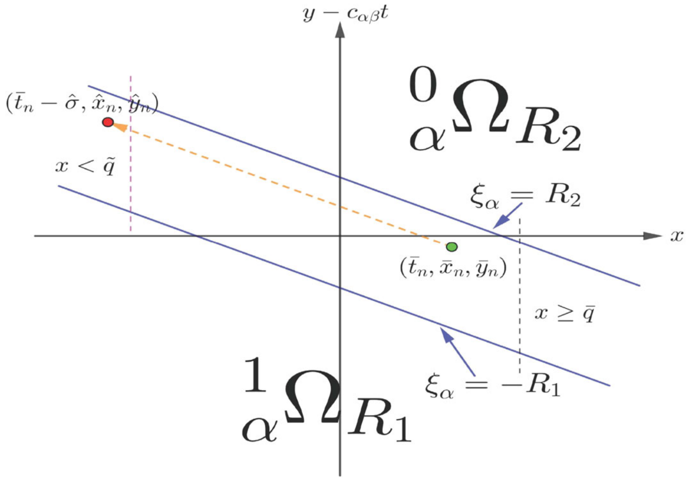

Fig. 3 The sequence $(\bar{t}_n, \bar{x}_n, \bar{y}_n)$

Proof of Theorem 2.12 By virtue of Proposition 7.3.1 of [39], let $w_n(t, x, y)$ be the unique solution of below Cauchy problem

$$\begin{cases} \partial_t w - \Delta_{x,y} w = f(x, y, w) & \text{when } t > -n,\ (x, y) \in \mathbb{R}^2 \\ w(t, x, y) = U^-_{\alpha\beta}(-n, x, y) & \text{when } t = -n,\ (x, y) \in \mathbb{R}^2 \end{cases} \tag{5.63}$$

for all $n \in \mathbb{N}$. By virtue of Lemma 5.4, since $U^-_{\alpha\beta}$ is a subsolution, using the comparison principle, for any $n \in \mathbb{N}$, one gets

$$U^-_{\alpha\beta}(t, x, y) \leq w_n(t, x, y) \leq U^+(t, x, y) \tag{5.64}$$

in $[-n, +\infty) \times \mathbb{R}^2$. Using the comparison principle again, it holds that the sequence of functions $\{w_n(t, x, y)\}_{n\in\mathbb{N}}$ is increasing in n. Applying Theorems 5.1.3 and 5.1.4 of [39] to (5.63), there exists a constant Λ independent of $n \in \mathbb{N}$ such that

$$\|w_n(\cdot, \cdot, \cdot)\|_{C^{1+\frac{\theta}{2}, 2+\theta}([-n+1, +\infty)\times\mathbb{R}^2)} \leq \Lambda$$

for some $\theta \in (0, 1)$ and all $n \in \mathbb{N}$. Then letting $n \to \infty$, the sequence $\{w_n(t, x, y)\}_{n\in\mathbb{N}}$ converges to an entire solution $V(t, x, y)$ of (1.1). Furthermore, it follows from (5.64) that

$$U^-_{\alpha\beta}(t, x, y) \leq V(t, x, y) \leq U^+(t, x, y) \ \text{ in } \mathbb{R} \times \mathbb{R}^2. \tag{5.65}$$

Thus by virtue of (5.40), we obtain

$$\sqrt{V(t, x, y) - U^-_{\alpha\beta}(t, x, y)} \leq C \min\left\{1, e^{-\nu_\star \min\{\xi_\alpha/\sin\alpha,\, \xi_\beta/\sin\beta\}}\right\}$$

in $\mathbb{R} \times \mathbb{R}^2$. Then letting $\varepsilon \to 0$ in U^+ yields $0 \leq V \leq 1$ and (2.13).

It follows from (5.63) that for any $\tau > 0$, $w_n(t + \tau, x, y)$ solves an equation of the type

$$\begin{cases} \partial_t w - \Delta_{x,y} w = f(x, y, w) & \text{when } t > -n,\ (x, y) \in \mathbb{R}^2 \\ w(t, x, y) = w_n(-n + \tau, x, y) & \text{when } t = -n,\ (x, y) \in \mathbb{R}^2 \end{cases}$$

for all $n \in \mathbb{N}$. In addition, (5.64) gives that

$$w_n(-n+\tau, x, y) \geq U^-_{\alpha\beta}(-n+\tau, x, y) \geq U^-_{\alpha\beta}(-n, x, y)$$

for all $(x, y) \in \mathbb{R}^2$. Then applying the comparison principle, one gets

$$w_n(t+\tau, x, y) \geq w_n(t, x, y), \;\; \forall (t, x, y) \in (-n, +\infty) \times \mathbb{R}^2, \; \forall \tau > 0,$$

which implies $\partial_t w_n(t, x, y) \geq 0$ in $(-n, +\infty) \times \mathbb{R}^2$. Letting $n \to \infty$ yields $\partial_t V(t, x, y) \geq 0$ in $\mathbb{R} \times \mathbb{R}^2$. By virtue of the strong maximum principle, we obtain (2.14). The proof of Theorem 2.12 is thereby complete. □

5.2 Uniqueness

Proof of Theorem 2.15 Let $V_1(t, x, y)$ be an entire solution of (1.1) satisfying (2.15) and $0 \leq V_1 \leq 1$. We will prove that $V \geq V_1$ in $\mathbb{R}^3$. It follows from (2.13) and (2.15) that

$$\lim_{R \to +\infty} \sup_{x^2+(y-c_{\alpha\beta}t)^2>R^2} \left| V(t, x, y) - U^-_{\alpha\beta}(t, x, y) \right| = 0 \tag{5.66}$$

and

$$\lim_{R \to +\infty} \sup_{x^2+(y-c_{\alpha\beta}t)^2>R^2} \left| V_1(t, x, y) - U^-_{\alpha\beta}(t, x, y) \right| = 0. \tag{5.67}$$

Denote

$$\begin{aligned} {}^0_\alpha\Omega_R &:= \left\{(t, x, y) \in \mathbb{R}^3 : \; \xi_\alpha / \sin\alpha > R\right\}, \quad {}^1_\alpha\Omega_R := \left\{(t, x, y) \in \mathbb{R}^3 : \; \xi_\alpha / \sin\alpha < -R\right\}, \\ {}^0_\beta\Omega_R &:= \left\{(t, x, y) \in \mathbb{R}^3 : \; \xi_\beta / \sin\beta > R\right\}, \quad {}^1_\beta\Omega_R := \left\{(t, x, y) \in \mathbb{R}^3 : \; \xi_\beta / \sin\beta < -R\right\}. \end{aligned}$$

Since the proof of Theorem 2.15 is almost same as *Step 3* of the proof of Lemma 5.4, we only give the outlines for the sake of saving space.

Step 1. By (5.5), (5.6), (5.66) and (5.67), there exist $R_1 > 1$ and $R_2 > 1$ such that

$$V(t, x, y) > 1 - \frac{\gamma_\star}{2}, \;\; V_1(t, x, y) > 1 - \frac{\gamma_\star}{2} \;\; \text{in } {}^1_\alpha\Omega_{R_1-1} \cup {}^1_\beta\Omega_{R_1-1} \tag{5.68}$$

and

$$V(t, x, y) < p, \;\; V_1(t, x, y) < p \;\; \text{in } {}^0_\alpha\Omega_{R_2-1} \cap {}^0_\beta\Omega_{R_2-1},$$

where p and $\gamma_\star$ are defined in (1.3) and (1.5). Denote $\Omega_{R_{12}} := \mathbb{R}^3 \backslash ({}^1_\alpha\Omega_{R_1} \cup {}^1_\beta\Omega_{R_1} \cup ({}^0_\alpha\Omega_{R_2} \cap {}^0_\beta\Omega_{R_2}))$. One can prove that there exists a number $\tau' > 0$ such that

$$V\big(t+\tau', x, y\big) > V_1(t, x, y) \;\; \text{in } \Omega_{R_{12}}. \tag{5.69}$$

Step 2. Define

$$\varepsilon^* := \inf\left\{\varepsilon > 0 : \; V\left(t+\tau', x, y\right) \geq V_1(t, x, y) - \varepsilon, \; \forall (t, x, y) \in {}^1_\alpha\Omega_{R_1} \cup {}^1_\beta\Omega_{R_1}\right\}.$$

One can prove that $\varepsilon^* = 0$, which means

$$V\left(t+\tau', x, y\right) \geq V_1(t, x, y) \;\; \text{in } {}^1_\alpha\Omega_{R_1} \cup {}^1_\beta\Omega_{R_1}. \tag{5.70}$$

Similarly, one can prove

$$V\left(t+\tau', x, y\right) \geq V_1(t, x, y) \;\; \text{in } {}^0_\alpha\Omega_{R_2} \cap {}^0_\beta\Omega_{R_2}. \tag{5.71}$$

Step 3. It follows from (5.69)–(5.71) that $V\left(t+\tau', x, y\right) \geq V_1(t, x, y)$ in $\mathbb{R}^3$. Define

$$\tau^* := \inf\left\{\tau > 0 :\ V(t+\tau, x, y) \geq V_1(t, x, y)\ \text{ in } \mathbb{R}^3\right\}.$$

One can prove that $\tau^* = 0$, which implies that $V \geq V_1$ in $\mathbb{R}^3$. With similar arguments as above, by permuting the roles of functions V and V_1, one can also prove that $V_1 \geq V$ in $\mathbb{R}^3$. Consequently $V_1 \equiv V$ in $\mathbb{R}^3$. The proof of Theorem 2.15 is thereby complete. □

Using the uniqueness result, we get Remark 2.14.

Proof of Remark 2.14 Apparently for any $k \in \mathbb{Z}$,

$$V_k(t, x, y) := V(t + L_2 k/c_{\alpha\beta}, x, y + L_2 k)$$

is an entire solution of (1.1) and $0 \leq V_k \leq 1$, where L_2 is the period of y. It follows from (5.5)–(5.8) that the values of $\xi_\alpha, \xi_\beta, \xi$ and η at points $(t + L_2 k/c_{\alpha\beta}, x, y + L_2 k)$ are invariant for all $k \in \mathbb{Z}$. Then since L_2 is the period of $U_e(s, x, y)$ in y for all $e \in \mathbb{S}^{N-1}$, one gets

$$U^+(t, x, y) = U^+(t + L_2 k/c_{\alpha\beta}, x, y + L_2 k),\quad U^-_{\alpha\beta}(t, x, y) = U^-_{\alpha\beta}(t + L_2 k/c_{\alpha\beta}, x, y + L_2 k)$$

in $\mathbb{R} \times \mathbb{R}^2$ for all $k \in \mathbb{Z}$. Thus we obtain from (5.10) that

$$\lim_{R\to+\infty} \sup_{x^2+(y-c_{\alpha\beta}t)^2>R^2} \left| U^+(t + L_2 k/c_{\alpha\beta}, x, y + L_2 k) - U^-_{\alpha\beta}(t + L_2 k/c_{\alpha\beta}, x, y + L_2 k)\right| \leq \varepsilon,$$

which implies by (5.65) that

$$\lim_{R\to+\infty} \sup_{x^2+(y-c_{\alpha\beta}t)^2>R^2} \left| V_k(t, x, y) - U^-_{\alpha\beta}(t, x, y)\right| = 0.$$

Therefore Theorem 2.15 yields that $V_k(t, x, y) \equiv V(t, x, y)$ in $\mathbb{R}^3$, which completes the proof. □

5.3 Stability

We construct super- and subsolutions for Cauchy problem (1.1) (ignore initial condition).

Lemma 5.6 *For each $\delta \in (0, \delta^*]$ and each $0 < \varepsilon < \varepsilon_0^+(\delta)$, there exist positive constants $\mu(\delta)$ and $\varrho(\delta, \mu)$ such that for any $0 < \lambda < \lambda_0^+(\delta, \varepsilon)$,*

$$W_\sigma^+(t, x, y) := U^+(\tau, x, y) + \sigma e^{-\mu t} \times \left[U_\alpha^\delta(\eta(\tau, x, y), x, y)\omega(\eta(\tau, x, y)) + (1 - \omega(\eta(\tau, x, y)))\right]$$

is a supersolution of (1.1) for $t \geq 0$ and $(x, y) \in \mathbb{R}^2$, for all $\sigma \in (0, \min\{p/4, \gamma_\star/4\}]$, where $\tau = \tau(t) := t - \varrho\sigma e^{-\mu t} + \varrho\sigma$, and δ^*, $\varepsilon_0^+(\delta)$, $\lambda_0^+(\delta, \varepsilon)$, U^+ are given in Lemma 5.4, and η, ω, p, $\gamma_\star$ are given in (5.8), (5.3), (1.3), (1.5), respectively.

Proof The strategy is to find two numbers $X' > 1$ and $X'' > 1$ and show the inequality

$$\mathcal{L}W_\sigma^+ := \partial_t W_\sigma^+ - \Delta_{x,y} W_\sigma^+ - f\left(x, y, W_\sigma^+\right) \geq 0,\quad \forall (t, x, y) \in [0, +\infty) \times \mathbb{R}^2,$$

by considering three cases $\eta(\tau, x, y) > X'$, $\eta(\tau, x, y) < -X''$, and $\eta(\tau, x, y) \in [-X'', X']$, respectively. Since U^+ is a supersolution of (1.1) by Lemma 5.4, one has

$$\begin{aligned} \mathcal{L}W_\sigma^+ \geq\ & \varrho\sigma\mu e^{-\mu t} U_\tau^+ + f\left(x, y, U^+\right) - f\left(x, y, W_\sigma^+\right) \\ & + (\partial_t - \Delta_{x,y})\left(\sigma e^{-\mu t} \times \left[U_\alpha^\delta(\eta, x, y)\omega(\eta) + (1 - \omega(\eta))\right]\right) \end{aligned} \tag{5.72}$$

in $\mathbb{R}\times\mathbb{R}^2$, where η, U^+ and all of its derivatives are evaluated at $(\tau(t),x,y)$.

Case 1: *$\eta(\tau(t),x,y)>X'$ and $t\geq 0$, where $X'>1$ is to be chosen.*

In this case, $\omega(\eta)\equiv 1$. Recalling (5.9) and (5.20), by virtue of Proposition 3.3, there exists a number $X_1'>1$ such that $W_\sigma^+<p$ in $\{(t,x,y):\ \eta(\tau(t),x,y)\in(X_1',+\infty),\ t\geq 0\}$. It follows from (5.72), (1.3) and (5.12) that

$$\begin{aligned}\mathcal{L}W_\sigma^+ &\geq\ \varrho\sigma\mu e^{-\mu t}U_\tau^+ +(\partial_t-\Delta_{x,y})\Big(\sigma e^{-\mu t}U_\alpha^\delta\Big)\\ &\geq\ -\sigma\mu e^{-\mu t}U_\alpha^\delta+\sigma e^{-\mu t}\times(\partial_\tau-\Delta_{x,y})\Big(U_\alpha^\delta\Big)-\delta U_\alpha^{\delta-1}\partial_s U_\alpha c_{\alpha\beta}\varrho\mu\left(\sigma e^{-\mu t}\right)^2\\ &\geq\ -\sigma\mu e^{-\mu t}U_\alpha^\delta+\sigma e^{-\mu t}\times(\partial_\tau-\Delta_{x,y})\Big(U_\alpha^\delta\Big)\end{aligned}\tag{5.73}$$

in $\{(t,x,y):\ \eta(\tau(t),x,y)\in(X_1',+\infty),\ t\geq 0\}$, where U_α and all of its derivatives are evaluated at $(\eta(\tau(t),x,y),x,y)$. Then one can obtain from (5.19), (5.21), (5.22) and (5.73) that there exists a sufficiently large number $X'>X_1'$ such that

$$\mathcal{L}W_\sigma^+\geq-\sigma\mu e^{-\mu t}U_\alpha^\delta+\sigma e^{-\mu t}U_\alpha^\delta\frac{\delta c_{\alpha\beta}c_\alpha}{4}$$

in $\{(t,x,y):\ \eta(\tau(t),x,y)\in(X',+\infty),\ t\geq 0\}$. Let $0<\mu<\delta c_{\alpha\beta}c_\alpha/4$. Thus we prove immediately that $\mathcal{L}W_\sigma^+>0$ in *Case 1*.

Case 2: *$\eta(\tau(t),x,y)<-X''$ and $t\geq 0$, where $X''>1$ is to be chosen.*

In this case, $\omega(\eta)\equiv 0$. Recalling (5.9), (5.20) and (5.29), by virtue of Proposition 3.3, there exists a number $X''>1$ such that $U^+,\ W_\sigma^+\in[1-\gamma_\star,1+\gamma_\star]$ in $\{(t,x,y):\ \eta(\tau(t),x,y)\in(-\infty,-X''),\ t\geq 0\}$. Then it follows from (5.72), (5.12) and (1.5) that

$$\begin{aligned}\mathcal{L}W_\sigma^+ &\geq\varrho\sigma\mu e^{-\mu t}U_\tau^+ + f\big(x,y,U^+\big)-f\big(x,y,W_\sigma^+\big)+(\partial_t-\Delta_{x,y})\big(\sigma e^{-\mu t}\big)\\ &\geq f\big(x,y,U^+\big)-f\big(x,y,W_\sigma^+\big)-\sigma\mu e^{-\mu t}\\ &\geq\frac{\kappa_1}{2}\sigma e^{-\mu t}-\sigma\mu e^{-\mu t}\end{aligned}$$

in $\{(t,x,y):\ \eta(\tau(t),x,y)\in(-\infty,-X''),\ t\geq 0\}$. Setting $0<\mu<\kappa_1/2$, one has that $\mathcal{L}W_\sigma^+>0$ in *Case 2*.

Case 3: *$-X''\leq\eta(\tau(t),x,y)\leq X'$ and $t\geq 0$.*

It holds from (5.9), (5.14) and (5.18) that

$$\begin{aligned}\frac{\partial}{\partial\tau}\Big(U^+(\tau,x,y)\Big)&=\partial_s U_{e(x)}(\xi,x,y)\xi_\tau+\varepsilon\mathrm{sech}(\lambda x)\frac{\partial}{\partial\tau}\Big[\Big(U_\alpha^\delta(\eta,x,y)-1\Big)\omega(\xi)\Big]\\ &\geq\partial_s U_{e(x)}(\xi,x,y)\frac{-c_{\alpha\beta}}{\sqrt{\psi'^2(\lambda x)+1}}\end{aligned}\tag{5.74}$$

in $\mathbb{R}^3$, where ξ and η are evaluated at (τ,x,y). Since η is bounded in this case, ξ is also bounded. Thus by (5.74), Proposition 3.9 and Lemma 5.2, there exists a number $r>0$ such that

$$U_\tau^+(\tau,x,y)>rc_{\alpha\beta}\ \text{ in }\{(t,x,y):\ \eta(\tau(t),x,y)\in[-X'',X']\}.\tag{5.75}$$

By Theorems 2.5 and 2.7, Lemma 5.2, (5.72), (5.8) and (5.75), we obtain that

$$\begin{aligned}\mathcal{L}W_\sigma^+ \geq\ &\varrho\sigma\mu e^{-\mu t}U_\tau^+ -\|f_u\|_{L^\infty}\sigma e^{-\mu t}+(\partial_t-\Delta_{x,y})\Big(\sigma e^{-\mu t}\Big[U_\alpha^\delta(\eta,x,y)\omega(\eta)+(1-\omega(\eta))\Big]\Big)\\ \geq\ &\varrho\sigma\mu e^{-\mu t}U_\tau^+ -\|f_u\|_{L^\infty}\sigma e^{-\mu t}-\sigma e^{-\mu t}\Delta_{x,y}\Big[\Big(U_\alpha^\delta(\eta,x,y)-1\Big)\omega(\eta)\Big]\\ &-\sigma\mu e^{-\mu t}+\sigma e^{-\mu t}\frac{\partial}{\partial\tau}\Big[\Big(U_\alpha^\delta(\eta,x,y)-1\Big)\omega(\eta)\Big]\tau'(t)\end{aligned}$$

$$\geq \varrho\sigma\mu e^{-\mu t}U_\tau^+ - \|f_u\|_{L^\infty}\sigma e^{-\mu t} - \sigma\mu e^{-\mu t} - \sigma e^{-\mu t}\Delta_{x,y}\Big[\Big(U_\alpha^\delta(\eta,x,y)-1\Big)\omega(\eta)\Big]$$
$$\geq \Big(\varrho\mu U_\tau^+ - \|f_u\|_{L^\infty} - \mu\Big)\sigma e^{-\mu t} - \sigma e^{-\mu t}\times C(\delta)$$

in $\{(t,x,y):\ \eta(\tau(t),x,y)\in[-X'',X'],\ t\geq 0\}$. Set

$$\varrho > \frac{\|f_u\|_{L^\infty}+\mu+C(\delta)}{\mu r c_{\alpha\beta}}.$$

Then we prove that $\mathcal{L}W_\sigma^+ > 0$ in *Case 3*. The proof of Lemma 5.6 is thereby complete. □

Lemma 5.7 *Assume that V is the solution defined in Theorem 2.12. Then for any $\delta\in(0,\delta^*]$, there is a positive constant $\tilde\lambda_0^+(\delta)$ such that, for each $0<\lambda<\tilde\lambda_0^+(\delta)$ there exist positive constants $\mu(\delta)$ and $\tilde\varrho(\delta,\mu,\lambda)$ such that*

$$V_\sigma^+(t,x,y;\ T) := V\big(T+\tilde\tau,x,y\big)+\sigma e^{-\mu t}\times\big[U_\alpha^\delta(\eta,x,y)\omega(\eta)+(1-\omega(\eta))\big]$$

is a supersolution of (1.1) for $t\geq 0$ and $(x,y)\in\mathbb{R}^2$, for all $T\in\mathbb{R}$ and $\sigma\in(0,\min\{p/4,\gamma_\star/4\}]$, where $\tilde\tau=\tilde\tau(t):=t-\tilde\varrho\sigma e^{-\mu t}+\tilde\varrho\sigma$, and η is evaluated at $(T+\tilde\tau,x,y)$, and δ^*, $\mu(\delta)$, η, ω, p, $\gamma_\star$ coincide with those in Lemma 5.6.

Moreover, for any $\delta\in(0,\delta^*]$, there is a positive constant $\hat\lambda_0^+(\delta)$ such that, for each $0<\lambda<\hat\lambda_0^+(\delta)$ there exist positive constants $\hat\mu(\delta)$, $\hat\varrho(\delta,\hat\mu,\lambda)$ and $\sigma^0(\delta,\hat\mu,\hat\varrho,\lambda)$ such that

$$V_\sigma^-(t,x,y;\ T) := V\big(T+\hat\tau,x,y\big)-\sigma e^{-\hat\mu t}\times\big[U_\alpha^\delta(\eta,x,y)\omega(\eta)+(1-\omega(\eta))\big]$$

is a subsolution of (1.1) for $t\geq 0$ and $(x,y)\in\mathbb{R}^2$, for all $T\in\mathbb{R}$ and $\sigma\in(0,\sigma^0]$, where $\hat\tau=\hat\tau(t):=t+\hat\varrho\sigma e^{-\hat\mu t}-\hat\varrho\sigma$, and η is evaluated at $(T+\hat\tau,x,y)$, and δ^*, η, ω, p, $\gamma_\star$ coincide with those in Lemma 5.6.

Proof *Step 1: we prove that $V_t\geq r$* in $\{(t,x,y):\ |\eta|\leq q\}$, *where $r=r(\lambda,q)>0$ is a constant.*

Assume by contradiction that there exists a sequence of points $\{(t_n,x_n,y_n)\}_{n\in\mathbb{N}}$ satisfying

$$V_t(t_n,x_n,y_n)\to 0 \text{ as } n\to\infty \ \text{ and } \ |\eta(t_n,x_n,y_n)|\leq q \text{ for all } n. \tag{5.76}$$

By (5.75), there is a positive number $r_1=r_1(q)$ independent of λ and ε, such that

$$\partial_t U^+ > r_1 \ \text{ in } \{(t,x,y):\ |\eta|\leq q+c_{\alpha\beta}\}. \tag{5.77}$$

Set $\varepsilon=r_1/16$, and fix arbitrary $0<\lambda<\lambda_0^+(\delta,\varepsilon)=:\bar\lambda_0^+(\delta,q)$, where $\lambda_0^+(\delta,\varepsilon)$ is given in Lemma 5.4. By virtue of (5.10), one gets that there is a number $\iota=\iota(\lambda)>0$ such that

$$\big|U^+(t,x,y)-V(t,x,y)\big|\leq\frac{r_1}{4} \tag{5.78}$$

for all $|x|\geq\iota$. Without loss of generality, assume that $x_n\leq 0$ for all $n\in\mathbb{N}$. Let

$$\bar y_n := y_n+\frac{\psi(\lambda(x_n-\iota))}{\lambda}-\frac{\psi(\lambda x_n)}{\lambda}.$$

Then $|\bar y_n-y_n|\leq\iota\|\psi'\|_{L^\infty}\leq\iota\max\{\cot\alpha,-\cot\beta\}$. It is trivial to check that $\eta(t_n,x_n-\iota,\bar y_n)=\eta(t_n,x_n,y_n)$, which implies

$$|\eta(t_n-\tau,x_n-\iota,\bar y_n)|\leq q+c_{\alpha\beta} \tag{5.79}$$

for all $\tau\in[0,1]$ and $n\in\mathbb{N}$. Since V is a solution of (1.1), it solves an equation of the type

$$(\partial_t-\Delta_{x,y})V_t-f_u(x,y,V)V_t=0 \tag{5.80}$$

in $\mathbb{R} \times \mathbb{R}^2$, where $f_u(x, y, V)$ is bounded by (1.2). Using (2.14), (5.76) and (5.80), we obtain from the Harnack inequality that

$$V_t(t_n - \tau, x_n - \iota, \bar{y}_n) \to 0 \;\text{ as } n \to \infty$$

uniformly for $\tau \in [0, 1]$, which yields

$$V(t_n, x_n - \iota, \bar{y}_n) - V(t_n - 1, x_n - \iota, \bar{y}_n) \to 0 \;\text{ as } n \to \infty.$$

By virtue of (5.77)–(5.79), we reach a contradiction

$$V(t_n, x_n - \iota, \bar{y}_n) - V(t_n - 1, x_n - \iota, \bar{y}_n) \geq \frac{r_1}{2}, \quad \forall n \in \mathbb{N}.$$

Step 2: *we prove that $V_\sigma^+(t, x, y;\ T)$ is a supersolution.*

By (5.10) (fix $\lambda_* < \lambda_0^+(\delta, \varepsilon_*)$ in U^+, where $\varepsilon_* < \min\{p/8, \gamma_\star/8\}$), one gets that there exist $X_1' > 1$ and $X'' > 1$ such that $V_\sigma^+ < p$ in $\left\{(t, x, y) :\ \eta(T + \tilde{\tau}(t), x, y) \in \left(X_1', +\infty\right),\ t \geq 0\right\}$, and

$$V,\ V_\sigma^+ \in [1 - \gamma_\star, 1 + \gamma_\star] \;\text{ in } \left\{(t, x, y) :\ \eta(T + \tilde{\tau}(t), x, y) \in \left(-\infty, -X''\right),\ t \geq 0\right\}.$$

With similar arguments as those in *Case 1* and *Case 2* of the proof of Lemma 5.6, there exists $X' > X_1'$ such that

$$\mathcal{L}V_\sigma^+ > 0 \;\text{ in } \left\{(t, x, y) :\ \eta(T + \tilde{\tau}(t), x, y) \in \left(-\infty, -X''\right) \cup \left(X', +\infty\right),\ t \geq 0\right\},$$

where X' and X'' are independent of λ, and $\mu < \min\{\delta c_{\alpha\beta} c_\alpha/4, \kappa_1/2\}$.

By *Step 1*, there is a positive constant $\tilde{\lambda}_0^+(\delta) := \bar{\lambda}_0^+(\delta, \max\{X', X''\})$ such that, for any $0 < \lambda < \tilde{\lambda}_0^+(\delta)$ there exists $r = r(\lambda)$ such that $V_t > r$ in $\{(t, x, y) :\ -X'' \leq \eta(T + \tilde{\tau}(t), x, y) \leq X'\}$. With similar arguments as those in *Case 3* of the proof of Lemma 5.6, there exists a constant $\tilde{\varrho} = \tilde{\varrho}(\delta, \mu, \lambda)$ such that $LV_\sigma^+ > 0$ in $\{(t, x, y) :\ \eta(T + \tilde{\tau}(t), x, y) \in [-X'', X'],\ t \geq 0\}$. In conclusion, V_σ^+ is a supersolution of (1.1) for $t \geq 0$ and $(x, y) \in \mathbb{R}^2$, for all $\sigma \in (0, \min\{p/4, \gamma_\star/4\}]$.

Step 3: *we prove that $V_\sigma^-(t, x, y;\ T)$ is a subsolution.*

Let $\sigma^0 \leq \sigma_1^0 := \min\{p/4, \gamma_\star/4\}$. With similar arguments as *Step 2*, one can get that there exist $\widehat{X}' > 1$ and $\widehat{X}'' > 1$ independent of λ, such that

$$\mathcal{L}V_\sigma^- < 0 \;\text{ in } \left\{(t, x, y) :\ \eta(T + \hat{\tau}(t), x, y) \in \left(-\infty, -\widehat{X}''\right),\ t \geq 0\right\}$$

(provided $\hat{\mu} < \kappa_1/2$), and

$$\begin{aligned} \mathcal{L}V_\sigma^- &\leq \sigma \hat{\mu} e^{-\hat{\mu}t} U_\alpha^\delta - \delta U_\alpha^{\delta-1} \partial_s U_\alpha c_{\alpha\beta} \hat{\varrho} \hat{\mu} \left(\sigma e^{-\hat{\mu}t}\right)^2 - \sigma e^{-\hat{\mu}t} U_\alpha^\delta \frac{\delta c_{\alpha\beta} c_\alpha}{4} \\ &\leq \left(\hat{\mu} + \hat{\mu} \frac{-\partial_s U_\alpha \delta c_{\alpha\beta}}{U_\alpha} \hat{\varrho} \sigma - \frac{\delta c_{\alpha\beta} c_\alpha}{4}\right) \sigma e^{-\hat{\mu}t} U_\alpha^\delta \end{aligned} \tag{5.81}$$

in $\left\{(t, x, y) :\ \eta(T + \hat{\tau}(t), x, y) \in \left(\widehat{X}', +\infty\right),\ t \geq 0\right\}$. It follows from Theorems 2.5 and 2.7 that $\partial_s U_\alpha / U_\alpha$ is bounded. If

$$\sigma \leq \sigma_2^0 := \frac{1}{\hat{\varrho}},$$

then by (5.81) there exists a constant $\hat{\mu}(\delta) > 0$ such that

$$\mathcal{L}V_\sigma^- < 0 \;\text{ in } \left\{(t, x, y) :\ \eta(T + \hat{\tau}(t), x, y) \in \left(\widehat{X}', +\infty\right),\ t \geq 0\right\}.$$

By *Step 1*, there is a positive constant $\hat{\lambda}_0^+(\delta) := \bar{\lambda}_0^+(\delta, \max\{\widehat{X}', \widehat{X}''\})$ such that, for any $0 < \lambda < \hat{\lambda}_0^+(\delta)$ there exists $\hat{r} = \hat{r}(\lambda)$ such that

$$\frac{\partial}{\partial \hat{\tau}}\big(V(T+\hat{\tau}, x, y)\big) > \hat{r} \ \text{ in } \Big\{(t,x,y): \ -\widehat{X}'' \le \eta(T+\hat{\tau}, x, y) \le \widehat{X}'\Big\}.$$

With similar arguments as those in *Case 3* of the proof of Lemma 5.6, we have that

$$\begin{aligned}
\mathcal{L}V_\sigma^- \le\ & -\hat{\varrho}\sigma\hat{\mu}e^{-\hat{\mu}t}\hat{r} + \|f_u\|_{L^\infty}\sigma e^{-\hat{\mu}t} + \sigma e^{-\hat{\mu}t}\Delta_{x,y}\Big[\Big(U_\alpha^\delta(\eta,x,y)-1\Big)\omega(\eta)\Big] \\
& + \sigma\hat{\mu}e^{-\hat{\mu}t} - \sigma e^{-\hat{\mu}t}\frac{\partial}{\partial\hat{\tau}}\Big[\Big(U_\alpha^\delta(\eta,x,y)-1\Big)\omega(\eta)\Big]\hat{\tau}'(t) \\
\le\ & \big(-\hat{\varrho}\hat{\mu}\hat{r} + \|f_u\|_{L^\infty} + \hat{\mu} + C_1(\delta)\big)\sigma e^{-\hat{\mu}t} - \sigma e^{-\hat{\mu}t}\frac{\partial}{\partial\hat{\tau}}\Big[\Big(U_\alpha^\delta(\eta,x,y)-1\Big)\omega(\eta)\Big]\hat{\tau}'(t) \\
\le\ & \big(-\hat{\varrho}\hat{\mu}\hat{r} + \|f_u\|_{L^\infty} + \hat{\mu} + C_1(\delta) + C_2(\delta)\times\hat{\varrho}\hat{\mu}\sigma\big)\sigma e^{-\hat{\mu}t}
\end{aligned} \tag{5.82}$$

in $\big\{(t,x,y): \ \eta(T+\hat{\tau}(t),x,y) \in \big[-\widehat{X}'', \widehat{X}'\big], \ t \ge 0\big\}$. If

$$\sigma \le \sigma_3^0 := \frac{\hat{r}}{2C_2(\delta)} \ \text{ and } \ \hat{\varrho} > \frac{2\big(\|f_u\|_{L^\infty} + \hat{\mu} + C_1(\delta)\big)}{\hat{\mu}\hat{r}},$$

then it follows from (5.82) that

$$\mathcal{L}V_\sigma^- < 0 \ \text{ in } \big\{(t,x,y): \ \eta(T+\hat{\tau}(t),x,y) \in \big[-\widehat{X}'', \widehat{X}'\big], \ t \ge 0\big\}.$$

At last, let $\sigma^0 := \min\{\sigma_1^0, \sigma_2^0, \sigma_3^0\}$.

All in all, V_σ^- is a subsolution of (1.1) for $t \ge 0$ and $(x,y) \in \mathbb{R}^2$, for all $\sigma \in (0, \sigma^0]$. □

Now, we are ready to prove the stability of the curved front $V(t,x,y)$ in Theorem 2.12.

Proof of Theorem 2.16 It follows from (5.5), (5.6), (5.8), Remark 5.3 and the fact that $-\cot\alpha < \psi' < -\cot\beta$ from Lemma 5.2, that

$$\eta(\tau(0),x,y) \le \xi_\alpha/\sin\alpha \ \text{ and } \ \eta(\tau(0),x,y) \le \xi_\beta/\sin\beta, \ \ \forall(0,x,y) \in \mathbb{R}^3, \ \forall\lambda > 0. \tag{5.83}$$

Step 1: *we construct supersolutions of Cauchy problem* (1.1) *with initial value* $u_0(x,y)$.

Set $\delta_1 := \min\{\nu/K, \delta^*\}$, where positive constants ν, K and δ^* are given in (2.17), Theorem 2.8 and Lemma 5.6, separately. For any $\sigma \in (0, \min\{p/4, \gamma_\star/4\}]$, there exists from (2.17) a number $R_\sigma > 0$ such that

$$u_0(x,y) \le U_{\alpha\beta}^-(0,x,y) + \sigma\left(\frac{C_1}{2}\right)^{\delta_1}\min\Big\{1, e^{-\nu\min\{\xi_\alpha/\sin\alpha, \xi_\beta/\sin\beta\}}\Big\} \tag{5.84}$$

for all $x^2+y^2 > R_\sigma$, where the constant $C_1 > 0$ is given in Theorem 2.2, and ξ_α, ξ_β are evaluated at $(0,x,y)$. We claim that

$$W_\sigma^+(0,x,y) \ge u_0(x,y) \ \text{ in } \mathbb{R}^2 \tag{5.85}$$

for all $\sigma \in (0, \min\{p/4, \gamma_\star/4\}]$, where parameters in Lemma 5.6 are taken as $\delta = \delta_1$, $\mu = \mu(\delta_1)$, $\varrho = \varrho(\delta_1, \mu)$, $\forall\varepsilon \in (0, \varepsilon_0^+(\delta_1))$, and $0 < \lambda < \lambda_0^+(\delta_1, \varepsilon)$ is to be determined.

Case 1: $\min\{\xi_\alpha, \xi_\beta\} > 0$. By Theorem 2.2, there exists a constant $X_* > 0$ such that

$$U_\alpha(\eta,x,y) \ge \frac{C_1}{2}e^{-c_\alpha\eta}, \ \ \forall(\eta,x,y) \in (X_*, +\infty)\times\mathbb{R}^2. \tag{5.86}$$

Recalling $\delta_1 \le \nu/K$, we obtain from Lemma 5.6, (5.83), (5.84) and (5.86) that

$$W_\sigma^+(0,x,y) \ge U^+(0,x,y) + \sigma U_\alpha^{\delta_1}(\eta,x,y)$$

$$\geq U^{-}_{\alpha\beta}(0,x,y)+\sigma\left(\frac{C_1}{2}\right)^{\delta_1}e^{-\delta_1 c_\alpha \eta}$$
$$\geq U^{-}_{\alpha\beta}(0,x,y)+\sigma\left(\frac{C_1}{2}\right)^{\delta_1}e^{-\nu\eta}$$
$$\geq U^{-}_{\alpha\beta}(0,x,y)+\sigma\left(\frac{C_1}{2}\right)^{\delta_1}e^{-\nu\min\{\xi_\alpha/\sin\alpha,\,\xi_\beta/\sin\beta\}}$$
$$\geq u_0(x,y)$$

in $\{(x,y):\ \eta(0,x,y)>X_*,\ x^2+y^2>R_\sigma\}$. Furthermore, one infers from Proposition 3.9 that

$$W^{+}_{\sigma}(0,x,y)\geq U^{+}(0,x,y)+\sigma U^{\delta_1}_{\alpha}(X_*,x,y)\geq U^{+}(0,x,y)+r_1 \tag{5.87}$$

in $\{(x,y):\ \eta(0,x,y)\leq X_*\}$ for some constant $r_1>0$. Thus even if it means increasing R_σ, we obtain from (2.17) that

$$W^{+}_{\sigma}(0,x,y)\geq u_0(x,y)\ \text{ in } \{(x,y):\ \eta(0,x,y)\leq X_*,\ x^2+y^2>R_\sigma\}.$$

Note that $\psi(0)>0$. It follows from (5.7) and Lemma 5.2 that

$$\xi(0,x,y)=\frac{y-\psi(\lambda x)/\lambda}{\sqrt{\psi'^2(\lambda x)+1}}\to-\infty\ \text{ as } \lambda\to 0$$

uniformly in $\{(x,y):\ x^2+y^2\leq R_\sigma\}$. Then by virtue of (5.83), Proposition 3.3 and Lemma 5.4, we get that there is a number $\lambda^{+}_{1}(\sigma)>0$ such that for any $0<\lambda<\lambda^{+}_{1}$,

$$W^{+}_{\sigma}(0,x,y)\geq U_{e(x)}(\xi,x,y)+\sigma U^{\delta_1}_{\alpha}(\xi_\alpha/\sin\alpha,x,y)\geq 1\geq u_0(x,y) \tag{5.88}$$

in $\{(x,y):\ x^2+y^2\leq R_\sigma\}$. Hence $W^{+}_{\sigma}(0,x,y)\geq u_0(x,y)$ is valid in *Case 1*.

Case 2: $\min\{\xi_\alpha,\xi_\beta\}\leq 0$. By virtue of (5.83) and Proposition 3.9, one has

$$W^{+}_{\sigma}(0,x,y)\geq U^{+}(0,x,y)+\sigma U^{\delta_1}_{\alpha}(0,x,y)\geq U^{-}_{\alpha\beta}(0,x,y)+r_2$$

for some constant $r_2>0$. By (2.17), even if it means increasing R_σ, we get

$$W^{+}_{\sigma}(0,x,y)\geq u_0(x,y)\ \text{ in } \left\{(x,y):\ x^2+y^2>R_\sigma\right\}.$$

With similar arguments as (5.88), there is a number $\lambda^{+}_{2}(\sigma)>0$ such that for any $0<\lambda<\lambda^{+}_{2}$,

$$W^{+}_{\sigma}(0,x,y)\geq u_0(x,y)\ \text{ in } \left\{(x,y):\ x^2+y^2\leq R_\sigma\right\}.$$

Therefore $W^{+}_{\sigma}(0,x,y)\geq u_0(x,y)$ is valid in *Case 2*.

In conclusion, claim (5.85) is true for all $0<\lambda<\min\{\lambda^{+}_{0}(\delta_1,\varepsilon),\ \lambda^{+}_{1}(\sigma),\ \lambda^{+}_{2}(\sigma)\}$.

Step 2: *we introduce a time sequence based on the periodicity of our problem, and prove that the Omega-limit set along the time sequence contains only $V(t,x,y)$.*

By virtue of (2.16), *Step 1* and Lemma 5.6, using the comparison principle, one gets

$$U^{-}_{\alpha\beta}(t,x,y)\leq u(t,x,y)\leq W^{+}_{\sigma}(t,x,y)\ \text{ in } [0,+\infty)\times\mathbb{R}^2 \tag{5.89}$$

for all $\sigma\in(0,\min\{p/4,\gamma_\star/4\}]$. Define $t_n:=L_2 n/c_{\alpha\beta}$ and

$$u_n(t,x,y):=u(t+t_n,x,y+L_2 n)\ \text{ in } \mathbb{R}\times\mathbb{R}^2$$

for all $n\in\mathbb{N}$, where L_2 is the period of y. Then $t_n\to+\infty$ as $n\to\infty$. By parabolic estimates, we have a sequence of functions $\{u_{n_k}\}_{k\in\mathbb{N}}$ converging locally uniformly to a function

$u_\infty(t,x,y)$ in $\mathbb{R}\times\mathbb{R}^2$, which is an entire solution of (1.1). Since $U^-_{\alpha\beta}(t+t_{n_k},x,y+L_2n_k)=U^-_{\alpha\beta}(t,x,y)$ and $U^+(t+t_{n_k},x,y+L_2n_k)=U^+(t,x,y)$, we obtain from (5.89) that

$$U^-_{\alpha\beta}(t,x,y)\le u_{n_k}(t,x,y)\le U^+\Big(t-\varrho\sigma e^{-\mu(t+t_{n_k})}+\varrho\sigma,x,y\Big)+\sigma e^{-\mu(t+t_{n_k})} \tag{5.90}$$

in $[-t_{n_k},+\infty)\times\mathbb{R}^2$ for all $k\in\mathbb{N}$. Passing to the limit $k\to\infty$ yields

$$U^-_{\alpha\beta}(t,x,y)\le u_\infty(t,x,y)\le U^+(t+\varrho\sigma,x,y)\ \text{ in } \mathbb{R}\times\mathbb{R}^2.$$

Let $w_m(t,x,y;\ g(x,y))$ be the unique solution of the following Cauchy problem

$$\begin{cases}\partial_t w-\Delta_{x,y}w=f(x,y,w) & \text{when } t>-m,\ (x,y)\in\mathbb{R}^2\\ w(t,x,y)=g(x,y) & \text{when } t=-m,\ (x,y)\in\mathbb{R}^2\end{cases}$$

for all $m\in\mathbb{N}$. It follows from the comparison principle that

$$w_{n_k}\Big(t,x,y;\ U^-_{\alpha\beta}(-n_k,x,y)\Big)\le u_\infty(t,x,y)\le w_{n_k,\varepsilon,\sigma}\big(t,x,y;\ U^+(-n_k+\varrho\sigma,x,y)\big) \tag{5.91}$$

in $[-n_k,+\infty)\times\mathbb{R}^2$ for all $k\in\mathbb{N}$, where parameters ε and σ are given in W^+_σ. Recalling the definition of $V(t,x,y)$ in the proof of Theorem 2.12, one has

$$V(t,x,y)\le u_\infty(t,x,y)\ \text{ in } \mathbb{R}^3. \tag{5.92}$$

Furthermore, the comparison principle yields that

$$U^-_{\alpha\beta}(t+\varrho\sigma,x,y)\le w_{n_k,\varepsilon,\sigma}\big(t,x,y;\ U^+(-n_k+\varrho\sigma,x,y)\big)\le U^+(t+\varrho\sigma,x,y)$$

in $[-n_k,+\infty)\times\mathbb{R}^2$ for all $k\in\mathbb{N}$. By parabolic estimates, the sequence $\{w_{n_k,\varepsilon,\sigma}\}$ converges, up to a subsequence, locally uniformly to an entire solution $w_\infty(t,x,y)$ of (1.1) as $k\to\infty$, $\varepsilon\to0$ and $\sigma\to0$, which satisfies by (5.10) that

$$\lim_{R\to+\infty}\ \sup_{x^2+(y-c_{\alpha\beta}t)^2>R^2}\Big|w_\infty(t,x,y)-U^-_{\alpha\beta}(t,x,y)\Big|=0 \tag{5.93}$$

and $0\le w_\infty\le1$. Then by virtue of Theorem 2.15 and (5.93), we obtain $w_\infty\equiv V$ in $\mathbb{R}^3$, which implies by (5.91) and (5.92) that $u_\infty\equiv V$ in $\mathbb{R}^3$.

Notice that the parameter $\varrho(\delta,\mu)$ is independent of σ in Lemma 5.6. For any $\vartheta>0$, choose a parameter $\sigma_1(\vartheta)\in(0,\min\{p/4,\gamma_\star/4\}]$ such that

$$\begin{aligned}\Big|U^+\Big(t-\varrho\sigma_1e^{-\mu(t+t_{n_k})}+\varrho\sigma_1,x,y\Big)+\sigma_1e^{-\mu(t+t_{n_k})}-U^+(t,x,y)\Big| &\le \sigma_1+\big\|\partial_tU^+\big\|_{L^\infty}\varrho\sigma_1\\ &<\frac{\vartheta^4}{2}\end{aligned} \tag{5.94}$$

for all $(t,x,y)\in[0,+\infty)\times\mathbb{R}^2$ and $k\in\mathbb{N}$. Together with (5.90) and (5.94), one has

$$U^-_{\alpha\beta}(t,x,y)\le u_{n_k}(t,x,y)\le U^+(t,x,y)+\frac{\vartheta^4}{2}\ \text{ in } [0,+\infty)\times\mathbb{R}^2 \tag{5.95}$$

for all $k\in\mathbb{N}$. It follows from (5.95), (5.38) and (5.65) that there exists a number $R_\vartheta>0$ such that

$$\big|u_{n_k}(0,x,y)-V(0,x,y)\big|\le\vartheta^4\ \text{ in } \big\{(x,y):\ x^2+y^2>R_\vartheta\big\} \tag{5.96}$$

for all $k \in \mathbb{N}$. Since $\{u_{n_k}\}_{k\in\mathbb{N}}$ converges locally uniformly to $V(t,x,y)$ in $\mathbb{R}\times\mathbb{R}^2$, there is a number $k_0(\vartheta)\in\mathbb{N}$ such that

$$\left|u_{n_k}(0,x,y)-V(0,x,y)\right| \le \vartheta^4 \ \text{ in } \left\{(x,y):\ x^2+y^2\le R_\vartheta\right\} \tag{5.97}$$

for all $k \ge k_0$. Denote $T_\vartheta := t_{n_{k_0}}$. Then by virtue of (5.96), (5.97) and Remark 2.14, one gets that for any $\vartheta > 0$,

$$|u(T_\vartheta,x,y)-V(T_\vartheta,x,y)| = \left|u_{n_{k_0}}(0,x,y-L_2 n_{k_0})-V(0,x,y-L_2 n_{k_0})\right| \le \vartheta^4 \tag{5.98}$$

for all $(x,y)\in\mathbb{R}^2$.

Step 3: *we construct super- and subsolutions, which are perturbations of* $V(t,x,y)$.

It holds from (5.89) and (5.65) that (fix arbitrary parameters of W_σ^+ in *Step 1*)

$$|u(t,x,y)-V(t,x,y)| \le \left|W_\sigma^+(t,x,y)-U_{\alpha\beta}^-(t,x,y)\right| \ \text{ in } [0,+\infty)\times\mathbb{R}^2.$$

With similar arguments as those in (5.40), we have

$$\sqrt{\left|W_\sigma^+(t,x,y)\right|+\left|U_{\alpha\beta}^-(t,x,y)\right|} \le \Lambda \min\left\{1, e^{-\tilde\nu\min\{\xi_\alpha/\sin\alpha,\ \xi_\beta/\sin\beta\}}\right\}$$

in $\mathbb{R}\times\mathbb{R}^2$, for some constants $\tilde\nu > 0$ and $\Lambda > 0$, which yields by (5.98) that for any $\vartheta > 0$,

$$|u(T_\vartheta,x,y)-V(T_\vartheta,x,y)| \le \Lambda\vartheta^2 \min\left\{1, e^{-\tilde\nu\min\{\xi_\alpha/\sin\alpha,\ \xi_\beta/\sin\beta\}}\right\}, \quad \forall (x,y)\in\mathbb{R}^2. \tag{5.99}$$

Let $\delta_2 := \min\{\tilde\nu/K, \delta^*\}$, where K is given in Theorem 2.8. It follows from Proposition 3.9 that there is a constant $r_3 > 0$ such that

$$U_\alpha^{\delta_2}(0,x,y) \ge U_\alpha^{\delta_2}(X_*,x,y) \ge r_3, \quad \forall (x,y)\in\mathbb{R}^2. \tag{5.100}$$

Denote

$$\Gamma := \min\left\{1, \left(\frac{C_1}{2}\right)^{\delta_2}\frac{\min\{p/4, \gamma_\star/4\}}{\Lambda}, \left(\frac{2}{C_1}\right)^{\delta_2} r_3\right\}, \tag{5.101}$$

where C_1 is given in Theorem 2.2.

We claim that

$$V_\sigma^-(0,x,y;\ T_\vartheta) \le u(T_\vartheta,x,y) \le V_\sigma^+(0,x,y;\ T_\vartheta) \tag{5.102}$$

in $\mathbb{R}^2$ for all $0 < \vartheta < \Gamma$, where V_σ^- and V_σ^+ are defined in Lemma 5.7, and parameters are taken as $\delta = \delta_2$, $\sigma = \bar\sigma(\vartheta) := \Lambda\vartheta\,(2/C_1)^{\delta_2}$ in V_σ^- and V_σ^+.

Case i: $\min\{\xi_\alpha,\xi_\beta\} > 0$. Recalling $\delta_2 \le \tilde\nu/K$, we obtain from Lemma 5.7, (5.83), (5.86), (5.99) and (5.101) that for any $0 < \vartheta < \Gamma$,

$$\begin{aligned} V_{\bar\sigma}^+(0,x,y;\ T_\vartheta) &\ge V(T_\vartheta,x,y) + \bar\sigma U_\alpha^{\delta_2}(\eta,x,y) \\ &\ge V(T_\vartheta,x,y) + \bar\sigma\left(\frac{C_1}{2}\right)^{\delta_2} e^{-\delta_2 c_\alpha \eta} \\ &\ge V(T_\vartheta,x,y) + \Lambda\vartheta^2 e^{-\tilde\nu\min\{\xi_\alpha/\sin\alpha,\ \xi_\beta/\sin\beta\}} \\ &\ge u(T_\vartheta,x,y) \end{aligned}$$

in $\{(x,y):\ \eta(T_\vartheta,x,y) > X_*\}$. Furthermore, one has by (5.99)–(5.101) that

$$V_{\bar\sigma}^+(0,x,y;\ T_\vartheta) \ge V(T_\vartheta,x,y) + \bar\sigma U_\alpha^{\delta_2}(X_*,x,y) \ge V(T_\vartheta,x,y) + \bar\sigma r_3 \ge u(T_\vartheta,x,y)$$

in $\{(x, y): \eta(T_\vartheta, x, y) \le X_*\}$ for all $0 < \vartheta < \Gamma$.

Case ii: $\min\{\xi_\alpha, \xi_\beta\} \le 0$. It follows from (5.83) and (5.99)–(5.101) that

$$V_{\bar{\sigma}}^{+}(0, x, y;\ T_\vartheta) \ge V(T_\vartheta, x, y) + \bar{\sigma} U_\alpha^{\delta_2}(0, x, y) \ge V(T_\vartheta, x, y) + \bar{\sigma} r_3 \ge u(T_\vartheta, x, y)$$

in *Case ii* for all $0 < \vartheta < \Gamma$.

All in all, $V_{\bar{\sigma}}^{+}(0, x, y;\ T_\vartheta) \ge u(T_\vartheta, x, y)$ in $\mathbb{R}^2$, where parameters in Lemma 5.7 are taken as $\delta = \delta_2$, $\lambda = \tilde{\lambda}_0^{+}(\delta_2)/2$, $\mu = \mu(\delta_2)$, $\varrho = \tilde{\varrho}(\delta_2, \mu, \lambda)$, $\sigma = \bar{\sigma} = \Lambda\vartheta\,(2/C_1)^{\delta_2}$. Similarly, one can get that $V_{\bar{\sigma}}^{-}(0, x, y;\ T_\vartheta) \le u(T_\vartheta, x, y)$ in $\mathbb{R}^2$. Thus claim (5.102) is valid. It follows from (5.102), Lemma 5.7 and the comparison principle that for any $0 < \vartheta < \Gamma$,

$$V_{\bar{\sigma}}^{-}(s, x, y;\ T_\vartheta) \le u(T_\vartheta + s, x, y) \le V_{\bar{\sigma}}^{+}(s, x, y;\ T_\vartheta),\quad \forall (s, x, y) \in [0, +\infty) \times \mathbb{R}^2.$$

Finally, letting $\vartheta \to 0$ which implies $\bar{\sigma} \to 0$, we obtain from Lemma 5.7 that

$$u(t, x, y) \to V(t, x, y)\ \text{ as } t \to +\infty$$

uniformly in $\mathbb{R} \times \mathbb{R}^2$. The proof of Theorem 2.16 is thereby complete. □

Proof of Theorem 2.17 Assume that there exists an entire solution $V(t, x, y)$ of (1.1) satisfying (2.13) for some constant $c_{\alpha\beta}$. Then it follows from the same proof of Theorem 1.7 in Guo et al. [26] that

$$\frac{c_\alpha}{\sin\alpha} = \frac{c_\beta}{\sin\beta} = c_{\alpha\beta}\ \text{ and }\ \frac{c_\theta}{\sin\theta} \ne c_{\alpha\beta},\ \forall \theta \in (\alpha, \beta).$$

Below we will rule out the case that $c_\theta/\sin\theta > c_{\alpha\beta}$, where $\theta \in (\alpha, \beta)$.

Fix any $\delta \in (0, 1)$. For any $\theta \in (\alpha, \beta)$ define

$$W_\theta^{-}(t, x, y;\ T) := U_\theta\,(\xi_\theta, x, y) - \sigma e^{-\mu t} \times \left[U_\theta^{\delta}(\xi_\theta, x, y)\omega(\xi_\theta) + (1 - \omega(\xi_\theta))\right]$$

in $[0, +\infty) \times \mathbb{R}^2$ for all $T \in \mathbb{R}$, where ξ_θ is evaluated at $(T + t + \varrho\sigma e^{-\mu t} - \varrho\sigma, x, y)$, and $\xi_\theta = \xi_\theta(t, x, y) := x\cot\theta + y\sin\theta - c_\theta t$, and μ, ϱ, σ are some positive constants. In fact, with similar arguments as those in (5.19), (5.21), (5.22) and in the proof of Lemma 5.7, we can get that

$$\mathcal{L}W_\theta^{-} := \partial_t W_\theta^{-} - \Delta_{x,y} W_\theta^{-} - f\big(x, y, W_\theta^{-}\big) \le 0\ \text{ in } [0, +\infty) \times \mathbb{R}^2,\ \forall \theta \in (\alpha, \beta),\ \forall T,$$

where $\mu = \mu(\delta, \theta)$, $\varrho = \varrho(\delta, \theta, \mu)$ and $\sigma = \sigma(\delta, \theta, \mu, \varrho)$. Since $\delta < 1$, there is a number $C = C(\sigma)$ independent of T such that

$$W_\theta^{-}(0, x, y;\ T) = U_\theta(\xi_\theta(T, x, y), x, y) - \sigma U_\theta^{\delta}(\xi_\theta(T, x, y), x, y) \le 0 \le V(0, x, y)$$

in $\{(0, x, y): \xi_\theta(T, x, y) \ge C\}$, for all $T \in \mathbb{R}$. Then by similar arguments as those in the proof of Claim 2.10 in [26], there exists a number $T_* \ll 0$ such that

$$W_\theta^{-}(0, x, y;\ T_*) \le V(0, x, y),\quad \forall (x, y) \in \mathbb{R}^2.$$

Thus it follows from the maximum principle that

$$V(t, x, y) \ge W_\theta^{-}(t, x, y;\ T_*),\ \ \forall (t, x, y) \in [0, +\infty) \times \mathbb{R}^2.$$

Finally taking a sequence $\{(t_n, 0, c_{\alpha\beta} t_n + R)\}_{n\in\mathbb{N}}$, where $t_n \to +\infty$ as $n \to \infty$ and the constant R is large enough, with the same arguments as in the proof of Theorem 1.7 in Guo et al. [26], we can obtain a contradiction with that $c_\theta/\sin\theta > c_{\alpha\beta}$ for some $\theta \in (\alpha, \beta)$. The proof of Theorem 2.17 is thereby complete. □

6 Appendix

In this section we give the proofs of Lemmas 4.6 and 4.8 respectively.

Proof of Lemma 4.6 *Step 1*: *we prove that the range of* $\mathcal{Q}_e$, *denoted by* $R(\mathcal{Q}_e)$, *is closed in* $H^1_\rho \times \mathbb{R}$.

Let $(\tilde{v}_n, \tilde{\gamma}_n),\ (\tilde{g}_n, \tilde{d}_n),\ (\tilde{g}, \tilde{d}) \in H^1_\rho \times \mathbb{R}$ satisfy

$$\mathcal{Q}_e(\tilde{v}_n, \tilde{\gamma}_n) = (\tilde{g}_n, \tilde{d}_n)\ \ \forall n \in \mathbb{N}, \ \ \text{and}\ \left\| (\tilde{g}_n, \tilde{d}_n) - (\tilde{g}, \tilde{d}) \right\|_{H^1_\rho \times \mathbb{R}} \to 0\ \ \text{as}\ n \to \infty. \tag{6.1}$$

Denote $v_n := \tilde{v}_n - \tilde{g}_n$. Then by virtue of (4.9) and (6.1), one has

$$v_n = -\mathcal{M}_e^{-1}\big(\tilde{\gamma}_n \partial_s U_e + f_u(z, U_e)\tilde{v}_n + \beta \tilde{v}_n\big) \in \mathcal{D}(\mathcal{M}_e) = \mathcal{D}(\mathcal{H}_e), \ \ \forall n \in \mathbb{N}.$$

Therefore, one further has

$$\mathcal{H}_e(v_n) = \mathcal{M}_e(v_n) + (f_u(z, U_e) + \beta)\, v_n = -\,(f_u(z, U_e) + \beta)\, \tilde{g}_n - \tilde{\gamma}_n \partial_s U_e \tag{6.2}$$

for all $n \in \mathbb{N}$. For any $w \in \ker(\mathcal{H}_e^*)\backslash\{0\}$, there holds

$$0 = \big(v_n, \mathcal{H}_e^*(w)\big)_{L^2_\rho} = (\mathcal{H}_e(v_n), w)_{L^2_\rho} = (-\,(f_u(z, U_e) + \beta)\, \tilde{g}_n - \tilde{\gamma}_n \partial_s U_e\,,\ w)_{L^2_\rho}$$

for all $n \in \mathbb{N}$, which implies

$$\tilde{\gamma}_n\, (\partial_s U_e, w)_{L^2_\rho} = -\,((f_u(z, U_e) + \beta)\, \tilde{g}_n\,,\ w)_{L^2_\rho}\,, \ \ \forall n \in \mathbb{N}. \tag{6.3}$$

We claim that $(\partial_s U_e, w)_{L^2_\rho} \neq 0$ for all $w \in \ker(\mathcal{H}_e^*)\backslash\{0\}$. Assume by contradiction that $(\partial_s U_e, w)_{L^2_\rho} = 0$ for some $w \in \ker(\mathcal{H}_e^*) \setminus \{0\}$. From Lemma 4.4, one knows

$$\dim\big(\ker(\mathcal{H}_e^*)\big) = 1 \ \ \text{and}\ \ L^2_\rho(\mathbb{R} \times \mathbb{L}^N) = R(\mathcal{H}_e) \oplus \ker(\mathcal{H}_e^*).$$

On the one hand, since $w \in \ker(\mathcal{H}_e^*)\backslash\{0\}$, one has $\partial_s U_e \in \big(\ker(\mathcal{H}_e^*)\big)^\perp = R(\mathcal{H}_e)$. On the other hand, it follows from Lemma 4.4 that the linear operator $\mathcal{H}_e$ has algebraically simple eigenvalue 0 and the kernel of $\mathcal{H}_e$ is generated by $\partial_s U_e$, thus $\partial_s U_e \notin R(\mathcal{H}_e)$, which is a contradiction.

It follows from (6.3) that

$$|\tilde{\gamma}_n - \tilde{\gamma}_m| \leq \frac{\|f_u\|_{L^\infty} + \beta}{\left|(\partial_s U_e, w)_{L^2_\rho}\right|}\, \|w\|_{L^2_\rho}\, \|\tilde{g}_n - \tilde{g}_m\|_{L^2_\rho}\,, \ \ \forall n,\ m \in \mathbb{N}.$$

Since $\{\tilde{g}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in H^1_ρ from (6.1), there holds that $\{\tilde{\gamma}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, then one can assume that $\{\tilde{\gamma}_n\}_{n\in\mathbb{N}}$ converges to $\tilde{\gamma}$. By (6.2), we have

$$\|\mathcal{H}_e(v_n) - \mathcal{H}_e(v_m)\|_{L^2_\rho} \leq \big(\|f_u\|_{L^\infty} + \beta\big)\, \|\tilde{g}_n - \tilde{g}_m\|_{L^2_\rho} + \|\partial_s U_e\|_{L^2_\rho}\, |\tilde{\gamma}_n - \tilde{\gamma}_m|$$

for all $n,\ m \in \mathbb{N}$, which together with (6.1) yield that $\{\mathcal{H}_e(v_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in L^2_ρ. Thus one can assume that $\mathcal{H}_e(v_n)$ converges to $g \in L^2_\rho$. One gets from (6.2) that

$$-\,(f_u(z, U_e) + \beta)\, \tilde{g} - \tilde{\gamma}\, \partial_s U_e = g. \tag{6.4}$$

Since $R(\mathcal{H}_e)$ is closed in L^2_ρ from Lemma 4.4, there exists a function $v \in \mathcal{D}(\mathcal{H}_e)$ such that $\mathcal{H}_e(v) = g$. Set $\tilde{v} := v + \alpha \partial_s U_e + \tilde{g} \in H^1_\rho$, where $\alpha \in \mathbb{R}$ is to be chosen. Clearly,

$$\int_{\mathbb{R}^+ \times \mathbb{L}^N} U_e \partial_s U_e \rho\, ds dz \neq 0, \tag{6.5}$$

because $U_e > 0,\ \partial_s U_e < 0$. Thus we can select $\alpha \in \mathbb{R}$ such that

$$2\int_{\mathbb{R}^+\times\mathbb{L}^N} U_e \tilde{v}\rho\, dsdz = \tilde{d}. \tag{6.6}$$

Since $\tilde{v} - \tilde{g} = v + \alpha\partial_s U_e \in \mathcal{D}(\mathcal{H}_e)$, it follows that $\mathcal{H}_e(\tilde{v} - \tilde{g}) = \mathcal{H}_e(v) = g \in L^2_\rho$, which implies

$$\mathcal{M}_e(\tilde{v} - \tilde{g}) + (f_u(z, U_e) + \beta)(\tilde{v} - \tilde{g}) = g \in L^2_\rho. \tag{6.7}$$

Composing (6.7) by $\mathcal{M}_e^{-1}$ and using (6.4), one concludes

$$\tilde{v} + \mathcal{M}_e^{-1}(\tilde{\gamma}\partial_s U_e + f_u(z, U_e)\tilde{v} + \beta\tilde{v}) = \tilde{g}. \tag{6.8}$$

Therefore, $\mathcal{Q}_e(\tilde{v}, \tilde{\gamma}) = (\tilde{g}, \tilde{d})$ by (6.6) and (6.8), where $(\tilde{v}, \tilde{\gamma}) \in H^1_\rho \times \mathbb{R}$. This completes *Step 1.*

Step 2: *we prove that* $\mathcal{Q}_e$ *is injective.*

Let $(\tilde{v}, \tilde{\gamma}) \in \ker(\mathcal{Q}_e) \subset H^1_\rho \times \mathbb{R}$, then

$$\tilde{v} = -\mathcal{M}_e^{-1}(\tilde{\gamma}\partial_s U_e + f_u(z, U_e)\tilde{v} + \beta\tilde{v}) \in \mathcal{D}(\mathcal{M}_e) = \mathcal{D}(\mathcal{H}_e),$$

which yields

$$\mathcal{H}_e(\tilde{v}) = \mathcal{M}_e\tilde{v} + f_u(z, U_e)\tilde{v} + \beta\tilde{v} = -\tilde{\gamma}\partial_s U_e.$$

Since the linear operator $\mathcal{H}_e$ has algebraically simple eigenvalue 0 and the kernel of $\mathcal{H}_e$ is generated by $\partial_s U_e$ from Lemma 4.4, one has $\mathcal{H}_e^2(\tilde{v}) = 0$, then $\tilde{v} = \sigma\partial_s U_e$, so $\mathcal{H}_e(\tilde{v}) = 0$, thus $\tilde{\gamma} = 0$. Furthermore, by virtue of $(\tilde{v}, \tilde{\gamma}) \in \ker(\mathcal{Q}_e)$, it holds that

$$0 = 2\int_{\mathbb{R}^+\times\mathbb{L}^N} U_e\tilde{v}\rho\, dsdz = 2\sigma\int_{\mathbb{R}^+\times\mathbb{L}^N} U_e\partial_s U_e\rho\, dsdz.$$

Combining with (6.5), one gets $\sigma = 0$, thus $\tilde{v} = 0$. Therefore, $\ker(\mathcal{Q}_e) = \theta$.

Step 3: *we prove that* $\ker(\mathcal{Q}_e^*) = \theta$.

Set $(\tilde{v}, \tilde{\gamma}) \in \ker(\mathcal{Q}_e^*) \subset H^1_\rho \times \mathbb{R}$, then for any $(\tilde{w}, \tilde{\mu}) \in H^1_\rho \times \mathbb{R}$, one has

$$\begin{aligned} 0 &= \big((\tilde{w}, \tilde{\mu}),\ \mathcal{Q}_e^*(\tilde{v}, \tilde{\gamma})\big)_{H^1_\rho\times\mathbb{R}} = (\mathcal{Q}_e(\tilde{w}, \tilde{\mu}),\ (\tilde{v}, \tilde{\gamma}))_{H^1_\rho\times\mathbb{R}} \\ &= \big(\tilde{w} + \mathcal{M}_e^{-1}(\tilde{\mu}\partial_s U_e + f_u(z, U_e)\tilde{w} + \beta\tilde{w})\ ,\ \tilde{v}\big)_{H^1_\rho} + 2\tilde{\gamma}\int_{\mathbb{R}^+\times\mathbb{L}^N} U_e\tilde{w}\rho\, dsdz. \end{aligned} \tag{6.9}$$

Choosing $(\tilde{w}, \tilde{\mu}) = (\partial_s U_e, 0)$ in (6.9), it follows from Lemma 4.4 that

$$0 = \big(\mathcal{M}_e^{-1}(\mathcal{H}_e(\partial_s U_e)),\ \tilde{v}\big)_{H^1_\rho} + 2\tilde{\gamma}\int_{\mathbb{R}^+\times\mathbb{L}^N} U_e\partial_s U_e\rho\, dsdz = 2\tilde{\gamma}\int_{\mathbb{R}^+\times\mathbb{L}^N} U_e\partial_s U_e\rho\, dsdz.$$

Recalling (6.5), one gets $\tilde{\gamma} = 0$. Choosing $(\tilde{w}, \tilde{\mu}) = (0, 1)$ in (6.9), one has

$$0 = \big(\mathcal{M}_e^{-1}(\partial_s U_e)\ ,\ \tilde{v}\big)_{H^1_\rho}. \tag{6.10}$$

Choosing $(\tilde{w}, \tilde{\mu}) = (\tilde{w}, 0) \in \mathcal{D}(\mathcal{H}_e) \times \mathbb{R}$ in (6.9), one has

$$0 = \big(\tilde{w} + \mathcal{M}_e^{-1}(f_u(z, U_e)\tilde{w} + \beta\tilde{w})\ ,\ \tilde{v}\big)_{H^1_\rho} = \big(\mathcal{M}_e^{-1}(\mathcal{H}_e(\tilde{w}))\ ,\ \tilde{v}\big)_{H^1_\rho}. \tag{6.11}$$

Since

$$L^2_\rho(\mathbb{R}\times\mathbb{L}^N) = R(\mathcal{H}_e) \oplus \ker(\mathcal{H}_e^*) \tag{6.12}$$

from Lemma 4.4, and since $\partial_s U_e \notin R(\mathcal{H}_e)$, it follows from (6.10) and (6.11) that

$$\left(\mathcal{M}_e^{-1}(\mathcal{P}(\partial_s U_e))\,,\ \tilde{v}\right)_{H_\rho^1} = 0 \tag{6.13}$$

where $\mathcal{P}(\partial_s U_e) \neq 0$ is the orthogonal projection of $\partial_s U_e$ onto $\ker(\mathcal{H}_e^*)$. Moreover, it follows from Lemma 4.4 that $\dim\left(\ker(\mathcal{H}_e^*)\right) = 1$, hence one gets that $\ker(\mathcal{H}_e^*)$ is generated by $\mathcal{P}(\partial_s U_e)$. Together with (6.11)–(6.13), one concludes

$$\left(\mathcal{M}_e^{-1}(w)\,,\ \tilde{v}\right)_{H_\rho^1} = 0 \ \text{ for all } w \in L_\rho^2(\mathbb{R} \times \mathbb{L}^N). \tag{6.14}$$

Now choosing $(\tilde{w}, \tilde{\mu}) = (\tilde{v}, 0) \in H_\rho^1 \times \mathbb{R}$ in (6.9), it follows from (6.14) that

$$0 = \left(\tilde{v} + \mathcal{M}_e^{-1}(f_u(z, U_e)\tilde{v} + \beta\tilde{v})\,,\ \tilde{v}\right)_{H_\rho^1} = (\tilde{v}, \tilde{v})_{H_\rho^1}.$$

As a consequence, $\tilde{v} = 0$, thus together with $\tilde{\gamma} = 0$, one gets $\ker(\mathcal{Q}_e^*) = \theta$.

Step 4: *we prove that $\mathcal{Q}_e$ is surjective.*

Since $\mathcal{M}_e^{-1}$ is a bounded linear operator from Lemma 4.2, it follows that $\mathcal{Q}_e : H_\rho^1 \times \mathbb{R} \to H_\rho^1 \times \mathbb{R}$ is also a bounded linear operator. Since $H_\rho^1 \times \mathbb{R}$ is a Hilbert space, since $R(\mathcal{Q}_e) = \overline{R(\mathcal{Q}_e)}$ in $H_\rho^1 \times \mathbb{R}$ from *Step 1*, and since $\ker(\mathcal{Q}_e^*) = \theta$ from *Step 3*, one concludes

$$H_\rho^1 \times \mathbb{R} = \overline{R(\mathcal{Q}_e)} \oplus \ker(\mathcal{Q}_e^*) = R(\mathcal{Q}_e) \oplus \theta = R(\mathcal{Q}_e).$$

Therefore, the linear operator $\mathcal{Q}_e$ is surjective.

All in all, the linear operator $\mathcal{Q}_e$ is injective and surjective, thus it is invertible. Furthermore, since $\mathcal{Q}_e$ is a bounded linear operator, we get that the inverse operator $\mathcal{Q}_e^{-1}$ is also bounded. □

Proof of Lemma 4.8 *Step 1*: *we prove that $\mathcal{Q}_e^{-1}$ is uniformly bounded with respect to $e \in \mathbb{S}^{N-1}$.*

Assume by contradiction that there exist $\{e_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^{N-1}$, e and $\{(g_n, d_n)\}_{n\in\mathbb{N}}$ such that

$$\|(g_n, d_n)\|_{H_\rho^1 \times \mathbb{R}} = 1 \ \ \forall n \in \mathbb{N}, \ \ \text{and } e_n \to e \text{ as } n \to \infty \tag{6.15}$$

and

$$\left\|\mathcal{Q}_{e_n}^{-1}(g_n, d_n)\right\|_{H_\rho^1 \times \mathbb{R}} \to +\infty \ \text{ as } n \to \infty. \tag{6.16}$$

Denote

$$(v_n, \gamma_n) := \mathcal{Q}_{e_n}^{-1}(g_n, d_n) \ \text{ and } \ (\tilde{v}_n, \tilde{\gamma}_n) := \frac{1}{\|(v_n, \gamma_n)\|_{H_\rho^1 \times \mathbb{R}}}(v_n, \gamma_n), \quad \forall n \in \mathbb{N}.$$

Clearly, $\|(\tilde{v}_n, \tilde{\gamma}_n)\|_{H_\rho^1 \times \mathbb{R}} = 1$. Using Lemma 4.2 and (4.8), one obtains from calculations that

$$\begin{aligned}
&\left\|\mathcal{Q}_e(\tilde{v}_n, \tilde{\gamma}_n) - \mathcal{Q}_{e_n}(\tilde{v}_n, \tilde{\gamma}_n)\right\|_{H_\rho^1 \times \mathbb{R}} \\
&= \left\|\mathcal{M}_e^{-1}(\tilde{\gamma}_n \partial_s U_e + f_u(z, U_e)\tilde{v}_n + \beta\tilde{v}_n) - \mathcal{M}_{e_n}^{-1}(\tilde{\gamma}_n \partial_s U_{e_n} + f_u(z, U_{e_n})\tilde{v}_n + \beta\tilde{v}_n)\right\|_{H_\rho^1} \\
&\quad + \left|2\int_{\mathbb{R}^+ \times \mathbb{L}^N} (U_e - U_{e_n})\tilde{v}_n \rho \, ds dz\right| \\
&\le \left\|\left(\mathcal{M}_e^{-1} - \mathcal{M}_{e_n}^{-1}\right)(\tilde{\gamma}_n \partial_s U_e + f_u(z, U_e)\tilde{v}_n + \beta\tilde{v}_n)\right\|_{H_\rho^1} \\
&\quad + M\left\|\tilde{\gamma}_n(\partial_s U_e - \partial_s U_{e_n}) + \tilde{v}_n\left(f_u(z, U_e) - f_u(z, U_{e_n})\right)\right\|_{L_\rho^2} + 2\left\|U_{e_n} - U_e\right\|_{L_\rho^2} \\
&\le \left\|\left(\mathcal{M}_e^{-1} - \mathcal{M}_{e_n}^{-1}\right)(\tilde{\gamma}_n \partial_s U_e + f_u(z, U_e)\tilde{v}_n + \beta\tilde{v}_n)\right\|_{H_\rho^1} + (M+2)\left\|U_{e_n} - U_e\right\|_{H_\rho^1}
\end{aligned}$$

$$
\begin{aligned}
&+ M \left\| \left(f_u(z, U_e) - f_u(z, U_{e_n})\right)^2 \tilde{v}_n^2 \rho \right\|_{L^1} \\
&\leq \left\| \left(\mathcal{M}_e^{-1} - \mathcal{M}_{e_n}^{-1}\right)(\tilde{\gamma}_n \partial_s U_e + f_u(z, U_e)\tilde{v}_n + \beta \tilde{v}_n) \right\|_{H_\rho^1} + (M+2) \left\| U_{e_n} - U_e \right\|_{H_\rho^1} \\
&+ M(2 \left\| f_u \right\|_{L^\infty})^{\frac{2N}{N+1}} C \left\| f_{uu} \right\|_{L^\infty}^{\frac{2}{N+1}} \left\| U_{e_n} - U_e \right\|_{L^2}^{\frac{2}{N+1}} \qquad (6.17)
\end{aligned}
$$

for all $n \in \mathbb{N}$, where C is a constant independent of n, and we have used the Sobolev imbedding theorem (one can refer to Theorem 4.12 of [1]). Note that $\|(\tilde{v}_n, \tilde{\gamma}_n)\|_{H_\rho^1 \times \mathbb{R}} = 1$ and $\partial_s U_e \in H_\rho^1$. Then $\tilde{\gamma}_n \partial_s U_e + f_u(z, U_e)\tilde{v}_n + \beta \tilde{v}_n$ belongs to $H_\rho^1(\mathbb{R} \times \mathbb{L}^N)$, and its H_ρ^1 norm is uniformly bounded with respect to n. By virtue of (6.17) and Lemmas 4.3 and 4.7, one gets

$$
\left\| \mathcal{Q}_e(\tilde{v}_n, \tilde{\gamma}_n) - \mathcal{Q}_{e_n}(\tilde{v}_n, \tilde{\gamma}_n) \right\|_{H_\rho^1 \times \mathbb{R}} \to 0 \;\text{ as } n \to \infty. \qquad (6.18)
$$

It follows from (6.15) and (6.16) that

$$
\left\| \mathcal{Q}_{e_n}(\tilde{v}_n, \tilde{\gamma}_n) \right\|_{H_\rho^1 \times \mathbb{R}} = \frac{\left\| \mathcal{Q}_{e_n}(v_n, \gamma_n) \right\|_{H_\rho^1 \times \mathbb{R}}}{\|(v_n, \gamma_n)\|_{H_\rho^1 \times \mathbb{R}}} = \frac{1}{\|(v_n, \gamma_n)\|_{H_\rho^1 \times \mathbb{R}}} \to 0 \;\text{ as } n \to \infty. \quad (6.19)
$$

Then, since $\mathcal{Q}_e^{-1}$ is bounded by Lemma 4.6, we obtain from (6.18) and (6.19) that

$$
\|(\tilde{v}_n, \tilde{\gamma}_n)\|_{H_\rho^1 \times \mathbb{R}} \to 0 \;\text{ as } n \to \infty,
$$

which contradicts the fact that $\|(\tilde{v}_n, \tilde{\gamma}_n)\|_{H_\rho^1 \times \mathbb{R}} = 1$.

Step 2. Let $\{e_m\}_{m \in \mathbb{N}} \subset \mathbb{S}^{N-1}$ satisfy $\lim_{m \to \infty} e_m = e$. For given $(g, d) \in H_\rho^1 \times \mathbb{R}$ satisfying $\|(g, d)\|_{H_\rho^1 \times \mathbb{R}} \leq 1$, denote

$$
(v_m, \gamma_m) := \mathcal{Q}_{e_m}^{-1}(g, d) \;\text{ and }\; (v, \gamma) := \mathcal{Q}_e^{-1}(g, d).
$$

Then *Step 1* yields that $\|(v_m, \gamma_m)\|_{H_\rho^1 \times \mathbb{R}} \leq \widetilde{C} \|(g, d)\|_{H_\rho^1 \times \mathbb{R}}$, where $\widetilde{C}$ is a constant independent of $m \in \mathbb{N}$. With similar arguments as (6.17) and (6.18), one gets

$$
\left\| \mathcal{Q}_e(v_m, \gamma_m) - \mathcal{Q}_{e_m}(v_m, \gamma_m) \right\|_{H_\rho^1 \times \mathbb{R}} \to 0 \;\text{ as } m \to \infty
$$

uniformly with respect to $(g, d) \in \{(g, d) \in H_\rho^1 \times \mathbb{R} : \; \|(g, d)\|_{H_\rho^1 \times \mathbb{R}} \leq 1\}$, which implies that

$$
\begin{aligned}
\|(v_m, \gamma_m) - (v, \gamma)\|_{H_\rho^1 \times \mathbb{R}} &\leq \left\| \mathcal{Q}_e^{-1} \right\| \|\mathcal{Q}_e(v_m, \gamma_m) - \mathcal{Q}_e(v, \gamma)\|_{H_\rho^1 \times \mathbb{R}} \\
&= \left\| \mathcal{Q}_e^{-1} \right\| \left\| \mathcal{Q}_e(v_m, \gamma_m) - \mathcal{Q}_{e_m}(v_m, \gamma_m) \right\|_{H_\rho^1 \times \mathbb{R}} \\
&\to 0 \;\text{ as }\; m \to \infty \qquad (6.20)
\end{aligned}
$$

uniformly with respect to $(g, d) \in \{(g, d) \in H_\rho^1 \times \mathbb{R} : \; \|(g, d)\|_{H_\rho^1 \times \mathbb{R}} \leq 1\}$. Eventually, one concludes from (6.20) that

$$
\left\| \mathcal{Q}_{e_m}^{-1} - \mathcal{Q}_e^{-1} \right\| = \sup_{\|(g,d)\|_{H_\rho^1 \times \mathbb{R}} \leq 1} \left\| \mathcal{Q}_{e_m}^{-1}(g, d) - \mathcal{Q}_e^{-1}(g, d) \right\|_{H_\rho^1 \times \mathbb{R}} \to 0
$$

as $m \to \infty$. The proof of Lemma 4.8 is thereby complete. □

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Declaration

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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