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Global Existence for Long Wave Hopf Unstable Spatially Extended Systems with a Conservation Law

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Abstract

We are interested in reaction–diffusion systems, with a conservation law, exhibiting a Hopf bifurcation at the spatial wave number $k = 0$. With the help of a multiple scaling perturbation ansatz a Ginzburg–Landau equation coupled to a scalar conservation law can be derived as an amplitude system for the approximate description of the dynamics of the original reaction–diffusion system near the first instability. We use the amplitude system to show the global existence of all solutions starting in a small neighborhood of the weakly unstable ground state for original systems posed on a large spatial interval with periodic boundary conditions.

Keywords Pattern formation · Conservation law · Amplitude equations · Justification

1 Introduction

We consider reaction–diffusion systems for u with $u(x,t) \in \mathbb{R}^d$ for $d \geq 2$ coupled to a diffusive conservation law for v with $v(x,t) \in \mathbb{R}$, namely

$$\partial_t u = D\partial_x^2 u + f(u, v), \tag{1}$$

$$\partial_t v = d_v \partial_x^2 v + \partial_x^2 g(u), \tag{2}$$

where $x \in \mathbb{R}$, $t \geq 0$, D a diagonal diffusion matrix with entries $d_j > 0$ for $j = 1, \ldots, d$, $d_v > 0$ a scalar diffusion coefficient, and $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}$ smooth reaction terms with

$$f(u, v) = \mathcal{O}(|u|(1 + |u| + |v|) \qquad \text{and} \qquad g(u) = \mathcal{O}(|u|^2)$$

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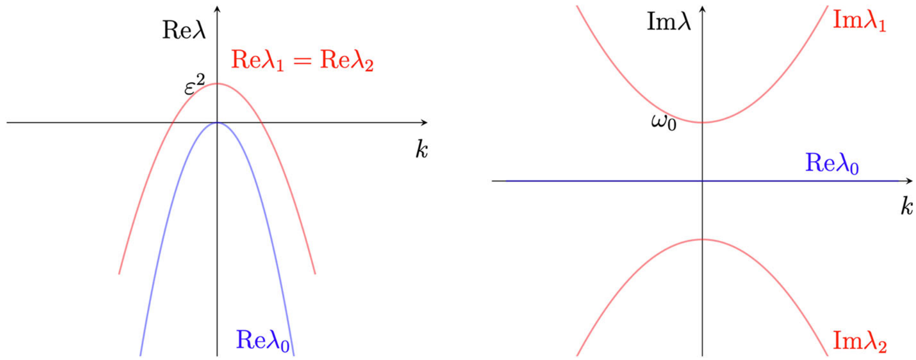

Fig. 1 The relevant spectral curves of the linearization around the trivial solution plotted as a function over the Fourier wave numbers for $\widetilde{\alpha} - \widetilde{\alpha}_c = \varepsilon^2 > 0$. The left panel shows the real part of the eigenvalue curves λ_0 (in blue), λ_1, and λ_2 (both in red), the right panel shows the imaginary part (Color figure online)

such that $(u, v) = (0, v^*)$ is a stationary solution for any constant $v^* \in \mathbb{R}$. As a consequence of the conservation law form the spatial integral of v is conserved in time. The fact that g only depends on u or that D is a diagonal matrix are no restrictions w.r.t. our purposes. For a detailed discussion about that see Sect. 6.

We are interested in the behavior of (1)–(2) close to the stationary solutions, w.l.o.g. for our purposes take $(u, v) = (0, 0)$. The linearization of (1)–(2) at $(0, 0)$,

$$\partial_t u = \Lambda_u u = D\partial_x^2 u + \partial_u f(0,0)u, \tag{3}$$

$$\partial_t v = \Lambda_v v = d_v \partial_x^2 v, \tag{4}$$

is solved by $u(x, t) = e^{ikx+\lambda t}\widehat{u}$ and $v(x, t) = e^{ikx+\lambda t}\widehat{v}$ where $\lambda \in \mathbb{C}$, $\widehat{u} \in \mathbb{C}^d$, and $\widehat{v} \in \mathbb{C}$ are determined by

$$\lambda\widehat{u} = -Dk^2\widehat{u} + \partial_u f(0,0)\widehat{u}, \tag{5}$$

$$\lambda\widehat{v} = -d_v k^2 \widehat{v}. \tag{6}$$

We find d curves of eigenvalues $\lambda_j = \lambda_j(k)$ ordered as $\mathrm{Re}\lambda_1(k) \geq \cdots \geq \mathrm{Re}\lambda_d(k)$ for (5) and $\lambda_0(k) = -d_v k^2$ for (6). The associated normalized eigenvectors or normalized generalized eigenvectors are denoted by $\widehat{U}_j \in \mathbb{C}^d$ for $j = 0, \ldots, d$.

We assume that (1)–(2) depends on a parameter $\widetilde{\alpha}$ and that for $\widetilde{\alpha} = \widetilde{\alpha}_c$ we have the following spectral situation.

(Spec) There is an $\omega_0 > 0$ such that $\mathrm{Re}\lambda_j(0)|_{\widetilde{\alpha}=\widetilde{\alpha}_c} = \lambda_j'(0)|_{\widetilde{\alpha}=\widetilde{\alpha}_c} = 0$, $\mathrm{Re}\lambda_j''(0)|_{\widetilde{\alpha}=\widetilde{\alpha}_c} < 0$ for $j = 1, 2$ and $\mathrm{Im}\lambda_1(0)|_{\widetilde{\alpha}=\widetilde{\alpha}_c} = -\mathrm{Im}\lambda_2(0)|_{\widetilde{\alpha}=\widetilde{\alpha}_c} = \omega_0$. Moreover, all other eigenvalues $\lambda_j|_{\widetilde{\alpha}=\widetilde{\alpha}_c}$ for $j = 1, \ldots, d$ have a negative real part. Finally, we assume that $\partial_{\widetilde{\alpha}}\mathrm{Re}\lambda_1(0)|_{\widetilde{\alpha}=\widetilde{\alpha}_c} > 0$.

For (1)–(2) from the assumption **(Spec)** a spectral situation follows as sketched in Fig. 1.

Notation. In order to make the notation more intuitive in the following we use the index -1 instead of 2, i.e., for example we write $\lambda_{-1} = \lambda_2$.

We introduce the bifurcation parameter $\varepsilon^2 = \widetilde{\alpha} - \widetilde{\alpha}_c$ and insert the ansatz

$$u(x,t) = \varepsilon A_1(X,T)e^{i\omega_0 t}\widehat{U}_1(0) + \text{c.c.} + \mathcal{O}(\varepsilon^2), \tag{7}$$

$$v(x,t) = \varepsilon^2 B_0(X,T), \tag{8}$$

with $X = \varepsilon x$, $T = \varepsilon^2 t$, $B_0(X, T) \in \mathbb{R}$, and $A_1(X, T) \in \mathbb{C}$ in (1)–(2). We obtain the system of amplitude equations

$$\partial_T A_1 = a_0 \partial_X^2 A_1 + a_1 A_1 + a_2 A_1 B_0 - a_3 A_1 |A_1|^2, \tag{9}$$

$$\partial_T B_0 = b_0 \partial_X^2 B_0 + b_1 \partial_X^2 (|A_1|^2), \tag{10}$$

with coefficients $a_0, a_3 \in \mathbb{C}$, $a_1, a_2, b_0, b_2 \in \mathbb{R}$, satisfying $\mathrm{Re} a_0 > 0$, $b_0 > 0$, $a_1 > 0$, and $\mathrm{Re} a_3 > 0$, consisting of a Ginzburg–Landau equation for A_1 coupled to a scalar conservation law for B_0. The amplitude function A_1 describes the oscillatory modes concentrated a $k = 0$ and B_0 the conservation law modes concentrated at $k = 0$.

Example 1.1 In order to make this introduction less abstract the derivation of the amplitude system will be explained for the following toy problem

$$\begin{aligned}
\partial_t u_1 &= \partial_x^2 u_1 + i\omega_0 u_1 + \varepsilon^2 u_1 + u_1^2 + u_1 u_{-1} + u_{-1}^2 + v u_1 + v u_{-1} - u_1^2 u_{-1}, \\
\partial_t u_{-1} &= \partial_x^2 u_{-1} - i\omega_0 u_{-1} + \varepsilon^2 u_{-1} + u_1^2 + u_1 u_{-1} + u_{-1}^2 + v u_1 + v u_{-1} - u_{-1}^2 u_1, \\
\partial_t v &= \partial_x^2 v + \partial_x^2 (u_1 u_{-1}),
\end{aligned}$$

with $u_{-1} = \overline{u_1}$. Although it is not of the form of (1)–(2), it shares essential properties with (1)–(2), in particular, it has qualitatively a spectral picture as plotted in Fig. 1. We make the ansatz

$$\begin{aligned}
u_1(x,t) &= \varepsilon A_1(X,T) e^{i\omega_0 t} + \varepsilon^2 A_{1,0}(X,T) \\
&\quad + \varepsilon^2 A_{1,2}(X,T) e^{2i\omega_0 t} + \varepsilon^2 A_{1,-2}(X,T) e^{-2i\omega_0 t}, \\
u_{-1}(x,t) &= \varepsilon A_{-1}(X,T) e^{-i\omega_0 t} + \varepsilon^2 A_{-1,0}(X,T) \\
&\quad + \varepsilon^2 A_{-1,2}(X,T) e^{2i\omega_0 t} + \varepsilon^2 A_{-1,-2}(X,T) e^{-2i\omega_0 t}, \\
v(x,t) &= \varepsilon^2 B_0(X,T),
\end{aligned}$$

with $A_{-1} = \overline{A_1}$, etc. For the u_1-equation we find:

$$\begin{aligned}
\varepsilon^3 e^{i\omega_0 t} &: \partial_T A_1 = \partial_X^2 A_1 + A_1 + B_0 A_1 \\
&\qquad + 2A_{1,0} A_1 + A_{1,2} A_{-1} + A_{-1,0} A_1 + 2A_{-1,2} A_{-1} - A_1^2 A_{-1}, \\
\varepsilon^2 e^{2i\omega_0 t} &: 2i\omega_0 A_{1,2} = i\omega_0 A_{1,2} + A_1^2, \\
\varepsilon^2 e^{0i\omega_0 t} &: 0 = i\omega_0 A_{1,0} + A_1 A_{-1}, \\
\varepsilon^2 e^{-2i\omega_0 t} &: -2i\omega_0 A_{1,-2} = i\omega_0 A_{1,-2} + A_{-1}^2.
\end{aligned}$$

For the u_{-1}-equation we find similar equations and for the v-equation we obtain:

$$\varepsilon^4 : \partial_T B_0 = \partial_X^2 B_0 + \partial_X^2 (A_1 A_{-1}).$$

If we eliminate the $A_{j,0}$ and $A_{j,2}$ by the above algebraic equations we find

$$\begin{aligned}
\partial_T A_1 &= \partial_X^2 A_1 + A_1 + B_0 A_1 - \gamma_3 |A_1|^2 A_1, \\
\partial_T B_0 &= \partial_X^2 B_0 + \partial_X^2 (|A_1|^2),
\end{aligned}$$

with

$$-\gamma_3 = -\frac{2}{i\omega_0} + \frac{1}{i\omega_0} + \frac{1}{i\omega_0} + \frac{2}{3i\omega_0} - 1 = -1 + \frac{2}{3i\omega_0}. \tag{11}$$

□

In order to establish the global existence and uniqueness for (9)–(10), in the following we assume

(Coeff) The coefficients $a_0, \ldots, b_1$ of (9)–(10) satisfy for the normalized System (18)–(19), subsequently computed in Remark 2.1, that $1 + \alpha^{-1}\beta > 0$.

Using the same multiple scaling analysis, in [13], in case of no conservation law, i.e., in case $v = 0$ and without the v-equation in (1)–(2), a Ginzburg–Landau equation was derived, and it was shown that all small solutions develop in such a way that they can be approximated after a certain time by the solutions of the Ginzburg–Landau equation. The proof differs essentially from the case when the bifurcating pattern is oscillatory in space which is based on mode-filters and a detailed analysis of the mode interactions. See [15, §10] for an overview. In contrast the proof of [13] is based on normal form methods. As a consequence of the results of [13], the global existence in time of all small bifurcating solutions and the upper-semicontinuity of the rescaled original system attractor towards the associated Ginzburg–Landau attractor follows. The result of [13] applies for instance to the Brusselator, the Schnakenberg, the Gray–Scott or the Gierer-Meinhardt model, cf. [16].

It is the purpose of this paper to prove a similar global existence result for (1)–(2), i.e., in case of an additional conservation law, with the help of the amplitude system (9)–(10).

This question turns out to be very challenging for the following reason. Since $(A_1, B_0) = (0, B^*)$, with constants $B^* \in \mathbb{R}$, is an unbounded family of stationary solutions for (9)–(10), this amplitude system does not possess an exponentially absorbing ball if posed on the real line, in contrast to a single Ginzburg–Landau equation if $\mathrm{Re} a_3 > 0$. However, assuming **(Coeff)** an exponentially attracting ball exists in case of periodic boundary conditions, say

$$A_1(X, T) = A_1(X + 2\pi, T) \qquad \text{and} \qquad B_0(X, T) = B_0(X + 2\pi, T). \tag{12}$$

Then we have the existence of an absorbing ball and the global existence and uniqueness of solutions.

Theorem 1.2 *Consider the amplitude system (9)–(10) with periodic boundary conditions (12) and assume that the coefficients $a_0, \ldots, b_1$ satisfy the condition* **(Coeff)**. *Then for all $s \in \mathbb{N}_0$ there exists a $C_R = C_R(s) > 0$ such that for all $C_1 > 0$ there exists a $T_0 = T_0(s, C_1) > 0$ such that to a given initial condition $(A_1(\cdot, 0), B_0(\cdot, 0)) \in H^{s+1} \times H^s$ with $\|A_1(\cdot, 0)\|_{H^{s+1}} + \|B_0(\cdot, 0)\|_{H^s} \leq C_1$ there exists a unique global solution $(A_1, B_0) \in C([0, \infty), H^{s+1} \times H^s)$ such that additionally $\|A_1(\cdot, T)\|_{H^{s+1}} + \|B_0(\cdot, T)\|_{H^s} \leq C_R$ for all $T \geq T_0$.*

Remark 1.3 In case of periodic boundary conditions the Sobolev space H^s can be embedded in the space $H^s_{l,u}$ of uniformly local Sobolev functions for $s \geq 0$ and so in case of periodic boundary conditions, the existence of an absorbing ball in $H^{s+1}_{l,u} \times H^s_{l,u}$ for (A_1, B_0) follows, too. For the definition of the space $H^s_{l,u}$ see the notations on Page 8.

As already said we are interested in a similar result for the original system (1)–(2) using the existence of an exponentially attracting absorbing ball for the amplitude system (9)–(10) and the fact that all solutions of (1)–(2) develop in such a way that after a certain time they can be approximated by the solutions of the amplitude system (9)–(10).

The 2π-spatially periodic boundary conditions for the amplitude system (9)–(10) correspond to $2\pi/\varepsilon$-spatially periodic boundary conditions for the original system (1)–(2), i.e.,

$$u(x, t) = u(x + 2\pi/\varepsilon, t) \qquad \text{and} \qquad v(x, t) = v(x + 2\pi/\varepsilon, t). \tag{13}$$

Then for these periodic boundary conditions and small $\varepsilon > 0$ we have the global existence and uniqueness of solutions for the original system (1)–(2).

Theorem 1.4 *Consider the original system (1)–(2) with periodic boundary conditions (13) and assume that the coefficients* $a_0, \ldots, b_1$ *of the associated amplitude system (9)–(10) satisfy the condition* **(Coeff)**. *Then for all* $n \geq 0$ *there exists a* $C_R > 0$ *and an* $\varepsilon_0 > 0$ *such that for all* $C_1 > 0$ *and all* $\varepsilon \in (0, \varepsilon_0)$*, there exists a* $t_0 = \mathcal{O}(1/\varepsilon^2) > 0$ *such that to a given initial condition* $(u(\cdot, 0), v(\cdot, 0)) \in H_{l,u}^{n+1} \times H_{l,u}^n$ *with* $\|u(\cdot, 0)\|_{H_{l,u}^{n+1}} + \varepsilon^{-1}\|v(\cdot, 0)\|_{H_{l,u}^n} \leq C_1\varepsilon$ *there exists a unique global solution* $(u, v) \in C([0, \infty), H_{l,u}^{n+1} \times H_{l,u}^n)$ *such that additionally* $\|u(\cdot, t)\|_{H_{l,u}^{n+1}} + \varepsilon^{-1}\|v(\cdot, t)\|_{H_{l,u}^n} \leq C_R\varepsilon$ *for all* $t \geq t_0$.

Remark 1.5 Hence, the global existence question can be answered positively at least for original systems with periodic boundary conditions (13) which correspond in the amplitude system (9)–(10) to periodic boundary conditions (12). Since the L^2-norm of $u = 1$ on the interval $[-\pi/\varepsilon, \pi/\varepsilon]$ grows as $\mathcal{O}(1/\sqrt{\varepsilon})$ with $\varepsilon \to 0$, Sobolev spaces are not adequate for controlling the norm and so spaces have to be used where functions such as $u = 1$ can be bounded independently of the small perturbation parameter $0 < \varepsilon \ll 1$.

Remark 1.6 The three main ingredients of the global existence proof are **(GL)**: the existence of an exponentially attracting absorbing ball of the amplitude system, **(APP)**: an approximation result which shows that solutions of the original system (1)–(2) can be approximated on the natural $\mathcal{O}(1/\varepsilon^2)$-time scale of (9)–(10) of the amplitude system via the solutions of (9)–(10), and **(ATT)**: an attractivity result, which shows that solutions of (1)–(2) to initial conditions of order $\mathcal{O}(\varepsilon)$ develop in such a way that after an $\mathcal{O}(1/\varepsilon^2)$-time scale they are of a form which allows us to approximate them afterwards by the solutions of (9)–(10).

Remark 1.7 Approximation and attractivity results have been established in [2, 7, 17] in case of a Turing pattern forming systems coupled to a conservation law. Attractivity and approximation results in case of a simultaneous Turing and a long wave Hopf bifurcation can be found in [16].

Remark 1.8 The idea is as follows. A neighborhood of the origin of the pattern forming system is mapped by the attractivity **(ATT)** into a set which can be described by the amplitude system. The amplitude system possesses an exponentially attracting absorbing ball **(GL)**. Therefore, by the approximation property **(APP)** the original neighborhood of the pattern forming system is mapped after a certain time into itself. These a priori estimates combined with the local existence and uniqueness gives the global existence and uniqueness of solutions of the pattern forming system in a neighborhood of the weakly unstable origin.

Remark 1.9 Examples of reaction–diffusion systems (1)–(2), falling into the class of systems we are interested in, are for instance the Brusselator, the Schnakenberg, the Gray–Scott and the Gierer-Meinhardt model coupled to a conservation law coming for instance from ecology. As an example we consider the Brusselator. The system, with the spatially homogeneous trivial equilibrium as origin, is given by

$$\partial_t u_1 = d_1 \partial_x^2 u_1 + (b-1)u_1 + a^2 u_2 + f(u_1, u_2), \tag{14}$$

$$\partial_t u_2 = d_2 \partial_x^2 u_2 - b u_1 - a^2 u_2 - f(u_1, u_2), \tag{15}$$

with nonlinear terms

$$f(u_1, u_2) = (b/a)u_1^2 + 2a u_1 u_2 + u_1^2 u_2.$$

The long-wave Hopf instability occurs at the critical wave number $k = 0$ for $b = b_{hopf}(a) = 1 + a^2$. For more details see [16]. This system can be brought into the form (1)–(2) by introducing a variable v satisfying

$$\partial_t v = d_v \partial_x^2 v + \partial_x^2 g(u_1, u_2),$$

with $g(0,0) = 0$, by replacing b by $v + \widetilde{b}$, and by introducing the small bifurcation parameter $\varepsilon^2 = (\widetilde{b} - b_{hopf})/b_{hopf}$.

The **plan of the paper** is as follows. In Sect. 2 we discuss the global existence and uniqueness of solutions of the amplitude system (9)–(10). The proof will be given in "Appendix D". Section 3 contains a number of preparations, in particular we eliminate a number of oscillatory terms from (1)–(2) by so called normal form transformations. In Sect. 4 we derive the amplitude equations and define the Ginzburg–Landau manifold, the set of solutions which can be approximated by our amplitude system. In Sect. 5 we formulate the attractivity result which is proven in "Appendix B", the approximation result which is proven in "Appendix C" and put them together to conclude on the global existence and uniqueness of solutions of the original reaction–diffusion system (1)–(2). In Sect. 6 a few further questions are discussed. Moreover, in "Appendix A" some estimates are provided which are used in the sequel.

Notation. The Sobolev space H^s is equipped with the norm $\|u\|_{H^s} = \sum_{j=0}^{s} \|\partial_x^j u\|_{L^2}$, where $\|u\|_{L^2}^2 = \int |u(x)|^2 dx$. The space $H^s_{l,u}$ of s-times locally uniformly weakly differentiable functions is equipped with the norm $\|u\|_{H^s_{l,u}} = \sum_{j=0}^{s} \|\partial_x^j u\|_{L^2_{l,u}}$, where $\|u\|_{L^2_{l,u}} = \sup_{x\in\mathbb{R}} (\int_x^{x+1} |u(y)|^2 dy)^{1/2}$, cf [15, §8.3.1]. Fourier transform w.r.t. the spatial variable is denoted by $\mathcal{F}$ and the inverse Fourier transform by $\mathcal{F}^{-1}$. Possibly different constants which can be chosen independently of the small perturbation parameter $0 < \varepsilon \ll 1$ are often denoted with the same symbol C.

2 Analysis of the Amplitude System

We consider

$$\partial_T A = a_0 \partial_X^2 A + a_1 A + a_2 AB - a_3 A|A|^2, \tag{16}$$

$$\partial_T B = b_0 \partial_X^2 B + b_1 \partial_X^2 (|A|^2), \tag{17}$$

where $T \geq 0$, $X \in \mathbb{R}$, $A(X,T) \in \mathbb{C}$, $B(X,T) \in \mathbb{R}$, and with coefficients having properties as specified below the Eqs. (9)–(10). We are interested in the situation of an unstable trivial solution, i.e., $a_1 > 0$. This is the general form of the amplitude system which appears for a long wave Hopf bifurcation in a pattern forming system with a conservation law. The system has been derived for pattern forming systems with a conservation law exhibiting a Turing instability, too, cf. [8]. In a singular limit spike solutions have been constructed in [10].

Remark 2.1 By rescaling A, B, T, and X and by possibly changing the sign of B, four of the coefficients can be eliminated. We set

$$A = c_A \widetilde{A}, \qquad B = c_B \widetilde{B}, \qquad T = c_T \widetilde{T}, \qquad \text{and} \qquad X = c_X \widetilde{X}.$$

We find

$$\partial_{\widetilde{T}} \widetilde{A} = c_T a_0 c_X^{-2} \partial_{\widetilde{X}}^2 \widetilde{A} + c_T a_1 \widetilde{A} + c_T a_2 c_B \widetilde{A}\widetilde{B} - c_T a_3 c_A^2 \widetilde{A}|\widetilde{A}|^2,$$

$$\partial_{\widetilde{T}} \widetilde{B} = c_T b_0 c_X^{-2} \partial_{\widetilde{X}}^2 \widetilde{B} + c_T b_1 c_A^2 c_X^{-2} c_B^{-1} \partial_{\widetilde{X}}^2 (|\widetilde{A}|^2).$$

We first choose $c_T \in \mathbb{R}$ such that $c_T a_1 = 1$. Next we set $c_A > 0$ such that $c_T(\mathrm{Re} a_3)c_A^2 = 1$. Then we choose $c_X > 0$ such that $c_T(\mathrm{Re} a_0)c_X^{-2} = 1$. Finally, we set $c_B \in \mathbb{R}$ such that $c_T b_1 c_A^2 c_X^{-2} c_B^{-1} = 1$ if $b_1 \neq 0$. If $b_1 = 0$, subsequently in (19) the term $\partial_X^2(|A|^2)$ will be away. Defining

$$\beta = c_T a_2 c_B, \quad \alpha = c_T b_0 c_X^{-2}, \quad \gamma_0 = \mathrm{Im}(c_T a_0 c_X^{-2}), \quad \gamma_3 = \mathrm{Im}(c_T a_3 c_A^2)$$

and dropping the tildes we finally consider

$$\partial_T A = (1 + i\gamma_0)\partial_X^2 A + A + \beta AB - (1 + i\gamma_3)A|A|^2, \tag{18}$$

$$\partial_T B = \alpha \partial_X^2 B + \partial_X^2(|A|^2), \tag{19}$$

with $\alpha > 0$ and $\beta, \gamma_0, \gamma_3 \in \mathbb{R}$.

Remark 2.2 Before we discuss the local and global existence of this system we have a short look at a family of special solutions. There are the X-independent time-periodic solutions $B = b$, $A = \widehat{A}e^{i\omega T}$ with $|\widehat{A}|^2 = 1 + \beta b$ and $\omega = -|\widehat{A}|^2\gamma_3$ for every b with $1 + \beta b > 0$. In case $1 + \beta b \leq 0$ we have the stationary solutions $B = b$ and $A = 0$.

Remark 2.3 Global existence for the classical Ginzburg–Landau equation on the real line, (18) in case $\beta = 0$, can be obtained in $C_b^0(\mathbb{R})$ with the maximum principle if $\gamma_0 = \gamma_3 = 0$. By the smoothing of the diffusion semigroup, global existence follows in all C_b^n-spaces and $H_{l,u}^m$-spaces for $m > 1/2$. An approach for general γ_0 and γ_3 is to work with weighted energies $\int_{\mathbb{R}} \rho_\delta(X)|A(X)|^2 dX$, where $\rho_\delta(X) = (1 + (\delta X)^2)^{-1}$ for $\delta > 0$, cf. [9].

Remark 2.4 However, so far, both approaches described in Remark 2.3 do not give global existence for the amplitude system (18)–(19) on the real line. Weighted energy estimates gives via the linear terms $\partial_X^2 A$ and $\alpha\partial_X^2 B$ some exponential growth of order $\mathcal{O}(\delta^2)$. For the classical Ginzburg–Landau equation one can get rid of these growth rates with the $-|A|^2 A$-term which allows for a point-wise estimate

$$\int_{\mathbb{R}} \rho_\delta(X)(|A(X)|^2 - |A(X)|^4)dX \leq \int_{\mathbb{R}} \rho_\delta(X)(1 - |A(X)|^2)dX. \tag{20}$$

However, there is no counterpart in (18)–(19) which can stop the growth of the weighted B-variable.

We help ourselves by considering the amplitude system (18)–(19) with periodic boundary conditions. 2π-periodicity for (16)–(17) corresponds to L-periodicity for (18)–(19) with $L = 2\pi\sqrt{a_1/a_0}$.

Remark 2.5 In case of periodic boundary conditions, the mean value of B is conserved in time. However, we could always further assume that the mean value b of B vanishes. If this would not be the case, we could set $B = b + \widetilde{B}$, with $\widetilde{B}$ having a vanishing mean value. Then we would obtain

$$\partial_T A = (1 + i\gamma_0)\partial_X^2 A + A + \beta A(b + \widetilde{B}) - (1 + i\gamma_3)A|A|^2,$$

$$\partial_T \widetilde{B} = \alpha\partial_X^2 \widetilde{B} + \partial_X^2(|A|^2).$$

Hence, by redefining the coefficient a_1 we could always come to a system, for which the mean value of B vanishes for all $T \geq 0$.

The choice of periodic boundary conditions allows us to use classical energy estimates without weights. In case $1 + \alpha^{-1}\beta > 0$ we have the following global existence result.

Theorem 2.6 *Assume that $1+\alpha^{-1}\beta > 0$ holds. Fix $s \geq 0$, $L > 0$ and consider (18)–(19) with L-periodic boundary conditions. Then there exists a $C_2 > 0$ such that for all $C_1 > 0$ there exists a $T_0 > 0$ such that the following holds. For initial conditions $(A(\cdot,0), B(\cdot,0)) \in H^{s+1} \times H^s$ with $\int_0^L B(X,0)dX = 0$ and*

$$\|A(\cdot,0)\|_{H^{s+1}} + \|B(\cdot,0)\|_{H^s} \leq C_1$$

the associated unique global solution $(A,B) \in C([0,\infty), H^{s+1} \times H^s)$ satisfies

$$\|A(\cdot,T)\|_{H^{s+1}} + \|B(\cdot,T)\|_{H^s} \leq C_2$$

for all $T \geq T_0$.

Proof See "Appendix D". □

3 Some Preparations

All operators appearing in the following are so called multipliers. A linear operator M is called multiplier if there exists a function $\widehat{M} : \mathbb{R} \to \mathbb{C}$ such that $Mu = \mathcal{F}^{-1}(\widehat{M}\mathcal{F}u)$, i.e., if the associated operator is a multiplication operator in Fourier space. Typical examples are differential operators, semigroups, or mode-filters, but also the normal form transformations at the end of this section can be interpreted as multilinear multipliers.

3.1 The Mode-Filters

For estimating the different parts of the solutions we use so called mode-filters. Since we work in $H^n_{l,u}$-spaces we cannot use cut-off functions in Fourier space to extract certain modes from the solutions. The associated operators in $H^n_{l,u}$ would not be smooth and so we take a $\widehat{\chi} \in C_0^\infty$ with

$$\widehat{\chi}(k) = \begin{cases} 1, & \text{for } |k| \leq 0.45\widetilde{\delta}, \\ 0, & \text{for } |k| \geq 0.55\widetilde{\delta}, \\ \in [0,1], & \text{else}, \end{cases} \tag{21}$$

for a $\widetilde{\delta} > 0$ sufficiently small but independent of the small perturbation parameter $0 < \varepsilon^2 \ll 1$. For extracting the modes around the Fourier wave number $k = 0$ we define a mode-filter E_0 by

$$\widehat{E}_0(k)\widehat{u}(k) = \widehat{\chi}(k)\widehat{u}(k).$$

This operator can be estimated as follows.

Lemma 3.1 *For every $m \in \mathbb{N}_0$ the operator E_0 is a bounded operator from $L^2_{l,u}$ to $H^m_{l,u}$, in detail, there exist constants C_m such that $\|E_0\|_{L^2_{l,u} \to H^m_{l,u}} \leq C_m$.*

Proof We use multiplier theory in $H^m_{l,u}$-spaces, cf. [15, §8.3.1]. We have

$$\|E_0 v\|_{H^m_{l,u}} \leq C\|\widehat{M}\|_{C_b^2}\|v\|_{L^2_{l,u}},$$

with $\widehat{M}(k) = (1+k^2)^{m/2}\widehat{\chi}(k)$. □

3.2 The Normal Form Transformation

For the subsequent analysis we need a separation of the u-modes into exponentially damped ($j = 3, \ldots, d$) and critical modes ($j = \pm 1$). In order to do so, we let

$$\tilde{P}_{\pm 1}(k, \varepsilon^2)u = \frac{1}{2\pi i} \int_{\Gamma_{\pm 1}} (\mu Id. - \tilde{\Lambda}_u(k, \varepsilon^2))^{-1} u d\mu,$$

where $\tilde{\Lambda}_u = \mathcal{F}\Lambda_u\mathcal{F}^{-1}$, with Λ_u defined in (3), and where $\Gamma_{\pm 1}$ is a closed curve surrounding the single eigenvalue $\lambda_{\pm 1}|_{\varepsilon=0,k=0} = \pm i\omega_0$ anti-clockwise. By the assumption (Spec) the projections $\tilde{P}_j$ can be defined for wave numbers in a neighborhood $U_\rho(0)$ for a $\rho > 0$ and so we set

$$E_{\pm 1} = E_0\tilde{P}_{\pm 1}, \quad E_c = E_1 + E_{-1}, \qquad \text{and} \qquad E_s = Id. - E_c,$$

choosing $\widetilde{\delta} < \rho/2$ in (21). Moreover, we define scalar-valued projections $\tilde{p}_{\pm 1}$ by

$$\tilde{P}_{\pm 1}(k, \varepsilon^2)u = (\tilde{p}_{\pm 1}(k, \varepsilon^2)u)\widehat{U}_{\pm 1}(k, \varepsilon^2)$$

and $e_{\pm 1} = E_0\tilde{p}_{\pm 1}$. With these operators we separate our linearized system (3)–(4) in critical and exponentially damped modes.

Then, in Fourier space, we write

$$\widehat{u}(k, t) = \widehat{c}_1(k, t)\widehat{U}_1(k) + \widehat{c}_{-1}(k, t)\widehat{U}_{-1}(k) + \widehat{u}_s(k, t),$$

with $\widehat{c}_{\pm 1}(k, t) \in \mathbb{C}$, and define $c_{\pm 1}$ and u_s to be solutions of

$$\partial_t c_1 = \lambda_1 c_1 + f_1(c_1, u_s, v), \tag{22}$$
$$\partial_t c_{-1} = \lambda_{-1} c_{-1} + f_{-1}(c_1, u_s, v), \tag{23}$$
$$\partial_t u_s = \Lambda_s u_s + f_s(c_1, u_s, v), \tag{24}$$
$$\partial_t v = \Lambda_v v + \partial_x^2 g(c_1, u_s), \tag{25}$$

with the additional assumption that the Fourier support of $c_{\pm 1}$ is contained in the Fourier support of E_0. Moreover, we assume that $\widehat{u}_s(k)$ projected on span$\{\widehat{U}_1(k), \widehat{U}_{-1}(k)\}$ vanishes for $|k| \leq 0.45\widetilde{\delta}$. In (22)–(25) the linear operator Λ_s is the restriction of Λ_u to the u_s-variable and

$$\begin{aligned} f_{\pm 1}(c_1, u_s, v) &= e_1 f(u, v) = \mathcal{O}(|c_1|^2 + |u_s|^2 + (|u_1| + |u_s|)|v|), \\ f_s(c_1, u_s, v) &= E_s f(u, v) = \mathcal{O}(|c_1|^2 + |u_s|^2 + (|u_1| + |u_s|)|v|), \\ g(c_1, u_s) &= \mathcal{O}(|c_1|^2 + |u_s|^2). \end{aligned}$$

Since $c_{-1} = \overline{c_1}$ we do not explicitly denote the appearance of c_{-1} in various places.

Since c_1 approximately oscillates as $e^{i\omega_0 t}$ all quadratic combinations of c_1 and c_{-1} can be eliminated from the c_1-equations by a near identity change of variables

$$\check{u}_1 = c_1 + \mathcal{O}(|c_1|^2).$$

A similar statement holds for the c_{-1}-equation. For details see the subsequent Remark 3.3.

Remark 3.2 In a similar way terms $vc_{\pm 1}$ in the v-equation could be eliminated in case of a more general nonlinearity in the v-equation.

After the transform we have a system of the form

$$\partial_t \check{u}_1 = \lambda_1 \check{u}_1 + \check{f}_1(\check{u}_1, u_s, v), \tag{26}$$

$$\partial_t u_s = \Lambda_s u_s + \check{f}_s(\check{u}_1, u_s, v), \tag{27}$$

$$\partial_t v = \Lambda_v v + \partial_x^2 \check{g}(\check{u}_1, u_s), \tag{28}$$

with

$$\begin{aligned} \check{f}_1(\check{u}_1, u_s, v) &= \mathcal{O}(|\check{u}_1|^3 + |\check{u}_1||u_s| + |u_s|^2 + (|\check{u}_1| + |u_s|)|v|), \\ \check{f}_s(\check{u}_1, u_s, v) &= \mathcal{O}(|\check{u}_1|^2 + |u_s|^2 + (|\check{u}_1| + |u_s|)|v|), \\ \check{g}(\check{u}_1, u_s) &= \mathcal{O}(|\check{u}_1|^2 + |u_s|^2). \end{aligned}$$

Detailed estimates about this transformation and the nonlinear terms are given below when needed.

Remark 3.3 In lowest order the equation for c_1 is of the form

$$\partial_t c_1 = \lambda_1 c_1 + N_{1,1}(c_1, c_1) + N_{1,-1}(c_1, c_{-1}) + N_{-1,-1}(c_{-1}, c_{-1}) + h.o.t.$$

where in Fourier space the $N_{i,j}$ have a representation

$$\widehat{N}_{i,j}(c_i, c_j)[k] = \int \widehat{n}_{i,j}(k, k-m, m)\widehat{c}_i(k-m)\widehat{c}_j(m)dm,$$

with kernel functions $\widehat{n}_{i,j} : \mathbb{R}^3 \to \mathbb{C}$. The quadratic terms can be eliminated by a transform

$$\check{u}_1 = c_1 + B_{1,1}(c_1, c_1) + B_{1,-1}(c_1, c_{-1}) + B_{-1,-1}(c_{-1}, c_{-1})$$

where in Fourier space the $B_{i,j}$ have a representation

$$\widehat{B}_{i,j}(c_i, c_j)[k] = \int \widehat{b}_{i,j}(k, k-m, m)\widehat{c}_i(k-m)\widehat{c}_j(m)dm.$$

The kernels $\widehat{b}_{i,j}(k, k-m, m)$ are solutions of

$$(\tilde{\lambda}_1(k) - \tilde{\lambda}_i(k-m) - \tilde{\lambda}_j(m))\widehat{b}_{i,j}(k, k-m, m) = \widehat{n}_{i,j}(k, k-m, m)$$

which are well-defined and bounded since

$$\inf_{k,m \in U_{4\rho}(0)} |\widehat{\lambda}_{j_1}(k) - \widehat{\lambda}_{j_2}(k-m) - \widehat{\lambda}_{j_3}(m)| \geq C > 0$$

for all $j_1, j_2, j_3 \in \{-1, 1\}$. For more details see [15, §11] or [13, §4].

Remark 3.4 After the transform we have a system of the form

$$\begin{aligned} \partial_t \check{u}_1 = \lambda_1 \check{u}_1 &+ N_{1,1,1}(\check{u}_1, \check{u}_1, \check{u}_1) + N_{1,1,-1}(\check{u}_1, \check{u}_1, \check{u}_{-1}) \\ &+ N_{1,-1,-1}(\check{u}_1, \check{u}_{-1}, \check{u}_{-1}) + N_{-1,-1,-1}(\check{u}_{-1}, \check{u}_{-1}, \check{u}_{-1}) + h.o.t. \end{aligned}$$

where in Fourier space the $N_{i,j,k}$ have a similar representation as above. Except of $N_{1,1,-1}$ the three other terms are non-resonant such that these can be eliminated by a second transformation.

Example 3.5 Applying the normal form transformation to the system from Example 1.1 yields a system of the form

$$\partial_t u_1 = \partial_x^2 u_1 + i\omega_0 u_1 + \varepsilon^2 u_1 + v u_1 - \gamma_3 u_1^2 u_{-1} + h.o.t.$$

$$\begin{aligned}\partial_t u_{-1} &= \partial_x^2 u_{-1} - i\omega_0 u_{-1} + \varepsilon^2 u_{-1} + v u_{-1} - \overline{\gamma_3} u_{-1}^2 u_1 + h.o.t.,\\ \partial_t v &= \partial_x^2 v + \partial_x^2 (u_1 u_{-1}) + h.o.t.,\end{aligned}$$

with γ_3 given by (11).

4 The Ginzburg–Landau Manifold

The notation Ginzburg–Landau manifold or Ginzburg–Landau set, cf. [4], was chosen to describe the set of initial conditions of the original system (1)–(2) which can be described by the Ginzburg–Landau approximation. In the non-conservation law case it was shown that this set is attractive, cf. [1, 4, 12]. In the conservation law case a first result was established in [2]. We will come back to this in Sect. B. It is the purpose of this section to derive the amplitude system, to compute a higher order approximation and to define what we will mean by Ginzburg–Landau manifold.

For possible future applications, similar to [9, 14], we introduce a new perturbation parameter δ with $0 < \varepsilon \leq \delta \ll 1$ and distinguish this parameter from the bifurcation parameter $0 < \varepsilon \ll 1$.

4.1 Derivation of the Amplitude System

Our starting point for the derivation of the amplitude system is System (22)–(25) which we write as

$$\begin{aligned}\mathrm{Res}_1 &= -\partial_t c_1 + \lambda_1 c_1 + f_1(c_1, u_s, v),\\ \mathrm{Res}_s &= -\partial_t u_s + \Lambda_s u_s + f_s(c_1, u_s, v),\\ \mathrm{Res}_v &= -\partial_t v + \Lambda_v v + \partial_x^2 g(c_1, u_s).\end{aligned}$$

The so called residuals Res_1, Res_s, and Res_v contain all terms which remain after inserting an approximation into System (22)–(25).

For the derivation of the amplitude system, cf. Example 1.1, we need an ansatz

$$\begin{aligned}c_1(x,t) &= \delta A_1(X,T)e^{i\omega_0 t} + \delta^2 A_{1,0}(X,T)\\ &\quad + \delta^2 A_{1,2}(X,T)e^{2i\omega_0 t} + \delta^2 A_{1,-2}(X,T)e^{-2i\omega_0 t},\\ c_{-1}(x,t) &= \delta A_{-1}(X,T)e^{-i\omega_0 t} + \delta^2 A_{-1,0}(X,T)\\ &\quad + \delta^2 A_{-1,2}(X,T)e^{2i\omega_0 t} + \delta^2 A_{-1,-2}(X,T)e^{-2i\omega_0 t},\\ u_s(x,t) &= \delta^2 A_{s,2}(X,T)e^{2i\omega_0 t} + \delta^2 A_{s,0}(X,T) + \delta^2 A_{s,-2}(X,T)e^{-2i\omega_0 t},\\ v(x,t) &= \delta^2 B_0(X,T),\end{aligned}$$

with $X = \delta x$ and $T = \delta^2 t$. By equating the coefficients in front of $\delta^2 e^{in\omega_0 t}$, with $n = 0, \pm 2$, to zero, we find $A_{j,2}$, $A_{j,0}$, $A_{j,-2}$ for $j = -1, 1, s$ as solutions of equations of the form

$$\begin{aligned}A_{j,2} &= \gamma_{j,2} A_1 A_1,\\ A_{j,0} &= \gamma_{j,0} A_1 A_{-1},\\ A_{j,-2} &= \gamma_{j,-2} A_{-1} A_{-1},\end{aligned}$$

with coefficients $\gamma_{j,i}$. The A_1, A_{-1}, and B_0 satisfy a system of the form

$$\begin{aligned}\partial_T A_1 &= a_0\partial_X^2 A_1 + \frac{\varepsilon^2}{\delta^2}a_1 A_1 + a_2 A_1 B_0 - \widetilde{a}_4 A_1|A_1|^2 \\ &\quad + \sum_{j=\pm 1,s} a_{6,j}A_{j,2}A_{-1} + \sum_{j=\pm 1,s} a_{7,j}A_{j,0}A_1, \\ \partial_T B_0 &= b_0\partial_X^2 B_0 + b_1\partial_X^2(|A_1|^2),\end{aligned}$$

Eliminating the $A_{j,2}$, $A_{j,0}$, $A_{j,-2}$ for $j = -1, 1, s$ through the above equations gives the amplitude system

$$\partial_T A_1 = a_0\partial_X^2 A_1 + \frac{\varepsilon^2}{\delta^2}a_1 A_1 + a_2 A_1 B_0 - a_3 A_1|A_1|^2, \tag{29}$$

$$\partial_T B_0 = b_0\partial_X^2 B_0 + b_1\partial_X^2(|A_1|^2), \tag{30}$$

similar to (9)–(10). We formally have

$$\mathrm{Res}_1 = \mathcal{O}(\delta^3), \quad \mathrm{Res}_s = \mathcal{O}(\delta^3), \quad \mathrm{Res}_v = \mathcal{O}(\delta^4)$$

for this approximation. In the residual of the c_1-equation we have for instance a term $\delta^3 A_1^3 e^{3i\omega_0 t}$ and in the residual of the v-equation we have for instance a term $\delta^4\partial_X^2(A_1^2)e^{2i\omega_0 t}$.

In order to show that the amplitude system (29)–(30) makes correct predictions about the original system (1)–(2) we establish subsequently the approximation Theorem 5.3.

4.2 Construction of a Higher Order Approximation

In order to obtain a more precise approximation we add higher order terms to the previous approximation. We insert

$$c_1 = \psi_1, \qquad c_{-1} = \psi_{-1}, \qquad u_s = \psi_s \qquad v = \psi_v$$

with

$$\begin{aligned}\psi_1(x,t) &= \sum_{m=-N}^{N}\sum_{n=0}^{M_1(N,m)} \delta^{\beta_1(m)+n}A_{+,m,n}(X,T)e^{im\omega_0 t}, \\ \psi_{-1}(x,t) &= \sum_{m=-N}^{N}\sum_{n=0}^{M_1(N,m)} \delta^{\beta_{-1}(m)+n}A_{-,m,n}(X,T)e^{im\omega_0 t}, \\ \psi_s(x,t) &= \sum_{m=-N}^{N}\sum_{n=0}^{M_s(N,m)} \delta^{\beta_s(m)+n}A_{s,m,n}(X,T)e^{im\omega_0 t}, \\ \psi_v(x,t) &= \sum_{m=-N}^{N}\sum_{n=0}^{M_v(N,m)} \delta^{\beta_v(m)+n}B_{m,n}(X,T)e^{im\omega_0 t},\end{aligned}$$

where N, $M_1(N,m)$, $M_s(N,m)$, and $M_v(N,m)$ are sufficiently large numbers such that

$$\mathrm{Res}_c = \mathcal{O}(\delta^{\theta+2}), \quad \mathrm{Res}_s = \mathcal{O}(\delta^{\theta+2}), \quad \mathrm{Res}_v = \mathcal{O}(\delta^{\theta+2})$$

for a given $\theta \in \mathbb{N}$ and where

m	-3	-2	-1	0	1	2	3	m
$\beta_1(m)$	3	2	3	2	1	2	3	m
$\beta_{-1}(m)$	3	2	1	2	3	2	3	m
$\beta_s(m)$	3	2	3	2	3	2	3	m
$\beta_v(m)$	5	4	5	2	5	4	5	$m+2$

The associated approximation is then denoted with Ψ_θ.

The coefficient functions are determined as follows. The functions $A_{+,1,0}$, $A_{-,-1,0}$, and $B_{0,0}$ satisfy the amplitude system from above. The $A_{+,1,n}$, $A_{-,-1,n}$, and $B_{0,n}$ for $n \geq 1$ satisfy linearisations of the amplitude system from above with some inhomogeneous terms which in the end depend on terms $A_{+,1,j}$, $A_{-,-1,j}$, and $B_{0,j}$ for $0 \leq j \leq n-1$. All other $A_{+,m,n}$, $A_{-,m,n}$, $A_{s,m,n}$, and $B_{m,n}$ satisfy algebraic equations and can be computed in terms of the $A_{+,1,j}$, $A_{-,-1,j}$, and $B_{0,j}$ for $0 \leq j \leq n$.

The solutions of this system are uniquely determined by the set of initial conditions $A_{+,1,j}|_{T=0}$, $A_{-,-1,j}|_{T=0}$, and $B_{0,j}|_{T=0}$ for $0 \leq j \leq n$.

Definition 4.1 For initial conditions

$$A_{+,1,0}|_{T=0} = A_1|_{T=0}, \qquad A_{-,-1,0}|_{T=0} = \overline{A_1}|_{T=0}, \qquad B_{0,0}|_{T=0} = B_0|_{T=0}$$

and

$$A_{+,1,j}|_{T=0}, \qquad A_{-,-1,j}|_{T=0}, \qquad B_{0,j}|_{T=0}$$

determined by the construction in "Appendix B.4" for $1 \leq j \leq n$ and (A_1, B_0) satisfying (29)–(30) we call the set of approximate solutions

$$(u, v)(\cdot, t) = \Psi_\theta(A_1(\cdot, T), B_0(\cdot, T))$$

for the original system (1)–(2) the Ginzburg–Landau manifold, where Ψ_θ is the associated higher order approximation defined above.

5 The Global Existence and Uniqueness Result

Throughout the rest of this paper we replace the boundary conditions (13) by the boundary conditions

$$u(x, t) = u(x + 2\pi/\delta, t) \qquad \text{and} \qquad v(x, t) = v(x + 2\pi/\delta, t). \tag{31}$$

with $0 < \varepsilon \leq \delta \ll 1$ and set later on $\delta = \varepsilon$.

Remark 5.1 There is local existence and uniqueness of (mild) solutions

$$(u, v) \in C([0, t_0], H^{n+1}_{l,u} \times H^n_{l,u})$$

of (1)–(2) for initial conditions $(u_0, v_0) \in H^{n+1}_{l,u} \times H^n_{l,u}$ if $n \geq 0$ where the existence time $t_0 > 0$ only depends on $\|u_0\|_{H^{n+1}_{l,u}} + \|v_0\|_{H^n_{l,u}}$. This can be established with the standard fixed point argument for semilinear parabolic equations, cf. [5]. For $n \geq 0$ the right-hand side of the variation of constant formula associated to (1)–(2) is a contraction in a ball in $C([0, t_0], H^{n+1}_{l,u} \times H^n_{l,u})$ for $t_0 > 0$ sufficiently small using that the nonlinear terms $(f(u, v), \partial_x g(u))$ are smooth mappings from $H^{n+1}_{l,u} \times H^n_{l,u}$ to $H^n_{l,u} \times H^n_{l,u}$ and that the linear semigroups $(e^{D\partial_x^2 t}, e^{d_v \partial_x^2 t}\partial_x)$ map $H^n_{l,u} \times H^n_{l,u}$ to $H^{n+1}_{l,u} \times H^n_{l,u}$ with an integrable singularity $t^{-1/2}$.

Hence, for establishing the global existence and uniqueness of (mild) solutions we need to bound the solutions in $H^{n+1}_{l,u} \times H^n_{l,u}$, i.e., if we establish an a priori bound

$$\sup_{t \in [0,\infty)} (\|u(t)\|_{H^{n+1}_{l,u}} + \|v(t)\|_{H^n_{l,u}}) \leq C_3 < \infty, \tag{32}$$

where C_3 is a constant only depending on $\|u_0\|_{H^{n+1}_{l,u}} + \|v_0\|_{H^n_{l,u}}$, then the local existence and uniqueness theorem can be applied again and again and the local solutions can be continued to global solutions

$$(u, v) \in C([0, \infty), H^{n+1}_{l,u} \times H^n_{l,u}).$$

The necessary a-priori estimates (32) for (1)–(2) can be obtained in a sufficiently small $\mathcal{O}(\delta)$-neighborhood of the weakly unstable origin with the help of an attractivity and approximation result for the Ginzburg–Landau manifold and the existence of an absorbing ball for the amplitude system.

The attractivity theorem is as follows

Theorem 5.2 *For all $R_0 > 0$, $n \geq 0$, and all $\theta \in \mathbb{N}_0$ the following holds. Consider (1)–(2) with initial conditions $(u_0, v_0) \in H^{n+1}_{l,u} \times H^n_{l,u}$ satisfying*

$$\|u_0\|_{H^{n+1}_{l,u}} + \delta^{-1}\|v_0\|_{H^n_{l,u}} \leq R_0\delta.$$

Then there exists a time $T_1 \in (0, 1)$, a $\delta_1 > 0$, an $R_1 > 0$ and a $C_1 > 0$, all only depending on R_0, θ, and n, such that for all $\delta \in (0, \delta_1)$, all $\varepsilon \in (0, \delta]$, and all $m > 1/2$ there are $(A_1(\cdot, 0), B_0(\cdot, 0)) \in H^{m+1}_{l,u} \times H^m_{l,u}$ with

$$\|A_1(\cdot, 0)\|_{H^{m+1}_{l,u}} + \|B_0(\cdot, 0)\|_{H^m_{l,u}} \leq R_1$$

such that the solution (u, v), with the initial conditions (u_0, v_0), satisfies at a time $t = T_1/\delta^2$ that

$$\|(u, \delta^{-1}v)|_{t=T_1/\delta^2} - (\Psi_{\theta,u}, \Psi_{\theta,v})(A_1(\cdot, 0), B_0(\cdot, 0))\|_{H^{m+1}_{l,u} \times H^m_{l,u}} \leq C\delta^\theta.$$

Proof See "Appendix B". □

The dynamics on the Ginzburg–Landau manifold is determined by the amplitude system (29)–(30). Although the Ginzburg–Landau manifold, constructed above, is not invariant under the flow of the original system (1)–(2), it is a good approximation of the flow near the Ginzburg–Landau manifold. This is documented in the following approximation theorem.

Theorem 5.3 *For all $R_2, T_0, C_2 > 0$, $n \geq 0$ and all $\theta \in \mathbb{N}_0$ there exists $C_3, \delta_0 > 0$ and $m \geq 0$ such that for all $0 \leq \varepsilon \leq \delta \leq \delta_0$ the following holds: Let (A_1, B_0) be a solution of (29)–(30) with*

$$\sup_{T \in [0, T_0]} (\|A_1(\cdot, T)\|_{H^{m+1}_{l,u}} + \|B_0(\cdot, T)\|_{H^m_{l,u}}) \leq R_2,$$

with initial conditions $(A_1, B_0)|_{T=0} = (A_1(\cdot, 0), B_0(\cdot, 0))$, and $(u_0, v_0) \in H^{n+1}_{l,u} \times H^n_{l,u}$ with

$$\|(u_0, \delta^{-1}v_0) - (\Psi_{\theta,u}, \Psi_{\theta,v})(A_1(\cdot, 0), B_0(\cdot, 0))\|_{H^{n+1}_{l,u} \times H^n_{l,u}} \leq C_2\delta^\theta.$$

Then there exists a solution (u, v) of (1)–(2) with initial condition $(u, v)|_{t=0} = (u_0, v_0)$ and

$$\sup_{0 \leq t \leq T_0/\delta^2} \|(u, \delta^{-1}v)(\cdot, t) - (\Psi_{\theta,u}, \Psi_{\theta,v})(A_1, B_0)(\cdot, t)\|_{H^{n+1}_{l,u} \times H^n_{l,u}} \leq C_3\delta^\theta.$$

Proof See "Appendix C". □

Now we have all ingredients for establishing a global existence result through some a-priori bound (32). For $\theta \geq 3$ the following holds:

(a) We start with the **attractivity**, cf. Theorem 5.2. For a sufficiently large $R_0 > 0$ we obtain $R_1 > 0$, $T_1 > 0$, and $A_1(\cdot, 0)$ and $B_0(\cdot, 0)$ with

$$\|A_1(\cdot, 0)\|_{H^{m+1}_{l,u}} + \|B_0(\cdot, 0)\|_{H^m_{l,u}} \leq R_1$$

such that the solution (u, v), with the initial conditions (u_0, v_0), satisfies

$$\|(u, \delta^{-1} v)|_{t=T_1/\delta^2} - (\Psi_{\theta,u}, \Psi_{\theta,v})(A_1(\cdot, 0), B_0(\cdot, 0))\|_{H^{n+1}_{l,u} \times H^n_{l,u}} \leq C\delta^\theta$$

for $\delta > 0$ sufficiently small.

(b) According to Theorem 1.2 and Remark 1.3, in case of periodic boundary conditions (12), the amplitude system (29)–(30) possesses an **absorbing ball** of radius C_R in $H^{m+1}_{l,u} \times H^m_{l,u}$. Solutions of (29)–(30) starting in the ball of the above radius R_1 need a time T_0 to come to the absorbing ball of radius C_R.

(c) We have to make sure that the original ball $R_0\delta$ for the original reaction–diffusion system (1)–(2) is so big that the Ginzburg–Landau embedding of the absorbing ball for the amplitude system (29)–(30) of radius C_R is contained in this ball. In detail, for A_1 and B_0 satisfying

$$\|A_1(\cdot, T_0)\|_{H^{m+1}_{l,u}} + \|B_0(\cdot, T_0)\|_{H^m_{l,u}} \leq C_R$$

we need that the starting radius R_0 is so big that

$$\|(\Psi_{\theta,u}, \Psi_{\theta,v})(A_1, B_0)(\cdot, T_0/\delta^2)\|_{H^{n+1}_{l,u} \times H^n_{l,u}} \leq R_0\delta/2.$$

(d) Finally we use the **approximation** property, i.e., that the amplitude systems (29)–(30) makes correct predictions about the dynamics of the original system, cf. Theorem 5.3. Then the triangle inequality guarantees that

$$\begin{aligned}
&\|(u, \delta^{-1} v)|_{(T_1+T_0)/\delta^2}\|_{H^{n+1}_{l,u} \times H^n_{l,u}} \\
&\leq \|(\Psi_{\theta,u}, \Psi_{\theta,v})(A_1, B_0)(\cdot, T_0/\delta^2)\|_{H^{n+1}_{l,u} \times H^n_{l,u}} \\
&\quad + \sup_{0 \leq t \leq T_0/\delta^2} \|(u, \delta^{-1} v)(\cdot, T_1/\delta^2 + t) - (\Psi_{\theta,u}, \Psi_{\theta,v})(A_1, B_0)(\cdot, t)\|_{H^{n+1}_{l,u} \times H^n_{l,u}} \\
&\leq R_0\delta/2 + C_3\delta^\theta \leq 3R_0\delta/4
\end{aligned}$$

for $\delta > 0$ sufficiently small. Thus, after a time $(T_1 + T_0)/\delta^2$ the flow of the original reaction–diffusion system (1)–(2) has mapped the rescaled initial ball of radius $R_0\delta$ into the smaller rescaled ball of radius $3R_0\delta/4$. Since the magnitude of the solution (u, v) is also controlled between $t = 0$ and $t = (T_1 + T_0)/\delta^2$ by our estimates, we established an a priori bound (32). Thus, with the above arguments the global existence and uniqueness of the solutions of (1)–(2) follows for $\delta > 0$ sufficiently small.

Remark 5.4 We remark that from a technical point of view, in contrast to previous approaches, we moved the first step of the approximation result as stated [9, 11, 14] to the attractivity result. This allows us to combine the attractivity and approximation result more easily.

6 Discussion

Before we give proofs of the attractivity theorem 5.2, the approximation theorem 5.3, and of Theorem 2.6 we would like to close the paper by discussing two other points, namely the restriction to a nonlinearity $g = g(u)$ and the global existence question in case that the periodic boundary conditions (12) and (13) are dropped.

Remark 6.1 For us, (1)–(2) is a toy model which already contains many features which are relevant for the global existence question addressed in this paper. The major restriction of our model (1)–(2) seems to be the assumption that $g(u) = \mathcal{O}(|u|^2)$ only depends on u. However, an additional dependence on v without further smoothing would lead to a quasilinear system and to functional analytic difficulties having to do with the quasilinearity of such a system, but not with the question addressed in this paper. Alternatively, instead of (2), one could consider the following semilinear toy problems

$$\partial_t v = d_v \partial_x^2 v + \partial_x^2 (1 - \partial_x^2)^{-1} g(u, v) \qquad \text{or} \qquad \partial_t v = -\partial_x^4 v + d_v \partial_x^2 v + \partial_x^2 g(u, v),$$

with $g(u, v) = \mathcal{O}(|u|^2 + |v|^2)$. Since we are not interested in the sideband unstable situation in the v-equation at the wave number $k = 0$, cf. [3], in these alternative models for notational simplicity we would assume $g(u, v) = \mathcal{O}(|u|^2+|v|^2))$ instead of $g(u, v) = \mathcal{O}(|u|^2+|v|)$. It is essential to remark that, w.r.t. the scaling used above, a term $|v|^2$ is of higher order than a term $|u|^2$ and will not appear in the amplitude system (29)–(30). In hydrodynamical applications the quasilinearity of the problem often cannot be avoided, cf. [18]. Global existence by the above approach is a problem which is unsolved in quasilinear situations even without a conservation law so far.

Remark 6.2 In this remark we would like to discuss a few observations about the global existence problem if the periodic boundary conditions (12) and (13) are dropped. We consider the situation when in lowest order in (9)–(10) the B-equation decouples from the A-equation, i.e., $b_1 = 0$. In this case the amplitude system in normal form is given by

$$\partial_T A = (1 + i\gamma_0)\partial_X^2 A + A + \beta AB - (1 + i\gamma_3)A|A|^2, \tag{33}$$

$$\partial_T B = \alpha \partial_X^2 B. \tag{34}$$

By the maximum principle B stays bounded and for A a uniform bound in time can be established with the weighted energy method explained in Remark 2.3. Hence, the solutions of the amplitude system exist globally in time and stay uniformly bounded. However, due to the B-equation the system does not possess an absorbing ball.

Adding the higher oder terms to the B-equation gives a system of the form

$$\partial_T B = d_v \partial_X^2 B + \partial_X^2 (\mathcal{O}(\varepsilon)).$$

With the variation of constant formula we obtain

$$B(T) = e^{d_v \partial_X^2 T} B(0) + \int_0^T e^{d_v \partial_X^2 (T-\tau)} \partial_X^2 (\mathcal{O}(\varepsilon)) d\tau$$

and using the estimate

$$\| e^{d_v \partial_X^2 T} \partial_X^{2-2\vartheta} \|_{H^n_{l,u} \to H^n_{l,u}} \leq C T^{\vartheta - 1}$$

we expect

$$B(T) - e^{d_v \partial_X^2 T} B(0) = \mathcal{O}(\varepsilon \int_0^T (T-\tau)^{\vartheta-1} d\tau) = \mathcal{O}(\varepsilon T^{\vartheta})$$

for a $\vartheta > 0$ arbitrarily small, but fixed. Hence, in the A-equation the term AB can grow as $\mathcal{O}(\varepsilon T^{\vartheta})A$. It can be expected that it can be balanced with the $-A|A|^2$-term as long $\varepsilon T^{\vartheta} \leq \mathcal{O}(1)$, i.e. for $T \leq \mathcal{O}(\varepsilon^{-1/\vartheta})$, i.e., on arbitrary long, but fixed, time scales w.r.t. ε. With more advanced estimates we even would obtain

$$B(T) - e^{d_v \partial_X^2 T} B(0) = \mathcal{O}(C + \varepsilon \int_0^{T-1} (T-\tau)^{-1} d\tau) = \mathcal{O}(C + \varepsilon \ln T),$$

respectively $T \leq \mathcal{O}(\exp(1/\varepsilon))$. It will be the subject of future research to make these arguments rigorous by iterating the attractivity and approximation result for a growing sequence of perturbation parameters δ. Note that an iteration, as used in [9, 14] with a sequence of suitable chosen δ_js, is not possible in case of periodic boundary conditions.

A The Analytic Set-Up

This section contains a few preparations for the subsequent proofs of the attractivity and approximation result.

(i) It turns out to be advantageous that all variables in (26)–(28) have the same regularity, i.e., we introduce the new variable $\check{v}$ by

$$v = \langle \partial_x \rangle \check{v} = (1 - \partial_x^2)^{1/2} \check{v}$$

such that (26)–(28) becomes

$$\partial_t \check{u}_1 = \lambda_1 \check{u}_1 + \check{f}_{1,n}(\check{u}_1, u_s, \check{v}), \tag{35}$$

$$\partial_t u_s = \Lambda_s u_s + \check{f}_{s,1}(\check{u}_1, u_s, \check{v}), \tag{36}$$

$$\partial_t \check{v} = \Lambda_v \check{v} + \partial_x^2 \check{g}_1(\check{u}_1, u_s), \tag{37}$$

with

$$\begin{aligned} \check{f}_{1,n}(\check{u}_1, u_s, \check{v}) &= \check{f}_1(\check{u}_1, u_s, \langle \partial_x \rangle \check{v}) \\ &= \mathcal{O}(|\check{u}_1|^3 + |\check{u}_1||u_s| + |u_s|^2 + (|\check{u}_1| + |u_s|)|\check{v}|), \\ \check{f}_{s,n}(\check{u}_1, u_s, \check{v}) &= \check{f}_s(\check{u}_1, u_s, \langle \partial_x \rangle \check{v}) \\ &= \mathcal{O}(|\check{u}_1|^2 + |u_s|^2 + (|\check{u}_1| + |u_s|)|\check{v}|), \\ \check{g}_1(\check{u}_1, u_s) &= \langle \partial_x \rangle^{-1} \check{g}(\check{u}_1, u_s) \\ &= \mathcal{O}(|\check{u}_1|^2 + |u_s|^2). \end{aligned}$$

As a consequence, the nonlinearities $\check{f}_{s,n}$ and $\partial_x^2 \check{g}_1$ are smooth mappings from $H^{s+1}_{l,u}$ to $H^s_{l,u}$. The mapping $\check{f}_{1,n}$ is arbitrarily smooth due to its compact support in Fourier space.

(ii) We introduce the scaling operator

$$(S_\delta u)(x) = u(\delta x)$$

and the scaled spaces $H^{s,\delta}_{l,u} = H^s_{l,u}$ equipped with the norm

$$\|u\|_{H^{s,\delta}_{l,u}} = \|S_{1/\delta} u\|_{H^s_{l,u}}$$

(iii) Before we start with estimating the linear semigroups we define the $H^s_{l,u}$-norm for $s \in (0,1)$ by

$$\|u\|_{H^s_{l,u}} = \|u\|_{L^2_{l,u}} + \|\partial_x \langle \partial_x \rangle^{s-1} u\|_{L^2_{l,u}}, \qquad (s \in (0,1)).$$

For $n \in \mathbb{N}_0$ and $s \in (0,1)$ we set

$$\|u\|_{H^{n+s}_{l,u}} = \|u\|_{H^n_{l,u}} + \|\partial_x^n u\|_{H^s_{l,u}}.$$

We need

Lemma A.1 *For $s, r \geq 0$ there exists a $\sigma > 0$, a $C \geq 1$ such that for $0 < \varepsilon \leq \delta \leq 1$ and all $t \geq 0$ the following estimates hold:*

$$\begin{aligned}
\|e^{\lambda_1 t}\|_{H^s_{l,u} \to H^{s+r}_{l,u}} &\leq C(1 + t^{-r/2}) e^{C\varepsilon^2 t}, \\
\|e^{\lambda_1 t}\|_{H^{s,\delta}_{l,u} \to H^{s+r,\delta}_{l,u}} &\leq C(1 + (\delta^2 t)^{-r/2})) e^{C\varepsilon^2 t}, \\
\|e^{\Lambda_s t}\|_{H^s_{l,u} \to H^{s+r}_{l,u}} &\leq C e^{-\sigma t}(1 + t^{-r/2}), \\
\|e^{\Lambda_s t}\|_{H^{s,\delta}_{l,u} \to H^{s+r,\delta}_{l,u}} &\leq C e^{-\sigma t}(1 + (\delta^2 t)^{-r/2})), \\
\|e^{\Lambda_v t}\|_{H^s_{l,u} \to H^{s+r}_{l,u}} &\leq C(1 + t^{-r/2}), \\
\|e^{\Lambda_v t}\|_{H^{s,\delta}_{l,u} \to H^{s+r,\delta}_{l,u}} &\leq C(1 + (\delta^2 t)^{-r/2})).
\end{aligned}$$

Proof These estimates have been established in a number of papers, cf. [15, §10]. □

Remark A.2 We refrain from recalling a complete proof of Lemma A.1. It is based on estimates like

$$\sup_{k \in \mathbb{R}} |e^{-k^2 t}(ik)^n| \leq C t^{-n/2}$$

for $n \in \mathbb{N}_0$ and on estimates like $\widehat{\lambda}_1(k) \leq -\alpha k^2$ for an $\alpha > 0$. For real-valued $n \geq 0$ with $n = n_0 + s$ with $n_0 \in \mathbb{N}$ and $s \in [0,1)$ we use

$$\sup_{k \in \mathbb{R}} |e^{-k^2 t} k \langle k \rangle^{s-1}| \leq \sup_{k \in \mathbb{R}} |e^{-k^2 t} |k|^s| \sup_{k \in \mathbb{R}} ||k|^{1-s} \langle k \rangle^{s-1}| \leq C t^{-s/2}.$$

B Proof of the Attractivity Theorem 5.2

In order to prove the attractivity result we have to show that the solution (u, v) of (1)–(2) to a small, but otherwise arbitrary initial condition $(u_0, v_0) \in H^{n+1}_{l,u} \times H^n_{l,u}$ develops in such a way that after a certain time it can be written in the form stated in (7)–(8), i.e., after that time we must be able to extract functions A_1 and B_0 which are functions of the long spatial variable $X = \delta x$. For the derivation of the amplitude system (29)–(30) we make a Taylor expansion w.r.t. the small perturbation parameter δ, with $0 < \varepsilon \leq \delta \ll 1$, and among other things we use that $\partial_x^m A_1(\delta x) = \mathcal{O}(\delta^m)$ and $\partial_x^m B_0(\delta x) = \mathcal{O}(\delta^m)$. In the end this means that we have to prove estimates such as $\partial_x^m((E_1 u(\cdot,t))) = \mathcal{O}(\delta^{m+1})$ and $\partial_x^m(E_0 v(\cdot,t)) = \mathcal{O}(\delta^{m+2})$ for $t > 0$ sufficiently large with initial conditions of (1)–(2) satisfying the estimates assumed in Theorem 5.2.

Remark B.1 By looking at the Fourier representation of the linearized problem we see that the u-solution and the v-solution are exponentially damped for all wave numbers except around $k = 0$ where in physical space the solutions are of order $\mathcal{O}(\delta)$. By nonlinear interaction no other modes of order $\mathcal{O}(\delta)$ are created.

B.1 The First Attractivity Step

We consider (1)–(2) after applying the normal form transformation from Sect. 3.2, i.e., in the following we consider (35)–(37).

(1) We start with solutions of order $\check{u}_1 = \mathcal{O}(\delta)$, $u_s = \mathcal{O}(\delta)$ and $\check{v} = \mathcal{O}(\delta^2)$. Setting $\check{u}_1 = \delta\widetilde{u}_1$, $u_s = \delta\widetilde{u}_s$, and $\check{v} = \delta^2\widetilde{v}$ gives

$$\partial_t\widetilde{u}_1 = \lambda_1\widetilde{u}_1 + \widetilde{f}_1(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}), \tag{38}$$

$$\partial_t\widetilde{u}_s = \Lambda_s\widetilde{u}_s + \widetilde{f}_s(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}), \tag{39}$$

$$\partial_t\widetilde{v} = \Lambda_v\widetilde{v} + \partial_x^2\widetilde{g}(\widetilde{u}_1, \widetilde{u}_s), \tag{40}$$

with

$$\widetilde{f}_1(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}) = \mathcal{O}(\delta^2|\widetilde{u}_1|^3 + \delta|\widetilde{u}_1||\widetilde{u}_s| + \delta|\widetilde{u}_s|^2 + \delta^2(|\widetilde{u}_1| + |\widetilde{u}_s|)|\widetilde{v}|),$$
$$\widetilde{f}_s(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}) = \mathcal{O}(\delta|\widetilde{u}_1|^2 + \delta|\widetilde{u}_s|^2 + \delta^2(|\widetilde{u}_1| + |\widetilde{u}_s|)|\widetilde{v}|),$$
$$\widetilde{g}(\widetilde{u}_1, \widetilde{u}_s) = \mathcal{O}(|\widetilde{u}_1|^2 + |\widetilde{u}_s|^2).$$

Considering the variation of constant formula

$$\widetilde{u}_s(t) = e^{\Lambda_s t}\widetilde{u}_s(0) + \int_0^t e^{\Lambda_s(t-\tau)}\widetilde{f}_s(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v})(\tau)d\tau,$$

it is easy to see that $u_s = \mathcal{O}(\delta^2)$ for instance for $t = 1/\delta^{1/4}$ using the exponential decay $\|e^{\Lambda_s t}\|_{H^{n+1}_{l,u}\to H^{n+1}_{l,u}} \leq Ce^{-\sigma t}$ for a $\sigma > 0$ independent of $0 < \varepsilon \leq \delta \ll 1$ under the assumption that $\widetilde{f}_s(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}) = \mathcal{O}(\delta)$ for $t \in [0, 1/\delta^{1/4}]$. However, since there is no $\delta^{1/4}$ in front of the nonlinear terms in the $\widetilde{v}$-equation we cannot guarantee that $\widetilde{v} = \mathcal{O}(1)$ for $t = \delta^{-1/4}$. In order to guarantee this, some extra work has to be done. Since the argument follows the arguments of next (more complicated) step 2) we assume for a moment that we have proved $\check{u}_c = \mathcal{O}(\delta)$, $u_s = \mathcal{O}(\delta^2)$ and $v = \mathcal{O}(\delta^2)$ for $t = 1/\delta^{1/4}$ and close the gap in the proof in Remark B.2 after completing step 2).

(2) We start (35)–(37) again, but now for initial conditions $\check{u}_c = \mathcal{O}(\delta)$, $u_s = \mathcal{O}(\delta^2)$, and $v = \mathcal{O}(\delta^2)$. Setting $\check{u}_1 = \delta\widetilde{u}_1$, $u_s = \delta^2\widetilde{u}_s$ and $v = \delta^2\widetilde{v}$. We find now

$$\partial_t\widetilde{u}_1 = \lambda_1\widetilde{u}_1 + \widetilde{f}_1(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}), \tag{41}$$

$$\partial_t\widetilde{u}_s = \Lambda_s\widetilde{u}_s + \widetilde{f}_s(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}), \tag{42}$$

$$\partial_t\widetilde{v} = \Lambda_v\widetilde{v} + \partial_x^2\widetilde{g}(\widetilde{u}_1, \widetilde{u}_s), \tag{43}$$

with

$$\widetilde{f}_1(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}) = \mathcal{O}(\delta^2|\widetilde{u}_1|^3 + \delta^2|\widetilde{u}_1||\widetilde{u}_s| + \delta^3|\widetilde{u}_s|^2 + \delta^2(|\widetilde{u}_1| + \delta|\widetilde{u}_s|)|\widetilde{v}|),$$
$$\widetilde{f}_s(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}) = \mathcal{O}(|\widetilde{u}_1|^2 + \delta|\widetilde{u}_1||\widetilde{u}_s| + \delta^2|\widetilde{u}_s|^2 + \delta(|\widetilde{u}_1| + \delta|\widetilde{u}_s|)|\widetilde{v}|),$$
$$\widetilde{g}(\widetilde{u}_1, \widetilde{u}_s) = \mathcal{O}(|\widetilde{u}_1|^2 + \delta|\widetilde{u}_1||\widetilde{u}_s| + \delta^2|\widetilde{u}_s|^2).$$

Since attractivity happens on an $\mathcal{O}(1/\delta^2)$-time scale we have to control the solutions of the last system on this long time scale. The first equation is not a problem since in front of all nonlinear terms there is a factor δ^2. In the second equation there is linear exponential damping which allows us to control all nonlinear terms in this equation. The main difficulty to control the solutions on the long $\mathcal{O}(1/\delta^2)$-time scale is the missing δ^2 in front of the nonlinear terms in the third equation. In order to get this missing δ^2 we need that $\widetilde{u}_1$ is two times differentiable w.r.t. the long space variable X. In detail, we need that these derivatives are $\mathcal{O}(1)$-bounded. However, this is a problem since this exactly what we are going to prove and what is not true for $t = 0$.

(a) We proceed as follows to get rid of this problem. We consider the variation of constant formula

$$\widetilde{u}_1(t) = e^{\lambda_1 t}\widetilde{u}_1(0) + \int_0^t e^{\lambda_1(t-\tau)}\widetilde{f}_1(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v})(\tau)d\tau,$$
$$\widetilde{u}_s(t) = e^{\Lambda_s t}\widetilde{u}_s(0) + \int_0^t e^{\Lambda_s(t-\tau)}\widetilde{f}_s(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v})(\tau)d\tau,$$
$$\widetilde{v}(t) = e^{\Lambda_v t}\widetilde{v}(0) + \int_0^t e^{\Lambda_v(t-\tau)}\partial_x^2\widetilde{g}(\widetilde{u}_1, \widetilde{u}_s)(\tau)d\tau.$$

For this system we are now going to establish a priori estimates which in combination with the local existence and uniqueness theorem will guarantee the long time existence of solutions on the long $\mathcal{O}(1/\delta^2)$-time scale.

We set

$$S_{c,0}(t) = \sup_{\tau\in[0,t]} \|\widetilde{u}_1(\tau)\|_{H^{n+1}_{l,u}}, \qquad S_{s,0}(t) = \sup_{\tau\in[0,t]} \|\widetilde{u}_s(\tau)\|_{H^{n+1}_{l,u}},$$

and

$$S_{v,0}(t) = \sup_{\tau\in[0,t]} \|\widetilde{v}(\tau)\|_{H^{n+1}_{l,u}}.$$

Moreover, we need the quantity

$$S_{c,1}(t) = S_{c,0}(t) + \sup_{\tau\in[0,t]} \tau^{1/2}\|\partial_x\widetilde{u}_1(\tau)\|_{L^2_{l,u}}.$$

Before we start, we remark that all $H^s_{l,u}$-norms for $\widetilde{u}_1$ are equivalent due to the compact support of $\widetilde{u}_1$ in Fourier space.

(i) We estimate

$$\begin{aligned}\|\widetilde{u}_1(t)\|_{H^{n+1}_{l,u}} &\leq \|e^{\lambda_1 t}\widetilde{u}_1(0)\|_{H^{n+1}_{l,u}} \\ &\quad + \int_0^t \|e^{\lambda_1(t-\tau)}\|_{H^{n+1}_{l,u}\to H^{n+1}_{l,u}}\|\widetilde{f}_1(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v})(\tau)\|_{H^{n+1}_{l,u}}d\tau \\ &\leq C\|\widetilde{u}_1(0)\|_{H^{n+1}_{l,u}} + C\int_0^t \|\widetilde{f}_1(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v})(\tau)\|_{H^{n+1}_{l,u}}d\tau \\ &\leq CS_{c,0}(0) + C\delta^2 t(S_{c,0}(t)^3 + S_{c,0}(t)S_{s,0}(t) + S_{c,0}(t)S_{v,0}(t) \\ &\quad +\delta(S_{s,0}(t)^2 + S_{s,0}(t)S_{v,0}(t))),\end{aligned}$$

where we used the semigroup estimate from Lemma A.1 and the bound on $\widetilde{f}_1$ after (43).

(ii) Next we find

$$\begin{aligned}&t^{1/2}\|\partial_x\widetilde{u}_1(t)\|_{L^2_{l,u}} \\ &\quad\leq t^{1/2}\|\partial_x e^{\lambda_1 t}\widetilde{u}_1(0)\|_{L^2_{l,u}}\end{aligned}$$

$$
\begin{aligned}
&+t^{1/2}\int_0^t \|\partial_x e^{\lambda_1(t-\tau)}\|_{H^{n+1}_{l,u}\to L^2_{l,u}} \|\widetilde{f}_1(\widetilde{u}_1,\widetilde{u}_s,\widetilde{v})(\tau)\|_{H^{n+1}_{l,u}} d\tau \\
&\le C\|\widetilde{u}_1(0)\|_{H^{n+1}_{l,u}} + t^{1/2}\int_0^t (t-\tau)^{-1/2}\|\widetilde{f}_1(\widetilde{u}_1,\widetilde{u}_s,\widetilde{v})(\tau)\|_{H^{n+1}_{l,u}} d\tau \\
&\le CS_{c,0}(0) + C\delta^2 t^{1/2}\int_0^t (t-\tau)^{-1/2} d\tau \\
&\quad\times(S_{c,0}(t)^3 + S_{c,0}(t)S_{s,0}(t) + S_{c,0}(t)S_{v,0}(t) + \delta(S_{s,0}(t)^2 + S_{s,0}(t)S_{v,0}(t))) \\
&\le CS_{c,0}(0) + C\delta^2 t(S_{c,0}(t)^3 + S_{c,0}(t)S_{s,0}(t) + S_{c,0}(t)S_{v,0}(t) \\
&\quad+\delta(S_{s,0}(t)^2 + S_{s,0}(t)S_{v,0}(t))),
\end{aligned}
$$

where we used the semigroup estimate from Lemma A.1 with $r = 1$ and again the bound on $\widetilde{f}_1$ after (43).

(iii) For the exponentially damped part we use that

$$
\int_0^t \|e^{\Lambda_s(t-\tau)}\|_{H^n_{l,u}\to H^{n+1}_{l,u}} d\tau \le \int_0^t e^{-\sigma(t-\tau)}(1+(t-\tau)^{-1/2})d\tau = \mathcal{O}(1)
$$

uniformly in $t \ge 0$, and the bound on $\widetilde{f}_s$ after (43), and so we find

$$
\begin{aligned}
\|\widetilde{u}_s(t)\|_{H^{n+1}_{l,u}} &\le \|e^{\Lambda_s t}\widetilde{u}_s(0)\|_{H^{n+1}_{l,u}} \\
&\quad+\int_0^t \|e^{\Lambda_s(t-\tau)}\|_{H^n_{l,u}\to H^{n+1}_{l,u}} \|\widetilde{f}_s(\widetilde{u}_1,\widetilde{u}_s,\widetilde{v})(\tau)\|_{H^n_{l,u}} d\tau \\
&\le \|\widetilde{u}_s(0)\|_{H^{n+1}_{l,u}} \\
&\quad+\int_0^t e^{-\sigma(t-\tau)}(1+(t-\tau)^{-1/2})\|\widetilde{f}_s(\widetilde{u}_1,\widetilde{u}_s,\widetilde{v})(\tau)\|_{H^n_{l,u}} d\tau \\
&\le CS_{s,0}(0) + C(S_{c,0}(t)^2 + \delta S_{c,0}(t)S_{s,0}(t) + \delta^2 S_{s,0}(t)^2 \\
&\quad+\delta S_{v,0}(t)(S_{c,0}(t) + \delta S_{s,0}(t))).
\end{aligned}
$$

(iv) The estimates for the $\widetilde{v}$-variable are obtained from

$$
\begin{aligned}
\|\widetilde{v}(t)\|_{H^{n+1}_{l,u}} &\le C\|\widetilde{v}(0)\|_{H^{n+1}_{l,u}} \\
&\quad+\int_0^t \|e^{\Lambda_v(t-\tau)}\partial_x\|_{H^{n+1}_{l,u}\to H^{n+1}_{l,u}} \|\partial_x \widetilde{g}(\widetilde{u}_1,\widetilde{u}_s)(\tau)\|_{H^{n+1}_{l,u}} d\tau \\
&\le CS_{v,0}(0) + C\int_0^t (t-\tau)^{-1/2}\tau^{-1/2} d\tau\, S_{c,0}(t)S_{c,1}(t) \\
&\quad+C\int_0^t (t-\tau)^{-1/2} d\tau(\delta S_{c,0}(t)S_{s,0}(t) + \delta^2 S_{s,0}(t)S_{s,0}(t)) \\
&\le CS_{v,0}(0) + CS_{c,0}(t)S_{c,1}(t) + C\delta t^{1/2}(S_{c,0}(t)S_{s,0}(t) + \delta S_{s,0}(t)^2),
\end{aligned}
$$

where the semigroup term is estimated with Lemma A.1 with $r = 1$, where we used that all $H^s_{l,u}$-norms for $\widetilde{u}_1$ are equivalent due to its compact support in Fourier space, and where we used estimates like

$$
|\tau^{-1/2}\tau^{1/2}\partial_x \widetilde{u}_1(\tau)| \le \tau^{-1/2} S_{c,1}(t).
$$

(v) Taking the sup w.r.t. t on the left-hand side gives the inequalities

$$
S_{c,0}(t) \le CS_{c,0}(0) + CT(S_{c,0}(t)^3 + S_{c,0}(t)S_{s,0}(t) + S_{c,0}(t)S_{v,0}(t)
$$

$$
\begin{aligned}
&\quad +\delta(S_{s,0}(t)^2 + S_{s,0}(t)S_{v,0}(t))),\\
S_{c,1}(t) &\leq CS_{c,0}(0) + CT(S_{c,0}(t)^3 + S_{c,0}(t)S_{s,0}(t) + S_{c,0}(t)S_{v,0}(t)\\
&\quad +\delta(S_{s,0}(t)^2 + S_{s,0}(t)S_{v,0}(t))),\\
S_{s,0}(t) &\leq CS_{s,0}(0) + C(S_{c,0}(t)^2 + \delta S_{c,0}(t)S_{s,0}(t) + \delta^2 S_{s,0}(t)^2\\
&\quad +\delta S_{v,0}(t)(S_{c,0}(t) + \delta S_{s,0}(t))),\\
S_{v,0}(t) &\leq CS_{v,0}(0) + CS_{c,0}(t)S_{c,1}(t) + CT^{1/2}(S_{c,0}(t)S_{s,0}(t) + \delta S_{s,0}(t)^2).
\end{aligned}
$$

For $\delta > 0$ and $T > 0$ sufficiently small the last two inequalities allow to estimate $S_{s,0}(t)$ and $S_{v,0}(t)$ in terms of $S_{s,0}(0)$, $S_{v,0}(0)$, $S_{c,0}(t)$, and $S_{c,1}(t)$. Replacing then $S_{s,0}(t)$ and $S_{v,0}(t)$ in the first two inequalities by these estimates and choosing $\delta_0 > 0$ and $T_1 = \mathcal{O}(1)$ sufficiently small, gives the existence of a $C_1 = \mathcal{O}(1)$ with

$$
S_{c,0}(t) + S_{c,1}(t) + S_{s,0}(t) + S_{v,0}(t) \leq C_1 \tag{44}
$$

for all $t \in [0, T_1/\delta^2]$ and $\delta \in (0, \delta_0)$.

Remark B.2 It remains to close Step 1), i.e., we have to prove that $\check{u}_1 = \mathcal{O}(\delta)$, $u_s = \mathcal{O}(\delta)$ and $\check{v} = \mathcal{O}(\delta^2)$ tor $t \in [0, 1/\delta^{1/4}]$. In order to do so, we follow the argument in Step 2) but now with $u_s = \delta\widetilde{u}_s$ instead of $u_s = \delta^2\widetilde{u}_s$. Moreover, we set

$$
S_{u,0}(t) = \sup_{\tau\in[0,t]} \|\widetilde{u}_1(\tau)\|_{H^{n+1}_{l,u}} + \sup_{\tau\in[0,t]} \|\widetilde{u}_s(\tau)\|_{H^{n+1}_{l,u}},
$$

and

$$
S_{u,1}(t) = S_{u,0}(t) + \sup_{\tau\in[0,t]} \tau^{1/2}\|\partial_x\widetilde{u}_1(\tau)\|_{L^2_{l,u}} + \sup_{\tau\in[0,t]} \tau^{1/2}\|\partial_x\widetilde{u}_s(\tau)\|_{L^2_{l,u}}
$$

With exactly the same calculations as in 2) we end up with the inequalities

$$
\begin{aligned}
S_{u,0}(t) &\leq CS_{u,0}(0) + C\delta t(S_{u,0}(t)^2 + \delta S_{u,0}(t)S_{v,0}(t)),\\
S_{u,1}(t) &\leq CS_{u,0}(0) + C\delta t(S_{u,0}(t)^2 + \delta S_{u,0}(t)S_{v,0}(t)),\\
S_{v,0}(t) &\leq CS_{v,0}(0) + CS_{c,0}(t)S_{c,1}(t).
\end{aligned}
$$

The last inequality allows to estimate $S_{v,0}(t)$ in terms of $S_{v,0}(0)$, $S_{u,0}(t)$, and $S_{u,1}(t)$. Replacing then $S_{v,0}(t)$ in the first two inequalities by this estimate and then choosing $\delta_0 > 0$ sufficiently small, gives the existence of a $C_1 = \mathcal{O}(1)$ with

$$
S_{u,0}(t) + S_{u,1}(t) + S_{v,0}(t) \leq C_1 \tag{45}
$$

for all $t \in [0, 1/\delta^{1/4}]$ and $\delta \in (0, \delta_0)$.

B.2 The Second Attractivity Step

Our estimates from the first attractivity step also guarantee that the solutions $\widetilde{u}_1$, $\widetilde{u}_s$, and $\widetilde{v}$ of (26)–(28) are $\mathcal{O}(1)$-bounded in $H^{1,\delta}_{l,u}$, $H^{n+1}_{l,u}$, and $H^{n+1}_{l,u}$, respectively, on time intervals of length $\mathcal{O}(1/\delta^2)$, for instance considering (45) for $t \in [T_1/(2\delta^2), T_1/\delta^2]$.

In the next step we prove that under these assumptions $\widetilde{u}_s$ and $\widetilde{v}$ will be in $H^{1/2,\delta}_{l,u}$ after an $\mathcal{O}(1/\delta^2)$-time scale. Since we have the existence and uniqueness of solutions it is sufficient to establish the bounds on this long time interval.

(i) We split

$$
\widetilde{f}_s(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}) = \widetilde{f}_{s,a}(\widetilde{u}_1) + \widetilde{f}_{s,b}(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}),
$$

with

$$\widetilde{f}_{s,a}(\widetilde{u}_1) = \mathcal{O}(|\widetilde{u}_1|^2),$$
$$\widetilde{f}_{s,b}(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v}) = \mathcal{O}(\delta|\widetilde{u}_1||\widetilde{u}_s| + \delta^2|\widetilde{u}_s|^2 + \delta(|\widetilde{u}_1| + \delta|\widetilde{u}_s|)|\widetilde{v}|),$$

and find for $t \leq T_1/\delta^2$ with $T_1 = \mathcal{O}(1)$ that

$$\begin{aligned}
& t^{1/4}\|\partial_x\langle\partial_x\rangle^{-1/2}\widetilde{u}_s(t)\|_{H^{n+1}_{l,u}} \\
&\leq t^{1/4}\|\partial_x\langle\partial_x\rangle^{-1/2}e^{\Lambda_s t}\widetilde{u}_s(0)\|_{H^{n+1}_{l,u}} \\
&\quad + t^{1/4}\int_0^t \|e^{\Lambda_s(t-\tau)}\|_{H^n_{l,u}\to H^{n+1}_{l,u}}\|\partial_x\langle\partial_x\rangle^{-1/2}\widetilde{f}_{s,a}(\widetilde{u}_1)(\tau)\|_{H^n_{l,u}}d\tau \\
&\quad + t^{1/4}\int_0^t \|\partial_x\langle\partial_x\rangle^{-1/2}e^{\Lambda_s(t-\tau)}\|_{H^n_{l,u}\to H^{n+1}_{l,u}}\|\widetilde{f}_{s,b}(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v})(\tau)\|_{H^n_{l,u}}d\tau \\
&\leq Ct^{1/4}t^{-1/4}e^{-\sigma t}\|\widetilde{u}_s(0)\|_{H^{n+1}_{l,u}} \\
&\quad + t^{1/4}\int_0^t e^{-\sigma(t-\tau)}(1+(t-\tau)^{-1/2})\|\partial_x\langle\partial_x\rangle^{-1/2}\widetilde{f}_{s,a}(\widetilde{u}_1)(\tau)\|_{H^n_{l,u}}d\tau \\
&\quad + t^{1/4}\int_0^t e^{-\sigma(t-\tau)}(1+(t-\tau)^{-3/4})\|\widetilde{f}_{s,b}(\widetilde{u}_1, \widetilde{u}_s, \widetilde{v})(\tau)\|_{H^n_{l,u}}d\tau \\
&\leq CS_{s,0}(0) + C\delta^{1/2}t^{1/4}S_{c,1}(T_1/\delta^2)^2 \\
&\quad + \delta t^{1/4}(S_{c,0}(t)S_{s,0}(t) + \delta S_{s,0}(t)^2 + S_{v,0}(t)(S_{c,0}(t) + \delta S_{s,0}(t)))
\end{aligned}$$

which is $\mathcal{O}(1)$ for $t = T_1/\delta^2$.

(ii) Similarly, we find for $t \leq T_1/\delta^2$ with $T_1 = \mathcal{O}(1)$ that

$$\begin{aligned}
& t^{1/4}\|\partial_x\langle\partial_x\rangle^{-1/2}\widetilde{v}(t)\|_{H^{n+1}_{l,u}} \\
&\leq Ct^{1/4}\|\partial_x\langle\partial_x\rangle^{-1/2}e^{\Lambda_v t}\widetilde{v}(0)\|_{H^{n+1}_{l,u}} \\
&\quad + t^{1/4}\int_0^t \|\partial_x\langle\partial_x\rangle^{-1/2}(e^{\Lambda_v(t-\tau)}\partial_x)\|_{H^{n+1}_{l,u}\to H^{n+1}_{l,u}}\|\partial_x\widetilde{g}_b(\widetilde{u}_1, \widetilde{u}_s)(\tau)\|_{H^{n+1}_{l,u}}d\tau \\
&\leq CS_{v,0}(0) + Ct^{1/4}\int_0^t (t-\tau)^{-3/4}d\tau\,\delta S_{c,1}(T_1/\delta^2)^2 \\
&\quad + Ct^{1/4}\int_0^t (t-\tau)^{-3/4}d\tau(\delta S_{c,0}(t)S_{s,0}(t) + \delta^2 S_{s,0}(t)S_{s,0}(t)) \\
&\leq CS_{v,0}(0) + C\delta t^{1/2}S_{c,1}(T_1/\delta^2)^2 + C\delta t^{1/2}(S_{c,0}(t)S_{s,0}(t) + \delta S_{s,0}(t)^2)
\end{aligned}$$

which is $\mathcal{O}(1)$ for $t = T_1/\delta^2$.

B.3 The Attractivity Induction Steps

Our estimates from the first two attractivity steps so far guarantee that the solutions $\widetilde{u}_1, \widetilde{u}_s$, and $\widetilde{v}$ of (26)–(28) are $\mathcal{O}(1)$-bounded in $H^{1,\delta}_{l,u}$, $H^{1/2,\delta}_{l,u} \cap H^{n+1}_{l,u}$, and $H^{1/2,\delta}_{l,u} \cap H^{n+1}_{l,u}$, respectively, on time intervals of length $\mathcal{O}(1/\delta^2)$.

In the next step we prove that under these assumptions $\widetilde{u}_1$ will be in $H^{3/2,\delta}_{l,u}$ after an $\mathcal{O}(1/\delta^2)$-time scale. After this we show that this implies that $\widetilde{u}_s$ and $\widetilde{v}$ will be in $H^{1,\delta}_{l,u}$ after an

$\mathcal{O}(1/\delta^2)$-time scale. In the next step we show that $\widetilde{u}_1$ will be in $H^{2,\delta}_{l,u}$ after an $\mathcal{O}(1/\delta^2)$-time scale, etc. We will do this by induction. Again it is sufficient to establish the bounds.

(i) In the first step we assume that

$$U_{m,c}(t) = \sup_{\tau\in[0,t]} \|\widetilde{u}_1(\tau)\|_{H^{m,\delta}_{l,u}}, \qquad U_{m-1/2,s}(t) = \sup_{\tau\in[0,t]} \|\widetilde{u}_s(\tau)\|_{H^{m-1/2,\delta}_{l,u}},$$

and

$$U_{m-1/2,v}(t) = \sup_{\tau\in[0,t]} \|\widetilde{v}(\tau)\|_{H^{m-1/2,\delta}_{l,u}}$$

are finite and of order $\mathcal{O}(1)$. We find for $t \leq T_1/\delta^2$ with $T_1 = \mathcal{O}(1)$ that

$$\begin{aligned}
&t^{1/4}\|\partial_x\langle\partial_x\rangle^{-1/2}\widetilde{u}_1(t)\|_{H^{m,\delta}_{l,u}} \\
&\quad\leq t^{1/4}\|\partial_x\langle\partial_x\rangle^{-1/2}e^{\lambda_1 t}\widetilde{u}_1(0)\|_{H^{m,\delta}_{l,u}} \\
&\qquad+t^{1/4}\int_0^t \|\partial_x\langle\partial_x\rangle^{-1/2}e^{\lambda_1(t-\tau)}\|_{H^{m-1/2,\delta}_{l,u}\to H^{m,\delta}_{l,u}}\|\widetilde{f}_1(\widetilde{u}_1,\widetilde{u}_s,\widetilde{v})(\tau)\|_{H^{m-1/2,\delta}_{l,u}}d\tau \\
&\quad\leq C\|\widetilde{u}_1(0)\|_{H^{m,\delta}_{l,u}} \\
&\qquad+t^{1/4}\int_0^t (t-\tau)^{-1/4}(\delta^2(t-\tau))^{-1/4}\|\widetilde{f}_1(\widetilde{u}_1,\widetilde{u}_s,\widetilde{v})(\tau)\|_{H^{m-1/2,\delta}_{l,u}}d\tau \\
&\quad\leq C\|\widetilde{u}_1(0)\|_{H^{m,\delta}_{l,u}} + C\delta^{3/2}t^{1/4}\int_0^t (t-\tau)^{-1/2}d\tau \\
&\qquad\times(U_{m,c}(t)^3 + U_{m,c}(t)U_{m-1/2,s}(t) + U_{m,c}(t)U_{m-1/2,v}(t) \\
&\qquad+\delta U_{m-1/2,s}(t)^2 + \delta U_{m-1/2,s}(t)U_{m-1/2,v}(t)) \\
&\quad\leq C\|\widetilde{u}_1(0)\|_{H^{m,\delta}_{l,u}} + C\delta^{3/2}t^{3/4}(U_{m,c}(t)^3 + U_{m,c}(t)U_{m-1/2,s}(t) \\
&\qquad+U_{m,c}(t)U_{m-1/2,v}(t) + \delta U_{m-1/2,s}(t)^2 + \delta U_{m-1/2,s}(t)U_{m-1/2,v}(t))
\end{aligned}$$

which is $\mathcal{O}(1)$ for $t = T_1/\delta^2$ and so $\widetilde{u}_1(t) \in H^{m+1/2,\delta}_{l,u}$ for $t = \mathcal{O}(1/\delta^2)$.

(ii) In the second induction step we assume that $U_{m+1/2,c}(t)$, $U_{m-1/2,s}(t)$, and $U_{m-1/2,v}(t)$ are finite and of order $\mathcal{O}(1)$. We find for $t \leq T_1/\delta^2$ with $T_1 = \mathcal{O}(1)$ that

$$\begin{aligned}
&t^{1/4}\|\partial_x\langle\partial_x\rangle^{-1/2}\widetilde{u}_s(t)\|_{H^{m-1/2,\delta}_{l,u}} \\
&\quad\leq t^{1/4}\|\partial_x\langle\partial_x\rangle^{-1/2}e^{\Lambda_s t}\widetilde{u}_s(0)\|_{H^{m-1/2,\delta}_{l,u}} \\
&\qquad+t^{1/4}\int_0^t \|e^{\Lambda_s(t-\tau)}\|_{H^{m-1/2,\delta}_{l,u}\to H^{m-1/2,\delta}_{l,u}}\|\partial_x\langle\partial_x\rangle^{-1/2}\widetilde{f}_{s,a}(\widetilde{u}_1)(\tau)\|_{H^{m-1/2,\delta}_{l,u}}d\tau \\
&\qquad+t^{1/4}\int_0^t \|\partial_x\langle\partial_x\rangle^{-1/2}e^{\Lambda_s(t-\tau)}\|_{H^{m-1/2,\delta}_{l,u}\to H^{m-1/2,\delta}_{l,u}}\|\widetilde{f}_{s,b}(\widetilde{u}_1,\widetilde{u}_s,\widetilde{v})(\tau)\|_{H^{m-1/2,\delta}_{l,u}}d\tau \\
&\quad\leq Ct^{1/4}t^{-1/4}e^{-\sigma t}\|\widetilde{u}_s(0)\|_{H^{m-1/2,\delta}_{l,u}} \\
&\qquad+t^{1/4}\int_0^t e^{-\sigma(t-\tau)}\|\partial_x\langle\partial_x\rangle^{-1/2}\widetilde{f}_{s,a}(\widetilde{u}_1)(\tau)\|_{H^{m-1/2,\delta}_{l,u}}d\tau \\
&\qquad+t^{1/4}\int_0^t e^{-\sigma(t-\tau)}(t-\tau)^{-1/4}\|\widetilde{f}_{s,b}(\widetilde{u}_1,\widetilde{u}_s,\widetilde{v})(\tau)\|_{H^{m-1/2,\delta}_{l,u}}d\tau \\
&\quad\leq C\|\widetilde{u}_s(0)\|_{H^{m-1/2,\delta}_{l,u}} + C\delta^{1/2}t^{1/4}U_{m+1/2,c}(T_1/\delta^2)^2
\end{aligned}$$

$$
\begin{aligned}
&+\delta t^{1/4}(U_{m+1/2,c}(t)U_{m-1/2,s}(t)+\delta U_{m-1/2,s}(t)^2\\
&+U_{m-1/2,v}(t)(U_{m+1/2,c}(t)(t)+\delta U_{m-1/2,s}(t)))
\end{aligned}
$$

which is $\mathcal{O}(1)$ for $t = T_1/\delta^2$ and so $\widetilde{u}_s(t) \in H^{m,\delta}_{l,u}$ for $t = \mathcal{O}(1/\delta^2)$.

(iii) In the second part of the second induction step we assume again that $U_{m+1/2,c}(t)$, $U_{m-1/2,s}(t)$, and $U_{m-1/2,v}(t)$ are finite and of order $\mathcal{O}(1)$. We find for $t \leq T_1/\delta^2$ with $T_1 = \mathcal{O}(1)$ that

$$
\begin{aligned}
&t^{1/4}\|\partial_x\langle\partial_x\rangle^{-1/2}\widetilde{v}(t)\|_{H^{n+1}_{l,u}}\\
&\leq Ct^{1/4}\|\partial_x\langle\partial_x\rangle^{-1/2}e^{\Lambda_v t}\widetilde{v}(0)\|_{H^{m-1/2,\delta}_{l,u}}\\
&\quad+t^{1/4}\int_0^t \|\partial_x\langle\partial_x\rangle^{-1/2}(e^{\Lambda_v(t-\tau)}\partial_x)\|_{H^{m-1/2,\delta}_{l,u}\to H^{m-1/2,\delta}_{l,u}}\|\partial_x\widetilde{g}_b(\widetilde{u}_1,\widetilde{u}_s)(\tau)\|_{H^{m-1/2,\delta}_{l,u}}d\tau\\
&\leq C\|\widetilde{v}(0)\|_{H^{m-1/2,\delta}_{l,u}}+Ct^{1/4}\int_0^t(t-\tau)^{-3/4}d\tau\,\delta U_{m+1/2,c}(T_1/\delta^2)^2\\
&\quad+Ct^{1/4}\int_0^t(t-\tau)^{-3/4}d\tau(\delta U_{m+1/2,c}(t)U_{m-1/2,s}(t)+\delta^2U_{m-1/2,s}(t)^2)\\
&\leq C\|\widetilde{v}(0)\|_{H^{m-1/2,\delta}_{l,u}}+C\delta t^{1/2}U_{m+1/2,c}(T_1/\delta^2)^2\\
&\quad+C\delta t^{1/2}(U_{m+1/2,c}(t)U_{m-1/2,s}(t)+\delta U_{m-1/2,s}(t)^2)
\end{aligned}
$$

which is $\mathcal{O}(1)$ for $t = T_1/\delta^2$ and so $\widetilde{v}(t) \in H^{m,\delta}_{l,u}$ for $t = \mathcal{O}(1/\delta^2)$.

B.4 Attractivity of the Ginzburg–Landau Manifold

In the first step we proved that the solutions of (22)–(25) develop in such a way that for arbitrary large but fixed m we have

$$
\delta^{-1}\|c_1|_{T_1/\delta^2}\|_{H^{m,\delta}_{l,u}\cap H^{n+1}_{l,u}}+\delta^{-2}\|u_s|_{T_1/\delta^2}\|_{H^{m,\delta}_{l,u}\cap H^{n+1}_{l,u}}+\delta^{-2}\|v|_{T_1/\delta^2}\|_{H^{m,\delta}_{l,u}\cap H^{n}_{l,u}}=\mathcal{O}(1).
$$

Then, we set

$$
\begin{aligned}
c_1&=\psi_1+\delta^2R_{1,1},\\
c_{-1}&=\psi_{-1}+\delta^2R_{-1,1},\\
u_s&=\psi_s+\delta^2R_{s,1},\\
v&=\psi_v+\delta^3R_{v,1}.
\end{aligned}
$$

where ψ_1, ψ_{-1}, ψ_s, and ψ_v were defined in Sect. 4.2. In the following we explain how to choose the $A_{\pm,m,n}$, $A_{s,m,n}$, and $B_{m,n}$ initially such that in the end the $\delta^2R_{\pm1,1}$, $\delta^2R_{s,1}$, and $\delta^3R_{v,1}$ will become smaller and smaller.

We start with $A_{+,1,0}|_{T=0} = \delta^{-1}c_1|_{t=T_1/\delta^2}$, $A_{-,1,0}|_{T=0} = \delta^{-1}c_{-1}|_{t=T_1/\delta^2}$, $A_{s,0}|_{T=0} = \delta^{-2}u_s|_{t=T_1/\delta^2}$, and $B_{0,0}|_{T=0} = \delta^{-2}E_0v|_{t=T_1/\delta^2}$. We choose the other $A_{\pm,m,n}$, $A_{s,m,n}$, and $B_{m,n}$ as in Sect. 4.2. However, by this choice we cannot guarantee that the remaining parts of the solution $\delta^2R_{1,1}$, $\delta^2R_{-1,1}$, $\delta^2R_{s,1}$, and $\delta^3R_{v,1}$ are smaller than the displayed orders w.r.t. δ.

These estimates can be improved by the following procedure. For $t = T_1/\delta^2$ we have

$$
\begin{aligned}
c_1(\delta x,T_1/\delta^2)&=(\delta A_{+,1,0}(\delta x,0)+\delta^2A_{+,1,1}(\delta x,0)+\delta^3A_{+,1,2}(\delta x,0)+\ldots)\\
&\quad+(\delta^2A_{+,2,0}(\delta x,0)+\delta^3A_{+,2,1}(\delta x,0)+\delta^4A_{+,2,2}(\delta x,0)+\ldots)
\end{aligned}
$$

$$
\begin{aligned}
&+(\delta^2 A_{+,0,0}(\delta x,0)+\delta^3 A_{+,0,1}(\delta x,0)+\delta^4 A_{+,0,2}(\delta x,0)+\ldots)\\
&+(\delta^2 A_{+,-2,0}(\delta x,0)+\delta^3 A_{+,-2,1}(\delta x,0)+\delta^4 A_{+,-2,2}(\delta x,0)+\ldots)\\
&+\mathcal{O}(\delta^3).
\end{aligned}
$$

We set

$$A_{+,1,0}(\delta x,0)=\delta^{-1}c_1(\delta x,T_1/\delta^2).$$

By this choice and the construction of the improved approximation in Sect. 4.2 we obtain initial conditions for $\delta^2 A_{+,2,0}(\delta x,0)$, $\delta^2 A_{+,0,0}(\delta x,0)$, and $\delta^2 A_{+,-2,0}(\delta x,0)$. Therefore, for a cancelation of the $\mathcal{O}(\delta^2)$-terms we set

$$\delta^2 A_{+,1,1}(\delta x,0)=-(\delta^2 A_{+,2,0}(\delta x,0)+\delta^2 A_{+,0,0}(\delta x,0)+\delta^2 A_{+,-2,0}(\delta x,0)).$$

Similarly, by the choice of $A_{+,1,0}(\delta x,0)$ and $B_{0,0}(\delta x,0)$ higher order $\mathcal{O}(\delta^{m+2})$-terms are determined. The $A_{+,1,m}(\delta x,0)$ can then be used to adjust the initial conditions at order $\mathcal{O}(\delta^{m+2})$.

Next we consider the B-equation. There we have

$$
\begin{aligned}
v(\delta x,T_1/\delta^2)=&(\delta^2 B_{0,0}(\delta x,0)+\delta^3 B_{0,1}(\delta x,0)+\delta^4 B_{0,2}(\delta x,0)+\ldots)\\
&+(\delta^4 B_{2,0}(\delta x,0)+\delta^5 B_{2,1}(\delta x,0)+\delta^6 B_{2,2}(\delta x,0)+\ldots)\\
&+(\delta^4 B_{-2,0}(\delta x,0)+\delta^5 B_{-2,1}(\delta x,0)+\delta^6 B_{-2,2}(\delta x,0)+\ldots)\\
&+\mathcal{O}(\delta^5).
\end{aligned}
$$

We set

$$B_{0,0}(\delta x,0)=\delta^{-2}v(\delta x,T_1/\delta^2).$$

By this choice, the choice of $A_{+,1,0}(\delta x,0)$, and the construction of the improved approximation in Sect. 4.2 we obtain initial conditions for $\delta^4 B_{2,0}(\delta x)$, $\delta^4 B_{-2,0}(\delta x)$, etc.. The $B_{0,m}(\delta x)$ can then be used to adjust the initial conditions at order $\mathcal{O}(\delta^{m+2})$.

Finally, we come to the u_s-equation. We have for $t=T_1/\delta^2$ that

$$
\begin{aligned}
u_s(\delta x,T_1/\delta^2)=&(\delta^2 A_{s,2,0}(\delta x,0)+\delta^3 A_{s,2,1}(\delta x,0)+\delta^4 A_{s,2,2}(\delta x,0)+\ldots)\\
&+(\delta^2 A_{s,0,0}(\delta x,0)+\delta^3 A_{s,0,1}(\delta x,0)+\delta^4 A_{s,0,2}(\delta x,0)+\ldots)\\
&+(\delta^2 A_{s,-2,0}(\delta x,0)+\delta^3 A_{s,-2,1}(\delta x,0)+\delta^4 A_{s,-2,2}(\delta x,0)+\ldots)\\
&+\mathcal{O}(\delta^3)+\delta^2 R_{s,0}(\delta x,0).
\end{aligned}
$$

By the choice of $A_{+,1,0}(\delta x,0)$ and $B_{0,0}(\delta x,0)$ the $A_{s,2,0}(\delta x,0)$, $A_{s,0,0}(\delta x,0)$, and $A_{s,-2,0}(\delta x,0)$ are determined. However, in general there is a mismatch between the solution on the left-hand side and the approximation terms on the right-hand side and so we need an initial correction $\delta^2 R_{s,0}(\delta x,0)$ on the right-hand side. Since the linear semigroup $e^{\Lambda_s t}$ decays with some exponential rate the variation of constant formula immediately yields

$$\delta^2 R_{s,0}(\delta x,1/\delta^{1/4})=\mathcal{O}(\delta^3).$$

Then we can go on and adjust the next order initial conditions in the c_1- and v-equation. An iteration of this procedure finally yields the statement of Theorem 5.2.

C Proof of the Approximation Theorem 5.3

We consider (1)–(2) after diagonalization and application of the normal form transformation from Sect. 3.2, i.e., we consider (26)–(28). We introduce the error functions by

$$(\check{u}_{\pm 1}, u_s, v) = (\delta \check{\Psi}_{\pm 1}, \delta^2 \Psi_s, \delta^2 \Psi_v) + (\delta^\theta R_{\pm 1}, \delta^{\theta+1} R_s, \delta^{\theta+1} R_v) \tag{46}$$

where $(\delta \check{\Psi}_{\pm 1}, \delta^2 \Psi_s, \delta^2 \Psi_v)$ are the components of the Ginzburg–Landau approximation for (26)–(28). We look for an $\mathcal{O}(1)$-bound for

$$\|R_{\pm 1}\|_{H^{1,\delta}_{l,u}} + \|R_s\|_{H^{n+1}_{l,u}} + \|R_v\|_{H^n_{l,u}}.$$

on the long $\mathcal{O}(1/\delta^2)$-time scale.

Remark C.1 This choice of norms allows us to use the ∂_x^2 in front of nonlinearity in the v-equation as follows. One ∂_x is transformed into a δ by using the smoothing of the linear semigroup, i.e.,

$$e^{d_v \partial_x^2 t} \partial_x = \mathcal{O}(t^{-1/2}) = \delta \mathcal{O}(T^{-1/2})$$

where $T = \delta^2 t$. The second ∂_x is transformed into a δ by using the estimate

$$\|\partial_x u\|_{H^{m,\delta}_{l,u}} \leq C\delta \|u\|_{H^{m+1,\delta}_{l,u}}.$$

Thus, in sum we obtain a factor δ^2 which allows us to bound the solutions on the long $\mathcal{O}(1/\delta^2)$-time scale.

Inserting the ansatz (46) into (26)–(28) and applying the variation of constant formula gives for the error (R_1, R_s, R_v) that

$$\begin{aligned}
R_1(t) &= e^{\Lambda_u t} E_1 R_1|_{t=0} + \int_0^t e^{\Lambda_u (t-\tau)} E_1 N_1(R(\tau)) d\tau, \\
R_s(t) &= e^{\Lambda_u t} E_s R_s|_{t=0} + \int_0^t e^{\Lambda_u (t-\tau)} E_s N_s(R(\tau)) d\tau, \\
R_v(t) &= e^{\Lambda_v t} R_v|_{t=0} + \int_0^t (e^{\Lambda_v (t-\tau)} \partial_x)(\partial_x N_v(R(\tau))) d\tau,
\end{aligned}$$

with

$$\begin{aligned}
\|N_1(R)\|_{H^{n+1}_{l,u}} &\leq C(\delta^2 \tilde{R} + \delta^3 \tilde{R}^2) + C_{res}\delta^2, \\
\|N_s(R)\|_{H^{n+1}_{l,u}} &\leq C\|R_1\|_{H^{1,\delta}_{l,u}} + C(\delta \tilde{R} + \delta \tilde{R}^2) + C_{res}, \\
\|\partial_x N_v(R)\|_{H^n_{l,u}} &\leq C\delta \|R_1\|_{H^{1,\delta}_{l,u}} + C(\delta \tilde{R} + \delta^2 \tilde{R}^2) + C_{res}\delta^2
\end{aligned}$$

where $\tilde{R} = \tilde{R}(t)$ is defined by

$$\tilde{R}(t) := \|R_1(t)\|_{H^{1,\delta}_{l,u}} + \|R_s(t)\|_{H^{n+1}_{l,u}} + \|R_v(t)\|_{H^n_{l,u}}, \tag{47}$$

and where C_{Res} stands for the $\mathcal{O}(1)$-constants coming from the residual terms.

In the following C_{IR} denotes $\mathcal{O}(1)$-constants which are obtained when integrating the residual terms or $\mathcal{O}(1)$-constants coming from the initial conditions. We obtain

$$\|R_1(t)\|_{H^{1,\delta}_{l,u}} \leq C_{\mathrm{IR}} + \int_0^t C\Big(\frac{\delta}{\sqrt{t-\tau}} + \delta^2\Big)(\tilde{R}(\tau) + \delta \tilde{R}(\tau)^2)\, \mathrm{d}\tau,$$

$$\begin{aligned}\|R_s(t)\|_{H^{n+1}_{l,u}} &\leq C_{\mathrm{IR}} + \int_0^t Ce^{-\sigma(t-\tau)}\left(1+(t-\tau)^{-1/2}\right) \\ &\quad \times(\|R_1(\tau)\|_{H^{1,\delta}_{l,u}} + \delta(\tilde{R}(\tau)+\tilde{R}(\tau)^2)\,\mathrm{d}\tau, \\ \|R_v(t)\|_{H^n_{l,u}} &\leq C_{\mathrm{IR}} + \int_0^t \frac{C\delta}{\sqrt{t-\tau}}\left(\tilde{R}(\tau)+\delta\tilde{R}(\tau)^2\right)\mathrm{d}\tau\end{aligned}$$

using Lemma A.1. Next we introduce

$$\begin{aligned}q_c(t) &= \sup_{\tau\in[0,t]} \|R_1(\tau)\|_{H^{1,\delta}_{l,u}}, \\ q_s(t) &= \sup_{\tau\in[0,t]} \|R_s(\tau)\|_{H^{n+1}_{l,u}}, \\ q_v(t) &= \sup_{\tau\in[0,t]} \|R_v(\tau)\|_{H^n_{l,u}}.\end{aligned}$$

We immediately obtain

$$q_s(t) \leq C_{\mathrm{IR}} + Cq(t) + C\delta((q(t)+q_s(t)) + (q(t)+q_s(t))^2),$$

where

$$q(t) = q_c(t) + q_v(t).$$

For $C\delta(1+(q(t)+q_s(t))) \leq 1/2$ this yields $q_s(t) \leq C(q(t)+C_{\mathrm{IR}})$ and then as a consequence

$$\begin{aligned}q_c(t) &\leq CC_{\mathrm{IR}} + \int_0^t C\left(\frac{\delta}{\sqrt{t-\tau}}+\delta^2\right)(q(\tau)+\delta q(\tau)^2)\,\mathrm{d}\tau, \\ q_v(t) &\leq CC_{\mathrm{IR}} + \int_0^t \frac{C\delta}{\sqrt{t-\tau}}(q(\tau)+\delta q(\tau)^2)\,\mathrm{d}\tau.\end{aligned}$$

Adding these two inequalities yields

$$\begin{aligned}q(t) &\leq CC_{\mathrm{IR}} + \int_0^t C\left(\frac{\delta}{\sqrt{t-\tau}}+\delta^2\right)(q(\tau)+\delta q(\tau)^2)\,\mathrm{d}\tau \\ &\leq CC_{\mathrm{IR}} + \int_0^t 2C\left(\frac{\delta}{\sqrt{t-\tau}}+\delta^2\right)q(\tau)\,\mathrm{d}\tau\end{aligned}$$

if $\delta q(\tau) \leq 1$. With $T = \delta^2 t$ and $\tilde{q}(T) = q(t)$ this can be written as

$$\tilde{q}(T) \leq CC_{\mathrm{IR}} + \int_0^T 2\left(\frac{C}{\sqrt{T-\tilde{\tau}}}+1\right)\tilde{q}(\tilde{\tau})\,\mathrm{d}\tilde{\tau}.$$

Since this equation is independent of δ, Gronwall's inequality immediately yields the existence of a constant $M_q = \mathcal{O}(1)$ such that

$$\sup_{T\in[0,T_0]} \tilde{q}(T) =: M_q < \infty$$

or equivalently

$$\sup_{t\in[0,T_0/\delta^2]} q(t) = M_q < \infty.$$

Then

$$q_s(t) \leq M_s := C(C_{\mathrm{IR}} + M_q).$$

Choosing $\delta_0 > 0$ so small that $\delta_0 M_q \leq 1$ and $C\delta_0(1 + M_q + M_s) \leq 1/2$ we proved the error estimates stated in Theorem 5.3. □

D Proof of Theorem 2.6

In the following $L = \mathcal{O}(1)$ is fixed. Therefore, it is not a problem that the subsequent estimates depend very badly on L for $L \to \infty$. For B, with vanishing mean value, we have Poincaré's inequality

$$\left(\int_0^L |\partial_X^{-1} B|^2 dX \right)^{1/2} \leq \frac{L}{2\pi} \left(\int_0^L |B|^2 dX \right)^{1/2}, \tag{48}$$

where $\partial_X^{-1} B$ is defined via its Fourier transform $\widehat{B}(k)/ik$ using $\widehat{B}(0) = 0$. Since $\mathrm{Re} \int_0^L i\gamma_0 |\partial_X A|^2 dX = 0$ and $\mathrm{Re} \int_0^L i\gamma_3 |A|^4 dX = 0$ we find

$$\frac{1}{2}\frac{d}{dT} \int_0^L |A|^2 dX = \int_0^L -|\partial_X A|^2 + |A|^2 - |A|^4 + \beta B|A|^2 dX,$$

$$\frac{1}{2}\frac{d}{dT} \int_0^L |\partial_X^{-1} B|^2 dX = \int_0^L -\alpha|B|^2 - B|A|^2 dX.$$

In case $\beta > 0$ we estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dT} \int_0^L |A|^2 + \beta|\partial_X^{-1} B|^2 dX \\
&\leq \int_0^L -|\partial_X A|^2 + |A|^2 - |A|^4 + \beta B|A|^2 - \alpha\beta|B|^2 - \beta B|A|^2 dX \\
&\leq \int_0^L |A|^2 - |A|^4 - \alpha\beta|B|^2 dX \\
&\leq \int_0^L 1 - |A|^2 - (2\pi)^2 L^{-2} \alpha\beta |\partial_X^{-1} B|^2 dX,
\end{aligned}$$

where we have used (48). Thus, we find

$$\limsup_{T\to\infty} \int_0^L |A|^2 + \beta|\partial_X^{-1} B|^2 dX \leq L \max\left(1, \frac{L^2}{(2\pi)^2\alpha}\right) =: C_{\infty,0}.$$

For estimating the higher order derivatives we keep some of the negative terms in the above calculations. Doing so, we also find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dT} \int_0^L |A|^2 + \beta|\partial_X^{-1} B|^2 dX \\
&\leq \int_0^L -|\partial_X A|^2 - \alpha\beta|B|^2 + 1 - |A|^2 - (2\pi)^2 L^{-2}\alpha\beta|\partial_X^{-1} B|^2 dX.
\end{aligned} \tag{49}$$

Next we compute

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dT} \int_0^L |\partial_X A|^2 + \beta|B|^2 dX \\
&\leq \int_0^L -|\partial_X^2 A|^2 + |\partial_X A|^2 - 2|A|^2|\partial_X A|^2 - \mathrm{Re}((1 + i\gamma_3)A^2(\partial_X \overline{A})^2)
\end{aligned}$$

$$
\begin{aligned}
&+\beta(\partial_X\overline{A})\partial_X(BA)+\beta(\partial_X A)\partial_X(B\overline{A})-\alpha\beta|\partial_X B|^2-\beta(\partial_X B)\partial_X|A|^2dX\\
\leq &\int_0^L -|\partial_X^2 A|^2+|\partial_X A|^2-\alpha\beta|\partial_X B|^2\\
&-\mathrm{Re}((1+i\gamma_3)A^2(\partial_X\overline{A})^2)+2\beta B|\partial_X A|^2dX.
\end{aligned}
$$

We add γ times the inequality (49) to the last inequality. and use that we already know that $\int_0^L |A|^2dX$ and $\int_0^L |\partial_X^{-1}B|^2dX$ are bounded. On the right hand side of the new inequality for $\gamma > 1$ we have the negative terms

$$
-(\gamma-1)\int_0^L |\partial_X A|^2dX,\quad -\int_0^L \alpha\beta\gamma|B|^2dX/2,\quad \text{and}\quad -\int_0^L |\partial_X^2 A|^2dX
$$

which we use to estimate the remaining non-negative terms on the right hand side of the new inequality.

Using Young's inequality, an interpolation inequality for $\|\partial_X A\|^2_{C_b^0}$, that $\int_0^L |A|^2dX \leq C$ and $\int_0^L |\partial_X^{-1}B|^2dX \leq C$ for a $C > 0$ uniformly in time, we estimate for every $\delta > 0$ that

$$
\begin{aligned}
\left|\int_0^L \mathrm{Re}((1+i\gamma_3)A^2(\partial_X\overline{A})^2)dX\right| &\leq (1+|\gamma_3|)\|A\|^2_{L^2}\|\partial_X A\|^2_{C_b^0}\\
&\leq C\|\partial_X A\|_{L^2}\|\partial_X^2 A\|_{L^2}\\
&\leq \frac{1}{2\delta}C^2\|\partial_X A\|^2_{L^2}+\frac{\delta}{2}\|\partial_X^2 A\|^2_{L^2},\\
\left|\int_0^L B|\partial_X A|^2dX\right| &= \left|\int_0^L |(\partial_X^{-1}B)\partial_X(|\partial_X A|^2)dX\right|\\
&\leq 2\|\partial_X^{-1}B\|_{L^2}\|\partial_X^2 A\|_{L^2}\|\partial_X A\|_{C_b^0}\\
&\leq C\|\partial_X^2 A\|^{3/2}_{L^2}\|\partial_X A\|^{1/2}_{L^2}\\
&\leq \frac{6}{\delta}C^{4/3}\|\partial_X A\|^2_{L^2}+\frac{\delta}{2}\|\partial_X^2 A\|^2_{L^2}.
\end{aligned}
$$

Therefore, by choosing γ sufficiently large, in case of periodic boundary conditions, we have established a-priori estimates for $A \in H^1$ and $B \in L^2$. Since we also have local existence and uniqueness in these spaces for (18)–(19) global existence in $H^1 \times L^2$ follows, too. The global existence for $A \in H^{s+1}$ and $B \in H^s$ follows by using the smoothing properties of the diffusion semigroup.

In case $\beta \leq 0$ with $1+\alpha^{-1}\beta > 0$ we proceed similarly. However, there is no cancelation and so we compute

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dT}\int_0^L |A|^2+q|\partial_X^{-1}B|^2dX\\
&\quad\leq \int_0^L -|\partial_X A|^2+|A|^2-|A|^4+\beta B|A|^2-\alpha q|B|^2-qB|A|^2dX\\
&\quad\leq \int_0^L |A|^2-r|A|^4-r\alpha\beta|B|^2dX
\end{aligned}
$$

under the assumption that we can establish an estimate

$$
(q-\beta)B|A|^2 \leq (1-r)\alpha q|B|^2+(1-r)|A|^4 \qquad (50)
$$

for an $r \in (0, 1]$. If we have established such an estimate we can proceed as above to establish the global existence of solutions However, the constant $C_{\infty,0}$ has to be modified since we no longer have $r = 1$. A simple calculation shows that the required estimate (50) can be established for α, β satisfying $1 + \alpha^{-1}\beta > 0$ if $r > 0$ is chosen sufficiently small and $q = 2\alpha + \beta$. We refrain from optimizing the bound around $\beta = 0$. □

Remark D.1 In case of periodic boundary conditions the H^s-space can be embedded in $H^s_{l,u}$. Together with the smoothing in case of periodic boundary conditions we have established the existence of an absorbing ball for spatially periodic $A \in H^{s+1}_{l,u}$ and $B \in H^s_{l,u}$, too.

Remark D.2 Dropping the periodic boundary conditions for the problem on the real line the global existence question remains an open problem.

Remark D.3 We expect that the condition $1 + \alpha^{-1}\beta > 0$ is sharp. The reason is as follows. In case $\gamma_0 = \gamma_3 = 0$ stationary solutions can be obtained by a simple integration of the conservation law (19) giving $\alpha B = -|A|^2 + b$, where $b \in \mathbb{R}$ is an arbitrary constant. Inserting this into (18) yields

$$0 = \partial_X^2 A + (1 + \alpha^{-1}\beta b)A - (1 + \alpha^{-1}\beta)A|A|^2.$$

Hence, the coefficient in front of the effective nonlinear terms is only negative for $1+\alpha^{-1}\beta > 0$. See also [6].

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