Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-024-10372-9

ORIGINAL PAPER

Semiflows Strongly Focusing Monotone with Respect to High-Rank Cones: I. Generic Dynamics

Lirui Feng[1]

Received: 26 April 2023 / Revised: 28 November 2023 / Accepted: 16 May 2024

Abstract
We consider a smooth semiflow strongly focusing monotone with respect to a cone of rank k on a Banach space. We obtain its generic dynamics, that is, semiorbits with initial data from an open and dense subset of any open bounded set either are pseudo-ordered or convergent to an equilibrium. For the case $k = 1$, it is the celebrated Hirsch's Generic Convergence Theorem. For the case $k = 2$, we obtain the generic Poincaré-Bendixson Theorem.

Keywords Strongly focusing monotone · High-rank cones · k-Exponential separation · k-Lyapunov exponents · Generic dynamics · Generic Poincaré-Bendixson theorem

1 Introduction

We investigate the generic dynamics of semiflows strongly focusing monotone with respect to a cone C of rank k on an infinite dimensional Banach space X. Roughly speaking, a cone of rank k (abbr. k-cone) is a closed subset of X containing a subspace of dimension k but no subspace of higher dimension, which is introduced by Krasnosel'skij, Lifshits and Sobolev [16] to obtain a Krein–Rutman theory on a Banach space, and also in the poineering works of Fusco and Oliva [5, 6] on the finite dimensional space. A convex cone K gives rise to a 1-cone, $K \cup (-K)$. Therefore, the class of semiflows strongly monotone with respect to k-cones includes the classical monotone semiflows originating from the groundbreaking works of Hirsch (see [8–14]). Due to the lack of convexity in $k(k \geqslant 2)$-cones, it is a very challenging task to study the behaviors of this general class of systems strongly monotone with respect to k-cones. Despite some progress being made (see [1–4]), their dynamics is far from being understood. Here, we introduce a slightly stronger property than strong monotonicity with respect to k-cones, that is, strongly focusing monotonicity with respect to k-cones, and study typical behaviors of most semiorbits in the topological sense in this paper.

A strongly focusing operator with respect to a k-cone C originated from Krasnosel'skij et.al [16] to prove a Krein–Rutman type theorem with respect to k-cones for a single operator, and also from Lian and Wang [19] to investigate the relationship between Multiplicative Ergodic Theorem and Krein–Rutman type Theorem for random linear dynamical systems.

✉ Lirui Feng
ruilif@ustc.edu.cn

1 School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, Anhui, People's Republic of China

Roughly speaking, the image of C for a strongly focusing operator is a subset of C such that unite vectors contained in it are uniformly separated from the boundary of C (see Definition 2.1(ii)). We should point out that any strongly positive operator R with respect to C (see Definition 2.1(i)) on a finite dimensional space is strongly focusing. Therefore, the smooth semiflows strongly focusing monotone with respect to the k-cone C on an infinite dimensional space is a kind of natural extension of the smooth flows with respect to k-cones on a finite dimensional space (refer to the flows in [2]). More precisely, this class of semiflows strongly monotone with respect to the k-cone C satisfies that for each compact invariant set Σ, one can find constants $\delta, T, \kappa > 0$ such that there is a strongly focusing operator $T_{(x,y)}$ with separation index greater than κ such that $T_{(x,y)}(x-y) = \Phi_T(x) - \Phi_T(y)$ for any $z \in \Sigma$ and $x, y \in B_\delta(z)$, where $B_\delta(z)$ is the ball centred at z with radius δ (see also Definition 2.2(ii)). This class of semiflows would have significant potential applications to the study of dynamics of nonlinear evolution equations.

There are exactly two types of nontrivial positive semiorbits for the semiflow Φ_t monotone with respect to the k-cone C (or a convex cone K): pseudo-ordered semiorbits and unordered semiorbits. A nontrivial positive semiorbit $O^+(x) = \{\Phi_t(x) : t \geq 0\}$ is pseudo-ordered if it contains a pair of different ordered points $\Phi_\tau(x)$ and $\Phi_s(x)$, i.e. $\Phi_\tau(x) - \Phi_s(x) \in C \setminus \{0\}$ (or $\Phi_\tau(x) - \Phi_s(x) \in K \setminus \{0\}$); otherwise, it is called unordered.

For the classical (in the sense of Hirsch's) monotone systems, the order reduced by a convex cone is a partial order relationship. Base on this fact, Monotone Convergence Criterion, the first key building block in Hirsch's theory, can be established. It is to say that every precompact pseudo-ordered semiorbit converges to an equilibrium. The partial order also plays an important role in the further developments from Monotone Convergence Criterion, that includes Nonordering of Limit Sets and Limit Set Dichotomy. These results consist of the key building blocks (see [24, Theorem 2.1, p.491]) for establishing Hirsch's Generic Convergence Theorem.

Compared with classical monotone systems, the order reduced by k-cones is a symmetric relationship, that causes the structure of the omega-limit set $\omega(x)$ of a pseudo-ordered semiorbit $O^+(x)$ is more complicated and new techniques are needed to analyze dynamics of the semiflows monotone with respect to high-rank cones. Sanchez [22] firstly treated the problem on the structure of the omega-limit set $\omega(x)$ of a pseudo-ordered orbit for flows on $\mathbb{R}^n$ strongly monotone with respect to a $k(k \geq 2)$-cone C. He used the C^1-closed lemma to prove that any orbit in $\omega(x)$ of a pseudo-ordered orbit is ordered, that is, the difference of any two points in any given orbit in $\omega(x)$ is in C; and further obtain a Poincaré-Bendixson theorem, that is, the omega-limit set $\omega(x)$ of a pseudo-ordered orbit containing no equilibrium is a closed orbit. For the total-ordering property of the entire set $\omega(x)$, he [22, p.1984] posed it as an open problem. In our previous work [1], we creatively utilized topological properties of continuous semiflows to study the total-ordering property for continuous semiflows strongly monotone with respect to a k-cone in a general Banach space and obtained the Order-Trichotomy (see [1, Theorem B]) for the omega-limit set $\omega(x)$ of a pseudo-ordered semiorbit. More precisely, we proved that either (a) $\omega(x)$ is ordered; or (b) $\omega(x)$ is an unordered set consisting of equilibria; or otherwise, (c) $\omega(x)$ possesses a certain ordered homoclinic property. In our previous work [1, Theorem A and C], we extended Sanchez's results to semiflows only continuous on an infinite dimensional space.

For semiflows strongly monotone with respect to high-rank cones, the symmetry of the order and the complexity of an omega-limit set Ω cause that it is difficult to reappear the key building blocks in Hirsch's theory. To treat the generic dynamics of flows Φ_t strongly monotone with respect to a k-cone on $\mathbb{R}^n$, we turned to analyze the local dynamics for each type of omega-limit set Ω in our previous works [2], where the types are classified by our approach of smooth ergodic arguments. More precisely, the linear skew-product flow $(\Phi_t, D\Phi_t)$ admits k-exponential separation along Ω associated with the k-cone C provided by the strong positivity of $D_x\Phi_t$ for any $x \in \Omega$ and $t > 0$. Roughly speaking, this property describes that there exist k-dimensional invariant subbundle $\Omega \times (E_x)$ and k-codimensional invariant subbundle $\Omega \times (F_x)$ with respect to $(\Phi_t, D\Phi_t)$ such that $\Omega \times \mathbb{R}^n = \Omega \times (E_x) \oplus \Omega \times (F_x)$; and more, the action of $(\Phi_t, D\Phi_t)$ on $\Omega \times (E_x)$ dominates the one on $\Omega \times (F_x)$ as $t \to \infty$ (see Definition 2.3 or its versions for random dynamics in [18, 19]). The related crucial tool is the k-Lyapunov exponent λ_{kx} of $x \in \Omega$ (defined as $\lambda_{kx} = \limsup_{t\to+\infty} \frac{\log m(D_x\Phi_t|_{E_x})}{t}$, see also Definition 2.4 and (2.1)), which describes the action's growth rate of $(\Phi_t, D\Phi_t)$ on the k-dimensional subbundle $\Omega \times (E_x)$. The theory on Lyapunov exponents and Multiplicative Ergodic Theorem ensures [7, 17–21, 26] that λ_{kx} is actually the limit for "most" points $x \in \Omega$; such points for which λ_{kx} is the limit are said to be regular and other points are said to be irregular. According to the sign of the k-Lyapunov exponents of the regular/irregular points on any given omega-limit set Ω, three are three types of Ω: (i) $\lambda_{kx} > 0$ for any point x in Ω; (ii) $\lambda_{kx} > 0$ for any regular point $x \in \Omega$ and $\lambda_{kz} \leq 0$ for some irregular point $z \in \Omega$; (iii) $\lambda_{kx} \leq 0$ for some regular point $x \in \Omega$. By discussing the local behaviors of each type of omega-limit sets, we obtain the finite dimensional version of Generic dynamics theorem (see [2, Theorem A]); and further combinate it with the Poincaré-Bendixson theorem (see Lemma 2.6 and also [1, Theorem C]) of the omega-limit set of a pseudo-ordered semiorbit to get the generic Poincaré-Bendixson theorem (see [2, Theorem B]) on $\mathbb{R}^n$.

Our purpose in this paper is to investigate the infinite dimensional version of generic dynamics of semiflow Φ_t strongly focusing monotone with respect to a k-cone C on a Banach space. We prove that

- For generic (open and dense) positive semiorbits either are pseudo-ordered or converge to an equilibrium.
- Whenever $k = 2$, for generic points, the omega-limit set containing no equilibrium is a periodic orbit.

By the strong positivity of $D_x\Phi_t$ in Definition 2.2(i), for each compact invariant set Σ on which Φ_t admits a flow extension, the linear skew-product semiflow $(\Phi_t, D\Phi_t)$ admits k-exponential separation $\Sigma \times X = \Sigma \times (E_x) \oplus \Sigma \times (F_x)$ along Σ associated with C. Here, Φ_t is said to *admit a flow extension on* Σ, if there is a flow $\tilde{\Phi}_t$ such that $\tilde{\Phi}_t(x) = \Phi_t(x)$ for any $x \in \Sigma$ and $t \geq 0$. Since the unite ball in an infinite dimensional Banach space is lack of compactness, unite vectors in the k-codimensional invariant subbundle $\Sigma \times (F_x)$ with respect to $(\Phi_t, D\Phi_t)$ are not uniformly far away from the boundary ∂C of the k-cone C. The method in [2] is not effective to estimate the proportion of the projection onto the fibres E_x and F_x with $x \in \Sigma$ for a nonzero vector $v \in C$ in an infinite dimensional space. We turn to estimate the proportion of the projection onto the fibres E_x and F_x with $x \in \Sigma$ for the difference $\Phi_t(\tilde{x}) - \Phi_t(\tilde{y})$ of a pair of ordered distinct points $\Phi_t(\tilde{x})$ and $\Phi_t(\tilde{y})$ by utilizing the strongly focusing monotonicity in Definition 2.2(ii). By this novel approach, we analyze the local dynamical features for each type of omega-limit sets and furthermore deduce the infinite dimensional version of generic dynamics. For case $k = 2$, the generic Poincaré-Bendixson theorem are also obtained.

The paper is organized as follows. In Sect. 2, we give some notations and summarize the preliminary results. In Sect. 3, we present the main results on the infinite dimensional version of Generic dynamics and generic Poincaré-Bendixson theorem for semiflows strongly focusing monotone with respect to the k-cone C. In Sect. 4, we discuss the local behaviors for each type of omega-limit sets. In Sect. 5, we prove our main results.

2 Notations and Preliminary Results

In this section, we give some preliminary knowledge to be used in the next sections. We start with basic nations and definitions on semiflows strongly focusing monotone with respect to high-rank cones. We then introduce the k-exponential separation and k-Lyapunov exponents with some crucial properties of them.

2.1 Semiflows Strongly Monotone with Respect to High-Rank Cones

Let $(X, \|\cdot\|)$ be a Banach space equipped with a norm $\|\cdot\|$. A *semiflow* on X is a continuous map $\Phi : \mathbb{R}^+ \times X \to X$ with $\Phi_0 = \mathrm{Id}$ and $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for $t, s \geq 0$. Here, $\mathbb{R}^+ = [0, +\infty)$, $\Phi_t(\cdot) = \Phi(t, \cdot)$ for $t \geq 0$, and Id is the identity map on X. A semiflow Φ_t on X is called $C^{1,\alpha}$*-smooth* if $\Phi|_{\mathbb{R}^+ \times X}$ is a $C^{1,\alpha}$-map (a C^1-map with a locally α-Hölder derivative) with $\alpha \in (0, 1]$. The derivative of Φ_t with respect to x, at (t, x), is denoted by $D_x\Phi_t$.

Let $x \in X$, the *positive semiorbit of* x is denoted by $O^+(x) = \{\Phi_t(x) : t \geq 0\}$. A *negative semiorbit* (resp. *full-orbit*) of x is a continuous function $\psi : \mathbb{R}^- = \{t \in \mathbb{R} | t \leq 0\} \to X$ (resp. $\psi : \mathbb{R} \to X$) such that $\psi(0) = x$ and, for any $s \leq 0$ (resp. $s \in \mathbb{R}$), $\Phi_t(\psi(s)) = \psi(t + s)$ holds for $0 \leq t \leq -s$ (resp. $0 \leq t$). Clearly, if ψ is a negative semiorbit of x, then ψ can be extended to a full-orbit $\tilde{\psi}(t)$ such that $\tilde{\psi}(t) = \psi(t)$ for $t \leq 0$ and $\tilde{\psi}(t) = \Phi_t(x)$ for $t \geq 0$. On the other hand, any full orbit of x when restricted on $\mathbb{R}^-$ is a negative semiorbit of x. Since Φ_t is just a semiflow, a negative semiorbit of x may not exist, and it is not necessary to be unique even if one exists.

An *equilibrium* (also called *a trivial orbit*) is a point x for which $O^+(x) = \{x\}$. Let E be the set of all equilibria w.r.t. Φ_t. A nontrivial positive semiorbit $O^+(x)$ is said to be a *periodic orbit* if $\Phi_T(x) = x$ for a $T > 0$. The nontrivial semiorbit $O^+(x)$ is said to be a T*-periodic orbit* if there is a $T > 0$ such that $\Phi_T(x) = x$ and $\Phi_t(x) \neq x$ for any $t \in (0, T)$, where T is called the minimal period of $O^+(x)$.

A subset $\Sigma \subset X$ is called *positively invariant with respect to* Φ_t *(for short, positively invariant)* if $\Phi_t(\Sigma) \subset \Sigma$ for any $t \in \mathbb{R}^+$, and is called *invariant* if $\Phi_t(\Sigma) = \Sigma$ for any $t \in \mathbb{R}^+$. Clearly, for any $x \in \Sigma$, there exists a negative semiorbit of x, provided that Σ is invariant. Let $\Sigma \subset X$ be an invariant set. Φ_t is said to *admit a flow extension on* Σ, if there is a flow $\tilde{\Phi}_t$ such that $\tilde{\Phi}_t(x) = \Phi_t(x)$ for any $x \in \Sigma$ and $t \geq 0$.

The omega-limit (abbr. ω*-limit*) *set* $\omega(x)$ of $x \in X$ is defined by $\omega(x) = \cap_{s \geq 0} \overline{\cup_{t \geq s} \Phi_t(x)}$. If $O^+(x)$ is precompact, then $\omega(x)$ is nonempty, compact, connected and invariant. Given a subset $D \subset X$, *the positive semiorbit* $O^+(D)$ *of* D is defined as $O^+(D) = \bigcup_{x \in D} O^+(x)$. A subset D is called ω*-compact* if $O^+(x)$ is precompact for each $x \in D$ and $\bigcup_{x \in D} \omega(x)$ is precompact. Clearly, D is ω-compact provided by the compactness of $\overline{O^+(D)}$.

A closed set $C \subset X$ is called a cone of rank-k (*abbr.* k*-cone*) if

(i) For any $v \in C$ and $l \in \mathbb{R}$, $lv \in C$;
(ii) $\max\{\dim W : C \supset W \text{ linear subspace}\} = k$.

Moreover, the integer $k (\geq 1)$ is called the rank of C. A k-cone $C \subset X$ is said to be *solid* if its interior $\mathrm{Int}C \neq \emptyset$; and C is called k*-solid* if there is a k-dimensional linear subspace W such that $W \setminus \{0\} \subset \mathrm{Int}C$. Given a k-cone $C \subset X$, we say that C is *complemented* if there exists a k-codimensional subspace $H^c \subset X$ such that $H^c \cap C = \{0\}$. For two points $x, y \in X$, we call that x *and* y *are ordered, denoted by* $x \sim y$, if $x - y \in C$. Otherwise, x, y are called to be *unordered*, denoted by $x \rightharpoondown y$. The pair $x, y \in X$ are said to be *strongly ordered*, denoted by $x \approx y$, if $x - y \in \mathrm{Int}C$. A nonempty set $W \subset X$ is called *ordered* if $x \sim y$ for any $x, y \in W$ and it is called *(resp. strongly ordered) unordered* if it is not a singleton and (resp. $x \approx y$) $x \rightharpoondown y$ for any two distinct points $x, y \in W$.

Let $d(x, y) = \|x - y\|$ for any $x, y \in X$ and $d(x, B) = \inf_{y \in B} d(x, y)$ for any $x \in X$, $B \subset X$.

Throughout this paper, we assume C is a complemented k-solid cone and Φ_t with compact x-derivative $D_x\Phi_t$ for any $x \in X$ and $t > 0$ admits a flow extension on each nonempty omega-limit set $\omega(x)$.

Definition 2.1 (i) A linear operator $R \in L(X)$ is called *strongly positive with respect to C* if $R\left(C \setminus \{0\}\right) \subset \mathrm{Int}C$.

(ii) A linear operator $R \in L(X)$ is called *strongly focusing with respect to C* if $0 \notin R(C \setminus \{0\})$ and there is a $\kappa > 0$ such that

$$\underline{\mathrm{dist}}(RC, X \setminus C) = \kappa,$$

where $\underline{\mathrm{dist}}(L_1, L_2)$ is the separation index between set L_1 and L_2 defined by

$$\underline{\mathrm{dist}}(L_1, L_2) = \inf_{v \in L_1, \|v\|=1} \{ \inf_{u \in L_2} \|v - u\|\}.$$

Here, κ is also called the separation index of R.

Remark 2.1 (i) A strongly focusing operator is automatically a strongly positive operator.

(ii) Let R be a strongly positive operator w.r.t. C on $\mathbb{R}^n$. Then, R is also strongly focusing w.r.t. C.

A semiflow Φ_t on X is called *monotone with respect to C* if

$$\Phi_t(x) \sim \Phi_t(y) \ \text{ whenever } x \sim y \text{ and } t \geq 0;$$

and Φ_t is called *strongly monotone with respect to C* if Φ_t is monotone with respect to C and

$$\Phi_t(x) \approx \Phi_t(y) \ \text{ whenever } x \neq y, x \sim y \text{ and } t > 0.$$

A nontrivial positive semiorbit $O^+(x)$ is called *pseudo-ordered* (also called *of Type-I*), if there exist two distinct points $\Phi_{t_1}(x), \Phi_{t_2}(x)$ in $O^+(x)$ such that $\Phi_{t_1}(x) \sim \Phi_{t_2}(x)$. Otherwise, $O^+(x)$ is called *unordered* (also called *of Type-II*). Hereafter, we let

$$Q = \{x \in X : O^+(x) \text{ is pseudo-ordered}\}.$$

Definition 2.2 A semiflow Φ_t is called strongly focusing monotone with respect to C, if it satisfies:

(i) It is C^1-smooth and strongly monotone with respect to C such that the x-derivative $D_x\Phi_t$ of Φ_t ($t > 0$) is strongly positive with respect to C for any $x \in X$;

(ii) For each compact invariant set Σ with respect to Φ_t, one can find constants $\delta, T, \kappa > 0$ such that there exists a strongly focusing operator $T_{(x,y)}$ with separation index greater than κ such that $T_{(x,y)}(x - y) = \Phi_T(x) - \Phi_T(y)$ for any $z \in \Sigma$ and $x, y \in B_\delta(z)$, where $B_\delta(z)$ is the closed ball centred at z with radius δ.

Remark 2.2 Let Φ_t be a C^1-smooth flow strongly monotone w.r.t. C on $\mathbb{R}^n$, whose x-derivative $D_x\Phi_t$ is strongly positive w.r.t. C for any $x \in \mathbb{R}^n$ and $t > 0$. Then, Φ_t is strongly focusing monotone w.r.t. C.

Remark 2.3 The condition "for each compact invariant set Σ" in the strongly focusing monotonicity in Definition 2.2(ii) can be relaxed and becomes "for each omega-limit set $\omega(x)$" in the proof of the results in this paper.

Remark 2.4 Let $\tilde{\Sigma} = \mathrm{Co}\{B_\delta(\Sigma)\} \times \mathrm{Co}\{B_\delta(\Sigma)\}$. Here, $B_\delta(\Sigma) = \{v \in X : d(v, \Sigma) \le \delta\}$ and $\mathrm{Co}\{B_\delta(\Sigma)\}$ is the convex hull of $B_\delta(\Sigma)$. Let $T_{(x,y)} = \int_0^1 D_{y+s(x-y)}\Phi_T ds$ for any $(x, y) \in \tilde{\Sigma}$. Then, one has $T_{(x,y)}(x-y) = \Phi_T(x) - \Phi_T(y)$. Let $\kappa > 0$. Compared with Definition 2.2(ii), the following condition has more restriction.

$$\{T_{(x,y)}\}_{(x,y)\in\tilde{\Sigma}} \text{ consists of strongly focusing operators with separation index greater than } \kappa. \tag{*}$$

Now, we give several useful results on semiflows strongly monotone with respect to C.

Lemma 2.5 *Assume that Φ_t is strongly monotone with respect to C. If $x \sim y$ and there is a sequence $t_n \to \infty$ such that $\Phi_{t_n}(x) \to z$ and $\Phi_{t_n}(y) \to z$, then $O(z)$ is nontrivial and ordered, or z is an equilibrium.*

Proof See [1, Lemma 4.3]. □

Lemma 2.6 *Assume that Φ_t is strongly monotone with respect to the k-cone C with $k = 2$ and $O^+(x)$ be a precompact pseudo-ordered semiorbit. If $\omega(x) \cap E = \emptyset$, $\omega(x)$ is a periodic orbit.*

Proof See [1, Theorem C]. □

2.2 *k*-Exponential Separation and *k*-Lyapunov Exponents

Let $G(k, X)$ be *the Grassmanian of k-dimensional linear subspaces of X*, which consists of all k-dimensional linear subspaces in X. $G(k, X)$ is a completed metric space by endowing *the gap metric* (see, for example, [15, 17]). More precisely, for any nontrivial closed subspaces $L_1, L_2 \subset X$, define that

$$d(L_1, L_2) = \max\left\{\sup_{v\in L_1\cap S}\inf_{u\in L_2\cap S}\|v-u\|, \sup_{v\in L_2\cap S}\inf_{u\in L_1\cap S}\|v-u\|\right\},$$

where $S = \{v \in X : \|v\| = 1\}$ is the unit sphere. For a solid k-cone $C \subset X$, we denote by $\Gamma_k(C)$ the set of k-dimensional subspaces inside C, that is,

$$\Gamma_k(C) = \{L \in G(k, X) : L \subset C\}.$$

Let $\Sigma \subset X$ be a compact invariant subset w.r.t. Φ_t on which Φ_t admits a flow extension. We consider the linear skew-product semiflow $(\Phi_t, D\Phi_t)$ on $\Sigma \times X$, which is defined as $(\Phi_t, D\Phi_t)(x, v) = (\Phi_t(x), D_x\Phi_t v)$ for any $(x, v) \in \Sigma \times X$ and $t \in \mathbb{R}^+$. Here, $D_x\Phi_t$ is the Fréchet derivative of Φ_t at $x \in \Sigma$. Let $\{E_x\}_{x\in\Sigma}$ be a family of k-dimensional subspaces of X. We call $\Sigma \times (E_x)$ a *k-dimensional continuous vector bundle on* Σ if the map $\Sigma \mapsto G(k, X) : x \mapsto E_x$ is continuous. Let $\{F_x\}_{x\in\Sigma}$ be a family of k-codimensional closed vector subspaces of X. We call $\Sigma \times (F_x)$ a *k-codimensional continuous vector bundle on* Σ if there is a k-dimensional continuous vector bundle $\Sigma \times (L_x) \subset \Sigma \times X^*$ such that the kernel $\mathrm{Ker}(L_x) = F_x$ for each $x \in \Sigma$. Here, X^* is the dual space of X.

Let $\Sigma \times (E_x)$ be a k-dimensional continuous vector bundle on Σ and $\Sigma \times (F_x)$ be a k-codimensional continuous vector bundle on Σ such that $X = E_x \oplus F_x$ for all $x \in \Sigma$. We define the *family of projections associated with the decomposition* $X = E_x \oplus F_x$ as $\{\Pi^{E_x}\}_{x\in\Sigma}$ where Π^{E_x} is the linear projection of X onto E_x along F_x for each $x \in \Sigma$. Write $\Pi^{F_x} = I - \Pi^{E_x}$ for each $x \in \Sigma$. Clearly, Π^{F_x} is the linear projection of X onto F_x along E_x. Moreover, both Π^{E_x} and Π^{F_x} are continuous with respect to $x \in \Sigma$. We say that the decomposition $X = E_x \oplus F_x$ is *invariant with respect to* $(\Phi_t,\ D\Phi_t)$ if $D_x\Phi_t E_x = E_{\Phi_t(x)}$, $D_x\Phi_t F_x \subset F_{\Phi_t(x)}$ for each $x \in \Sigma$ and $t \geq 0$.

Definition 2.3 Let $\Sigma \subset X$ be a compact invariant subset w.r.t. Φ_t on which Φ_t admits a flow extension. The linear skew-product semiflow $(\Phi_t,\ D\Phi_t)$ admits a k-**exponential separation along** Σ (for short, k-exponential separation), if there are k-dimensional continuous vector bundle $\Sigma \times (E_x)$ and k-codimensional continuous vector bundle $\Sigma \times (F_x)$ such that

(i) $X = E_x \oplus F_x$ for any $x \in \Sigma$;
(ii) $D_x\Phi_t E_x = E_{\Phi_t(x)}$, $D_x\Phi_t F_x \subset F_{\Phi_t(x)}$ for any $x \in \Sigma$ and $t > 0$;
(iii) there are constants $M > 0$ and $0 < \gamma < 1$ such that

$$\|D_x\Phi_t w\| \leq M\gamma^t \|D_x\Phi_t v\|$$

for all $x \in \Sigma$, $w \in F_x \cap S$, $v \in E_x \cap S$ and $t \geq 0$, where $S = \{v \in X : \|v\| = 1\}$. Let $C \subset X$ be a complemented k-solid cone. If, in addition,
(iv) $E_x \subset \mathrm{Int}C \cup \{0\}$ and $F_x \cap C = \{0\}$ for any $x \in \Sigma$,

then $(\Phi_t,\ D\Phi_t)$ is said to admit a k-**exponential separation along** Σ **associated with** C.

Since E_x is k dimensional for any $x \in \Sigma$, one can define *the infimum norm* $m(D_x\Phi_t|_{E_x})$ *of* $D_x\Phi_t$ *restricted on* E_x for each $x \in \Sigma$ and $t \geq 0$ as follows:

$$m(D_x\Phi_t|_{E_x}) = \inf_{v\in E_x\cap S} \|D_x\Phi_t v\|, \tag{2.1}$$

where $S = \{v \in X : \|v\| = 1\}$.

Definition 2.4 For each $x \in \Sigma$, *the k-Lyapunov exponent* is defined as

$$\lambda_{kx} = \limsup_{t\to+\infty} \frac{\log m(D_x\Phi_t|_{E_x})}{t}. \tag{2.2}$$

A point $x \in \Sigma$ is called *a regular point* if $\lambda_{kx} = \lim_{t\to+\infty} \dfrac{\log m(D_x\Phi_t|_{E_x})}{t}$.

Lemma 2.7 *Assume that* $\Sigma \subset X$ *be a compact invariant subset with respect to* Φ_t *on which* Φ_t *admits a flow extension. Assume that* Φ_t *is* C^1*-smooth such that* $D_x\Phi_t(C \setminus \{0\}) \subset \mathrm{Int}\,C$ *for any* $x \in \Sigma$ *and* $t > 0$*. Then,* $(\Phi_t, D\Phi_t)$ *admits a k-exponential separation along* Σ *associated with* C*.*

Proof See Tereščák [25, Corollary 2.2]. One may also refer to Tereščák [25, Theorem 4.1]. □

Now, we give some crucial lemmas for $(\Phi_t, D\Phi_t)$ admitting a k-exponential separation $X = E_x \oplus F_x$ along a compact invariant set Σ associated with C, which satisfies (i)-(iv) in Definition 2.3.

Lemma 2.8 *There exists a constant $\delta' > 0$ such that*

$$\{v \in X : d(v,\ E_x \cap S) \leq \delta'\} \subset Int\, C \ \ for\ any\ x \in \Sigma.$$

Proof See [2, Lemma 3.3]. We here point out that all arguments in [2, Lemma 3.3] still remain valid for C^1-smooth semiflow Φ_t on a Banach space. □

Lemma 2.9 *(i) The projections Π^{E_x} and Π^{F_x} are bounded uniformly for $x \in \Sigma$.*
(ii) There exists a constant $C_1 > 0$ such that, if $v \in X \backslash \{0\}$ satisfies $\|\Pi^{E_x}(v)\| \geq C_1 \|\Pi^{F_x}(v)\|$ for some $x \in \Sigma$, then $v \in Int\, C$.

Proof See [2, Lemma 3.5(i) and (ii)]. The arguments in [2, Lemma 3.5(i) and (ii)] are also effective for the semiflow Φ_t on a Banach space. □

Lemma 2.10 *Let $x \in \Sigma$. Then*

(i) If $w \in F_x \setminus \{0\}$, then $\lambda(x, w) \leq \lambda_{kx} + \log(\gamma)$, where $\lambda(x, w) = \limsup_{t\to\infty} \frac{\log\|D_x\Phi_t w\|}{t}$.
(ii) Let x be a regular point. If $\lambda_{kx} \leq 0$, then there exists a number $\beta \in (\gamma, 1)$ such that for any $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that

$$\|D_{\Phi_{t_1}(x)}\Phi_{t_2} w\| \leq C_\epsilon e^{\epsilon t_1} \beta^{t_2} \|w\|$$

for any $w \in F_{\Phi_{t_1}(x)} \setminus \{0\}$ and $t_1, t_2 > 0$.

Proof The results are directly implied by repeating all arguments in [2, Lemma 3.6] □

Remark 2.11 Lemma 2.8–2.10 are the infinite demensional version of [2, Lemma 3.3, 3.5(i)–(ii), 3.6]. Lemma 2.5 and 2.8–2.10 are crucial tools for the arguments of Theorem 4.4 and Lemma 4.5.

3 Main Results

Let $C_E = \{x \in X :\ \omega(x)\ \text{is a singleton}\}$.

Theorem A *(Generic dynamics thoerem) Assume that Φ_t is a $C^{1,\alpha}$-smooth semiflow strongly focusing monotone with respect the k-cone C. Let $\mathcal{D} \subset X$ be an open bounded set such that $\mathcal{O}^+(\mathcal{D})$ is precompact. Then* $\text{Int}(Q \cup C_E)$ *(interior in X)* is dense in $\mathcal{D}$.

Remark 3.1 Theorem A states that, for smooth semiflow Φ_t strongly focusing monotone with respect to the k-cone C, generic (open and dense) positive semiorbits either are pseudo-ordered or convergent to equilibria. If the rank $k = 1$, Theorem A automatically implies Hirsch's Generic Convergence Theorem due to the Monotone Convergence Criterion.

Theorem B *(Generic Poincaré-Bendixson theorem) Assume that Φ_t is a $C^{1,\alpha}$-smooth semiflow strongly focusing monotone with respect to the k-cone C. Let $k = 2$ and $\mathcal{D} \subset X$ be an open bounded set such that $\mathcal{O}^+(\mathcal{D})$ is precomact. Then, for generic (open and dense) points $x \in \mathcal{D}$, the omega-limit set $\omega(x)$ containing no equilibria is a periodic orbit.*

4 Local Behaviors of Omega-Limit Sets

Due to Lemma 2.7, we hereafter always assume that for any compact invariant set Σ on which Φ_t admits a flow extension, the linear skew-product semiflow $(\Phi_t, D\Phi_t)$ admits a k-exponential separation along Σ such that $X = E_x \oplus F_x$ for any $x \in \Sigma$. $\Sigma \times (E_x)$ and $\Sigma \times (F_x)$ are the corresponding k-dimensional and k-codimensional continuous invariant vector subbundles. In this paper, we attempt to extend the works on generic dynamics from classical monotone systems w.r.t. convex cones (see [23]) and flows strongly monotone w.r.t. k-cones (see [2]) to the semiflows strongly focusing monotone w.r.t. the k-cone C on an infinite dimensional Banach space.

We define the set of *regular points* on nonempty compact $\omega(x)$ as:

$$\omega_0(x) = \{z \in \omega(x) : z \text{ is a regular point}\}. \tag{4.1}$$

Due to the Multiplicative Ergodic Theorem (cf. [20, Theorem A]) and the similiar arguments for [26, Proposition 4.11], $\omega_0(x)$ is non-empty. Moreover, it is easy to see that any equilibrium in $\omega(x)$ is regular and hence, is contained in $\omega_0(x)$. By utilizing the k-Lyapunov exponents on $\omega(x)$, we classify the omega-limit sets into three types and obtain their local behaviors.

Firstly, we prove that if $\lambda_{kz} > 0$ for any $z \in \omega(x)$, then x is highly unstable (see Lemma 4.2), and meanwhile, it belongs to the closure $\overline{Q}$ (see Theorem 4.3). We secondly show that if $\lambda_{k\tilde{z}} > 0$ for any regular point $\tilde{z} \in \omega_0(x)$ and there is an irregular point z with $\lambda_{kz} \leq 0$, then $x \in \overline{Q}$ (see Theorem 4.4). We finally show that if $\omega(x)$ contains a regular point z such that $\lambda_{kz} \leq 0$, then either $x \in Q$ or $\omega(x)$ is a singleton (see Theorem 4.6).

We start with discussion on the case that $\lambda_{kz} > 0$ for any point $z \in \omega(x)$. Before going further, we give two technical lemmas.

Lemma 4.1 *If $\lambda_{kz} > 0$ for any $z \in \omega(x)$, then for any constant $\kappa > 0$, there is a local constant (hence bounded) function $\nu_\kappa(z)$ on $\omega(x)$ (depending on κ) such that*

$$\begin{aligned} \frac{\|D_z\Phi_{\nu_\kappa(z)} w_F\|}{\|D_z\Phi_{\nu_\kappa(z)} w_E\|} &< \frac{\kappa}{2(1+\kappa)}, \\ \|D_z\Phi_{\nu_\kappa(z)} w_E\| &> \frac{4}{\kappa} \end{aligned} \tag{4.2}$$

for any $z \in \omega(x)$ and $w_E \in E_z \cap S$ and $w_F \in F_z \cap S$, where $S = \{v \in X : \|v\| = 1\}$.

Proof By the definition of λ_{kz}, for each $z \in \omega(x)$, there is a sequence $t_n \to +\infty$ such that

$$\|D_z\Phi_{t_n} w_E\| > e^{\frac{\lambda_{kz}}{2} t_n}$$

for any $w_E \in E_z \cap S$. Furthermore, the definition of k-exponential separation along Σ indicates that there exist $M > 0$ and $\gamma \in (0, 1)$ such that

$$\frac{\|D_z\Phi_t w_F\|}{\|D_z\Phi_t w_E\|} < M\gamma^t$$

for any $t > 0$ and $w_E \in E_z \cap S$, $w_F \in F_z \cap S$. Since $\lambda_{kz} > 0$, one can find a $N_\kappa(z) > 0$ such that

$$\frac{\|D_z\Phi_{t_n} w_F\|}{\|D_z\Phi_{t_n} w_E\|} < \frac{\kappa}{2(1+\kappa)}$$

and

$$\|D_z\Phi_{t_n}w_E\| > \frac{4}{\kappa}$$

for any $t_n > N_\kappa(z)$ and $w_E \in E_z \cap S$.

Therefore, for each $z \in \omega(x)$, one can associate with a number $\nu_\kappa(z) \geq N_\kappa(z)$ such that (4.2) holds for any $w_E \in E_z \cap S$ and $w_F \in F_z \cap S$. Moreover, together with the compactness of $\omega(x)$ and the smoothness of Φ_t, one can further take such $\nu_\kappa(z)$ as a local constant (hence bounded) function. We have completed the proof. □

Lemma 4.2 *Assume that $\lambda_{kz} > 0$ for any $z \in \omega(x)$. There exists a constant $\delta'' > 0$ such that*

$$\limsup_{t\to+\infty}\|\Phi_t(y) - \Phi_t(x)\| \geq \delta'',$$

whenever y satisfies $y \neq x$ and $y \sim x$.

Proof Since Φ_t is strongly focusing monotone w.r.t. C, one can find constants $\delta, T, \kappa > 0$ such that there exists a strongly focusing operator $T_{(x,y)}$ with separation index greater than κ such that $T_{(x,y)}(x-y) = \Phi_T(x) - \Phi_T(y)$ for any $z \in \omega(x)$ and $x, y \in B_\delta(z)$, where $B_\delta(z)$ is the closed ball centred at z with radius δ.

For any given $y \in X$ such that $y \neq x$ and $y \sim x$, let $\tilde{y}_t = \Phi_t(y)$ and $\tilde{x}_t = \Phi_t(x)$ for $t > 0$. By the strongly focusing monotonicity of Φ_t, one has $\tilde{y}_t \neq \tilde{x}_t$ for any $t \in \mathbb{R}^+$. Since $\omega(x)$ attracts x, one can take a curve $\{z_t\}_{t>0} \subset \omega(x)$ such that

$$\lim_{t\to\infty}\|\tilde{x}_t - z_t\| = 0. \tag{4.3}$$

Hence, there exists a $\tilde{T}_y > T$ such that $\|\tilde{x}_{t-T} - z_{t-T}\| < \frac{\delta}{2}$ for any $t > \tilde{T}_y$. Moreover, if $\tilde{y}_{t-T} \in B_\delta(z_{t-T})$ for some $t > \tilde{T}_y$, then there is a strongly focusing operator $T_{\tilde{y}_{t-T},\tilde{x}_{t-T}}$ with separation index greater than κ such that

$$\tilde{y}_t - \tilde{x}_t = T_{\tilde{y}_{t-T},\tilde{x}_{t-T}}(\tilde{y}_{t-T} - \tilde{x}_{t-T}).$$

By Lemma 2.7, $(\Phi_t, D\Phi_t)$ admits a k-exponential separation along $\omega(x)$ associated with C. Denoted by $\omega(x) \times (E_z)$ and $\omega(x) \times (F_z)$ the corresponding k-dimensional invariant subbundle k-codimensional invariant subbundle respectively. Recall that Π^{E_z} is the projection onto E_z along F_z and $\Pi^{F_z} = I - \Pi^{E_z}$. Clearly, for $t > \tilde{T}_y$ and $\tilde{y}_{t-T} \in B_\delta(z_{t-T})$, one has $d(\frac{\tilde{y}_t - \tilde{x}_t}{\|\tilde{y}_t - \tilde{x}_t\|}, X \setminus C) \geq \kappa$ and hence,

$$\begin{aligned}\frac{\|\Pi^{E_{z_t}}(\tilde{y}_t - \tilde{x}_t)\|}{\|\tilde{y}_t - \tilde{x}_t\|} &\geq \kappa,\\ \frac{\|\Pi^{F_{z_t}}(\tilde{y}_t - \tilde{x}_t)\|}{\|\Pi^{E_{z_t}}(\tilde{y}_t - \tilde{x}_t)\|} &\leq \frac{1+\kappa}{\kappa}.\end{aligned} \tag{4.4}$$

Let ν_κ be the local constant function mentioned in Lemma 4.1 and $m_\kappa = \max_{z\in\omega(x)}\{\nu_\kappa(z)\}$. By the smoothness of Φ_t and the compactness of $\omega(x)$, there is a $\delta^{''} \in (0, \frac{\delta}{2})$ such that

$$\|D_{y_1}\Phi_t - D_{y_2}\Phi_t\| < \frac{1}{2} \tag{4.5}$$

for any $t \in [0, m_\kappa]$ and $y_1, y_2 \in \mathcal{B}_{2\delta''}(\omega(x))$ satisfying $\|y_1 - y_2\| < 2\delta''$, where $\mathcal{B}_{2\delta''}(\omega(x)) = \{v \in X : d(v, \omega(x)) \le 2\delta''\}$.

Now, we will prove that δ'' is the desired constant. Prove by contrary. Suppose that

$$\limsup_{t\to+\infty} \|\tilde{y}_t - \tilde{x}_t\| < \delta''$$

for some $y \in X$ satisfying $y \neq x$ and $y \sim x$. Then, one can find a $N_1 > \tilde{T}_y$ such that

$$\|\tilde{y}_{t-T} - \tilde{x}_{t-T}\| < \delta'' \text{ and } \|\tilde{x}_{t-T} - z_{t-T}\| < \delta''$$

for any $t \ge N_1$. Hence, $\tilde{y}_{t-T}, \tilde{x}_{t-T} \in \mathcal{B}_\delta(z_{t-T})$ and (4.4) hold for any $t > N_1$. Take $\tau_1 \ge N_1$ and $\tau_{n+1} = \tau_n + \nu_\kappa(z_{\tau_n})$ for $n = 1, 2, \cdots$. Denoted by $y_{\tau_n} = \Phi_{\tau_n}(y)$, $x_{\tau_n} = \Phi_{\tau_n}(x)$ and $\Pi^{E_{\tau_n}} = \Pi^{E_{z_{\tau_n}}}$, $\Pi^{F_{\tau_n}} = \Pi^{F_{z_{\tau_n}}}$. Then, one has

$$\begin{aligned}
\|D_{z_{\tau_n}}\Phi_{\nu_\kappa(z_{\tau_n})}(y_{\tau_n} - x_{\tau_n})\| &\ge \|D_{z_{\tau_n}}\Phi_{\nu_\kappa(z_{\tau_n})}\Pi^{E_{\tau_n}}(y_{\tau_n} - x_{\tau_n})\| \cdot \Big[1 - \frac{\|D_{z_{\tau_n}}\Phi_{\nu_\kappa(z_{\tau_n})}\Pi^{F_{\tau_n}}(y_{\tau_n} - x_{\tau_n})\|}{\|D_{z_{\tau_n}}\Phi_{\nu_\kappa(z_{\tau_n})}\Pi^{E_{\tau_n}}(y_{\tau_n} - x_{\tau_n})\|}\Big] \\
&\overset{(4.2)}{\ge} \|D_{z_{\tau_n}}\Phi_{\nu_\kappa(z_{\tau_n})}\Pi^{E_{\tau_n}}(y_{\tau_n} - x_{\tau_n})\| \cdot \Big[1 - \frac{\kappa}{2(1+\kappa)} \cdot \frac{\|\Pi^{F_{\tau_n}}(y_{\tau_n} - x_{\tau_n})\|}{\|\Pi^{E_{\tau_n}}(y_{\tau_n} - x_{\tau_n})\|}\Big] \\
&\overset{(4.4)}{\ge} \|D_{z_{\tau_n}}\Phi_{\nu_\kappa(z_{\tau_n})}\Pi^{E_{\tau_n}}(y_{\tau_n} - x_{\tau_n})\| \cdot \Big[1 - \frac{\kappa}{2(1+\kappa)} \cdot \frac{1+\kappa}{\kappa}\Big] \\
&= \frac{1}{2}\|D_{z_{\tau_n}}\Phi_{\nu_\kappa(z_{\tau_n})}\Pi^{E_{\tau_n}}(y_{\tau_n} - x_{\tau_n})\| \\
&\overset{(4.2)}{\ge} \frac{2}{\kappa}\|\Pi^{E_{\tau_n}}(y_{\tau_n} - x_{\tau_n})\| \\
&\overset{(4.4)}{\ge} 2\|y_{\tau_n} - x_{\tau_n}\|
\end{aligned}$$

for any $n > 0$. Notice that

$$\begin{aligned}
y_{\tau_{n+1}} - x_{\tau_{n+1}} &= D_{z_{\tau_n}}\Phi_{\nu_\kappa(z_{\tau_n})}(y_{\tau_n} - x_{\tau_n}) + \int_0^1 [D_{x_{\tau_n}+s(y_{\tau_n}-x_{\tau_n})}\Phi_{\nu_\kappa(z_{\tau_n})} - D_{z_{\tau_n}}\Phi_{\nu_\kappa(z_{\tau_n})}]ds \\
&\quad \cdot (y_{\tau_n} - x_{\tau_n})
\end{aligned}$$

for any $n > 0$. Together with (4.5), one has

$$\|y_{\tau_{n+1}} - x_{\tau_{n+1}}\| \ge \frac{3}{2} \cdot \|y_{\tau_n} - x_{\tau_n}\|$$

for any $n > 0$. Hence, $\lim_{n\to\infty} \|y_{\tau_n} - x_{\tau_n})\| = \infty$, a contradiction.

Therefore, we have completed the proof. □

Theorem 4.3 *Let $\mathcal{D}$ be an open ω-compact set and $x \in \mathcal{D}$. If $\lambda_{kz} > 0$ for any $z \in \omega(x)$, then one has $x \in \overline{Q}$.*

Proof See [2, Theorem 4.5]. We here point out that all arguments in [2, Theorem 4.5] still remain valid for semiflow Φ_t strongly focusing monotone w.r.t. C on an infinite dimensional Banach space. □

We now consider the case that $\lambda_{k\tilde{z}} > 0$ for any $\tilde{z} \in \omega_0(x)$ and $\lambda_{kz} \le 0$ for some $z \in \omega(x) \setminus \omega_0(x)$.

Theorem 4.4 *Let $\mathcal{D}$ be an open ω-compact set and $x \in \mathcal{D}$. Assume that $\lambda_{k\tilde{z}} > 0$ for any $\tilde{z} \in \omega_0(x)$. If there exists some $z \in \omega(x) \setminus \omega_0(x)$ such that $\lambda_{kz} \le 0$, then $x \in \overline{Q}$.*

Proof Since Φ_t admits a flow extension on $\omega(x)$, one can define a vector $v_y = \frac{d}{dt}\mid_{t=0} \Phi_t(y) \in X$ for any $y \in \omega(x)$. The smoothness of Φ_t implies that the map $y :\mapsto v_y$ is continuous. Hence, all arguments in [2, Theorem 4.6] are still effective for semiflow Φ_t strongly focusing monotone with respect to C on an infinite dimensional Banach space. □

Before discussing the case that $\lambda_{kz} \le 0$ for a regular point $z \in \omega_0(x)$, we present the following lemma, which describes the nonlinear dynamics nearby a regular point.

Lemma 4.5 *Let $x \in \Sigma$ be a regular point. If $\lambda_{kx} \le 0$, then there exists an open neighborhood $\mathcal{V}$ of x such that for any $y \in \mathcal{V}$, one of two following properties holds:*

(a) $\|\Phi_t(x) - \Phi_t(y)\| \to 0$ as $t \to +\infty$;

(b) There exists a $T > 0$ such that $\Phi_T(x) - \Phi_T(y) \in C \setminus \{0\}$; and hence, $\Phi_t(x) - \Phi_t(y) \in \mathrm{Int}\, C$ for any $t > T$.

Proof By repeating all arguments in [2, Lemma 4.1], the conclusion in this lemma can be obtained for Φ_t strongly focusing monotone w.r.t. C. □

Theorem 4.6 *Assume that there exists a regular point $z \in \omega(x)$ satisfying $\lambda_{kz} \le 0$. Then either $x \in Q$, or $\omega(x) = \{z\}$ consists of a singleton.*

Proof The result is implied by repeating all arguments in [2, Theorem 4.2]. □

5 Proofs of Theorem A and B

Due to the local behaviors in the last section, we can describe the *generic dynamics* of the semiflow Φ_t strongly focusing monotone w.r.t. C (see Theorem A) on a general Banach space, which concludes that generic (open and dense) positive semiorbits either are pseudo-ordered or convergent to an equilibrium. When $k = 2$, together with the results in Lemma 2.6, we will further show the Poincaré-Bendixson Theorem (see Theorem B), that is to say, for generic (open and dense) points, its ω-limit set containing no equilibria is a periodic orbit.

Proof of Theorem A. We note that $\mathcal{D}$ is ω-compact because $\mathcal{O}^+(\mathcal{D})$ is precompact. Then, for any $x \in \mathcal{D}$, one of the following three alternatives holds:

(a) $\lambda_{kz} > 0$ for any $z \in \omega(x)$;
(b) $\lambda_{k\tilde{z}} > 0$ for any regular point $\tilde{z} \in \omega_0(x)$ and $\lambda_{kz} \le 0$ for some irregular point $z \in \omega(x) \setminus \omega_0(x)$;
(c) $\lambda_{kz} \le 0$ for some regular point $z \in \omega(x)$.

By virtue of Theorem 4.3, 4.4 and 4.6, one has $x \in \overline{Q} \cup C_E$ for any $x \in \mathcal{D}$. Thus, $Q \cup C_E$ is dense in $\mathcal{D}$.

To prove that $\mathrm{Int}(Q \cup C_E)$ is dense in $\mathcal{D}$ by contrary, we suppose that there is an open subset U of $\mathcal{D}$ such that $U \cap \mathrm{Int}(Q \cup C_E) = \emptyset$. By strong monotonicity of Φ_t, Q is open. Together with Theorem 4.3 and 4.4, case (a) and (b) will not occur for any point in U. Then, only case (c) can occur for any point in U. Moreover, by virtue of Theorem 4.6, one has that $U \subset C_E$. Recall that U is open. Thus, $U \subset \mathrm{Int}(Q \cup C_E)$, a contradiction.

Therefore, we have completed the proof.

Proof of Theorem B. By Theorem A, $\mathrm{Int}(Q \cup C_E)$ is open and dense in $\mathcal{D}$. Now, given any $x \in \mathrm{Int}(Q \cup C_E) \cap \mathcal{D}$, if $\omega(x) \cap E = \emptyset$, then $x \in Q$. It then follows from Lemma 2.6, $\omega(x)$ is a periodic orbit. Therefore, we have completed the proof.

Remark 5.1 In this paper, we introduce the concept of strongly focusing monotonicity and extend our previous works on the generic dynamics of flows strongly monotone with respect to high-rank cones on $\mathbb{R}^n$ (see [2]) to the class of general semiflows strongly focusing monotone with respect to the k-cone C on an infinite dimensional Banach space. In the sequel, we intend to investigate the "typical" behaviors of this class of semiflows in the measure theoretic sense, and establish the total-ordering property and Poincaré-Bendixson property of omega-limit sets. In future work, we will extend the strongly focusing monotonicity from semiflows to discrete systems, cocycles and skew-product semiflows.

Acknowledgements The author would like to appreciate Prof. Yi Wang and Prof. Jianhong Wu for many useful suggestions. The author is greatly indebted to an anonymous referee for careful reading and comments which led to much improvement of the earlier version of this paper.

Author Contributions The unique author contributes to this work.

Funding The author was supported by NSF of China No.12101583, No.12331006, Wu Wen-Tsun Key Laboratory of Mathematics and the Fundamental Research Funds of the Central Universities through grant WK0010000067.

Data and Materials Availability This article describes entirely theoretical research. There is no dataset generated or analysed during the current study.

Declarations

Conflict of interest The work has no Conflict of interest.

Ethical Approval The submitted work is original and have not been published elsewhere in any form or language.

References

1. Feng, L., Wang, Y., Wu, J.: Semiflows "monotone with respect to high-rank cones" on a Banach space. SIAM J. Math. Anal. **49**, 142–161 (2017)
2. Feng, L., Wang, Y., Wu, J.: Generic behavior of flows strongly monotone with respect to high-rank cones. J. Differ. Equ. **275**, 858–881 (2021)
3. Feng, L., Wang, Y., Wu, J.: Generic convergence to periodic orbits for monotone flows with respect to 2-cones and applications to SEIRS-modles. SIAM J. Appl. Math. **82**(5), 1829–1850 (2022)
4. Feng, L., Wang, Y., Wu, J.: Pseudo-ordered principle for flows strongly monotone with respect to high-rank cones, submitted
5. Fusco, G., Oliva, W.: Jacobi matrices and transversality. Proc. R. Soc. Edinburg Sect. A Math. **109**, 231–243 (1988)
6. Fusco, G., Oliva, W.: A Perron theorem for the existence of invariant subspaces. Annali di Matematica **160**, 63–76 (1991)
7. Johnson, R.A., Palmer, K.J., Sell, G.R.: Ergodic properties of linear dynamical systems. SIAM J. Math. Anal. **18**, 1–33 (1987)
8. Hirsch, M.W.: Systems of differential equations which are competitive or cooperative: I. Limit sets. SIAM J. Math. Anal. **13**, 167–179 (1982)
9. Hirsch, M.W.: Systems of differential equations that are competitive or cooperative. II. Convergence almost everywhere. SIAM J. Math. Anal. **16**, 423–439 (1985)
10. Hirsch, M.W.: Systems of differential equations which are competitive or cooperative: III. Competing species. Nonlinearity **1**, 51–71 (1988)

11. Hirsch, M.W.: Stability and convergence in strongly monotone dynamical systems. J. Reine Angew. Math. **383**, 1–53 (1988)
12. Hirsch, M.W.: Systems of differential equations that are competitive or cooperative. IV: Structural stability in three-dimensional systems. SIAM J. Math. Anal. **21**, 1225–1234 (1990)
13. Hirsch, M.W.: Systems of differential equations that are competitive or cooperative. V: Convergence in 3-dimensional systems. J. Differ. Equ. **80**, 94–106 (1989)
14. Hirsch, M.W.: Systems of differential equations that are competitive or cooperative. VI: A local C^r closing lemma for 3-dimensional systems. Ergod. Theory Dyn. Syst. **11**, 443–454 (1991)
15. Kato, T.: Pertubation theory for linear operators. Springer, New York (1995)
16. Krasnosel'skij, M.A., Je, A., Sobolev, A.V.: Positive linear systems: the method of positive operators. Heldermann Verlag, Berlin (1989)
17. Lian, Z., Lu, K.: Lyapunov exponents and invariant manifolds for random dynamical systems on a Banach space, vol. 206. American Mathematical Society, Providence (2010)
18. Lian, Z., Wang, Y.: K-dimensional invariant cones of random dynamical system in $\mathbb{R}^n$ with applications. J. Differ. Equ. **259**, 2807–2832 (2015)
19. Lian, Z., Wang, Y.: On linear random dynamical systems in a Banach space: I. Multiplicative ergodic theorem and Krein-Rutmann type theorems. Adv. Math. **312**, 374–424 (2017)
20. Mañé, R.: Lyapunov exponents and stable manifolds for compact transformations. In: Geometric dynamics, of lecture notes in mathematics, Vol. 1007, pp. 522–577. Springer (1983)
21. Oseledec, V.I.: A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. Trans. Mosc. Math. Soc. **19**, 197–231 (1968)
22. Sanchez, L.A.: Cones of rank 2 and the Poincaré-Bendixson property for a new class of monotone systems. J. Differ. Equ. **246**, 1178–1190 (2009)
23. Smith, H.: Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems. In: Mathematical Surveys and Monographs, vol. 41. American Mathematical Society, Providence(1995)
24. Smith, H.: Monotone dynamical systems: reflections on new advances and applications. Discret. Contin. Dyn. Syst. **37**, 485–504 (2017)
25. Tereščák, I.: Dynamical systems with discrete Lyapunov functionals. In: Ph.D. thesis, Comenius University, Bratislava (1994)
26. Viana, M.: Lectures on Lyapunov Exponents. In: Cambridge Studies in Advanced Mathematics, Vol. 145. Cambridge University Press (2014)