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Periodic Generalized Birkhoff Solutions and Farey Intervals for Monotone Recurrence Relations

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Abstract
The aim of this paper is to extend the results associated with periodic orbits from two-dimensions to higher-dimensions. Because of the one-to-one correspondence between solutions for the monotone recurrence relation and orbits for the induced high-dimensional cylinder twist map, we consider the system of solutions for monotone recurrence relations instead. By introducing intersections of type (k, l), we propose the definition of generalized Birkhoff solutions, generalizing the concept of Birkhoff solutions. We show that if there is a (p, q)-periodic solution which is not a generalized Birkhoff solution, then the system has positive topological entropy and the Farey interval of p/q is contained in the rotation set.

Keywords Rotation set · Generalized Birkhoff solution · Topological entropy · Farey interval · Monotone recurrence relation

Mathematics Subject Classification 37B40 · 37C65 · 37E40 · 37E45

1 Introduction

There have been extensive and profound investigations of periodic orbits for maps of the annulus or two-dimensional cylinder. The main purpose of this paper is to study the relation among periodic orbits, rotation sets and the complexity of systems on the high-dimensional cylinder.

Before proceeding futher, we need to review the definition of monotone recurrence relations of type (k, l) for $k, l \in \mathbb{N}$, which are equations as follows:

$$\Delta(x_{i-k}, \cdots, x_i, \cdots, x_{i+l}) = 0, \text{ for all } i \in \mathbb{Z}, \tag{1.1}$$

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where $\Delta : \mathbb{R}^{k+l+1} \to \mathbb{R}$ is a continuous function with following hypotheses:

(H1) $\Delta(x_{-k}, \cdots, x_0, \cdots, x_l)$ is a nondecreasing function of all the x_j except x_0. Moreover, it is strictly increasing in x_{-k} and x_l,
(H2) $\Delta(x_{-k}+1, \cdots, x_l+1) = \Delta(x_{-k}, \cdots, x_l)$,
(H3) $\lim_{x_{-k}\to\pm\infty} \Delta(x_{-k}, \cdots, x_l) = \pm\infty$ and $\lim_{x_l\to\pm\infty} \Delta(x_{-k}, \cdots, x_l) = \pm\infty$.

Hypotheses (H1) and (H3) implies that we can solve (1.1) for x_{i+l} if $(x_{i-k}, \cdots, x_{i+l-1})$ is given, or x_{i-k} if $(x_{i-k+1}, \cdots, x_{i+l})$ is given, thereby generating a homeomorphism map $F_\Delta : \mathbb{R}^{k+l} \to \mathbb{R}^{k+l}$ by

$$F_\Delta(x_{i-k}, \cdots, x_{i+l-1}) = (x_{i-k+1}, \cdots, x_{i+l}).$$

According to the periodicity hypothesis (H2), we obtain on the high-dimensional cylinder $\mathbb{T}^1 \times \mathbb{R}^{k+l-1}$ a homeomorphism φ_Δ which is a generalization of the class of monotone twist maps of the annulus or two-dimensional cylinder [2]. Especially for $k = l = 1$, the induced map φ_Δ coincides with the classical monotone twist map.

For instance,

$$\Delta(x_{i-1}, x_i, x_{i+1}) = x_{i-1} - 2x_i + x_{i+1} + K \sin 2\pi x_i, \quad i \in \mathbb{Z}, \ K \in \mathbb{R},$$

corresponds to the standard map φ_Δ as

$$\varphi_\Delta(x \ (\mathrm{mod}\ 1), y) = (x + y - K \sin \pi x \ (\mathrm{mod}\ 1), \ y - K \sin \pi x).$$

If a configuration $\mathbf{x} = (x_i)$ satisfies (1.1), then we say $\mathbf{x}$ is a solution of (1.1). Further, a solution $\mathbf{x} = (x_i)$ is said to be with bounded action if there exists a constant $L > 0$ such that $|x_{i+1} - x_i| \le L$ for all $i \in \mathbb{Z}$. Then the system generated by solutions for (1.1) with bounded action is equivalent to that by φ_Δ on the high-dimensional cylinder restricted to orbits with bounded action. Therefore, we would study the dynamical behavior of solutions for (1.1) rather than φ_Δ.

Let $\mathbf{x} = (x_i)$ be a solution of (1.1) with bounded action. Define the forward rotation interval of $\mathbf{x}$ to be [11, 22, 23, 25]

$$\rho(\mathbf{x}) = \left[\liminf_{n\to+\infty} \frac{x_n}{n}, \ \limsup_{n\to+\infty} \frac{x_n}{n} \right],$$

and the backward rotation interval $\rho^*(\mathbf{x})$ similarly for $n \to -\infty$. If $\rho(\mathbf{x})$ ($\rho^*(\mathbf{x})$) is a single point, i.e., the limit $\lim_{n\to+\infty} x_n/n$ ($\lim_{n\to-\infty} x_n/n$) exists, then $\mathbf{x}$ is said to have a forward (backward) rotation number. We say $\mathbf{x}$ has a rotation number if $\rho(\mathbf{x}) = \rho^*(\mathbf{x})$ is a singleton.

The rotation set $\rho(\Delta)$ is defined to be the union of $\rho(\mathbf{x})$, where $\mathbf{x} = (x_i)$ is a solution of (1.1) with bounded action.

Our first concern will be the relation between periodic orbits and topological entropy. With the help of topological methods, Hall showed [18] that each monotone twist map which has a (p, q)-periodic orbit certainly has a (p, q)-periodic Birkhoff orbit. After that, an analogous result [10] has been derived by Boyland for more general homeomorphisms of the annulus.

The question is whenever (p, q)-periodic orbits for the monotone twist map are exactly (p, q)-periodic Birkhoff orbits. Using Thurston's classification of surface diffeomorphisms [12, 13, 24], it was originally proven by Boyland that each (p, q)-periodic orbit is Birkhoff provided the system has zero topological entropy [6]. Taking into account the fact that Birkhoff solutions of (1.1) with $k = l = 1$ correspond to the Birkhoff orbits of the monotone twist map, Angenent reproduced [2] the conclusion by constructing a subsolution and a supersolution of (1.1) exchanging rotation numbers (See Definition 2.3). We remark that his approach strongly depended on the assumption that $k = l = 1$, therefore it is nontrivial and interesting to extend the preceding conclusion to higher-dimensions (See (i) of Main Theorem).

Another issue that we are concerned with will involve Farey intervals of rational numbers. We recall the definition as below.

Suppose $p, q \in \mathbb{N}$ are relatively prime. $I(p/q) = [r/s, r'/s']$ is called the Farey interval of p/q if r/s denotes the maximum of $\{n/m \mid n/m < p/q, m < q, \text{and } (m, n) = 1\}$ and r'/s' denotes the minimum of $\{n/m \mid n/m > p/q, m < q, \text{and } (m, n) = 1\}$. In fact, the endpoints of the Farey interval are the last two convergents of the continued fraction of p/q if the last partial quotient is made equal to one [17].

Let f be the monotone twist map of the annulus and p/q be the rational number mentioned above, Boyland and Hall showed [8] that the existence of a (p, q)-periodic non-Birkhoff orbit implies the non-existence of certain invariant circles for f, and they also proved that the Farey interval $I(p/q)$ is contained in the rotation band [7, 9] of the periodic orbit, which measures the fastest and slowest rates of rotation associated with the periodic orbit. Furthermore, Boyland demonstrated in [9] that the rotation band of a chain transitive set [14] must lies in the rotation set $\rho(f)$. There is no doubt that periodic orbits are special chain transitive sets, so the Farey interval $I(p/q) \subseteq \rho(f)$. It's worth pointing out that the Thurston-Nielsen theory played an important role in the process. Similar results for orientation and boundary preserving homeomorphisms can be found in [10].

Accordingly, a natural question arises: What is the version of the high-dimensional case? We shall answer this question in (ii) of Main Theorem.

We briefly describe our exploration to derive the main theorem. For the two-dimensional case, we emphasize that the usual tool is the Thurston-Nielson theory of surface automorphisms, and relevant proofs benefit from plane topology more or less. In view of this, we turn to look for alternative methods for high-dimensional cases. Inspired by the idea of Angenent [2], we intend to construct a subsolution and a supersolution of (1.1) which exchange rotation numbers. Unfortunately, it's not hard to see that Angenent's discussion relies on a concept of intersections (See Definition 2.1), and this concept could only be applied in the situation that $k = l = 1$.

In order to deal with general cases, we need to introduce a 'new intersection' in Section 3 to develop the concept of intersections. Taking advantage of the 'new intersection', we could construct by the monotonicity hypothesis (H1) a subsolution and a supersolution of (1.1) exchanging rotation numbers, and a periodic solution of (1.1) with a certain period. We stress that these constructions forms the basis for the proof of our main conclusion.

For our main result, we shall use generalized Birkhoff solutions defined in Sect. 3, which are generalizations of Birkhoff solutions.

Main Theorem *Assume that $p, q \in \mathbb{N}$ are relatively prime. If (1.1) has a (p, q)-periodic solution $\mathbf{x}$ which is not a generalized Birkhoff solution, then the following conclusions hold:*

(i) φ_Δ has positive topological entropy.
(ii) The Farey interval $I(p/q) \subseteq \rho(\Delta)$.

In addition, we conclude from the remark following Definition 3.1 that (p, q)-periodic generalized Birkhoff solutions are exactly Birkhoff solutions for $k = l = 1$. Since (1.1) with $k = l = 1$ will induce a monotone twist map and (p, q)-periodic Birkhoff solutions of (1.1) correspond to (p, q)-periodic Birkhoff orbits of the map, our main theorem can be regarded as an extension of the results in [2, 6, 7, 10].

2 Preliminaries

In this section, we set up basic notation and terminology, and then present some necessary propositions used in later sections.

We consider the configuration space $\mathbb{R}^{\mathbb{Z}}$ equipped with the product topology and define a family of translations $\{\tau_{m,n} \mid (m,n) \in \mathbb{Z}^2\}$ on $\mathbb{R}^{\mathbb{Z}}$ by

$$(\tau_{m,n}\mathbf{x})_i = x_{i-m} + n, \quad \text{for } \mathbf{x} = (x_i) \in \mathbb{R}^{\mathbb{Z}} \text{ and } i \in \mathbb{Z}.$$

Next, we state two kinds of configurations both with bounded action and rotation numbers.

A configuration $\mathbf{x} = (x_i)$ is called (p,q)-periodic if $\tau_{p,q}\mathbf{x} = \mathbf{x}$, i.e., $x_{i-q} + p = x_i$ for all i. It's easy to check that (p,q)-periodic configurations are configurations with bounded action and rotation number p/q. We note that (p,q)-periodic solutions of (1.1) correspond to the (p,q)-periodic orbits of F_Δ generated by (1.1).

Another kind of well-known configurations are called Birkhoff configurations. We begin with the definition of intersections [1, 2, 5].

Definition 2.1 We say $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ intersect at the integer j if either

$$(y_{j-1} - x_{j-1})(y_j - x_j) < 0$$

or

$$y_j = x_j \text{ and } (y_{j-1} - x_{j-1})(y_{j+1} - x_{j+1}) < 0.$$

Let $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}}$. $\mathbf{x}$ is said to be Birkhoff if $\mathbf{x}$ and its any translation $\tau_{m,n}\mathbf{x}$ do not intersect. The following lemma reflects the principle features of Birkhoff configurations [15, 20, 21].

Lemma 2.2 *Let $\mathbf{x} = (x_i)$ be a Birkhoff configuration. Then it has a rotation number ρ with*

$$|x_j - x_i - (j-i)\rho| \le 1, \quad \text{for all } i, j \in \mathbb{Z}.$$

Moreover, the map $\mathbf{x} \mapsto \rho(\mathbf{x})$ is continuous in the product topology.

It follows immediately from Lemma 2.2 that Birkhoff configurations have bounded action and rotation numbers. It is noted that Birkhoff solutions of (1.1) correspond to Birkhoff orbits of the induced high-dimensional cylinder map [2].

Definition 2.3 If two configurations $\underline{\mathbf{x}} = (\underline{x}_i)$ and $\overline{\mathbf{x}} = (\overline{x}_i)$ satisfying

$$\Delta(\underline{x}_{i-k}, \cdots, \underline{x}_i, \cdots, \underline{x}_{i+l}) \ge 0 \text{ and } \Delta(\overline{x}_{i-k}, \cdots, \overline{x}_i, \cdots, \overline{x}_{i+l}) \le 0 \text{ for all } i \in \mathbb{Z},$$

then they are called a subsolution and a supersolution of (1.1), respectively. For $\omega_1 < \omega_2$, it is said that they exchange rotation numbers ω_1 and ω_2 if

$$\limsup_{n\to+\infty} \frac{\underline{x}_n}{n} \le \omega_1, \ \liminf_{n\to-\infty} \frac{\underline{x}_n}{n} \ge \omega_2, \ \liminf_{n\to+\infty} \frac{\overline{x}_n}{n} \ge \omega_2, \ \limsup_{n\to-\infty} \frac{\overline{x}_n}{n} \le \omega_1.$$

Finally, we give some relevant propositions, their proofs can be found in [2] and is omitted here.

Proposition 2.4 (Theorem 4.2 and Addendum 4.4 in [2]) *Let $\underline{\mathbf{x}} = (\underline{x}_i)$, $\overline{\mathbf{x}} = (\overline{x}_i)$ be a subsolution and a supersolution of (1.1), respectively, satisfying $\underline{x}_i \le \overline{x}_i$ for all $i \in \mathbb{Z}$. If $\underline{\mathbf{x}}$ or $\overline{\mathbf{x}}$ is (p,q)-periodic, then there exists a (p,q)-periodic solution $\mathbf{x} = (x_i)$ satisfying $\underline{x}_i \le x_i \le \overline{x}_i$ for all $i \in \mathbb{Z}$.*

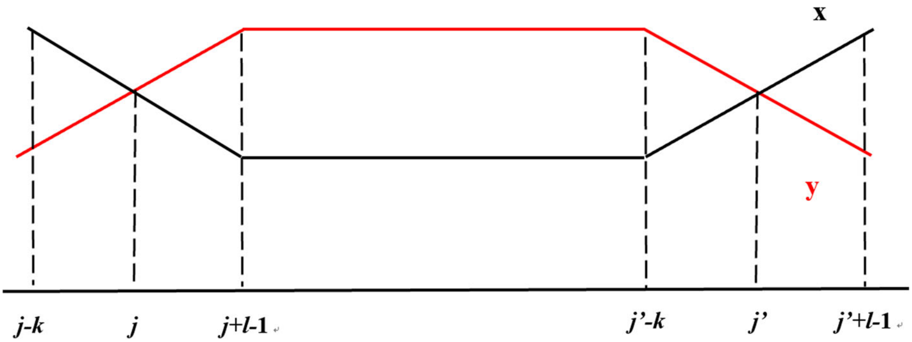

Fig. 1 Intersection of type (k, l)

Proposition 2.5 (Theorem 6.1 and Theorem 7.1 in [2]) *Let $\omega_1 < \omega_2$. If (1.1) have a subsolution $\underline{\mathbf{x}} = (\underline{x}_i)$ and a supersolution $\overline{\mathbf{x}} = (\overline{x}_i)$ exchanging rotation numbers ω_1 and ω_2, then φ_Δ has positive topological entropy and for each $\rho \in (\omega_1, \omega_2)$, there is a Birkhoff solution with rotation number ρ.*

It should be pointed out that Proposition 2.5 has provided great convenience for judging whether the system has positive topological entropy, which has been applied in [2–4, 16, 19, 22, 23, 25] under different background and conditions.

3 Intersections of Type (k, l)

In this section, we will define a new type of intersections for solutions of (1.1), which lead to the definition of generalized Birkhoff solutions. Later, we shall show our main lemmas by utilizing this type of intersections.

Definition 3.1 We say $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ have an intersection of type (k, l) if they intersect at $j, j' \in \mathbb{Z}$ with

$$y_{j+n} \leq x_{j+n}, \ n = -k, \cdots, -1; \ \ y_{j+n} \geq x_{j+n}, \ n = 0, \cdots, l-1,$$

and

$$y_{j'+n} \geq x_{j'+n}, \ n = -k, \cdots, -1; \ \ y_{j'+n} \leq x_{j'+n}, \ n = 0, \cdots, l-1.$$

Remark Let $k = l = 1$ in (1.1) and $\mathbf{x}, \mathbf{y}$ be two solutions of (1.1) with the same periodicity. It's obvious that they intersect if and only if they have an intersection of type (k, l), which means these two definitions are equivalent in this case.

Furthermore, we shall define generalized Birkhoff solutions for monotone recurrence relation (1.1).

Definition 3.2 A solution $\mathbf{x}$ of (1.1) is called a generalized Birkhoff solution if $\mathbf{x}$ and its any translation $\tau_{m,n}\mathbf{x}$ do not have an intersection of type (k, l).

It is easily seen that each Birkhoff solution is a generalized Birkhoff solution. We now proceed to divide this section into two subsections, each subsection consists of an essential lemma.

3.1 Constructions of Subsolutions and Supersolutions

Lemma 3.3 *Assume* $\mathbf{x}$ *is a* (p, q)*-periodic solution of (1.1) and* $a/b < c/d$ *with* $b, d > 0$*. If* $\tau_{b,a}\mathbf{x}$*,* $\tau_{d,c}\mathbf{x}$ *have an intersection of type* (k, l) *with* $\mathbf{x}$*, respectively, then there exist a subsolution and a supersolution exchanging rotation numbers* a/b *and* c/d*.*

Proof We may assume without loss of generality that $b, d > k + l$ for simplicity of presentation and proofs for $b, d \le k + l$ can be obtained by similar arguments. Our first goal is to construct a subsolution $\underline{\mathbf{x}} = (\underline{x}_i)$ with forward rotation number a/b and backward rotation number c/d.

Since $\tau_{d,c}\mathbf{x}$ have an intersection of type (k, l) with $\mathbf{x}$, there exists $j \in \mathbb{Z}$ such that $\tau_{d,c}\mathbf{x}$ intersects with $\mathbf{x}$ at j satisfying

$$x_{j-d+n} + c = (\tau_{d,c}\mathbf{x})_{j+n} \le x_{j+n}, \quad n = -k, \cdots, -1, \tag{3.1}$$

and

$$x_{j-d+n} + c = (\tau_{d,c}\mathbf{x})_{j+n} \ge x_{j+n}, \quad n = 0, \cdots, l - 1. \tag{3.2}$$

Let $\underline{x}_i = x_i$ for $j - d \le i \le j - 1$, and for $s \ge 1$, $1 \le t \le d$, define

$$\underline{x}_{j-sd-t} = \underline{x}_{j-t} - sc.$$

Claim. $\delta_i := \Delta(\underline{x}_{i-k}, \cdots, \underline{x}_i, \cdots, \underline{x}_{i+l}) \ge 0$ for all $i < j - l$.

Indeed, it suffices to consider $i = j - l - d, \cdots, i = j - l - 1$ by periodicity, and we shall prove it as follows due to the monotonicity hypothesis (H1).

For $i = j - l - d$ (See Fig. 2), we obtain by (3.2) with $n = 0$ that

$$\begin{aligned}
\delta_i &= \Delta(\underline{x}_{j-l-d-k}, \cdots, \underline{x}_{j-l-d}, \cdots, \underline{x}_{j-d}) \\
&= \Delta(x_{j-l-k} - c, \cdots, x_{j-l} - c, \cdots, x_{j-1} - c, x_{j-d}) \\
&\ge \Delta(x_{j-l-k} - c, \cdots, x_{j-l} - c, \cdots, x_{j-1} - c, x_j - c) \\
&= 0.
\end{aligned}$$

$\cdots\cdots$

For $i = j - d - 1$, we derive by (3.2) with $n = 0, \cdots, l - 1$ that

$$\begin{aligned}
\delta_i &= \Delta(\underline{x}_{j-d-1-k}, \cdots, \underline{x}_{j-d-1}, \cdots, \underline{x}_{j-d-1+l}) \\
&= \Delta(x_{j-k-1} - c, \cdots, x_{j-1} - c, x_{j-d}, \cdots, x_{j-d+l-1}) \\
&\ge \Delta(x_{j-k-1} - c, \cdots, x_{j-1} - c, x_j - c, \cdots, x_{j+l-1} - c) \\
&= 0.
\end{aligned}$$

For $i = j - d$, we derive by (3.2) with $n = -k, \cdots, -1$ that

$$\begin{aligned}
\delta_i &= \Delta(\underline{x}_{j-d-k}, \cdots, \underline{x}_{j-d}, \cdots, \underline{x}_{j-d+l}) \\
&= \Delta(x_{j-k} - c, \cdots, x_{j-1} - c, x_{j-d}, \cdots, x_{j-d+l}) \\
&\ge \Delta(x_{j-d-k}, \cdots, x_{j-d-1}, x_{j-d}, \cdots, x_{j-d+l}) \\
&= 0.
\end{aligned}$$

$\cdots\cdots$

For $i = j - d + k - 1$, we have by (3.1) with $n = -1$ that

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{j-d-1}, \cdots, \underline{x}_{j-d+k-1}, \cdots, \underline{x}_{j-d+k+l-1}) \\ &= \Delta(x_{j-1} - c, x_{j-d}, \cdots, x_{j-d+k-1}, \cdots, x_{j-d+k+l-1}) \\ &\geq \Delta(x_{j-d-1}, x_{j-d}, \cdots, x_{j-d+k-1}, \cdots, x_{j-d+k+l-1}) \\ &= 0.\end{aligned}$$

For $i = j - d + k$, we have

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{j-d}, \cdots, \underline{x}_{j-d+k}, \cdots, \underline{x}_{j-d+k+l}) \\ &= \Delta(x_{j-d}, \cdots, x_{j-d+k}, \cdots, x_{j-d+k+l}) \\ &= 0.\end{aligned}$$

$$\cdots\cdots$$

For $i = j - l - 1$, we have

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{j-l-1-k}, \cdots, \underline{x}_{j-l-1}, \cdots, \underline{x}_{j-1}) \\ &= \Delta(x_{j-l-1-k}, \cdots, x_{j-l-1}, \cdots, x_{j-1}) \\ &= 0.\end{aligned}$$

Consequently, the claim is true.

Since $\tau_{b,a}\mathbf{x}$ have an intersection of type (k, l) with $\mathbf{x}$ and $\mathbf{x}$ is (p, q)-periodic, there exists $m \gg j$ such that $\tau_{b,a}\mathbf{x}$ intersects with $\mathbf{x}$ at $m + b$ satisfying

$$x_{m+n} + a = (\tau_{b,a}\mathbf{x})_{m+b+n} \leq x_{m+b+n}, \quad n = -k, \cdots, -1, \tag{3.3}$$

and

$$x_{m+n} + a = (\tau_{b,a}\mathbf{x})_{m+b+n} \geq x_{m+b+n}, \quad n = 0, \cdots, l - 1. \tag{3.4}$$

Let $\underline{x}_i = x_i$ for $j \leq i \leq m + b - 1$. Moreover, we define

$$\underline{x}_{m+sb+t} = \underline{x}_{m+t} + sa \quad \text{for } s \geq 1 \text{ and } \quad 0 \leq t < b.$$

Note that

$$\Delta(\underline{x}_{i-k}, \cdots, \underline{x}_i, \cdots, \underline{x}_{i+l}) = \Delta(x_{i-k}, \cdots, x_i, \cdots, x_{i+l}) = 0 \text{ for } j - l \leq i < m + k.$$

To obtain $\underline{\mathbf{x}}$ is a subsolution, we only have to show

$$\delta_i := \Delta(\underline{x}_{i-k}, \cdots, \underline{x}_i, \cdots, \underline{x}_{i+l}) \geq 0 \quad \text{for all } \ i \geq m + k.$$

Taking the periodicity into consideration, it is sufficient to verify by (H1) the above inequalities hold for $i = m + k, \cdots, m + k + b - 1$.

For $i = m + k$, we have

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_m, \cdots, \underline{x}_{m+k}, \cdots, \underline{x}_{m+k+l}) \\ &= \Delta(x_m, \cdots, x_{m+k}, \cdots, x_{m+k+l}) \\ &= 0.\end{aligned}$$

$$\cdots\cdots$$

For $i = m + b - l - 1$, we have

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{m+b-l-1-k}, \cdots, \underline{x}_{m+b-l-1}, \cdots, \underline{x}_{m+b-1}) \\ &= \Delta(x_{m+b-l-1-k}, \cdots, x_{m+b-l-1}, \cdots, x_{m+b-1}) \\ &= 0.\end{aligned}$$

For $i = m + b - l$, we derive by (3.4) with $n = 0$ that

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{m+b-l-k}, \cdots, \underline{x}_{m+b-l}, \cdots, \underline{x}_{m+b}) \\ &= \Delta(x_{m+b-l-k}, \cdots, x_{m+b-l}, \cdots, x_{m+b-1}, x_m + a) \\ &\geq \Delta(x_{m+b-l-k}, \cdots, x_{m+b-l}, \cdots, x_{m+b-1}, x_{m+b}) \\ &= 0.\end{aligned}$$

$$\cdots\cdots$$

For $i = m + b - 1$, we deduce from (3.4) with $n = 0, \cdots, l - 1$ that

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{m+b-1-k}, \cdots, \underline{x}_{m+b-1}, \cdots, \underline{x}_{m+b+l-1}) \\ &= \Delta(x_{m+b-1-k}, \cdots, x_{m+b-1}, x_m + a, \cdots, x_{m+l-1} + a) \\ &\geq \Delta(x_{m+b-1-k}, \cdots, x_{m+b-1}, x_{m+b}, \cdots, x_{m+b+l-1}) \\ &= 0.\end{aligned}$$

For $i = m + b$ (See Fig. 2), we deduce from (3.3) with $n = -k, \cdots, -1$ that

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{m+b-k}, \cdots, \underline{x}_{m+b}, \cdots, \underline{x}_{m+b+l}) \\ &= \Delta(x_{m+b-k}, \cdots, x_{m+b-1}, x_m + a, \cdots, x_{m+l} + a) \\ &\geq \Delta(x_{m-k} + a, \cdots, x_{m-1} + a, x_m + a, \cdots, x_{m+l} + a) \\ &= 0.\end{aligned}$$

$$\cdots\cdots$$

For $i = m + b + k - 1$, we have by (3.3) with $n = -1$ that

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{m+b-1}, \cdots, \underline{x}_{m+b+k-1}, \cdots, \underline{x}_{m+b+k+l-1}) \\ &= \Delta(x_{m+b-1}, x_m + a, \cdots, x_{m+k-1} + a, \cdots, x_{m+k+l-1} + a) \\ &\geq \Delta(x_{m-1} + a, x_m + a, \cdots, x_{m+k-1} + a, \cdots, x_{m+k+l-1} + a) \\ &= 0.\end{aligned}$$

In conclusion, we have constructed a subsolution $\underline{\mathbf{x}} = (\underline{x}_i)$ with forward rotation number a/b and backward rotation number c/d. Analogously, we can also construct a supersolution $\overline{\mathbf{x}} = (\overline{x}_i)$ with forward rotation number c/d and backward rotation number a/b. This completes the proof of Lemma 3.3. □

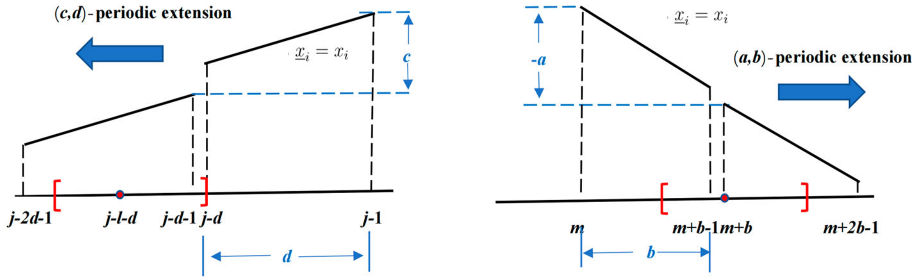

Fig. 2 $\underline{\mathbf{x}}$ at $i = j - l - d$ and $i = m + b$

3.2 Constructions of (r, s)-Periodic Solutions

Lemma 3.4 *Let $r \in \mathbb{Z}$, $s \in \mathbb{N}$ and $\mathbf{x}$ be a solution of (1.1). If $\tau_{s,r}\mathbf{x}$ and $\mathbf{x}$ have an intersection of type (k, l), then there exists a (r, s)-periodic solution of (1.1).*

Proof For convenience, we may assume $s > k + l$ as in Lemma 3.3. Since $\tau_{s,r}\mathbf{x}$ and $\mathbf{x}$ have an intersection of type (k, l), there exist $j, j' \in \mathbb{Z}$ satisfying:

$$x_{j-s+n} + r = (\tau_{s,r}\mathbf{x})_{j+n} \le x_{j+n}, \ n = -k, \cdots, -1, \tag{3.5}$$

$$x_{j-s+n} + r = (\tau_{s,r}\mathbf{x})_{j+n} \ge x_{j+n}, \ n = 0, \cdots, l-1, \tag{3.6}$$

$$x_{j'-s+n} + r = (\tau_{s,r}\mathbf{x})_{j'+n} \ge x_{j'+n}, \ n = -k, \cdots, -1, \tag{3.7}$$

and

$$x_{j'-s+n} + r = (\tau_{s,r}\mathbf{x})_{j'+n} \le x_{j'+n}, \ n = 0, \cdots, l-1. \tag{3.8}$$

Our task is now to construct by (3.5) and (3.6) a (r, s)-periodic subsolution $\underline{\mathbf{x}} = (\underline{x}_i)$. We define

$$\underline{x}_i = x_i \ \text{ and } \ \underline{x}_{i+ms} = \underline{x}_i + mr, \quad \text{for } \ j - s \le i \le j - 1, \ m \in \mathbb{Z}.$$

Then $\underline{\mathbf{x}}$ is (r, s)-periodic, and it remains to prove by (H1) that

$$\delta_i := \Delta(\underline{x}_{i-k}, \cdots, \underline{x}_i, \cdots, \underline{x}_{i+l}) \ge 0, \ \text{ for } i = j - s, \cdots, j - 1.$$

Indeed, for $i = j - s$ (See Figure 3), we conclude from (3.5) with $n = -k, \cdots, -1$ that

$$\begin{aligned}
\delta_i &= \Delta(\underline{x}_{j-s-k}, \cdots, \underline{x}_{j-s}, \cdots, \underline{x}_{j-s+l}) \\
&= \Delta(x_{j-k} - r, \cdots, x_{j-1} - r, x_{j-s}, \cdots, x_{j-s+l}) \\
&\ge \Delta(x_{j-s-k}, \cdots, x_{j-s-1}, x_{j-s}, \cdots, x_{j-s+l}) \\
&= 0.
\end{aligned}$$

$\cdots\cdots$

For $i = j - s + k - 1$, we conclude from (3.5) with $n = -1$ that

$$\begin{aligned}
\delta_i &= \Delta(\underline{x}_{j-s-1}, \cdots, \underline{x}_{j-s+k-1}, \cdots, \underline{x}_{j-s+k+l-1}) \\
&= \Delta(x_{j-1} - r, x_{j-s}, \cdots, x_{j-s+k-1}, \cdots, x_{j-s+k+l-1}) \\
&\ge \Delta(x_{j-s-1}, x_{j-s}, \cdots, x_{j-s+k-1}, \cdots, x_{j-s+k+l-1})
\end{aligned}$$

$$= 0.$$

For $i = j - s + k$, we get

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{j-s}, \cdots, \underline{x}_{j-s+k}, \cdots, \underline{x}_{j-s+k+l})\\ &= \Delta(x_{j-s}, \cdots, x_{j-s+k}, \cdots, x_{j-s+k+l})\\ &= 0.\end{aligned}$$

$$\cdots\cdots$$

For $i = j - l - 1$, we get

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{j-l-1-k}, \cdots, \underline{x}_{j-l-1}, \cdots, \underline{x}_{j-1})\\ &= \Delta(x_{j-l-1-k}, \cdots, x_{j-l-1}, \cdots, x_{j-1})\\ &= 0.\end{aligned}$$

For $i = j - l$, we get from (3.6) with $n = 0$ that

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{j-l-k}, \cdots, \underline{x}_{j-l}, \cdots, \underline{x}_{j})\\ &= \Delta(x_{j-l-k}, \cdots, x_{j-l}, \cdots, x_{j-1}, x_{j-s} + r)\\ &\geq \Delta(x_{j-l-k}, \cdots, x_{j-l}, \cdots, x_{j-1}, x_{j})\\ &= 0.\end{aligned}$$

$$\cdots\cdots$$

For $i = j - 1$, we have by (3.6) with $n = 0, \cdots, l - 1$ that

$$\begin{aligned}\delta_i &= \Delta(\underline{x}_{j-1-k}, \cdots, \underline{x}_{j-1}, \cdots, \underline{x}_{j-1+l})\\ &= \Delta(x_{j-1-k}, \cdots, x_{j-1}, x_{j-s} + r, \cdots, x_{j-s+l-1} + r)\\ &\geq \Delta(x_{j-1-k}, \cdots, x_{j-1}, x_{j}, \cdots, x_{j+l-1})\\ &= 0.\end{aligned}$$

Therefore, we have constructed a (r, s)-periodic subsolution $\underline{\mathbf{x}}$ of (1.1). Using the similar argument as above, we can obtain by (3.7) and (3.8) a (r, s)-periodic supersolution $\overline{\mathbf{x}}$ of (1.1). Because both $\underline{\mathbf{x}}$ and $\overline{\mathbf{x}}$ are (r, s)-periodic, we can assume, by making an integer translation if necessary, that $\underline{x}_i \leq \overline{x}_i$ for all i. It follows from Proposition 2.4 that there is a (r, s)-periodic solution of (1.1). □

4 Proof of Main Theorem

By means of preceding lemmas, we are now in a position to present the proof of Main Theorem.

Proof of Main Theorem: Since $\mathbf{x}$ is a (p, q)-periodic solution which is not a generalized Birkhoff solution, there exist $0 < m < q$ and $n \in \mathbb{Z}$ such that $\tau_{m,n}\mathbf{x}$ and $\mathbf{x}$ have an intersection of type (k, l).

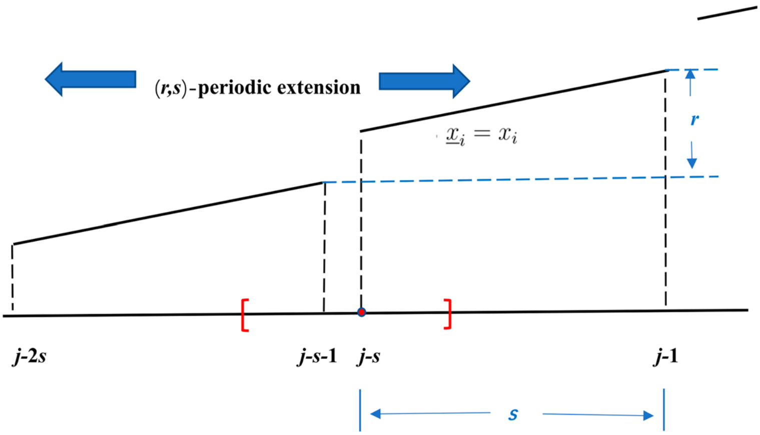

Fig. 3 $\underline{\mathbf{x}}$ at $i = j - s$

Let $m' = q - m, n' = p - n$. Since p, q are relatively prime, we obtain $n/m \neq n'/m'$. Otherwise, a simple calculation gives that $p/q = n/m$. Then there exists $N \geq 1$ such that $(n, m) = N(p, q)$, which contradicts to $0 < m < q$.

Now we denote

$$(a, b) = (n, m) \text{ and } (c, d) = (n', m') \text{ if } \frac{n}{m} < \frac{n'}{m'}, \tag{4.1}$$

while

$$(a, b) = (n', m') \text{ and } (c, d) = (n, m) \text{ if } \frac{n}{m} > \frac{n'}{m'}. \tag{4.2}$$

By definition, p/q is between n/m and n'/m'. As a result, $p/q \in (a/b, c/d)$ and $I(p/q) \subseteq [a/b, c/d]$.

Note that $\tau_{m,n}\mathbf{x}$ and $\mathbf{x}$ have an intersection of type (k, l). It follows that $\mathbf{x}$ and $\tau_{-m,-n}\mathbf{x}$ have an intersection of type (k, l) as well. We derive by periodicity that

$$\tau_{m',n'}\mathbf{x} = \tau_{q-m,p-n}\mathbf{x} = \tau_{-m,-n}(\tau_{q,p}\mathbf{x}) = \tau_{-m,-n}\mathbf{x},$$

hence $\tau_{m',n'}\mathbf{x}$ and $\mathbf{x}$ have an intersection of type (k, l).

Due to (4.1) and (4.2), $\tau_{b,a}\mathbf{x}$ and $\tau_{d,c}\mathbf{x}$ have an intersection of type (k, l) with $\mathbf{x}$, respectively. It follows from Lemma 3.3 that there exist a subsolution and a supersolution exchanging rotation numbers a/b and c/d, which yields φ_Δ has positive topological entropy (We finish the proof of (i)) and each $\rho \in (a/b, c/d)$ can be realized by a Birkhoff solution according to Proposition 2.5. As a consequence, $(a/b, c/d) \subseteq \rho(\Delta)$.

In order to verify $I(p/q) \subseteq \rho(\Delta)$, what is left is to show $a/b, c/d \in \rho(\Delta)$, and it is a straightforward result from Lemma 3.4 (The proof of (ii) is complete). □

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