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Non-collision Orbits for a Class of Singular Hamiltonian Systems on the Plane with Weak Force Potentials

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Abstract

We study the existence of non-collision orbits for a class of singular Hamiltonian systems

$$\ddot{q} + V'(q) = 0$$

where $q : \mathbb{R} \longrightarrow \mathbb{R}^2$ and $V \in C^2(\mathbb{R}^2 \backslash \{e\}, \mathbb{R})$ is a potential with a singularity at a point $e \neq 0$. We consider V which behaves like $-1/|q-e|^\alpha$ as $q \to e$ with $\alpha \in]0, 2[$. Under the assumption that 0 is a strict global maximum for V, we establish the existence of a homoclinic orbit emanating from 0. Moreover, in case $V(q) \longrightarrow 0$ as $|q| \to +\infty$, we prove the existence of a heteroclinic orbit "at infinity" i.e. a solution q such that

$$\lim_{t\to-\infty} q(t) = 0, \quad \lim_{t\to+\infty} |q(t)| = +\infty \text{ and } \lim_{t\to\pm\infty} \dot{q}(t) = 0.$$

Keywords Hamiltonian systems · Homoclinic and heteroclinic orbits · Minimization methods

Mathematics Subject Classification 34C37 · 37C29

1 Introduction

In this paper we consider the second order Hamiltonian system

$$\ddot{q} + V'(q) = 0 \tag{HS}$$

where $q : \mathbb{R} \longrightarrow \mathbb{R}^2$ and $V \in C^2(\mathbb{R}^2 \backslash \{e\}, \mathbb{R})$ has a singularity at a point $e \neq 0$ such that

$$V(q) \sim -\frac{1}{|q-e|^\alpha} \quad \text{as } q \to e \text{ with } \alpha \in]0, 2[. \tag{1}$$

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We will assume that V has a strict global maximum at $q = 0$. So 0 is a equilibrium point for (HS).
Our goal in the first part is studying the existence of nontrivial homoclinic solutions to 0 of (HS), i.e. solutions q of (HS) such that

$$q \neq 0 \quad \text{and} \quad \lim_{t\to\pm\infty} q(t) \equiv q(\pm\infty) = 0 = \dot{q}(\pm\infty).$$

We note that the order α in (1) plays an important role and we consider the existence of homoclinic solutions of (HS) under weak force case ($\alpha \in]0, 2[$). This case has been studied in several works which deal via variational methods with the periodic problem. See, e.g., [1], [2], [5], [9], [15], [17]. Let us now define the strong force condition:

(SF) There exists a neighbourhood Ω of e in $\mathbb{R}^2$ and $U \in C^1(\Omega \setminus \{e\}, \mathbb{R})$ such that

$$|U(q)| \to \infty \qquad \text{as } q \to e,$$
$$-V(q) \geq |U'(q)|^2 \quad \text{for all } q \in \Omega \setminus \{e\}.$$

Condition (SF) was introduced by Gordon [10]. For a potential $V(q) \sim -\dfrac{1}{|q-e|^\alpha}$ as $q \to e$, (SF) is satisfied if and only if $\alpha \geq 2$. In fact, for $\alpha \geq 2$ we can take $U(q) = -2^{-1} \ln |q - e|$. The major role of (SF) is the following property.

Lemma 1.1 *Assume (SF) and $V(q) \to -\infty$ as $q \to e$. Let $a < b \in R$ and $(q_m) \subset H^1([a, b], \Omega \setminus \{e\})$ which converges weakly in $H^1([a, b], \mathbb{R}^2)$ to q such that $q(t_0) = e$ for some $t_0 \in [a, b]$. Then*

$$-\int_a^b V(q_m)dt \longrightarrow +\infty$$

(and therefore $\displaystyle\int_a^b \left[\frac{1}{2}|\dot{q}_m|^2 - V(q_m)\right]dt \longrightarrow +\infty$*).*

The proof of this lemma can be found in ([11], Lemma 2.1) or in [13]. As a consequence, if (SF) holds then functions with bounded energy are uniformly away from the singularity e. Therefore, in such case, a standard variational arguments in [13] provided the existence of a a pair of homoclinic orbits that wind respectively around the singularity e in a positive and negative sense. These solutions were obtained by minimizing the energy functional

$$I(q) = \int_{\mathbb{R}} \left[\frac{1}{2}|\dot{q}|^2 - V(q)\right]dt$$

on classes of sets with a fixed winding number around e (see also [6, 7] for multiplicity results). If this condition is dropped (weak force case), Rabinowitz [13] proved the existence of a "generalized" homoclinic solution of (HS) which may pass through the singularity.

In $\mathbb{R}^N$ with $N \geq 3$, the existence of homoclinic solutions of (HS) was proved in [16] for strong force potentials (see also [8] in the case of time periodic potentials) and [3, 14] for weak force potentials like (1.1). In [3, 14], the authors introduced a strong force perturbed potential V_ε for $\varepsilon \in]0, 1]$ such that $V_\varepsilon(q) = V(q) - \varepsilon/|q - e|^2$ near $q = e$ and proved through a min-max method from Bahri-Rabinowitz [4] the existence of non-collision solutions for approximated differential problems. Then they passed to the limit as $\varepsilon \to 0$ with the aid of appropriate estimates to obtain a generalized homoclinic solution. In [3] we studied the Morse index of approximated functionals at critical points to estimate the number of collisions. In particular we established the existence of non-collision homoclinic solution for $\alpha \in]1, 2[$ i.e.

$q(t) \neq e$ for all $t \in \mathbb{R}$, while in [14] this result is obtained by assuming that $V(q)$ is radially symmetric near $q = e$.

The main purpose of Sect. 2 is to prove the existence of non-collision homoclinic orbits of (HS) in $\mathbb{R}^2$ for weak force potentials. By exploiting the topology of the plane and using a minimization method, we first show the existence of a generalized homoclinic solution of (HS) as a limit of solutions of perturbed problems with boundary conditions. Then and for the regularity of this solution, we will use a Tanaka's rescaling argument to prove some additional properties of approximated solutions near collisions, and we will prove how the generalized homoclinic solution obtained is actually a non-collision orbit in the case $\alpha \in]1, 2[$.

In Sect. 3, we assume that V has another global maximum at infinity i.e. $\lim_{|x| \to +\infty} V(x) = V(0)$ and we study the existence of a heteroclinic orbit "at infinity" i.e. a solution q of (HS) satisfying

$$q(-\infty) = 0, \ |q(+\infty)| = +\infty \quad \text{and} \quad \dot{q}(\pm\infty) = 0.$$

The problem in $\mathbb{R}^N$ was treated by Serra in [14] for regular potentials where $V(q) \sim -a/|q|^b$ as $|q| \to +\infty$ with $a, \ b > 0$. He also treated the case of singular potentials which behaves like (1) when $N \geq 3$ and established the existence of non-collision orbits using some results from [15] on the analysis of collisions solutions of minimization problems. In the present paper we deal with the case $N = 2$ and we will perturb V near e with a strong force term to get the existence of sequence (q_n) of heteroclinic orbits at infinity for perturbed problems. We obtain uniform estimates to show that (q_n) converges to a generalized solution. Some local properties of q_n near collisions and the fact that q_n is obtained via a minimization procedure permit us to obtain a non-collision heteroclinic solution at infinity.

2 Existence of Homoclinic Orbits

In this section, we consider the existence of a homoclinic solution of (HS) where the potential V satisfies the following assumptions:

(V1) $V \in C^2(\mathbb{R}^2 \setminus \{e\}, \mathbb{R})$ for some $e \neq 0$;
(V2) $V(q) < V(0) = 0$ for all $q \in \mathbb{R}^2 \setminus \{0, e\}$;
(V3) V is of the form

$$V(q) = -\frac{1}{|q - e|^\alpha} + W(q),$$

with $\alpha \in]0, 2[$ and W is such that

$$|q - e|^{\alpha - \nu} W(q), \ \ |q - e|^{\alpha - \nu + 1} W'(q) \ \text{ and } \ |q - e|^{\alpha - \nu + 2} W''(q) \longrightarrow 0 \text{ as } q \to e$$

for some $\nu \in]0, \alpha[$;

(V4) There are $R > 2|e|$ and a function $W_\infty \in C^1(\mathbb{R}^2, \mathbb{R})$ such that

$|W_\infty(q)| \longrightarrow +\infty$ as $|q| \to +\infty$ and $-V(q) \geq |W'_\infty(q)|$ for $|q| \geq R$.

Remark 2.1 i) The condition (V3) remains valid when $\nu = 0$. In particular it involves that $V \sim -1/|q - e|^\alpha$ near $q = e$ with $\alpha \in]0, 2[$.
ii) The condition (V4) concerns the behavior of the potential at infinity. It will be satisfied if for example $V(q) \sim -a|q|^\beta$ as $|q| \to +\infty$ where $a > 0$ and $\beta \geq -2$.

Our main result of this section is

Theorem 2.2 *Assume (V1)-(V4).*

1) *If* $\alpha \in]1, 2[$*, then (HS) possesses at least one non-collision homoclinic solution.*
2) *If* $\alpha \in]0, 1]$*, then (HS) possesses a non trivial generalized homoclinic solution* q *having at most one collision. Moreover, if* $q(t_0) = e$ *then* $q(t)$ *is a collision brake orbit, i.e.* $q(t + t_0) = q(t_0 - t)$ *for all* $t \in \mathbb{R}$*.*

Here, similarly as in [4], [17] for the periodic problem, we define a generalized homoclinic solution as a continuous function $q : \mathbb{R} \longrightarrow \mathbb{R}^2$ such that

(i) $\dot{q} \in L^2(\mathbb{R}, \mathbb{R}^2)$ and $I(q) < \infty$;
(ii) $D = \{t \in \mathbb{R},\ q(t) = e\}$ is a set of measure 0;
(iii) $q \in C^2(\mathbb{R} \setminus D, \mathbb{R}^2)$ and satisfies (HS) on $\mathbb{R} \setminus D$;
(iv) $\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = 0$ for $t \in \mathbb{R}\backslash D$;
(v) $q(t) \longrightarrow 0$ and $\dot{q}(t) \longrightarrow 0$ as $t \to \pm\infty$.

If $D = \emptyset$, q is a classical (non-collision) homoclinic solution.

Remark 2.3 Since V is independent of t, $q(-t)$ is a homoclinic solution of (HS) whenever $q(t)$ is a homoclinic solution.

The proof of Theorem 2.2. is divided in various steps. We shall construct a homoclinic solution of (HS) as a limit of solutions of approximate value problems. We started by modifying the potential V near e. For $\varepsilon \in]0, 1]$, we define $V_\varepsilon \in C^2(\mathbb{R}^2\backslash\{e\}, \mathbb{R})$ such that $V_1 \leq V_\varepsilon \leq V$ and

$$V_\varepsilon(q) = \begin{cases} V(q) - \dfrac{\varepsilon}{|q-e|^2} & \text{if } \ 0 < |q - e| \leq |e|/4, \\ 0 & \text{if } \ |q - e| \geq |e|/2. \end{cases}$$

Remark that $V_\varepsilon(q) \sim -\dfrac{\varepsilon}{|q-e|^2}$ as $q \to e$. So V_ε satisfies the strong force condition.
Let $(\varepsilon_n)_{n\in\mathbb{N}^*} \subset]0, 1]$ be a non-increasing sequence converging to 0. We consider for each $n \in \mathbb{N}^*$ the Dirichlet boundary value problem

$$\begin{cases} \ddot{q} + V'_{\varepsilon_n}(q) = 0 \quad \text{in } \ ]0, n[, \\ q(0) = q(n) = 0. \end{cases} \tag{D$_n$}$$

The corresponding functional is

$$I_{0,n}(q) = \int_0^n \Big[\frac{1}{2}|\dot{q}|^2 - V_{\varepsilon_n}(q)\Big]dt \in C^1(\Lambda_n, \mathbb{R})$$

where

$$\Lambda_n = \{q \in H_0^1([0, n], \mathbb{R}^2); \quad q(t) \neq e,\ \forall\, t \in [0, n]\}.$$

Let $\mathrm{ind}_{z_0}(\gamma)$ denote the winding number of a closed curve in $\mathbb{C}$ around a point z_0. That is

$$\mathrm{ind}_{z_0}(\gamma) = \frac{1}{2i\pi}\int_\gamma \frac{dz}{z - z_0}$$

which is a integer representing the number of counterclockwise turns that γ makes around z_0.

A critical point of $I_{0,n}$ will be found as a minimizer of $I_{0,n}$ over the set

$$\Gamma_n^{\pm 1} = \{q \in \Lambda_n,\ \mathrm{ind}_e(q) = \pm 1\}.$$

Clearly $\Gamma_n^{\pm 1} \neq \emptyset$, so we can define

$$c_n^{\pm 1} = \inf_{q \in \Gamma_n^{\pm 1}} I_{0,n}(q). \tag{2}$$

We remark that, since $I_{0,n}(q) = I_{0,n}(q(n-.))$ for all $q \in \Lambda_n$, then $c_n^1 = c_n^{-1}$.

Proposition 2.4 *1) There exist $M_1,\ M_2 > 0$ such that*

$$0 < M_1 \leq c_n^1 \leq M_2, \quad \forall n \in \mathbb{N}^*. \tag{3}$$

2) For every $n \in \mathbb{N}^*$, there is $q_n \in \Gamma_n^1$ such that $I_{0,n}(q_n) = c_n^1$. Moreover q_n is a non trivial classical solution of (D_n).

Proof 1) Let $q \in \Gamma_n^1$. The fact that $\mathrm{ind}_e(q) = 1$ implies that $||q||_{L^\infty([0,n],\,\mathbb{R}^2)} \geq |e|$. Since $q(0) = q(n) = 0$, there exist $s_q < t_q$ such that

$$|q(s_q)| = \frac{|e|}{3},\ \ |q(t_q)| = \frac{2|e|}{3} \text{ and } \frac{|e|}{3} \leq |q(t)| \leq \frac{2|e|}{3} \text{ for all } t \in [s_q, t_q].$$

Using the Cauchy-Schwartz inequality, we have the general formula

$$\begin{aligned}\int_{t_1}^{t_2} \Big[\frac{1}{2}|\dot{u}|^2 - V(u)\Big]dt &\geq \frac{|u(t_2) - u(t_1)|^2}{2(t_2 - t_1)} + (t_2 - t_1) \min_{t \in [t_1, t_2]} -V(u(t)) \\ &\geq |u(t_2) - u(t_1)| \sqrt{2 \min_{t \in [t_1, t_2]} -V(u(t))}\end{aligned} \tag{4}$$

where $u \in H^1([t_1, t_2], \mathbb{R}^2)$.
We denote $c = \min_{\frac{|e|}{3} \leq |x| \leq \frac{2|e|}{3}} -V(x) > 0$. Then from (4), we get

$$\begin{aligned}I_{0,n}(q) &\geq \int_{s_q}^{t_q} \Big[\frac{1}{2}|\dot{q}|^2 - V(q)\Big]dt \\ &\geq \frac{|e|}{3}\sqrt{2c} = M_1.\end{aligned}$$

Thus by the arbitrariness of q, we obtain $c_n^1 \geq M_1 > 0$ for any $n \in \mathbb{N}^*$.
In order to prove that c_n^1 is bounded from above, let $q \in \Gamma_1^1$ and define

$$v_n(t) = \begin{cases} q(t) & \text{if } t \in [0, 1], \\ 0 & \text{if } t \in]1, n]. \end{cases}$$

Clearly $v_n \in \Gamma_n^1$ and then

$$\begin{aligned}c_n^1 \leq I_{0,n}(v_n) &= \int_0^1 \Big[\frac{1}{2}|\dot{q}|^2 - V_{\varepsilon_n}(q)\Big]dt \\ &\leq I_{0,1}(q).\end{aligned}$$

Therefore

$$c_n^1 \leq \inf_{q \in \Gamma_1^1} I_{0,1}(q) = M_2.$$

2) Let (u_m) be a minimizing sequence for c_n^1. We have from (3), (u_m) is bounded in $H_0^1([0,n],\ \mathbb{R}^2)$. It follows that along a subsequence (u_m) converge weakly in $H_0^1([0,n],\ \mathbb{R}^2)$ and uniformly in $[0,n]$ to a function q_n. Since $\int_0^n -V_{\varepsilon_n}(u_m)dt$ is bounded independently of

m and V_{ε_n} is a strong force, Lemma 1.1 shows that $q_n \in \Lambda_n$. Moreover we know that the winding number is continuous with respect to uniform convergence of curves. Therefore $\mathrm{ind}_e(q_n) = \lim_{m\to+\infty} \mathrm{ind}_e(u_m) = 1$ and so $q_n \in \Gamma_n^1$. Using the lower semi continuity of $I_{0,n}$, we get $I_{0,n}(q_n) \leq \liminf_{m\to+\infty} I_{0,n}(u_m) = c_n^1$. That is $I_{0,n}(q_n) = c_n^1$. Now in a standard way, we can see that q_n is a critical point of $I_{0,n}$ and then a nontrivial classical solution of (D_n). □

As a consequence of Proposition 2.4, we get the following estimates:

Lemma 2.5 *(i) There is a constant $C > 0$ which is independent of n such that for any $n \in \mathbb{N}^*$,*

$$||\dot{q}_n||_{L^2([0,n],\,\mathbb{R}^2)} \leq C; \quad \int_0^n -V(q_n)dt \leq C; \quad ||q_n||_{L^\infty([0,n],\,\mathbb{R}^2)} \leq C.$$

(ii) For every $n \in \mathbb{N}^*$, there is a constant $h_n > 0$ such that

$$\frac{1}{2}|\dot{q}_n(t)|^2 + V_{\varepsilon_n}(q_n(t)) = h_n, \quad \forall\, t \in [0, n].$$

Moreover, $h_n = \frac{1}{2}|\dot{q}_n(0)|^2 = \frac{1}{2}|\dot{q}_n(n)|^2 \longrightarrow 0.$

Since $q_n \in \Gamma_n^1$, we have $\max_{t\in[0,n]} |q_n(t)| > |e|/4$. Otherwise we would have $\mathrm{ind}_e(q_n) = 0$. Then we can find numbers $\tau_n^1,\ \tau_n^2 \in]0, n[$ such that

$$|q_n(\tau_n^1)| = |q_n(\tau_n^2)| = |e|/4 \quad \text{and } |q_n(t)| < |e|/4 \quad \text{if} \quad t \in [0, \tau_n^1[\cup]\tau_n^2, n].$$

Note that in [3], it was also proved the existence of approximated solution q_n of (D_n) in $\mathbb{R}^N$ $(N \geq 3)$ such that

* $\max_{t\in[0,n]} |q_n(t)| > \rho$ where $\rho > 0$ is a constant;
* $|\dot{q}_n(0)| \to 0$ and $|\dot{q}_n(n)| \to 0$.

Using the continuity theorem of solutions with respect to initials conditions, we can see in a similar way to Lemma 2.7 in [3],

$$\tau_n^1 \to \infty \quad \text{and } n - \tau_n^2 \to \infty.$$

Next we define

$$\tilde{q}_n(t) = \begin{cases} q_n(t + \tau_n^1) & \text{if } t \in [-\tau_n^1, n - \tau_n^1], \\ 0 & \text{if } t \in \mathbb{R} \setminus [-\tau_n^1, n - \tau_n^1]. \end{cases} \tag{5}$$

Clearly $|\tilde{q}_n(0)| = |e|/4$ and $\tilde{q}_n$ verifies

$$\ddot{\tilde{q}}_n + V'_{\varepsilon_n}(\tilde{q}_n) = 0 \quad \text{in }] - \tau_n^1,\ n - \tau_n^1[,$$
$$\frac{1}{2}|\dot{\tilde{q}}_n|^2 + V_{\varepsilon_n}(\tilde{q}_n) = h_n \quad \text{in }] - \tau_n^1,\ n - \tau_n^1[.$$

By (i) of Lemma 2.5, we can extract a subsequence -still denoted by $\tilde{q}_n$- which converges in $C_{loc}(\mathbb{R},\ \mathbb{R}^2)$ to some function $\tilde{q} \in C(\mathbb{R},\ \mathbb{R}^2) \cap L^\infty(\mathbb{R},\ \mathbb{R}^2)$ with $\dot{\tilde{q}} \in L^2(\mathbb{R},\ \mathbb{R}^2)$. Since $-\tau_n^1 \to -\infty$ and $n - \tau_n^1 \to +\infty$, we can see $\tilde{q}$ is a non trivial generalized homoclinic solution of (HS). The complete proofs to Lemma 2.5 and the last statements are ommited as they are similar to its analogues in [3].

In what follows, we focus our attention to study the regularity of $\tilde{q}$. First we state some further properties of $\tilde{q}_n$ and $\tilde{q}$ near the singularity.
Let $t \in \mathbb{R}$ such that $|\tilde{q}_n(t) - e| < |e|/4$. From the definition of V_{ε_n}, $\tilde{q}_n(t)$ verifies

$$\ddot{\tilde{q}}_n + \alpha \frac{\tilde{q}_n - e}{|\tilde{q}_n - e|^{\alpha+2}} + W'(\tilde{q}_n) + 2\varepsilon_n \frac{\tilde{q}_n - e}{|\tilde{q}_n - e|^4} = 0, \tag{6}$$

$$\frac{1}{2}|\dot{\tilde{q}}_n|^2 - \frac{1}{|\tilde{q}_n - e|^\alpha} + W(\tilde{q}_n) - \frac{\varepsilon_n}{|\tilde{q}_n - e|^2} = h_n. \tag{7}$$

Then

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}|\tilde{q}_n(t) - e|^2 &=< \ddot{\tilde{q}}_n, \tilde{q}_n - e > + |\dot{\tilde{q}}_n|^2 \\ &= \frac{2-\alpha}{|\tilde{q}_n - e|^\alpha} - W'(\tilde{q}_n)(\tilde{q}_n - e) - 2W(\tilde{q}_n) + 2h_n \\ &= \frac{1}{|\tilde{q}_n - e|^\alpha}[2 - \alpha - |\tilde{q}_n - e|^\alpha W'(\tilde{q}_n)(\tilde{q}_n - e) - 2|\tilde{q}_n - e|^\alpha W(\tilde{q}_n) \\ &\quad + 2h_n|\tilde{q}_n - e|^\alpha].\end{aligned}$$

By (V3) (see Remark 2.1 i)) and the fact that $h_n \to 0$, we can find $0 < \delta < |e|/4$ such that for sufficiently large n,

$$\frac{1}{2}\frac{d^2}{dt^2}|\tilde{q}_n(t) - e|^2 > 0 \quad \text{if} \quad |\tilde{q}_n(t) - e| < \delta. \tag{8}$$

Similarly, if $\tilde{q}(t) \neq e$ then $\tilde{q}(t)$ satisfies (HS) and of energy 0. From this, we obtain

$$\frac{1}{2}\frac{d^2}{dt^2}|\tilde{q}(t) - e|^2 = \frac{1}{|\tilde{q} - e|^\alpha}[2 - \alpha - |\tilde{q} - e|^\alpha W'(\tilde{q})(\tilde{q} - e) - 2|\tilde{q} - e|^\alpha W(\tilde{q})].$$

Thus the property (8) holds also for $\tilde{q}$, i.e.

$$\frac{1}{2}\frac{d^2}{dt^2}|\tilde{q}(t) - e|^2 > 0 \quad \text{if} \quad 0 < |\tilde{q}(t) - e| < \delta. \tag{9}$$

Taking into account the property (ii) of a generalized solution, (9) implies that the collisions times of $\tilde{q}$ (if they exist) are isolated.

Now we suppose that $\tilde{q}$ has a collision at $t = \tilde{t}$ i.e. $\tilde{q}(\tilde{t}) = e$ for some $\tilde{t} \in \mathbb{R}$. We will study the angle which describes $\tilde{q}_n(t)$ around e when t is near $\tilde{t}$. In particular we will show that $\tilde{q}_n$ have one self intersection if $\alpha \in]1, 2[$.

Since $\tilde{q}(t) \longrightarrow 0$ as $t \to \pm\infty$, there exist $\tau_1 < \tilde{t} < \tau_2$ such that

$$|\tilde{q}(\tau_1) - e| = |\tilde{q}(\tau_2) - e| = \frac{\delta}{2} \quad \text{and} \quad 0 < |\tilde{q}(t) - e| < \frac{\delta}{2} \quad \forall\, t \in]\tau_1, \tau_2[\setminus\{\tilde{t}\}.$$

Thus for sufficiently large n, we have

$$|\tilde{q}_n(\tau_i) - e| \geq \frac{\delta}{4} \quad \text{for} \quad i = 1, 2 \tag{10}$$

and

$$|\tilde{q}_n(t) - e| < \delta \quad \forall\, t \in [\tau_1, \tau_2]. \tag{11}$$

Let $t_n \in [\tau_1, \tau_2]$ and $\delta_n > 0$ such that $\delta_n = |\tilde{q}_n(t_n) - e| = \min_{t \in [\tau_1, \tau_2]} |\tilde{q}_n(t) - e|$.

Clearly $\delta_n \leq |\tilde{q}_n(\tilde{t}) - e| \longrightarrow |\tilde{q}(\tilde{t}) - e| = 0$. So $\delta_n \longrightarrow 0$. Moreover, up a subsequence, we have $t_n \longrightarrow \tilde{t}$.

By (8), we have

$$\frac{d}{dt}|\tilde{q}_n(t) - e| < 0 \quad \forall t \in [\tau_1, t_n[, \tag{12}$$

$$\frac{d}{dt}|\tilde{q}_n(t) - e| > 0 \quad \forall t \in]t_n, \tau_2]. \tag{13}$$

In the sequel we use a rescaling argument as in [17] and we introduce the function

$$x_n(s) = \delta_n^{-1}\Big[\tilde{q}_n\left(\delta_n^{\frac{\alpha+2}{2}} s + t_n\right) - e\Big], \quad s \in \mathbb{R}.$$

Remark that

$$|x_n(0)| = 1 \quad \text{and} \quad (x_n(0),\ \dot{x}_n(0)) = 0. \tag{14}$$

Let $l > 0$. For sufficiently large n, since $\delta_n^{\frac{\alpha+2}{2}} s + t_n \in [\tau_1, \tau_2]$ for $s \in [-l, l]$, we have from (11) and (6)-(7),

$$\ddot{x}_n(s) + \alpha \frac{x_n}{|x_n|^{\alpha+2}} + \delta_n^{\alpha+1} W'(\delta_n x_n + e) + \frac{2\varepsilon_n}{\delta_n^{2-\alpha}} \frac{x_n}{|x_n|^4} = 0 \quad \text{in } [-l, l], \tag{15}$$

$$\frac{1}{2}|\dot{x}_n|^2 - \frac{1}{|x_n|^\alpha} + \delta_n^\alpha W(\delta_n x_n + e) - \frac{\varepsilon_n}{\delta_n^{2-\alpha}} \frac{1}{|x_n|^2} = \delta_n^\alpha h_n \quad \text{in } [-l, l]. \tag{16}$$

We extract a subsequence still indexed by n such that

$$d = \lim_{n\to+\infty} \frac{\varepsilon_n}{\delta_n^{2-\alpha}} \in [0, +\infty] \tag{17}$$

exists. For d we need to show that

Lemma 2.6 *The quantity d defined in (17) is a finite one.*

Proof On the contrary, we assume that $d = +\infty$. We will prove that $\tilde{q}_n$ has a self intersection around e to find a contradiction. Let consider another rescaling of $\tilde{q}_n$:

$$y_n(s) = \delta_n^{-1}\Big[\tilde{q}_n\left(\varepsilon_n^{-\frac{1}{2}} \delta_n^2 s + t_n\right) - e\Big], \quad s \in \mathbb{R}. \tag{18}$$

Since $\varepsilon_n^{-\frac{1}{2}} \delta_n^2 = \left(\varepsilon_n^{-1} \delta_n^{2-\alpha}\right)^{\frac{1}{2}} \delta_n^{1+\frac{\alpha}{2}} \longrightarrow 0$, then for sufficiently large n, we have $\varepsilon_n^{-\frac{1}{2}} \delta_n^2\, s + t_n \in [\tau_1, \tau_2]$, $\forall\, s \in [-l, l]$. From (12)-(13), we get

$$\frac{d}{ds}|y_n(s)| < 0 \quad \forall\, s \in [-l, 0[,$$

$$\frac{d}{ds}|y_n(s)| > 0 \quad \forall\, s \in]0, l].$$

As in [3], we can see that -up a subsequence-

$$y_n \longrightarrow \cos(\sqrt{2}s)e_1 + \sin(\sqrt{2}s)e_2 \quad \text{in } C^2_{\text{loc}}(\mathbb{R},\ \mathbb{R}^2)$$

where (e_1, e_2) is an orthonormal basis of $\mathbb{R}^2$. Using polar coordinates, there exists a function $\alpha_n \in C^2(\mathbb{R},\ \mathbb{R})$ such that

$$y_n(s) = |y_n(s)|\Big(\cos(\alpha_n(s))e_1 + \sin(\alpha_n(s))e_2\Big).$$

We take $l > \sqrt{2}\pi$. Since $\dot{\alpha}_n \longrightarrow \sqrt{2}$ uniformly on $[-l, l]$, then for sufficiently large n, there exist $s_1 < 0 < s_2$ such that

$$\alpha_n(0) - \alpha_n(s_1) = \alpha_n(s_2) - \alpha_n(0) = 2\pi. \tag{19}$$

We may assume that $1 < |y_n(s_1)| \leq |y_n(s_2)|$. By continuity, there exists $s_3 \in]0, s_2]$ such that $|y_n(s_1)| = |y_n(s_3)|$. Since $\dot{\alpha}_n > 0$, it follows from (19) that

$$\begin{aligned}\alpha_n(s_3) - \alpha_n(s_1) &= \alpha_n(s_3) - \alpha_n(0) + \alpha_n(0) - \alpha_n(s_1)\\ &> 2\pi.\end{aligned}$$

This implies the existence of $s_1', s_2' \in [s_1, s_3]$ such that $y_n(s_1') = y_n(s_2')$ and $\mathrm{ind}_0 y_n|_{[s_1', s_2']} = 1$. From (5) and (18), it follows the existence of $t', t'' \in]0, n[$ such that $q_n(t') = q_n(t'')$ and $\mathrm{ind}_e q_n|_{[t', t'']} = 1$. But this contradicts the fact that q_n is a minimum of $I_{0,n}$ over Γ_n^1. Indeed, let consider the function

$$\underline{q_n}(t) = \begin{cases} q_n(t) & \text{if } t \in [0, n] \setminus [t', t''], \\ q_n(t' + t'' - t) & \text{if } t \in [t', t'']. \end{cases}$$

Then $\underline{q_n} \in \Gamma_n^{-1}$ and $I_{0,n}(\underline{q_n}) = I_{0,n}(q_n) = c_n^1 = c_n^{-1}$. Therefore $\underline{q_n}$ is a classical solution of (D_n). By the uniqueness of the solution of ordinary differential equation, we deduce that $\underline{q_n} = q_n$: clearly this is a contradiction since $\mathrm{ind}_e(\underline{q_n}) = -1$ and $\mathrm{ind}_e(q_n) = 1$. □

Since $d < +\infty$, by the continuity dependence of solutions on initial data and equation, we can see from (14)-(16) and (V3) the existence of an orthonormal basis (e_1, e_2) of $\mathbb{R}^2$ such that

$$x_n(s) \longrightarrow x_{\alpha,d}(s) \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^2)$$

where $x_{\alpha,d}(s)$ is the solution of the initial value problem

$$\begin{cases} \ddot{x} + \dfrac{\alpha x}{|x|^{\alpha+2}} + 2d\dfrac{x}{|x|^4} = 0, \\ x(0) = e_1, \quad \dot{x}(0) = \sqrt{2(1+d)}e_2. \end{cases}$$

We use polar coordinates and write

$$\tilde{q}_n(t) - e = |\tilde{q}_n(t) - e|\Big(\cos(\tilde{\theta}_n(t))e_1 + \sin(\tilde{\theta}_n(t))e_2\Big),$$

$$x_{\alpha,d}(s) = |x_{\alpha,d}(s)|\Big(\cos(\theta_{\alpha,d}(s))e_1 + \sin(\theta_{\alpha,d}(s))e_2\Big),$$

where $\tilde{\theta}_n(s),\ \theta_{\alpha,d}(s) \in \mathbb{R}$ with $\theta_{\alpha,d}(0) = 0$. In [18] we observed the following properties for $\theta_{\alpha,d}$

$$\dot{\theta}_{\alpha,d}(s) > 0 \quad \forall\, s \in \mathbb{R}, \tag{20}$$

$$\Delta\theta_{\alpha,d} = \lim_{s\to+\infty}(\theta_{\alpha,d}(s) - \theta_{\alpha,d}(-s)) = \frac{2\pi\sqrt{1+d}}{2-\alpha}. \tag{21}$$

We remark that $\Delta\theta_{\alpha,d} > \pi\ \forall\alpha \in]0, 1]$ and if $\alpha \in]1, 2[$ then $\Delta\theta_{\alpha,d} > 2\pi$.
Let $B_r(e)$ denote the open ball of radius r about e. We will give a estimate of $\tilde{\theta}_n(t)$ when $\tilde{q}_n(t) \in B_\mu(e) \setminus B_{L\delta_n}(e)$ for sufficiently small $\mu > 0$ and large $L > 1$ and n.

We have for $t < t'$,

$$|\tilde{\theta}_n(t') - \tilde{\theta}_n(t)| \leq \int_t^{t'} |\dot{\tilde{\theta}}_n(\tau)| d\tau$$
$$= \int_t^{t'} \left| \frac{d}{dt} \frac{\tilde{q}_n(\tau) - e}{|\tilde{q}_n(\tau) - e|} \right| d\tau \tag{22}$$

On the other hand, Tanaka [18] studied under the condition (V3) with $e = 0$ the behavior of generalized periodic solutions of singular Hamiltonian systems in $\mathbb{R}^N$. In a neighborhood of the singularity, the generalized solution is a limit of classical solutions of perturbed problems with potentials V_ε as in our case, so we can apply some locally property of approximated solutions near a collision. More precisely, modifying the argument in Proposition 1.5 slightly, we can see that for any $\eta > 0$ there exist constants $\mu,\ S > 0$ and $n_0 \in \mathbb{N}^*$ such that for $n \geq n_0$,

$$\int_t^{t'} \left| \frac{d}{dt} \frac{\tilde{q}_n(\tau) - e}{|\tilde{q}_n(\tau) - e|} \right| d\tau \leq \frac{\eta}{2} \quad \text{if} \quad \tilde{q}_n(t),\ \ \tilde{q}_n(t') \in B_\mu(e) \quad \text{and}$$
$$\tau_1 < t < t' < t_n - S\delta_n^{\frac{\alpha+2}{2}} \quad \text{or} \quad t_n + S\delta_n^{\frac{\alpha+2}{2}} < t < t' < \tau_2. \tag{23}$$

Combining (22) and (23), we get

Lemma 2.7 *For any $\eta > 0$, there are constants $\mu \in]0,\ \delta/4[,\ S > 0$ such that for sufficiently large n, if $\tilde{q}_n(t),\ \tilde{q}_n(t') \in B_\mu(e)$ and*

$$\tau_1 < t < t' < t_n - S\delta_n^{\frac{\alpha+2}{2}} \quad or \quad t_n + S\delta_n^{\frac{\alpha+2}{2}} < t < t' < \tau_2,$$

then

$$|\tilde{\theta}_n(t') - \tilde{\theta}_n(t)| \leq \frac{\eta}{2}.$$

End of the proof of Theorem 2.2. 1) If $\alpha \in]1, 2[$, there exists from (21) $\eta > 0$ such that $\Delta\theta_{\alpha,d} > 2\pi + \eta$. For this η, we choose $\mu \in]0,\ \delta/4[,\ \ S > 0$ and n sufficiently large as in Lemma 2.7.
From (21) again we can take $S_1 > S$ such that

$$\theta_{\alpha,d}(S_1) - \theta_{\alpha,d}(-S_1) > 2\pi + \eta.$$

Then we obtain for sufficiently large n,

$$\tilde{\theta}_n\left(t_n + \delta_n^{\frac{\alpha+2}{2}} S_1\right) - \tilde{\theta}_n\left(t_n - \delta_n^{\frac{\alpha+2}{2}} S_1\right) > 2\pi + \eta. \tag{24}$$

On the other hand, since $|\tilde{q}_n(t_n \pm S_1\delta_n^{\frac{\alpha+2}{2}}) - e| \longrightarrow |\tilde{q}(\tilde{t}) - e| = 0$, we can assume that

$$|\tilde{q}_n\left(t_n \pm S_1\delta_n^{\frac{\alpha+2}{2}}\right) - e| < \mu.$$

We set $t'_{1,n} = t_n - S_1\delta_n^{\frac{\alpha+2}{2}}$. Then we have from (10)

$$|\tilde{q}_n(t'_{1,n}) - e| < \mu < \frac{\delta}{4} \leq |\tilde{q}_n(\tau_1) - e|.$$

Therefore there exists $t_{1,n} \in]\tau_1,\ t'_{1,n}[$ such that

$$|\tilde{q}_n(t_{1,n}) - e| = \mu.$$

Similarly we set $t_{2,n} = t_n + S_1\delta_n^{\frac{\alpha+2}{2}}$. Since $|\tilde{q}_n(t_{2,n}) - e| < \mu < \dfrac{\delta}{4} \leq |\tilde{q}_n(\tau_2) - e|$, there exists $t'_{2,n} \in]t_{2,n},\ \tau_2[$ such that

$$|\tilde{q}_n(t'_{2,n}) - e| = \mu.$$

Applying lemma 2.7 for $t = t_{i,n}$ and $t' = t'_{i,n}$ $(i = 1, 2)$, we obtain

$$|\tilde{\theta}_n(t'_{i,n}) - \tilde{\theta}_n(t_{i,n})| \leq \frac{\eta}{2} \quad \text{for} \quad i = 1,\ 2. \tag{25}$$

It follows from (24)-(25),

$$\begin{aligned}\tilde{\theta}_n(t'_{2,n}) - \tilde{\theta}_n(t_{1,n}) &= \tilde{\theta}_n(t'_{2,n}) - \tilde{\theta}_n(t_{2,n}) + \tilde{\theta}_n(t_{2,n}) - \tilde{\theta}_n(t'_{1,n}) + \tilde{\theta}_n(t'_{1,n}) - \tilde{\theta}_n(t_{1,n}) \\ &> -\frac{\eta}{2} + 2\pi + \eta - \frac{\eta}{2} = 2\pi.\end{aligned}$$

That is $\tilde{q}_n$ describes an angle greater than 2π in going from $\partial B_\mu(e)$ back to $\partial B_\mu(e)$ which implies the existence of $t''_{1,n},\ t''_{2,n}$ with $\tau_1 < t''_{1,n} < t''_{2,n} < \tau_2$ such that

$$\tilde{q}_n(t''_{1,n}) = \tilde{q}_n(t''_{2,n}) \quad \text{and } \mathrm{ind}_e \tilde{q}_n|_{[t''_{1,n}, t''_{2,n}]} = 1.$$

Thus we deduce that q_n has a self intersection around e for sufficiently large n. As in the proof of Lemma 2.6, we get a contradiction and then we conclude that $\tilde{q}$ is a non collision homoclinic solution of (HS).

2) In the case $\alpha \in]0, 1]$, the angle which describes $\tilde{q}_n$ near e is greater than π and $\tilde{q}_n$ cannot have a self intersection. The fact that the collisions times of $\tilde{q}$ are isolated and since $\tilde{q}(t) \longrightarrow 0$ as $t \to \pm\infty$, we get that the number of collisions of $\tilde{q}$ is finite. Assume $\tilde{q}(t)$ enters the singularity e and let

$$t_0 = \min\{t \in \mathbb{R},\ \tilde{q}(t) = e\}.$$

Since (HS) is time reversible, the function

$$q(t) = \begin{cases} \tilde{q}(t) & \text{if } t \leq t_0, \\ \tilde{q}(2t_0 - t) & \text{if } t \geq t_0, \end{cases}$$

is a generalized homoclinic solution of (HS) and satisfies $q(t + t_0) = q(t_0 - t)$ for all t. Moreover q has one collision in $\mathbb{R}$. The proof of Theorem 2.2 is finally complete.

Remark 2.8 The assumption (V3) is far too restrictive in the case $\alpha \in]0, 1]$ and the existence of a generalized homoclinic solution with finite number of collisions and then a solution as in Theorem 2.2 2) still holds if we replace (V3) by

(V'3) (i) $V(q) \to -\infty$ as $q \to e$;
(ii) There exists a constant $\delta \in]0, |e|/4[$ such that

$$V(q) + \frac{1}{2}V'(q)(q - e) < 0 \quad \text{for } 0 < |q - e| \leq \delta.$$

We have kept (V3) in the case $\alpha \in]0, 1]$, on the one hand to obtain a certain symmetry in the statements of Theorem 2.2, on the other hand the study of approximated solutions near collisions under (V3) will be useful in Sect. 3 to prove the existence of a non-collision heteroclinic orbit at infinity for every $\alpha \in]0, 2[$ (see Theorem 3.1 below).

3 Existence of Heteroclinic Orbits

In this section, the existence of non-collision heteroclinic orbits at infinity for (HS) will be established. Consider the problem

$$\begin{cases} \ddot{q} + V'(q) = 0, \\ q(-\infty) = 0, \quad |q(+\infty)| = +\infty, \\ \dot{q}(\pm\infty) = 0, \end{cases} \tag{P}$$

where V behaves like (1) near e and satisfies the assumptions (V1)-(V3) of Theorem 2.2.

The natural condition for V at infinity for (P) is $\lim_{|q|\to+\infty} V(q) = 0$. More precisely, we assume

(V'4) $V(q) \sim -\dfrac{a}{|q|^b}$ as $|q| \to +\infty$ for some $a > 0,\ b > 2$.

When $\alpha \in]0, 1]$, we need a further property of V near e

(V5) there exists $\phi \in C^2(]0, r[, \mathbb{R})$ for some $r \in]0, |e|/4[$ such that

$$V(q) = \phi(|q - e|) \quad \forall\, q \in B_r(e).$$

Theorem 3.1 *Suppose (V1)-(V3), (V'4) and (V5)(only when $\alpha \in]0, 1]$). Then there exists at least one non-collision orbit of (P).*

We now pass to the proof of Theorem 3.1. Solutions of (P) can be found as critical points of the functional

$$I(q) = \int_{\mathbb{R}} \left[\frac{1}{2}|\dot{q}|^2 - V(q)\right] dt$$

defined on the set

$$\Lambda_0^\infty = \{q \in H;\ \ q(-\infty) = 0,\ |q(+\infty)| = +\infty,\ q(t) \neq e\,\forall\, t \in \mathbb{R}\}$$

where

$$H = \left\{q \in H^1_{loc}(\mathbb{R}, \mathbb{R}^N),\ \int_{\mathbb{R}} |\dot{q}|^2 dt < +\infty\right\}.$$

In [14] the case $\alpha \geq 2$ (strong force case) has been studied and the existence of one classical solution of (P) has been found as a minimizer of I on Λ_0^∞. In our situation where $0 < \alpha < 2$, we make a perturbation to the potential as in Theorem 2.2 and we consider for every n the problem

$$\begin{cases} \ddot{q} + V'_{\varepsilon_n}(q) = 0, \\ q(-\infty) = 0, \quad |q(+\infty)| = +\infty, \\ \dot{q}(\pm\infty) = 0. \end{cases} \tag{P$_n$}$$

Since V_{ε_n} is a strong force, we can use Lemma 1.1, and a standard compactness argument provides the existence of a classical (non-collision) solution q_n of (P_n) as a minimizer of the functional

$$I_n(q) = \int_{\mathbb{R}} \left[\frac{1}{2}|\dot{q}|^2 - V_{\varepsilon_n}(q)\right] dt$$

on Λ_0^∞, i.e. $q_n \in \Lambda_0^\infty$ such that

$$I_n(q_n) = \inf_{q \in \Lambda_0^\infty} I_n(q). \tag{26}$$

By normalization, we can assume that

$$|q_n(0)| = \frac{|e|}{4} \quad \text{and} \quad |q_n(t)| < \frac{|e|}{4} \quad \forall\, t < 0.$$

Remark also that q_n has energy zero.

Now we observe that $I_n(q_n) \leq \inf_{q\in\Lambda_0^\infty} I_1(q) = c_1 < +\infty$. We deduce then the existence of a constant $C > 0$ independent of n such that $||q_n||_H \leq C$ and $\int_{\mathbb{R}} -V(q_n)dt \leq C$. Thus there is a subsequence still denoted by (q_n) and a function $q \in H$ such that q_n converges weakly in H and uniformly in $C_{loc}(\mathbb{R}, \mathbb{R}^2)$ to q. By Fatou's lemma $\int_{\mathbb{R}} -V(q)dt \leq C$, so the set of collisions $D = \{t \in \mathbb{R},\ q(t) = e\}$ is of measure 0. In a standard way, we can see that $q \in C^2(\mathbb{R} \setminus D, \mathbb{R}^2)$, satisfies (HS) and has energy zero in $\mathbb{R} \setminus D$, that is q is a generalized solution of (HS).

Lemma 3.2 *$q(t) \neq e$ for all $t \in \mathbb{R}$.*

Proof We prove by contradiction assuming $q(\tilde{t}) = e$ for some $\tilde{t} \in \mathbb{R}$. From (V3) and the conservation of the energy, q satisfies the property (9) and then we can see that the collisions times of q are isolated. Moreover there is a sequence (t_n) such that $t_n \longrightarrow \tilde{t}$ and $|q_n(t) - e|$ takes its local minimum at $t = t_n$.

As in Theorem 2.2 we define $\delta_n = |q_n(t_n) - e|$ and $d = \lim_{n\to+\infty} \frac{\varepsilon_n}{\delta_n^{2-\alpha}} \in [0, +\infty]$ (we extract a subsequence if necessary).
If we suppose that $d = +\infty$, we can see as in Lemma 2.6 that q_n has a self intersection i.e. there exist $\sigma_1 < \sigma_2$ such that $q_n(\sigma_1) = q_n(\sigma_2)$ and $\text{ind}_e q_n|_{[\sigma_1,\sigma_2]} = 1$. Here we consider the function

$$u_n(t) = \begin{cases} q_n(t + \sigma_1 - \sigma_2) & \text{if } \ t \leq \sigma_2, \\ q_n(t) & \text{if } \ t \geq \sigma_2. \end{cases}$$

Then $u_n \in \Lambda_0^\infty$ and it is easy to see that $I_n(u_n) < I_n(q_n)$, which contradicts (26).

Therefore we get $d < +\infty$. In that case, there is a function $x_{\alpha,d}$ such that after taking a subsequence still denoted by n,

$$\delta_n^{-1}\Big[q_n\Big(\delta_n^{\frac{\alpha+2}{2}} s + t_n\Big) - e\Big] \longrightarrow x_{\alpha,d}(s) = |x_{\alpha,d}(s)|\Big(\cos(\theta_{\alpha,d}(s))e_1 + \sin(\theta_{\alpha,d}(s))e_2\Big)$$

in $C^2_{loc}(\mathbb{R}, \mathbb{R}^2)$ where (e_1, e_2) is an orthonormal basis of $\mathbb{R}^2$ and $\theta_{\alpha,d} : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $\theta_{\alpha,d}(0) = 0$ and the properties (20)-(21).

In polar coordinates, there exists $\theta_n : \mathbb{R} \to \mathbb{R}$ such that

$$q_n(t) - e = |q_n(t) - e|\Big(\cos(\theta_n(t))e_1 + \sin(\theta_n(t))e_2\Big).$$

For $\alpha \in]1, 2[$, we have from (21) $\Delta\theta_{\alpha,d} > 2\pi$. Repeating the argument of Theorem 2.2, we get that q_n has a self intersection around e which is a contradiction as above.
For $\alpha \in]0, 1]$, we will use (V5) to get a contradiction. Here $\Delta\theta_{\alpha,d} > \pi$ and q_n cannot have a self intersection. However there exists $L > 0$ such that $\theta_{\alpha,d}(L) - \theta_{\alpha,d}(-L) > \pi$. Setting $\sigma_{1,n} = t_n - \delta_n^{\frac{\alpha+2}{2}} L$ and $\sigma_{2,n} = t_n + \delta_n^{\frac{\alpha+2}{2}} L$, for sufficiently large n we have

$$\begin{aligned} &|q_n(t) - e| \leq r, \quad \forall\ t \in [\sigma_{1,n}, \sigma_{2,n}], \\ &\theta_n(\sigma_{2,n}) - \theta_n(\sigma_{1,n}) > \pi, \\ &\dot{\theta}_n(t) > 0 \quad \forall t \in [\sigma_{1,n}, \sigma_{2,n}]. \end{aligned} \tag{27}$$

Let $\sigma'_{1,n}, \sigma'_{2,n} \in [\sigma_{1,n}, \sigma_{2,n}]$ such that

$$\theta_n(\sigma'_{2,n}) - \theta_n(\sigma'_{1,n}) = \pi.$$

We consider the function $\hat{q}_n$ defined by

$$\hat{q}_n(t) = q_n(t) \quad \text{if} \quad t \in \mathbb{R} \setminus [\sigma'_{1,n}, \sigma'_{2,n}],$$
$$\hat{q}_n(t) - e = |q_n(t) - e|\Big(\cos\big(-\theta_n(t) + 2\theta_n(\sigma'_{1,n})\big)e_1 + \sin\big(-\theta_n(t) + 2\theta_n(\sigma'_{1,n})\big)e_2\Big)$$
$$\text{if} \quad t \in [\sigma'_{1,n}, \sigma'_{2,n}].$$

That is $\hat{q}_n|_{[\sigma'_{1,n},\sigma'_{2,n}]}$ and $q_n|_{[\sigma'_{1,n},\sigma'_{2,n}]}$ are axially symmetric with respect to the axis joining the two points $q_n(\sigma'_{1,n})$ and $q_n(\sigma'_{2,n})$.

Clearly $\hat{q}_n \in \Lambda_0^\infty$ and from (V5), since V is radially symmetric about e in $B_r(e)$, we get that $I_n(q_n) = I_n(\hat{q}_n) = \inf_{q \in \Lambda_0^\infty} I_n(q)$. It follows that $\hat{q}_n$ is of class C^2 and satisfies the equation $\ddot{q} + V'_{\varepsilon_n}(q) = 0$. By the uniqueness of solution of ordinary differential equation, we deduce that $q_n = \hat{q}_n$, which enters in contradiction with (27). Therefore we conclude that $q(t) \neq e$ for all $t \in \mathbb{R}$. □

End of the proof of Theorem 3.1. To prove that q is a solution of (P), it remains to show that $q(-\infty) = 0$, $|q(+\infty)| = +\infty$ and $\dot{q}(\pm\infty) = 0$. Using the formula (4) and the fact that $I(q) < +\infty$ one can see that $|q(-\infty)|$ and $|q(+\infty)|$ exist and they are 0 or $+\infty$. Since $|q(t)| = \lim |q_n(t)| \leq |e|/4 \; \forall t \leq 0$, then $q(-\infty) = 0$.

To show that $|q(+\infty)| = +\infty$, we suppose that $|q(+\infty)| = 0$. We will construct as in [12] a function $Q_n \in \Lambda_0^\infty$ such that $I_n(Q_n) < I_n(q_n)$. Indeed, let $\varepsilon \in]0, |e|/16[$ and $T_\varepsilon > 0$ such that $q(T_\varepsilon) \in B_\varepsilon$ the open ball of radius ε about 0. For sufficiently large n we have $q_n(T_\varepsilon) \in B_{2\varepsilon}$. We consider the function $Q_n \in \Lambda_0^\infty$ different from q_n for $t < T_\varepsilon$ such that

$$Q_n(t) = \begin{cases} 0 & \text{if } t < T_\varepsilon - 1, \\ (t - T_\varepsilon + 1)q_n(T_\varepsilon) & \text{if } t \in [T_\varepsilon - 1, T_\varepsilon], \\ q_n(t) & \text{if } t \geq T_\varepsilon. \end{cases}$$

Since $V_{\varepsilon_n} = V$ in $B_{2\varepsilon}$ and $V_{\varepsilon_n} \leq V$, we have

$$I_n(Q_n) - I_n(q_n) \leq 2\varepsilon^2 + \max_{x \in B_{2\varepsilon}} -V(x) - \int_{-\infty}^{T_\varepsilon} \Big[\frac{1}{2}|\dot{q}_n|^2 - V(q_n)\Big]dt. \tag{28}$$

On the other hand, since $|q_n(0)| = |e|/4$ and $|q_n(T_\varepsilon)| \leq 2\varepsilon < |e|/8$, there are $t_1 < t_2$ in $[0, T_\varepsilon]$ such that

$$|q_n(t_1)| = \frac{|e|}{4}, \;\; |q_n(t_2)| = \frac{|e|}{8} \text{ and } \frac{|e|}{8} \leq |q_n(t)| \leq \frac{|e|}{4} \text{ for all } t \in [t_1, t_2].$$

By the formula (4), it holds that

$$\begin{aligned} \int_{-\infty}^{T_\varepsilon} \Big[\frac{1}{2}|\dot{q}_n|^2 - V(q_n)\Big]dt &\geq \int_{t_1}^{t_2} \Big[\frac{1}{2}|\dot{q}_n|^2 - V(q_n)\Big]dt \\ &\geq \frac{|e|}{8}\sqrt{2m_0} \end{aligned} \tag{29}$$

where $m_0 = \min_{\frac{|e|}{8} \leq |x| \leq \frac{|e|}{4}} -V(x) > 0$.

Then combining (28) and (29), we get $I_n(Q_n) - I_n(q_n) < 0$ for sufficiently small ε, which contradicts (26). We conclude that $|q(+\infty)| = +\infty$.

From the conservation of energy and the fact that $V(q(t)) \longrightarrow 0$ as $t \to \pm\infty$, it follows that $\frac{1}{2}|\dot{q}(t)|^2 = -V(q(t)) \longrightarrow 0$ as $t \to \pm\infty$, that is $\dot{q}(\pm\infty) = 0$. The proof is complete.

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