Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-024-10356-9

Global Attractors for a Class of Discrete Dynamical Systems

Everaldo de Mello Bonotto[1] · José Manuel Uzal[2]

Received: 20 June 2023 / Revised: 13 December 2023 / Accepted: 7 February 2024

Abstract
In this paper, we study the existence of global attractors for a class of discrete dynamical systems naturally originated from impulsive dynamical systems. We establish sufficient conditions for the existence of a discrete global attractor. Moreover, we investigate the relationship among different types of global attractors, i.e., the attractor $\mathcal{A}$ of a continuous dynamical system, the attractor $\tilde{\mathcal{A}}$ of an impulsive dynamical system and the attractor $\hat{\mathcal{A}}$ of a discrete dynamical system. Two applications are presented, one involving an integrate-and-fire neuron model, and the other involving a nonlinear reaction-diffusion initial boundary value problem.

Keywords Discrete dynamical systems · Impulsive dynamical systems · Global attractors

Mathematics Subject Classification Primary: 34D45 · 35B41; Secondary: 34A37

1 Introduction

The theory of impulsive dynamical systems describes the evolution of processes where the continuous dynamics are interrupted by abrupt changes of state, i.e., the system can experience a sudden "impulse". For example, the introduction of a new predator or the removal of a food source can cause a sudden change in the population of a species, which can be modeled using an impulsive dynamical system. The new phenomena presented in impulsive dynamical systems have been drawn attention because of their irregularity. Besides, these

E. M. Bonotto: Research partially supported by FAPESP # 2020/14075-6 and CNPq # 310540/2019-4. J. M. Uzal: Research partially supported by BES-2017-082334.

✉ Everaldo de Mello Bonotto
ebonotto@icmc.usp.br

José Manuel Uzal
juzal@ucm.es

1 Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Campus de São Carlos, Caixa Postal 668, São Carlos, SP 13560-970, Brazil

2 Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

systems admit a more complex structure than the non-impulsive systems and have many real-world applications. The reader may consult [4, 5, 8, 9, 11, 15, 17] for more details.

In 1990, Saroop Kaul [12] constructed the theory of impulsive dynamical systems where the impulses depend on the state, that is, there exists a set in the phase space which is responsible by the discontinuities of the system. Although this theory is well-developed, there is no study of the long-term behavior of discrete dynamical systems that arise naturally from a given impulsive dynamical system. Given an impulsive dynamical system (X, π, I, M) it is possible, under certain conditions, to construct an associated discrete dynamical system $(\hat{X}, g)$. Some recursive properties as periodicity, minimality and recurrence are developed for this new class of discrete dynamical systems in [13]. However, the theory of global attractors has not been explored for this new class of discrete dynamical systems. Therefore, in this paper, we aim to investigate the existence of global attractors for such discrete systems. In what follows, we describe the organization of the paper.

In Sect. 2, we present the basis of the theory of impulsive dynamical systems. In particular, we exhibit some results on global attractors that will be useful in the main results of this paper.

Section 3 is dedicated to studying the long-term behavior of the class of discrete dynamical systems of type $(\hat{X}, g)$ associated with a class of impulsive dynamical systems of type (X, π, I, M). We define the concept of discrete global attractors, and we exhibit sufficient conditions for the existence of a such attractor, see Theorem 3.11. Some characterizations of the discrete global attractor are given in Theorems 3.12 and 3.14.

In general, there is no relation among the existence of the attractors $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ of the systems (X, π), (X, π, M, I) and $(\hat{X}, g)$, respectively. This fact is illustrated in Sect. 4, based on some examples. Moreover, in Subsection 4.1, we provide some conditions to relate these attractors (see Theorem 4.14). In Subsection 4.2, we establish the existence of the discrete global attractor of $(\hat{X}, g)$ provided (X, π) and (X, π, M, I) admits their attractors, see Theorem 4.17.

In Sect. 5, we present two applications. Subsection 5.1 deals with the existence and the relationship among the global attractors $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ of an integrate-and-fire neuron model. In Sect. 5.2, we consider the nonlinear reaction-diffusion initial boundary value problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{for } (x,t) \in \Omega \times (0, \infty), \\ u(x,t) = 0, & \text{for } (x,t) \in \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x), & \text{for } x \in \Omega, \end{cases}$$

under impulse perturbation, where Ω is a bounded smooth domain of $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary, Δ is the Laplace operator in Ω, and $u_0 \in L^2(\Omega)$. The nonlinearity f satisfies some general conditions. We investigated the existence and the relationship among the global attractors $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$.

2 Preliminaries

Consider a metric space (X, d). Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We represent by $\mathcal{B}(X)$ the set of all bounded subsets from X.

A *semidynamical system* (or *semiflow*) on X is a family of maps $\{\pi(t) \colon t \in \mathbb{R}_+\}$ acting from X to X satisfying the following conditions:

(a) $\pi(0) = I$, where $I \colon X \to X$ is the identity operator;
(b) $\pi(t+s) = \pi(t)\pi(s)$ for all $t, s \in \mathbb{R}_+$;
(c) $\mathbb{R}_+ \times X \ni (t, x) \mapsto \pi(t)x \in X$ is continuous.

A semiflow on X will be denoted simply by (X, π).

Let $Z \subset X$ and $\Delta \subset \mathbb{R}_+$ be given. The *past* of Z with respect to the set Δ is given by

$$F(Z, \Delta) = \bigcup_{t \in \Delta} \pi(t)^{-1}(Z).$$

For each fixed $x \in X$ and $t \in \mathbb{R}_+$, the set $F(x, t)$ in the context of semiflows is not singleton in general. See [3] for more details.

Given a semiflow (X, π), a nonempty closed subset $M \subset X$ is called an *impulsive set* if for each $x \in M$ there exists $\epsilon_x > 0$ such that

$$\bigcup_{t \in (0, \epsilon_x)} \{\pi(t)x\} \cap M = \emptyset, \tag{2.1}$$

i.e., the trajectories of (X, π) are in some sense "transversal" to the set M.

Definition 2.1 An *impulsive dynamical system* (X, π, M, I) consists of a semiflow (X, π), an impulsive set $M \subset X$ and a continuous function $I : M \to X$ called impulsive function.

Remark 2.2 In [4, 5, 8, 9], an impulsive set M satisfies the following property: for each $x \in M$ there exists $\epsilon_x > 0$ such that

$$F(x, (0, \epsilon_x)) \cap M = \emptyset \quad \text{and} \quad \bigcup_{t \in (0, \epsilon_x)} \{\pi(t)x\} \cap M = \emptyset.$$

However, the condition

$$F(x, (0, \epsilon_x)) \cap M = \emptyset \tag{2.2}$$

is not necessary to obtain many properties of attractors, as discussed throughout this paper. In this way, we consider just condition (2.1) to define an impulsive set.

An important tool to study the evolution of an impulsive dynamical system is the *impact function*, i.e., the function $\phi : X \to (0, \infty]$ given by

$$\phi(x) = \begin{cases} s, & \text{if } \pi(s)x \in M \text{ and } \pi(t)x \notin M \text{ for } 0 < t < s, \\ \infty, & \text{if } \pi(t)x \notin M \text{ for all } t > 0. \end{cases}$$

If $\phi(x) < \infty$, then $\phi(x)$ stands for the smallest positive time such that the trajectory of x meets M. The function ϕ is not continuous in general (see [8]). Using the impact function, we can describe the *impulsive positive trajectory* of $x \in X$ in (X, π, M, I) that is represented by a map

$$\tilde{\pi}(\cdot)x : J_x \to X$$

defined on some interval $J_x \subseteq \mathbb{R}_+$ containing 0, given inductively by the following way: if $\phi(x) = \infty$ then $\tilde{\pi}(t)x = \pi(t)x$ for all $t \in \mathbb{R}_+$. On the other hand, if $\phi(x) < \infty$ then we set $x = x_0^+$ and we define $\tilde{\pi}(\cdot)x$ on $[0, \phi(x_0^+)]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t)x_0^+, & \text{if } 0 \le t < \phi(x_0^+), \\ I(\pi(\phi(x_0^+))x_0^+), & \text{if } t = \phi(x_0^+). \end{cases}$$

In order to simplify the notation, write $s_0 = \phi(x_0^+)$, $x_1 = \pi(s_0)x_0^+$ and $x_1^+ = I(\pi(s_0)x_0^+)$. Since $s_0 < \infty$, the previous process can go on, but now starting at x_1^+. If $\phi(x_1^+) = \infty$ then we define $\tilde{\pi}(t)x = \pi(t-s_0)x_1^+$ for all $t \geq s_0$. But, if $s_1 = \phi(x_1^+) < \infty$ i.e., $x_2 = \pi(s_1)x_1^+ \in M$ then we define $\tilde{\pi}(\cdot)x$ on $[s_0, s_0+s_1]$ by

$$\tilde{\pi}(t)x = \begin{cases} \pi(t-s_0)x_1^+, & \text{if} \quad s_0 \leq t < s_0+s_1, \\ I(x_2), & \text{if} \quad t = s_0+s_1. \end{cases}$$

Here, denote $x_2^+ = I(x_2)$. This process ends after a finite number of steps if $\phi(x_n^+) = \infty$ for some $n \in \mathbb{N}_0$, or it may proceed indefinitely, if $\phi(x_n^+) < \infty$ for all $n \in \mathbb{N}_0$ and, in this case, $\tilde{\pi}(\cdot)x$ is defined in the interval $[0, T(x))$, where $T(x) = \displaystyle\sum_{i=0}^{\infty} s_i$ can be finite or infinite. The reader may consult [4, 5, 8, 9, 12] for more details.

Note that

$$\tilde{\pi}(t)x = \pi(t-t_k)x_k^+, \quad t_k \leq t < t_{k+1}, \tag{2.3}$$

where $x_0^+ = x$, $t_0 = 0$ and $t_k = \sum_{j=0}^{k-1} \phi(x_j^+)$, $k \geq 1$.

In order to study the long-term behavior of impulsive dynamical systems, we shall consider the following condition:

(H) There exists $\xi > 0$ such that $\phi(x) \geq \xi$ for every $x \in I(M)$.

This condition guarantees that an impulsive dynamical system is defined for all positive times. Note that, if $I(M)$ is a compact set and $I(M) \cap M = \emptyset$ then condition **(H)** holds.

Next, we recall the concepts of invariance, impulsive $\tilde{\omega}$-limit sets, asymptotic compactness and dissipativeness.

Definition 2.3 A subset $A \subset X$ is called:

(a) *positively* $\tilde{\pi}$*-invariant*, if $\tilde{\pi}(t)A \subset A$ for all $t \in \mathbb{R}_+$;
(b) *negatively* $\tilde{\pi}$*-invariant*, if $\tilde{\pi}(t)A \supset A$ for all $t \in \mathbb{R}_+$;
(c) $\tilde{\pi}$*-invariant*, if it is both positively $\tilde{\pi}$-invariant and negatively $\tilde{\pi}$-invariant.

Definition 2.4 Let $B \in \mathcal{B}(X)$. The *impulsive ω-limit set* of B in (X, π, M, I) is defined as

$$\tilde{\omega}(B) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} \tilde{\pi}(s)B} = \{x \in X : \text{ there exist sequences } \{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+ \text{ and } \{x_n\}_{n\in\mathbb{N}} \subset B \text{ such that } t_n \overset{n\to\infty}{\longrightarrow} \infty \text{ and } \tilde{\pi}(t_n)x_n \overset{n\to\infty}{\longrightarrow} x\}.$$

Definition 2.5 An impulsive dynamical system (X, π, M, I) is called *asymptotically compact*, if given a set $B \in \mathcal{B}(X)$, a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $t_n \overset{n\to\infty}{\longrightarrow} \infty$, and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset B$, then the sequence $\{\tilde{\pi}(t_n)x_n\}_{n\in\mathbb{N}}$ possesses a convergent subsequence in X.

Lemma 2.6 *[4, Lemma 3.3] Let $B \in \mathcal{B}(X)$. Assume that (X, π, M, I) is asymptotically compact satisfying condition* **(H)**. *Then $\tilde{\omega}(B)$ is nonempty, compact and attracts the set B.*

Remark 2.7 The proof of Lemma 2.6 does not require condition (2.2).

Definition 2.8 An impulsive dynamical system (X, π, M, I) is called *dissipative*, if there exists a set $B_0 \in \mathcal{B}(X)$, called absorbing set, such that for every $B \in \mathcal{B}(X)$ there exists a time $T_B \geq 0$ such that $\tilde{\pi}(t)B \subset B_0$ for all $t \geq T_B$.

As described in [4], for the purpose of obtain a well behavior of the evolution of impulsive dynamical systems, we shall consider condition **(T)**:

(T) If $x \in M$, $\{z_n\}_{n\in\mathbb{N}} \subset X$ is a sequence that converges to z and $t > 0$ are such that $\pi(t)z_n \overset{n\to\infty}{\longrightarrow} x$, then there exist a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ and a sequence $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$, $\alpha_k \overset{k\to\infty}{\longrightarrow} 0$, such that $t + \alpha_k \geq 0$ and $\pi(t+\alpha_k)z_{n_k} \in M$.

Condition **(T)** implies the following result on the continuity of ϕ, which does not require condition (2.2).

Theorem 2.9 *Let (X, π, M, I) be an impulsive dynamical system satisfying condition* **(T)**. *Then ϕ is upper semicontinuous in X and it is continuous in $X\backslash M$.*

Proof The continuity of ϕ in $X\backslash M$ is a particular case of [4, Theorem 5.2]. The upper semicontinuity of ϕ in X follows by the last part of the proof of [4, Theorem 5.2]. □

In Lemma 2.10, under conditions **(H)** and **(T)**, we present sufficient conditions for an impulsive dynamical system to be asymptotically compact. This result is a consequence of [4, Lemma 6.3] and its proof does not require condition (2.2).

Lemma 2.10 *[4, Lemma 6.3] Let (X, π, M, I) be an impulsive dynamical system satisfying conditions* **(H)** *and* **(T)**. *If the semiflow (X, π) is compact and (X, π, M, I) is dissipative, then (X, π, M, I) is asymptotically compact.*

Lemma 2.11 deals with an important property that is used in the proof of the existence of a global attractor. This result is presented in [4, Lemma 6.7] for multivalued impulsive systems. However, the authors provide a proof using condition (2.2). In contrast, in the paper [6], the authors consider a version of [4, Lemma 6.7] under weaker conditions but for positive invariant sets. Since our result holds for any bounded set, we rewrite the proof of [6, Lemma 2.9] for the case of single-valued impulsive dynamical systems, using condition **(T)**.

Lemma 2.11 *Let (X, π, M, I) be an impulsive dynamical system satisfying conditions* **(H)** *and* **(T)**. *Assume that (X, π, M, I) is asymptotically compact and let $B \subset X$ be a bounded set. Then $\tilde{\omega}(B) \cap M \subset \overline{\tilde{\omega}(B)\backslash M}$.*

Proof Let $x \in \tilde{\omega}(B) \cap M$. Then there exist sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and $\{x_n\}_{n\in\mathbb{N}} \subset B$ such that $t_n \overset{n\to\infty}{\longrightarrow} \infty$ and

$$\tilde{\pi}(t_n)x_n \overset{n\to\infty}{\longrightarrow} x.$$

Using (2.3), for each $n \in \mathbb{N}$, there exists an integer $k_n \geq 0$ such that $\tau_{k_n}^n \leq t_n < \tau_{k_n+1}^n$ and

$$\tilde{\pi}(t_n)x_n = \pi(t_n - \tau_{n_k}^n)(x_n)_{k_n}^+,$$

where $\tilde{\pi}(t)x_n = \pi(t - \tau_{k_n}^n)(x_n)_{k_n}^+$ for $\tau_{k_n}^n \leq t < \tau_{k_n+1}^n$ (if $t_n < \phi(x_n)$, then we may just take $\tau_{k_n}^n = 0$ and $k_n = 0$, and if the number of jumps is finite and equal to k_n, then we set $\tau_{k_n+1}^n = \infty$) and τ_j^n, $j \in \mathbb{N}$, are the jump times in the trajectory starting at x_n. Up to subsequences, we may consider the following three cases:

(i) $t_n - \tau_{k_n}^n \overset{n\to\infty}{\longrightarrow} 0^+$,

(ii) $t_n - \tau_{k_n}^n \overset{n\to\infty}{\longrightarrow} r > 0$,

(iii) $t_n - \tau_{k_n}^n \overset{n\to\infty}{\longrightarrow} \infty$.

Case (i). Let $m_0 \in \mathbb{N}$ be such that $\frac{1}{m_0} < \epsilon_x$, where $\epsilon_x > 0$ comes from condition (2.1). Define $w_n^m = \tilde{\pi}(t_n + \frac{1}{m})x_n$ for $m \geq m_0$ and $n \in \mathbb{N}$. By condition **(H)**, we may assume that $t_n + \frac{1}{m} \in (\tau_{k_n}^n, \tau_{k_n+1}^n)$ for all $m \geq m_0$ and $n \in \mathbb{N}$. Using the asymptotic compactness, we also may assume that $w_n^m \overset{n\to\infty}{\longrightarrow} y_m$ for every $m \geq m_0$. Note that $y_m \in \tilde{\omega}(B)$, $m \geq m_0$. We claim that $y_m \notin M$ for all $m \geq m_0$. In fact, note that

$$w_n^m = \tilde{\pi}\left(t_n + \frac{1}{m}\right)x_n = \pi\left(t_n + \frac{1}{m} - \tau_{k_n}^n\right)(x_n)_{k_n}^+ = \pi\left(\frac{1}{m}\right)\tilde{\pi}(t_n)x_n \overset{n\to\infty}{\longrightarrow} \pi\left(\frac{1}{m}\right)x,$$

which implies that $y_m = \pi(\frac{1}{m})x$, $m \geq m_0$. Since $\frac{1}{m} < \epsilon_x$ for all $m \geq m_0$, we conclude that $y_m \notin M$, i.e., $y_m \in \tilde{\omega}(B)\backslash M$ for all $m \geq m_0$. Hence, $y_m = \pi\left(\frac{1}{m}\right)x \overset{m\to\infty}{\longrightarrow} x$ and the proof of this case is complete.

Case (ii). Let $m_1 \in \mathbb{N}$ be such that $\tau_{k_n}^n < t_n - \frac{1}{m} - \frac{r}{2} < \tau_{k_n+1}^n$ for all $m \geq m_1$. Using the asymptotic compactness of (X, π, M, I), up to subsequences, we have

$$\tilde{\pi}(\tau_{k_n}^n)x_n = (x_n)_{k_n}^+ \overset{n\to\infty}{\longrightarrow} z,$$

for some $z \in \tilde{\omega}(B)$. Now, define $w_n^m = \tilde{\pi}(t_n - \frac{1}{m})x_n$ for $m \geq m_1$ and $n \in \mathbb{N}$. Then

$$w_n^m = \pi\left(\frac{r}{2}\right)\pi\left(t_n - \frac{1}{m} - \tau_{k_n}^n - \frac{r}{2}\right)(x_n)_{k_n}^+ \overset{n\to\infty}{\longrightarrow} \pi\left(r - \frac{1}{m}\right)z := y_m \in \tilde{\omega}(B).$$

We claim that $y_m \notin M$ for all $m \geq m_1$. Indeed, if $y_m \in M$ for some $m \geq m_1$, it follows by condition **(T)** that, up to a subsequence, there exists $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\alpha_n \overset{n\to\infty}{\longrightarrow} 0$ and $\pi\left(t_n - \frac{1}{m} - \tau_{k_n}^n + \alpha_n\right)(x_n)_{k_n}^+ \in M$ which implies

$$\begin{aligned} r = \limsup_{n\to\infty}(t_n - \tau_{k_n}^n) &\leq \limsup_{n\to\infty}(\tau_{k_n+1}^n - \tau_{k_n}^n) = \limsup_{n\to\infty}\phi((x_n)_{k_n}^+) \\ &\leq \limsup_{n\to\infty}\left(t_n - \frac{1}{m} - \tau_{k_n}^n + \alpha_n\right) = r - \frac{1}{m}, \end{aligned}$$

which is a contradiction. Hence, the claim follows.

On the other hand, $x = \pi(r)z$ as $\tilde{\pi}(t_n)x_n = \pi(t_n - \tau_{k_n}^n)(x_n)_{k_n}^+ \overset{n\to\infty}{\longrightarrow} \pi(r)z$. Hence,

$$y_m = \pi\left(r - \frac{1}{m}\right)z \overset{m\to\infty}{\longrightarrow} \pi(r)z = x$$

which completes the proof of case (ii).

Case (iii). Using again the asymptotic compactness of (X, π, M, I), up to a subsequence, we may assume that $\tilde{\pi}\left(t_n - 1 - \frac{1}{m}\right)x_n = \pi\left(t_n - \tau_{k_n}^n - 1 - \frac{1}{m}\right)(x_n)_{k_n}^+$ converges and, for each $m \in \mathbb{N}$, there exists $y_m \in \tilde{\omega}(B)$ such that

$$\tilde{\pi}\left(t_n - \frac{1}{m}\right)x_n = \pi(1)\pi\left(t_n - \tau_{k_n}^n - 1 - \frac{1}{m}\right)(x_n)_{k_n}^+ \overset{n\to\infty}{\longrightarrow} y_m.$$

We claim that $y_m \notin M$ for all $m \in \mathbb{N}$. If $y_m \in M$ for some $m \in \mathbb{N}$ then, by condition **(T)**, up to a subsequence, there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\alpha_n \overset{n\to\infty}{\longrightarrow} 0$ and $\pi\left(t_n - \frac{1}{m} - \tau_{k_n}^n + \alpha_n\right)(x_n)_{k_n}^+ \in M$ which implies

$$\tilde{\pi}\left(t_n - \frac{1}{m} + \alpha_n\right) \in M$$

which is a contradiction. Hence, $y_m \notin M$ for all $m \in \mathbb{N}$.

The compactness of $\tilde{\omega}(B)$ implies that $y_m \overset{m\to\infty}{\longrightarrow} x_0$ (passing to a subsequence if necessary). Since

$$\tilde{\pi}\,(t_n)\,x_n = \pi\left(\frac{1}{m}\right)\tilde{\pi}\left(t_n - \frac{1}{m}\right)x_n \overset{n\to\infty}{\longrightarrow} \pi\left(\frac{1}{m}\right)y_m = x,$$

as $m \to \infty$, we obtain $x = x_0$. It follows that y_m converges to x and the proof of case (iii) is complete. □

Given two nonempty subsets $A, B \subseteq X$, we denote the *Hausdorff semidistance* between A and B (in this order) by

$$\mathrm{d_H}(A, B) := \sup_{a\in A}\inf_{b\in B} d(a, b).$$

Definition 2.12 A nonempty set $\tilde{\mathcal{A}} \subset X$ is called a *global attractor* for (X, π, M, I) if:

(i) $\tilde{\mathcal{A}}$ is pre-compact and $\tilde{\mathcal{A}} = \overline{\tilde{\mathcal{A}}}\backslash M$;
(ii) $\tilde{\mathcal{A}}$ is $\tilde{\pi}$-invariant;
(iii) $\mathrm{d_H}(\tilde{\pi}(t)B, \tilde{\mathcal{A}}) \overset{n\to\infty}{\longrightarrow} 0$ for every $B \in \mathcal{B}(X)$.

By [5, Proposition 4.1], if the global attractor exists, then it is uniquely determined.

The next result deals with the existence of the global attractor. The proof of Theorem 2.13 follows by [4, Theorem3.9] and [4, Corollary 4.8], and condition (2.2) is not needed as we have Lemma 2.11.

Theorem 2.13 *Let (X, π, M, I) be an impulsive dynamical system satisfying* **(H)**.

(i) *If (X, π, M, I) has a global attractor $\tilde{\mathcal{A}}$ then it is asymptotically compact and dissipative.*
(ii) *If (X, π, M, I) is asymptotically compact, dissipative with absorbing set B_0, and it satisfies* **(T)**, *then it has a global attractor $\tilde{\mathcal{A}}$.*

As in the non-impulsive case, we can characterize the global attractor through global solution.

Definition 2.14 A function $\psi : \mathbb{R} \to X$ is called a *global solution* of $\tilde{\pi}$ if

$$\tilde{\pi}(t)\psi(s) = \psi(t + s), \quad \text{for all} \ \ t \geq 0 \ \ \text{and} \ \ s \in \mathbb{R}.$$

If $\psi(0) = x$ then we say that ψ is a global solution through x. Moreover, if $\psi(\mathbb{R})$ is bounded in X then ψ is said to be a bounded global solution.

Theorem 2.15 *If (X, π, M, I) has a global attractor $\tilde{\mathcal{A}}$ and $I(M) \cap M = \emptyset$, then*

$$\tilde{\mathcal{A}} = \{x \in X : \textit{there exists a bounded global solution of } \tilde{\pi} \textit{ through } x\}.$$

Proof The proof is analogous to the proof of [5, Proposition 4.3] and condition (2.2) is not required. □

Remark 2.16 If $M = \emptyset$, then the previous results are valid for the continuous semidynamical system (X, π). The definitions of invariance, ω-limit sets, asymptotic compactness, and dissipativeness in (X, π) are the same as those previously defined, where we replace $\tilde{\pi}$ with π. The global attractor of (X, π) is a compact set $\mathcal{A} \subset X$ that is π-invariant and satisfies $\mathrm{d_H}(\pi(t)B, \mathcal{A}) \overset{n\to\infty}{\longrightarrow} 0$ for every $B \in \mathcal{B}(X)$. A global solution of π will be represented by $\varphi : \mathbb{R} \to X$, that is, a map such that $\pi(t)\varphi(s) = \varphi(t + s)$, for all $t \geq 0$ and $s \in \mathbb{R}$. Since conditions **(H)** and **(T)** are related to the impulse set M, Lemma 2.6, Lemma 2.10, Theorem 2.13 and Theorem 2.15 hold for the semidynamical system (X, π) without these conditions.

3 The Discrete Global Attractor

In [13], Saroop Kaul introduced a new class of discrete dynamical systems that arise naturally from a given impulsive dynamical system. More specifically, consider an impulsive dynamical system (X, π, M, I) satisfying the following general conditions:

(H1) (X, π, M, I) satisfies conditions **(H)** and **(T)**;
(H2) there exists $z \in I(M)$ such that $\phi(z_k^+) < \infty$ for all $k \in \mathbb{N}_0$;
(H3) $I(M) \cap M = \emptyset$.

Now, define the set

$$\hat{X} = \{x \in I(M) \colon \phi(x_k^+) < \infty \text{ for all } k \in \mathbb{N}_0\}$$

and the map $g \colon \hat{X} \to \hat{X}$ by

$$g(x) = I(\pi(\phi(x))x). \tag{3.1}$$

Firstly, note by condition (H2) that the set $\hat{X}$ is nonempty. Also, g maps $\hat{X}$ to $\hat{X}$, hence, $(\hat{X}, g)$ defines a discrete dynamical system on $\hat{X}$ associated with the impulsive dynamical system (X, π, M, I). Note that $g^0(x) = x$ and $g^n(x) = x_n^+$ for all $x \in \hat{X}$ and $n \in \mathbb{N}_0$. Consequently, $g(x_n^+) = x_{n+1}^+$ for all $x \in \hat{X}$ and $n \in \mathbb{N}_0$. The *positive orbit* of a point x in $(\hat{X}, g)$ is represented by

$$\hat{\mathcal{O}}(x) = \{g^n(x) \colon n \in \mathbb{N}_0\}.$$

The map $g \colon \hat{X} \to \hat{X}$ defined in (3.1) depends on the impact function ϕ, the impulsive function I and the semiflow π. Under conditions (H1), (H2) and (H3), we have the following result.

Lemma 3.1 *Assume that* (X, π, M, I) *satisfies conditions* (H1)-(H3). *Then the map* g *is continuous on* $\hat{X}$.

Proof Since condition **(T)** holds, it follows by Theorem 2.9 that ϕ is continuous on $X \backslash M$. By (H3), we obtain $\hat{X} \cap M = \emptyset$. Moreover, I is continuous on M and $\mathbb{R}_+ \times X \ni (t, x) \mapsto \pi(t)x \in X$ is continuous. Hence, g is continuous on $\hat{X}$. □

The following definitions are established based on concepts already known in the theory of attractors for discrete dynamical systems, as presented in [10].

Definition 3.2 A subset $\hat{B} \subset \hat{X}$ is said to be:

(i) *positively g-invariant* w.r.t. $(\hat{X}, g)$, if $g(\hat{B}) \subset \hat{B}$;
(ii) *negatively g-invariant* w.r.t. $(\hat{X}, g)$, if $g(\hat{B}) \supset \hat{B}$;
(iii) *g-invariant* if it is both positively and negatively g-invariant w.r.t. $(\hat{X}, g)$.

The positive orbit of a point $x \in \hat{X}$ is positively g-invariant, but it is not generally negatively g-invariant.

Example 3.3 Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, $X = S^1 \times \mathbb{R}$ and $\lambda \colon \mathbb{C} \to \mathbb{C}$ be a map given by $\lambda(z) = \mathrm{e}^{i(\theta + 2\pi\alpha)}$ for $z = \mathrm{e}^{i\theta}$, where α defines an irrational rotation. Now, let us consider the semiflow $\{\pi(t) \colon t \geq 0\}$ given by

$$\pi(t)(z, s) = (z, t + s),$$

for all $z \in S^1$, $s \in \mathbb{R}$ and $t \in \mathbb{R}_+$. Define the impulsive set $M = S^1 \times \{2\}$ and the impulsive function $I : M \to S^1 \times \{0\}$ by $I(z, s) = (\lambda(z), 0)$, $(z, s) \in M$. Since $\phi(z, 0) < \infty$ for all $z \in S^1$, we have $\hat{X} = I(M)$. Note that $\overline{\hat{\mathcal{O}}(z, 0)} = \hat{X}$ for every $z \in S^1$. Although $\hat{\mathcal{O}}(z, 0)$ is positively g-invariant for every $z \in S^1$, it is not negatively g-invariant since $e^{2\pi n i\alpha} z \neq z$ for every $n \in \mathbb{N}$.

Definition 3.4 The *omega limit set* of a subset $\hat{B} \subset \hat{X}$ is given by

$$\hat{\omega}(\hat{B}) = \{x \in \hat{X} : \text{ there exist sequences } \{x_k\}_{k\in\mathbb{N}} \subset \hat{B} \text{ and } \{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N} \\ \text{with } n_k \stackrel{k\to\infty}{\longrightarrow} \infty \text{ such that } g^{n_k}(x_k) \stackrel{k\to\infty}{\longrightarrow} x\}.$$

In Example 3.3, $\hat{\omega}(z, 0) = \hat{X}$ for every $z \in S^1$.

Next, we provide some properties of omega limit sets.

Lemma 3.5 *Assume that* (X, π, M, I) *satisfies conditions* (H1)-(H3).

(*a*) *If* $\hat{B} \subset \hat{X}$ *is compact and positively* g*-invariant, then* $\hat{\omega}(\hat{B})$ *is nonempty and compact.*
(*b*) *Given* $\hat{B} \subset \hat{X}$*, the omega limit set* $\hat{\omega}(\hat{B})$ *is positively* g*-invariant.*

Proof Let us prove item (*b*). Suppose that $\hat{\omega}(\hat{B}) \neq \emptyset$ and let $x \in \hat{\omega}(\hat{B})$. Then there are sequences $\{x_k\}_{k\in\mathbb{N}} \subset \hat{B}$ and $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ with $n_k \stackrel{k\to\infty}{\longrightarrow} \infty$ such that $g^{n_k}(x_k) \stackrel{k\to\infty}{\longrightarrow} x$. By the continuity of g (see Lemma 3.1), we have $g^{n_k+1}(x_k) = g(g^{n_k}(x_k)) \stackrel{k\to\infty}{\longrightarrow} g(x)$, i.e., $g(x) \in \hat{\omega}(\hat{B})$. □

Let $\mathcal{B}(\hat{X})$ denote the set of all bounded subsets from $\hat{X}$. Next, we present the concept of a global attractor for the system $(\hat{X}, g)$.

Definition 3.6 A set $\hat{\mathcal{A}} \subset \hat{X}$ is called a *discrete global attractor* for $(\hat{X}, g)$ if:

(i) $\hat{\mathcal{A}}$ is compact;
(ii) $\hat{\mathcal{A}}$ is g-invariant;
(iii) $\mathrm{d_H}(g^n(\hat{B}), \hat{\mathcal{A}}) \stackrel{n\to\infty}{\longrightarrow} 0$ for every $\hat{B} \in \mathcal{B}(\hat{X})$.

The property (iii) means that the discrete global attractor $\hat{\mathcal{A}}$ g-attracts all the bounded sets from $\hat{X}$.

If a discrete global attractor exists, then it is uniquely determined. Indeed, suppose that $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{A}}_2$ are discrete global attractors for $(\hat{X}, g)$. By invariance, $g^n(\hat{\mathcal{A}}_i) = \hat{\mathcal{A}}_i$, $i = 1, 2$, for all $n \in \mathbb{N}_0$. Consequently,

$$\mathrm{d_H}(\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2) = \mathrm{d_H}(g^n(\hat{\mathcal{A}}_1), \hat{\mathcal{A}}_2) \stackrel{n\to\infty}{\longrightarrow} 0$$

and

$$\mathrm{d_H}(\hat{\mathcal{A}}_2, \hat{\mathcal{A}}_1) = \mathrm{d_H}(g^n(\hat{\mathcal{A}}_2), \hat{\mathcal{A}}_1) \stackrel{n\to\infty}{\longrightarrow} 0.$$

Hence, $\hat{\mathcal{A}}_1 = \hat{\mathcal{A}}_2$.

The notions of asymptotic compactness and dissipativeness are presented in the sequel. These conditions will play an important role for the existence of the discrete global attractor.

Definition 3.7 A discrete dynamical system $(\hat{X}, g)$ is called *asymptotically compact* if, given a set $\hat{B} \in \mathcal{B}(\hat{X})$, a sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ with $n_k \stackrel{k\to\infty}{\longrightarrow} \infty$, and a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \hat{B}$, then the sequence $\{g^{n_k}(x_k)\}_{k\in\mathbb{N}}$ admits a convergent subsequence in $\hat{X}$.

Definition 3.8 A discrete dynamical system $(\hat{X}, g)$ is called *bounded dissipative* if, there exists a set $\hat{B}_0 \in \mathcal{B}(\hat{X})$, called *absorbing set*, such that for every $\hat{B} \in \mathcal{B}(\hat{X})$ there exists an integer $n_{\hat{B}} \geq 0$ such that $g^n(\hat{B}) \subset \hat{B}_0$ for all $n \geq n_{\hat{B}}$.

Lemma 3.9 *Assume that* (X, π, M, I) *satisfies condition* **(H)**, $I(M) \cap M = \emptyset$, $\hat{X} \neq \emptyset$, *and* $(\hat{X}, g)$ *is asymptotically compact. Then for any bounded set* $\hat{B} \subset \hat{X}$*, the omega limit set* $\hat{\omega}(\hat{B})$ *is nonempty and compact. Further,* $\mathrm{d_H}(g^n(\hat{B}), \hat{\omega}(\hat{B})) \overset{n\to\infty}{\longrightarrow} 0$.

Proof Let $\hat{B} \subset \hat{X}$ be a bounded set. Given $x \in \hat{B}$, it follows by the asymptotic compactness of $(\hat{X}, g)$ that $\{g^k(x)\}_{k\in\mathbb{N}}$ admits a convergent subsequence in $\hat{X}$. Hence, $\hat{\omega}(\hat{B}) \neq \emptyset$.

Now, let $\{x_m\}_{m\in\mathbb{N}} \subset \hat{\omega}(\hat{B})$ be a sequence. For each $m \in \mathbb{N}$, there exist sequences $\{w_k^m\}_{k\in\mathbb{N}} \subset \hat{B}$ and $\{n_k^m\}_{k\in\mathbb{N}} \subset \mathbb{N}$ with $n_k^m \overset{k\to\infty}{\longrightarrow} \infty$ such that $g^{n_k^m}(w_k^m) \overset{k\to\infty}{\longrightarrow} x_m$. Thus, for each $m \in \mathbb{N}$, one can obtain $k_m > m$ such that

$$d\left(g^{n_{k_m}^m}(w_{k_m}^m), x_m\right) < \frac{1}{m}.$$

Using the asymptotic compactness of $(\hat{X}, g)$, we may assume up to a subsequence that $g^{n_{k_m}^m}(w_{k_m}^m) \overset{m\to\infty}{\longrightarrow} z \in \hat{\omega}(\hat{B})$. Hence, $x_m \overset{m\to\infty}{\longrightarrow} z$ and $\hat{\omega}(\hat{B})$ is compact.

Lastly, suppose to the contrary that there are $\epsilon > 0$, $\{x_k\}_{k\in\mathbb{N}} \subset \hat{B}$ and $n_k \overset{k\to\infty}{\longrightarrow} \infty$ such that

$$d(g^{n_k}(x_k), \hat{\omega}(\hat{B})) \geq \epsilon,$$

for all $k \in \mathbb{N}$. Again, by the asymptotic compactness, there exists $w \in \hat{X}$ such that $d(g^{n_k}(x_k), w) \overset{k\to\infty}{\longrightarrow} 0$. This means that $w \in \hat{\omega}(\hat{B})$ and we obtain a contradiction. □

Lemma 3.10 *Assume that* (X, π, M, I) *satisfies conditions* (H1)-(H3)*, and* $(\hat{X}, g)$ *is asymptotically compact. Then the omega limit set* $\hat{\omega}(\hat{B})$ *is negatively* g*-invariant for every* $B \in \hat{\mathcal{B}}(\hat{X})$.

Proof Let $\hat{B} \in \mathcal{B}(\hat{X})$ and $x \in \hat{\omega}(\hat{B})$. Then there are sequences $\{x_k\}_{k\in\mathbb{N}} \subset \hat{B}$ and $n_k \overset{k\to\infty}{\longrightarrow} \infty$ such that $g^{n_k}(x_k) \overset{k\to\infty}{\longrightarrow} x$. Since $(\hat{X}, g)$ is asymptotically compact, there exists $z \in \hat{X}$ such that, taking a subsequence if necessary,

$$g^{n_k-1}(x_k) \overset{k\to\infty}{\longrightarrow} z.$$

Note that $z \in \hat{\omega}(\hat{B})$. Using Lemma 3.1, we obtain

$$g^{n_k}(x_k) = g(g^{n_k-1}(x_k)) \overset{k\to\infty}{\longrightarrow} g(z).$$

By uniqueness, $x = g(z) \in g(\hat{\omega}(\hat{B}))$. Hence, $\hat{\omega}(\hat{B}) \subset g(\hat{\omega}(\hat{B}))$. □

In Theorem 3.11, we establish sufficient conditions for the existence of a discrete global attractor.

Theorem 3.11 *Assume that* (X, π, M, I) *satisfies conditions* (H1)-(H3) *and* $(\hat{X}, g)$ *is asymptotically compact and bounded dissipative with absorbing set* $\hat{B}_0$*. Then* $(\hat{X}, g)$ *has a discrete global attractor* $\hat{\mathcal{A}}$ *given by* $\hat{\mathcal{A}} = \hat{\omega}(\hat{B}_0)$.

Proof By Lemma 3.9, $\hat{\omega}(\hat{B}_0)$ is nonempty and compact. By Lemmas 3.5 and 3.10, $\hat{\omega}(\hat{B}_0)$ is g-invariant. Let us show that $\hat{\omega}(\hat{B}_0)$ attracts all the bounded sets from $\hat{X}$. Indeed, let $\hat{B} \subset \hat{X}$ be a bounded set. Note that $\hat{\omega}(\hat{B}) \subset \hat{\omega}(\hat{B}_0)$. Thus, using Lemma 3.9, we get

$$\mathrm{d_H}(g^n(\hat{B}), \hat{\omega}(\hat{B}_0)) \leq \mathrm{d_H}(g^n(\hat{B}), \hat{\omega}(\hat{B})) \stackrel{n\to\infty}{\longrightarrow} 0.$$

By the uniqueness of the discrete global attractor, we conclude that $\hat{\mathcal{A}} = \hat{\omega}(\hat{B}_0)$ is the discrete global attractor of $(\hat{X}, g)$. □

Next, we give some characterizations of a discrete global attractor.

Theorem 3.12 *Let $\hat{\mathcal{A}}$ be the discrete global attractor of $(\hat{X}, g)$. Then*

(*a*) *$\hat{\mathcal{A}}$ is the minimal subset of $\hat{X}$ which is closed and g-attracts bounded sets from $\hat{X}$;*
(*b*) *$\hat{\mathcal{A}} = \bigcup_{\hat{B}\in\mathcal{B}(\hat{X})} \hat{\omega}(\hat{B})$.*

Proof (*a*) Let $\hat{K}$ be a closed set in $\hat{X}$ which g-attracts bounded sets from $\hat{X}$. Then

$$\mathrm{d_H}(\hat{\mathcal{A}}, \hat{K}) = \mathrm{d_H}(g^n(\hat{\mathcal{A}}), \hat{K}) \stackrel{n\to\infty}{\longrightarrow} 0,$$

that is, $\hat{\mathcal{A}} \subset \hat{K}$.

(*b*) Since $\hat{\mathcal{A}}$ is a discrete global attractor, we have $\bigcup_{\hat{B}\in\mathcal{B}(\hat{X})} \hat{\omega}(\hat{B}) \subset \hat{\mathcal{A}}$. On the other hand, let $x \in \hat{\mathcal{A}}$. Since $g(\hat{\mathcal{A}}) = \hat{\mathcal{A}}$, there exists $a_1 \in \hat{\mathcal{A}}$ such that $g(a_1) = x$. Now, we can take $a_2 \in \hat{\mathcal{A}}$ such that $g(a_2) = a_1$. Continuing with this process, one can obtain $a_{k+1} \in \hat{\mathcal{A}}$ such that $g(a_{k+1}) = a_k$, $k \in \mathbb{N}$. Thus,

$$d(x, \hat{\omega}(\hat{\mathcal{A}})) = d(g^k(a_k), \hat{\omega}(\hat{\mathcal{A}})) \stackrel{k\to\infty}{\longrightarrow} 0.$$

By the boundedness of $\hat{\mathcal{A}}$, we conclude the other set inclusion. □

Definition 3.13 A function $\hat{\psi} : \mathbb{Z} \to \hat{X}$ is called a *discrete global solution* of g if

$$g^n(\hat{\psi}(k)) = \hat{\psi}(k+n)$$

for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $\hat{\psi}(0) = x$, we say that $\hat{\psi}$ is a discrete global solution through x. Further, $\hat{\psi}$ is said to be bounded if there exists a bounded set $\hat{B} \subset \hat{X}$ such that $\hat{\psi}(k) \subset \hat{B}$ for all $k \in \mathbb{Z}$.

We end this section, characterizing the discrete global attractor through the bounded discrete global solutions.

Theorem 3.14 *Let $\hat{\mathcal{A}}$ be the discrete global attractor of $(\hat{X}, g)$. Then*

$$\hat{\mathcal{A}} = \{x \in \hat{X} : \textit{ there exists a bounded discrete global solution of } g \textit{ through } x\}.$$

Proof Let $x \in \hat{X}$ and $\hat{\psi}$ be a bounded discrete global solution of g through x. Then $x = \hat{\psi}(0) = g^k(\hat{\psi}(-k))$ for all $k \in \mathbb{N}_0$. Since $\{\hat{\psi}(-k)\}_{k\in\mathbb{N}}$ is bounded, we have

$$d(x, \hat{\mathcal{A}}) = d(g^k(\hat{\psi}(-k)), \hat{\mathcal{A}}) \stackrel{k\to\infty}{\longrightarrow} 0,$$

that is, $x \in \hat{\mathcal{A}}$.

On the other hand, let $x \in \hat{\mathcal{A}}$. Since $g(\hat{\mathcal{A}}) = \hat{\mathcal{A}}$, there is $a_{-1} \in \hat{\mathcal{A}}$ such that $g(a_{-1}) = x$. Also, there exists $a_{-2} \in \hat{\mathcal{A}}$ such that $g(a_{-2}) = a_{-1}$. Continuing with this process, one can obtain $a_{-k-1} \in \hat{\mathcal{A}}$ such that $g(a_{-k-1}) = a_{-k}$ for every $k \in \mathbb{N}$. Note that $g^k(a_{-k}) = x$ for all $k \in \mathbb{N}$ and

$$g^{m+n}(a_{-k}) = a_{m+n-k},$$

whenever $m, n, k \in \mathbb{N}$ and $m + n - k < 0$. Define the map $\hat{\psi} : \mathbb{Z} \to \hat{X}$ by

$$\hat{\psi}(k) = \begin{cases} a_k & \text{if } k < 0,\ k \in \mathbb{Z}, \\ g^k(x) & \text{if } k \in \mathbb{N}_0. \end{cases}$$

Let $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. If $k \geq 0$ then

$$g^n(\hat{\psi}(k)) = g^n(g^k(x)) = g^{n+k}(x) = \hat{\psi}(k + n).$$

If $k < 0$ and $k + n \geq 0$ then

$$g^n(\hat{\psi}(k)) = g^n(a_k) = g^{n+k}(g^{-k}(a_k)) = g^{n+k}(x) = \hat{\psi}(k + n).$$

If $k < 0$ and $k + n < 0$ then

$$g^n(\hat{\psi}(k)) = g^n(g(a_{k-1})) = g^{n+1}(a_{k-1}) = a_{n+k} = \hat{\psi}(n + k).$$

Thus, $\hat{\psi}$ is a discrete global solution of g through x. By construction, $\hat{\psi}(k) \subset \hat{\mathcal{A}}$ for all $k \in \mathbb{Z}$. Therefore, $\hat{\psi}$ is a bounded discrete global solution of g through x. □

4 Relationship Among the Attractors $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$

Let (X, π) be a continuous semidynamical system, (X, π, M, I) be an associated impulsive dynamical system and $(\hat{X}, g)$ be its associated discrete dynamical system. Does the existence of a global attractor in one of these systems imply the existence of a global attractor in the others? As presented in the next examples, we show that there is no relationship between the existence of the attractors of these systems. When it exists, we will denote by $\mathcal{A}$ the global attractor of (X, π), by $\tilde{\mathcal{A}}$ the global attractor of (X, π, M, I) and by $\hat{\mathcal{A}}$ the discrete global attractor of $(\hat{X}, g)$.

Example 4.1 Consider the system of differential equations

$$\begin{cases} x' = -x, \\ y' = -y, \end{cases}$$

in $X = \mathbb{R}^2$. In this simple example, $\mathcal{A} = \{(0, 0)\}$.

(a) If $M = \bigcup_{n \in \mathbb{N}} M_n$ with $M_n = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = n^2\}$, $n = 1, 2, \ldots$, and $I(x, y) = \left(x(1 + \frac{1}{2n}), y(1 + \frac{1}{2n})\right)$ for $(x, y) \in M_n$, $n = 1, 2, \ldots$, then the systems (X, π, M, I) and $(\hat{X}, g)$ do not admit global attractors.

(b) If $M = \bigcup_{n \in \mathbb{N}} M_n$ with $M_n = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = n^2\}$, $n = 1, 2, \ldots$, and $I(x, y) = (\frac{x}{2n}, \frac{y}{2n})$ for all $(x, y) \in M_n$, $n = 1, 2, \ldots$. Then $\tilde{\mathcal{A}} = \mathcal{A}$ and $\hat{X} = \emptyset$.

(c) If $M = \mathbb{R} \times \{1\}$ and $I(x, 1) = (\arctan(x), 2)$, $x \in \mathbb{R}$. Then $\hat{\mathcal{A}} = \{(0, 2)\}$ and $\tilde{\mathcal{A}} = \{(0, y) : 1 < y \leq 2\} \cup \{(0, 0)\}$. Note that $\hat{X} = I(M) = (-\frac{\pi}{2}, \frac{\pi}{2}) \times \{2\}$. Moreover,

$$\tilde{\mathcal{A}} = \pi([0, \ln 2))\hat{\mathcal{A}} \cup \mathcal{A}.$$

Example 4.2 Consider the system $x' = |x|$ in $X = \mathbb{R}$. The semiflow $\{\pi(t) : t \in \mathbb{R}_+\}$ is given by

$$\pi(t)x = \begin{cases} xe^t, & x > 0, \\ 0, & x = 0, \\ xe^{-t}, & x < 0. \end{cases}$$

There is no global attractor $\mathcal{A}$ since the solutions with positive initial data are not bounded.

(a) If $M = \mathbb{N}$ and $I(n) = -1$, $n \in \mathbb{N}$, then $\hat{X} = \emptyset$ and $\tilde{\mathcal{A}} = [-1, 1)$.
(b) If $M = \mathbb{N}$ and $I(n) = \frac{1}{2}$, $n \in \mathbb{N}$, then $\tilde{\mathcal{A}} = [0, 1)$ and $\hat{\mathcal{A}} = \{\frac{1}{2}\}$.
(c) If $M = \mathbb{N}$ and $I(n) = n + \frac{1}{2}$, $n \in \mathbb{N}$, then the systems (X, π, M, I) and $(\hat{X}, g)$ do not admit global attractors.

Example 4.3 Consider the semiflow $\pi(t)f = e^{-t}f$ in $X = L^2([0, 1])$ defined for all $t \geq 0$. Note that $\mathcal{A} = \{0\}$. Let $h \in L^2(\Omega)$ be such that $\|h\|^2_{L^2} = 2$. If $M = \{g \in L^2(\Omega) : \|g\|^2_{L^2} = 1\}$ and $I(g) = h$ for all $g \in M$, then $\hat{\mathcal{A}} = \hat{X} = \{h\}$. Here, the global attractor $\tilde{\mathcal{A}}$ does not exist due to a lack of pre-compactness.

Example 4.4 Consider the semiflow $\pi(t)f = e^t f$ in $X = L^2([0, 1])$ defined for all $t \geq 0$. There is no global attractor $\mathcal{A}$ for this system. Let $h \in L^2(\Omega)$ be such that $\|h\|^2_{L^2} = \frac{1}{2}$. If $M = \{g \in L^2(\Omega) : \|g\|^2_{L^2} = 1\}$ and $I(g) = h$ for all $g \in M$, then $\hat{\mathcal{A}} = \hat{X} = \{h\}$. The global attractor $\tilde{\mathcal{A}}$ does not exist.

When the global attractors $\mathcal{A}$ and $\tilde{\mathcal{A}}$ exist, and $\mathcal{A} \cap M = \emptyset$, then the impulsive attractor $\tilde{\mathcal{A}}$ contains $\mathcal{A}$. This fact is shown in the next result.

Proposition 4.5 *Assume that (X, π) has a global attractor $\mathcal{A}$ with $\mathcal{A} \cap M = \emptyset$ and (X, π, M, I) has a global attractor $\tilde{\mathcal{A}}$ satisfying $I(M) \cap M = \emptyset$. Then $\mathcal{A} \subset \tilde{\mathcal{A}}$.*

Proof Let $x \in \mathcal{A}$. By Theorem 2.15 and Remark 2.16, there exists a bounded global solution $\varphi : \mathbb{R} \longrightarrow X$ of π such that $\varphi(0) = x$. Using the invariance of $\mathcal{A}$ in (X, π), we obtain $\varphi(\mathbb{R}) \subset \mathcal{A}$. Since $\mathcal{A} \cap M = \emptyset$, we also obtain $\varphi(\mathbb{R}) \cap M = \emptyset$. Therefore, for any $s \in \mathbb{R}$ and $t \geq 0$, we deduce

$$\tilde{\pi}(t)\varphi(s) = \pi(t)\varphi(s) = \varphi(t + s),$$

i.e., φ is a bounded global solution of $\tilde{\pi}$ through x. By Theorem 2.15, we conclude that $x \in \tilde{\mathcal{A}}$. □

If $\mathcal{A} \cap M \neq \emptyset$, then the result established in Proposition 4.5 can be not true. In fact, consider the semiflow $\{\pi(t) : t \in \mathbb{R}_+\}$ generated by the solutions of the system

$$\begin{cases} \theta' = 1, \\ r' = 1 - r, \end{cases}$$

in $X = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 : r \in [1, 2], \theta \in [0, 2\pi]\}$. Let $M = \{(x, 0) : x \in [1, 2]\}$ and $I(x, 0) = (-x, 0)$ for $1 \leq x \leq 2$. In this case, $\mathcal{A} = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 : \theta \in [0, 2\pi]\}$ and $\tilde{\mathcal{A}} = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 : \theta \in [\pi, 2\pi)\}$. Moreover, $\{(-1, 0)\} = \hat{\mathcal{A}} \subset \tilde{\mathcal{A}} \subset \mathcal{A}$.

However, by defining the sets

$$S_x = \{\varphi : \mathbb{R} \longrightarrow X : \varphi \text{ is a bounded global solution of } \pi \text{ through } x\}, \quad x \in \mathcal{A},$$

and

$$\mathcal{S} = \{x \in \mathcal{A} : \varphi((-\infty, 0]) \cap M \neq \emptyset \text{ for every } \varphi \in S_x\},$$

we have $\mathcal{A}\backslash\mathcal{S} \subset \tilde{\mathcal{A}}$. This fact is presented next.

Proposition 4.6 *Assume that* (X, π) *has a global attractor* $\mathcal{A}$ *and* (X, π, M, I) *has a global attractor* $\tilde{\mathcal{A}}$ *satisfying* $I(M) \cap M = \emptyset$. *Then* $\mathcal{A}\backslash\mathcal{S} \subset \tilde{\mathcal{A}}$.

Proof Let $x \in \mathcal{A}\backslash\mathcal{S}$. Then there exists a bounded global solution $\varphi\colon \mathbb{R} \longrightarrow X$ of π such that $\varphi(0) = x$ (see Theorem 2.15 and Remark 2.16). Moreover, we know that $\varphi((-\infty, 0]) \cap M = \emptyset$ because $x \notin \mathcal{S}$. In this way, define the function $\psi : \mathbb{R} \longrightarrow X$ by

$$\psi(t) = \begin{cases} \varphi(t), & t < 0, \\ \tilde{\pi}(t)x, & t \geq 0. \end{cases}$$

Let $s \in \mathbb{R}$ and $t \geq 0$ be given.

- If $s < 0$ and $t + s \leq 0$, then

$$\tilde{\pi}(t)\psi(s) = \tilde{\pi}(t)\varphi(s) = \pi(t)\varphi(s) = \varphi(s + t) = \psi(s + t).$$

- If $s < 0$ and $t + s > 0$, then

$$\begin{aligned}\psi(t + s) =& \tilde{\pi}(t + s)x = \tilde{\pi}(t + s)\varphi(0) = \tilde{\pi}(t + s)\pi(-s)\varphi(s) \\ &= \tilde{\pi}(t + s)\tilde{\pi}(-s)\varphi(s) = \tilde{\pi}(t)\varphi(s) = \tilde{\pi}(t)\psi(s).\end{aligned}$$

- If $s \geq 0$, then

$$\tilde{\pi}(t)\psi(s) = \tilde{\pi}(t)\tilde{\pi}(s)x = \tilde{\pi}(t + s)x = \psi(t + s).$$

Thus, ψ is a bounded global solution of $\tilde{\pi}$ through x. It remains to check that $\psi(\mathbb{R})$ is bounded. The set $\{\psi(t) : t \leq 0\}$ is bounded since φ is a bounded global solution. Since (X, π, M, I) is dissipative, there exists $t_x > 0$ such that $\{\tilde{\pi}(t)x : t \geq t_x\} \subset \mathcal{B}_0$, where $\mathcal{B}_0$ is the absorbing set. Finally, on the interval $[0, t_x]$, there are $0 \leq N < \infty$ jump times. Since $\pi([0, t_1])x, \pi([0, t_2 - t_1])x, \ldots, \pi([0, t_x - t_N])x$ are compact sets, where $t_1, \ldots, t_N$ are the possible jump times, then $\psi([0, t_x))$ is bounded. Hence, $\psi(\mathbb{R})$ is bounded in X and ψ defines a bounded global solution of $\tilde{\pi}$ through x. Therefore, $x \in \tilde{\mathcal{A}}$. □

On the other hand, when the attractors $\hat{\mathcal{A}}$ and $\tilde{\mathcal{A}}$ exist, then the impulsive attractor $\tilde{\mathcal{A}}$ also contains $\hat{\mathcal{A}}$ as shown in Proposition 4.7.

Proposition 4.7 *Assume that* (X, π, M, I) *satisfies conditions* (H1)-(H3)*, it has a global attractor* $\tilde{\mathcal{A}}$ *and* $(\hat{X}, g)$ *has a discrete global attractor* $\hat{\mathcal{A}}$. *Then* $\hat{\mathcal{A}} \subset \tilde{\mathcal{A}}$.

Proof Let $x \in \hat{\mathcal{A}}$. By Theorem 3.14, there exists a bounded discrete global solution $\hat{\psi} : \mathbb{Z} \to \hat{X}$ of g through x. Since

$$x = \hat{\psi}(0) = g^k(\hat{\psi}(-k)) = \tilde{\pi}\left(\sum_{j=1}^{k} \phi(\hat{\psi}(-j))\right)\hat{\psi}(-k)$$

for all $k \in \mathbb{N}_0$, $\{\hat{\psi}(-k)\}_{k\in\mathbb{N}} \subset \hat{\mathcal{A}}$ and $T_k := \sum_{j=1}^{k} \phi(\hat{\psi}(-j)) \stackrel{k\to\infty}{\longrightarrow} \infty$ (as condition (H) from (H1) holds), we obtain

$$d(x, \tilde{\mathcal{A}}) = d(\tilde{\pi}(T_k)\hat{\psi}(-k), \tilde{\mathcal{A}}) \stackrel{k\to\infty}{\longrightarrow} 0.$$

Hence, $x \in \tilde{\mathcal{A}}$. Using condition (H3), we obtain $\hat{X} \cap M = \emptyset$. Hence, $\hat{\mathcal{A}} \subset \tilde{\mathcal{A}} \backslash M$. □

As a consequence of Propositions 4.5 and 4.7, we deduce the following result.

Corollary 4.8 *Assume that* (X, π) *has a global attractor* $\mathcal{A}$ *with* $\mathcal{A} \cap M = \emptyset$, (X, π, M, I) *satisfies* (H1)-(H3) *and admits a global attractor* $\tilde{\mathcal{A}}$, *and* $(\hat{X}, g)$ *has a discrete global attractor* $\hat{\mathcal{A}}$. *Then it holds* $\hat{\mathcal{A}} \cup \mathcal{A} \subset \tilde{\mathcal{A}}$.

4.1 The Relation $\tilde{\mathcal{A}} = \mathcal{A} \cup \left(\bigcup_{a \in \hat{\mathcal{A}}} \pi([0, \phi(a)))a \right)$.

Throughout this subsection, we shall assume that conditions (H1)-(H3) hold. Let $\mathcal{A}$ be the global attractor of (X, π) and $\hat{\mathcal{A}}$ be the discrete global attractor of $(\hat{X}, g)$. Our aim in this section is to prove that if $\mathcal{A} \cap M = \emptyset$, then the global attractor $\tilde{\mathcal{A}}$ exists and the attractors $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ are related by the equality

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \left(\bigcup_{a \in \hat{\mathcal{A}}} \pi([0, \phi(a)))a \right).$$

For that, let $\mathcal{A}_1 = \bigcup_{a \in \hat{\mathcal{A}}} \pi([0, \phi(a)))a$. Before to present the existence result, we point out in the next remark that if the attractors $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ exist then $\mathcal{A} \cup \mathcal{A}_1 \subset \tilde{\mathcal{A}}$.

Remark 4.9 (i) If the attractors $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ exist, then $\mathcal{A}_1 \subset \tilde{\mathcal{A}}$. Indeed, let $x \in \hat{\mathcal{A}}$ and $r \in [0, \phi(x))$. We aim to construct a bounded global solution of $\tilde{\pi}$ through $\pi(r)x$. Since $x \in \hat{\mathcal{A}}$, it follows by Theorem 3.14 that there exists a bounded discrete global solution $\hat{\psi}: \mathbb{Z} \longrightarrow X$ of g with $\hat{\psi}(0) = x$. Now, consider the notations

$$t_0 = 0,\ t_1 = \phi(x),\ x_1 = \pi(t_1)x,\ \text{and } x_1^+ = I(x_1).$$

For each $n \geq 1$, let us define

$$t_{n+1} = t_n + \phi(x_n^+),\ x_{n+1} = \pi(t_{n+1} - t_n)x_n^+ \text{ and } x_{n+1}^+ = I(x_{n+1}),$$

and, for each $n \leq -1$, set

$$t_{-n} = t_{-n+1} - \phi(\hat{\psi}(-n)).$$

Thus, define the map $\psi_1: \mathbb{R} \longrightarrow X$ by

$$\psi_1(t) = \begin{cases} \pi(t - t_n)\hat{\psi}(n), & t \in [t_n, t_{n+1}),\ n \geq 0, \\ \pi(t - t_{-n})\hat{\psi}(-n), & t \in [t_{-n}, t_{-n+1}),\ n \geq 1. \end{cases}$$

By construction, ψ_1 is a global solution of $\tilde{\pi}$ through x ($\psi_1(0) = \hat{\psi}(0) = x$). Set $T = \sup_{x \in \hat{\mathcal{A}}} \phi(x)$. Since conditions (H1)-(H3) hold, we have ϕ is continuous on the compact set $\hat{\mathcal{A}}$, consequently, $T < \infty$. Now, note that $\psi_1(\mathbb{R}) \subset \pi([0, T])\hat{\mathcal{A}}$, which implies that ψ_1 is bounded. Therefore, the map $\psi: \mathbb{R} \longrightarrow X$ defined by $\psi(t) = \psi_1(t + r)$ is a bounded global solution of $\tilde{\pi}$ through $\pi(r)x$. Thus, by Theorem 2.15, $\mathcal{A}_1 \subset \tilde{\mathcal{A}}$.

(ii) If the attractors $\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ exist and $\mathcal{A} \cap M = \emptyset$, then by the previous item (i) and Proposition 4.5, $\mathcal{A} \cup \mathcal{A}_1 \subset \tilde{\mathcal{A}}$.

In what follows, we exhibit some auxiliary results.

Lemma 4.10 *Assume that (X,π) admits a global attractor $\mathcal{A}$ with $\mathcal{A}\cap M=\emptyset$. Let $B\in\mathcal{B}(X)$ be such that $\phi(x)<\infty$ for every $x\in B$. Then there exists $K>0$, depending on B, such that $\phi(x)\le K$ for all $x\in B$.*

Proof Suppose that for each $n\in\mathbb{N}$, there exists $x_n\in B$ such that $\phi(x_n)>n$. Since $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence, $\phi(x_n)\longrightarrow+\infty$ and (X,π) is asymptotically compact (see Theorem 2.13 and Remark 2.16), we have $\{\pi(\phi(x_n))x_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence, which will be denoted by the same, with limit x. Note that $x\in M$ as $\pi(\phi(x_n))x_n\in M$, for every $n\in\mathbb{N}$. But, we also have $x\in\mathcal{A}$. Thus, we conclude that $x\in\mathcal{A}\cap M$ which is a contradiction. Hence, ϕ is bounded on B. □

Lemma 4.11 *Assume that $(\hat{X},g)$ has a global attractor $\hat{\mathcal{A}}$. Then*

(i) $\overline{\mathcal{A}}_1=\bigcup_{a\in\hat{\mathcal{A}}}\pi([0,\phi(a)])a$ *is compact.*

(ii) $\mathcal{A}_1$ *is* $\tilde{\pi}$*-invariant.*

Proof (i) Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_1$ such that $x_n\overset{n\to\infty}{\longrightarrow}x$. For each $n\in\mathbb{N}$, there exist $a_n\in\hat{\mathcal{A}}$ and $t_n\in[0,\phi(a_n))$ such that $x_n=\pi(t_n)a_n$. Since $t_n<\phi(a_n)$ for every $n\in\mathbb{N}$, $\hat{\mathcal{A}}$ is compact and ϕ is continuous in $\hat{\mathcal{A}}$, we may assume without loss of generality that $t_n\overset{n\to\infty}{\longrightarrow}t$ and $a_n\overset{n\to\infty}{\longrightarrow}a\in\hat{\mathcal{A}}$ with $t\le\phi(a)$. Hence, $x=\pi(t)a\in\pi([0,\phi(a)])a$ with $a\in\hat{\mathcal{A}}$. Thus, the equality $\overline{\mathcal{A}}_1=\bigcup_{a\in\hat{\mathcal{A}}}\pi([0,\phi(a)])a$ holds. Using again the compactness of $\hat{\mathcal{A}}$ and the continuity of ϕ on $\hat{\mathcal{A}}$, we conclude that $\overline{\mathcal{A}}_1$ is compact.

(ii) First, let us prove that $\mathcal{A}_1$ is positively $\tilde{\pi}$-invariant. Let $a\in\hat{\mathcal{A}}$ and $s\in[0,\phi(a))$. We will prove that $\tilde{\pi}(t)\pi(s)a\in\mathcal{A}_1$ for every $t\ge0$. For that, denote

$$t_0=0,\quad t_1=\phi(\pi(s)a),\quad a_1^+=g(a),$$

and for any integer $n\ge1$, $t_{n+1}=t_n+\phi(a_n^+)$ and $a_{n+1}^+=g^{n+1}(a)$.

Given $t\ge0$, there exists $n\in\mathbb{N}_0$ such that $t\in[t_n,t_{n+1})$. Note that

$$\tilde{\pi}(t)\pi(s)a=\tilde{\pi}(t-t_n)\tilde{\pi}(t_n+s)a=\pi(t-t_n)g^n(a).$$

Since $g^n(a)\in\hat{\mathcal{A}}$ and $t-t_n<\phi(g^n(a))$, we obtain $\tilde{\pi}(t)\pi(s)a\in\mathcal{A}_1$.

Now, let us prove that $\mathcal{A}_1$ is negatively $\tilde{\pi}$-invariant. Let $a\in\hat{\mathcal{A}}$, $s\in[0,\phi(a))$ and fix an arbitrary $t\ge0$. We need to prove that there exist $x\in\hat{\mathcal{A}}$ and $r\in[0,\phi(x))$ such that $\tilde{\pi}(t)\pi(r)x=\pi(s)a$. In fact, if $t\le s$, then take $x=a$ and $r=s-t$. Thus, $t<\phi(a)-r=\phi(\pi(r)a)$ and

$$\tilde{\pi}(t)\pi(r)x=\pi(t)\pi(s-t)a=\pi(s)a.$$

However, if $t>s$, then by Theorem 3.14 there exists a bounded discrete global solution $\hat{\psi}:\mathbb{Z}\longrightarrow X$ of g through a. Set $a_n=\hat{\psi}(n)$ for all $n\in\mathbb{Z}$, and define $t_0=0$, $t_{-1}=-\phi(a_{-1})$ and $t_{-n}=t_{-n+1}-\phi(a_{-n})$ for $n>2$. There exists $n\in\mathbb{N}$ such that $s-t\in[t_{-n},t_{-n+1})$. Take $x=a_{-n}$ and $r=s-t-t_{-n}\ge0$. Then

$$t=s+\phi(a_{-1})+\cdots+\phi(a_{-n})-r$$

and

$$\begin{aligned}\tilde{\pi}(t)\pi(r)x&=\tilde{\pi}(s)\tilde{\pi}(\phi(a_{-1}))\cdots\tilde{\pi}(\phi(a_{-n+1}))\tilde{\pi}(\phi(a_{-n})-r)\pi(r)a_{-n}\\&=\cdots=\tilde{\pi}(s)\tilde{\pi}(\phi(a_{-1}))a_{-1}=\tilde{\pi}(s)a=\pi(s)a.\end{aligned}$$

In conclusion, $\mathcal{A}_1$ is $\tilde{\pi}$-invariant. □

Lemma 4.12 *Assume that (X,π) admits a global attractor $\mathcal{A}$ with $\mathcal{A}\cap M=\emptyset$ and $(\hat{X},g)$ has a global attractor $\hat{\mathcal{A}}$. Then $\mathcal{A}\cup\mathcal{A}_1$ is pre-compact, $\tilde{\pi}$-invariant and $\mathcal{A}\cup\mathcal{A}_1=\overline{\mathcal{A}\cup\mathcal{A}_1}\backslash M$.*

Proof Since the global attractor $\mathcal{A}$ is compact and $\tilde{\pi}$-invariant as $\mathcal{A}\cap M=\emptyset$, it follows by Lemma 4.11 that $\mathcal{A}\cup\mathcal{A}_1$ is pre-compact and $\tilde{\pi}$-invariant. Moreover, by Lemma 4.11 we have $\overline{\mathcal{A}}_1\backslash M=\mathcal{A}_1$, i.e., $\mathcal{A}\cup\mathcal{A}_1=\overline{\mathcal{A}\cup\mathcal{A}_1}\backslash M$. □

Lemma 4.13 *Under the conditions of Lemma 4.12, assume also that (X,π,M,I) is dissipative and $\phi(x)<\infty$ for all $x\in I(M)$. Then $\mathcal{A}\cup\mathcal{A}_1$ $\tilde{\pi}$-attracts bounded sets from X.*

Proof Let $B\in\mathcal{B}(X)$. By dissipativeness, there exists an absorbing set $\mathcal{B}_0\subset X$, consequently, there exists $T_B\geq 0$ such that $\tilde{\pi}(t)B\subset\mathcal{B}_0$ for all $t\geq T_B$. Now, let us denote

$$B_\infty:=\{x\in\tilde{\pi}(T_B)B:\ \phi(x)=\infty\}\ \text{ and }\ B_{\text{fin}}:=\{x\in\tilde{\pi}(T_B)B:\ \phi(x)<\infty\}.$$

Clearly, both sets are bounded.

Let $\epsilon>0$ be arbitrary. Since $\tilde{\pi}(t)x=\pi(t)x$ for every $t\geq 0$ and every $x\in B_\infty$, and $d_H(\pi(t)B_\infty,\mathcal{A})\overset{t\to\infty}{\longrightarrow}0$, there exists $T_1=T_1(B_\infty)\geq 0$ such that $\mathrm{d_H}(\tilde{\pi}(t)B_\infty,\mathcal{A}\cup\mathcal{A}_1)<\epsilon$ for all $t\geq T_1$.

Now, we claim there exists $T_2=T_2(B_{\text{fin}})\geq 0$ such that $\mathrm{d_H}(\tilde{\pi}(t)B_{\text{fin}},\mathcal{A}\cup\mathcal{A}_1)<\epsilon$ for all $t\geq T_2$. In fact, define the set

$$B_1=\{\tilde{\pi}(\phi(y))y:y\in B_{\text{fin}}\}.$$

Since $B_1\subset I(M)\cap\mathcal{B}_0$, it is bounded. Note that $g^n(B_1)$ is bounded for every $n\in\mathbb{N}_0$ according to its definition. By Lemma 4.10, for each n there exists $K_n>0$ such that $\phi(y)\leq K_n$ for every $y\in g^n(B_1)$. Moreover, there is $K_{-1}>0$ such that $\phi(y)\leq K_{-1}$ for every $y\in B_{\text{fin}}$.

Using the compactness of $\hat{\mathcal{A}}$, the continuity of π and the continuity of ϕ on $\hat{\mathcal{A}}$, we obtain:

(I) $T:=\max\{\phi(a):a\in\hat{\mathcal{A}}\}+1<\infty$;
(II) there exists $\delta_1=\delta_1(\epsilon)\in(0,1)$ such that if $s_1,s_2\in[0,T]$, $y\in X$, $a\in\hat{\mathcal{A}}$, with $|s_1-s_2|<\delta_1$ and $d(y,a)<\delta_1$, then $d(\pi(s_1)y,\pi(s_2)a)<\epsilon$;
(III) there exists $\delta_2=\delta_2(\delta_1)>0$ such that if $y\in X$, $a\in\hat{\mathcal{A}}$ with $d(y,a)<\delta_2$, then $|\phi(y)-\phi(a)|<\delta_1$.

Take $\delta=\frac{1}{2}\min\{\delta_1(\epsilon),\delta_2(\delta_1),\epsilon\}$. Since $\mathrm{d_H}(g^n(B_1),\hat{\mathcal{A}})\overset{n\to\infty}{\longrightarrow}0$, there exists $N\in\mathbb{N}$ such that $\mathrm{d_H}(g^n(B_1),\hat{\mathcal{A}})<\delta$ whenever $n\geq N$.

Besides, if $t\geq K_{-1}+K_0+\cdots+K_N$, then every point in B_{fin} suffered at least $N+2$ jump times under $\tilde{\pi}$ until time t. Thus, if $t\geq K_{-1}+K_0+\cdots+K_N$ and $x\in B_{\text{fin}}$, then

$$\tilde{\pi}(t)x=\pi(r)g^{n_0}(y),\ \text{ for some } n_0\geq N+2,\ \ y\in B_1 \text{ and } 0\leq r<\phi(g^{n_0}(y)).$$

Note that there exists $a\in\hat{\mathcal{A}}$ such that $d(g^{n_0}(y),a)<\delta$.

Case 1: If $r\leq\phi(a)$, then using (II) we obtain $d(\pi(r)g^{n_0}(y),\pi(r)a)<\epsilon$.

Case 2: If $r>\phi(a)$, then $\phi(a)<r<\phi(g^{n_0}(y))$. By (III), we have $|\phi(g^{n_0}(y))-\phi(a)|<\delta_1$. Thus, $\phi(a)<r<\phi(g^{n_0}(y))<\phi(a)+\delta_1<T$ which implies, by using (II), that $d(\pi(r)g^{n_0}(y),\pi(\phi(a))a)<\epsilon$.

By taking $T_2=K_{-1}+K_0+\cdots+K_N$, we conclude that $\mathrm{d_H}(\tilde{\pi}(t)B_{\text{fin}},\mathcal{A}\cup\mathcal{A}_1)<\epsilon$ for all $t\geq T_2$.

In conclusion, $\mathrm{d_H}(\tilde{\pi}(t)B,\mathcal{A}\cup\mathcal{A}_1)<\epsilon$ for all $t\geq\max\{T_B+T_1,T_B+T_2\}$. Since $\epsilon>0$ is arbitrary, $\mathcal{A}\cup\mathcal{A}_1$ indeed $\tilde{\pi}$-attracts bounded sets from X. □

As a consequence of the previous Lemmas 4.12 and 4.13, we may state the following result.

Theorem 4.14 *Under conditions* (H1)-(H3)*, assume that* (X, π) *admits a global attractor* $\mathcal{A}$ *with* $\mathcal{A} \cap M = \emptyset$*,* $(\hat{X}, g)$ *has a global attractor* $\hat{\mathcal{A}}$*,* (X, π, M, I) *is dissipative and* $\phi(x) < \infty$ *for all* $x \in I(M)$*. Then* (X, π, M, I) *admits a global attractor* $\tilde{\mathcal{A}}$ *given by*

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \left(\bigcup_{a \in \hat{\mathcal{A}}} \pi([0, \phi(a)))a \right).$$

Remark 4.15 Lemma 4.13 still holds if we replace the dissipativeness of (X, π, M, I) by the boundedness of $I(M)$. Indeed, let $B \in \mathcal{B}(X)$ and define the bounded sets

$$B_\infty = \{x \in B : \phi(x) = \infty\} \quad \text{and} \quad B_{\text{fin}} = \{x \in B : \phi(x) < \infty\}.$$

As in the proof of Lemma 4.13, given $\epsilon > 0$, there exists $T_1 = T_1(B_\infty) \geq 0$ such that $\mathrm{d_H}(\tilde{\pi}(t)B_\infty, \mathcal{A} \cup \mathcal{A}_1) < \epsilon$ for all $t \geq T_1$.

For the set B_{fin}, we also define $B_1 = \{\tilde{\pi}(\phi(y))y : y \in B_{\text{fin}}\}$. But now, B_1 and $g^n(B_1)$ are bounded since $I(M)$ is assumed to be bounded. The rest of the proof is exactly the same as in the proof of Lemma 4.13.

4.2 Existence of the Discrete Global Attractor $\hat{\mathcal{A}}$

In this section, we provide sufficient conditions for the existence of the discrete global attractor $\hat{\mathcal{A}}$ when (X, π) and (X, π, M, I) admit global attractors. We shall assume that conditions (H1)-(H3) hold.

Let (X, π) be a semidynamical system with global attractor $\mathcal{A}$ and (X, π, M, I) be the associated impulsive dynamical system with global attractor $\tilde{\mathcal{A}}$.

Lemma 4.16 *Assume that* $\mathcal{A} \cap M = \emptyset$ *and let* $\psi : \mathbb{R} \longrightarrow X$ *be a bounded global solution of* $\tilde{\pi}$*. If* ψ *has one jump time, then there exists a sequence of times* $\{t_n\}_{n \in \mathbb{N}}$ *with* $t_n \overset{n \to \infty}{\longrightarrow} -\infty$ *such that each* t_n *is a jump time of* ψ*.*

Proof Suppose to the contrary that there exists a jump time $t_* \in \mathbb{R}$ of ψ such that there are no jump times before t_*. Thus, let us define the set $B = \{\psi(t) : t \leq t_* - 1\}$, which is bounded. Then

$$\lim_{t \to \infty} \mathrm{d_H}(\pi(t)B, \mathcal{A}) = 0.$$

On the other hand, define the continuous map $\varphi \colon \mathbb{R} \longrightarrow X$ by

$$\varphi(t) = \begin{cases} \psi(t), & t \leq t_* - 1, \\ \pi(t - (t_* - 1))\psi(t_* - 1), & t \geq t_* - 1. \end{cases}$$

By construction φ is a bounded global solution of π, consequently, $\varphi(\mathbb{R}) \subset \mathcal{A}$. But we know that $\varphi(t_*) = \pi(1)\psi(t_* - 1) \in M$, because ψ has a jump time at t_*. Thus, $\varphi(t_*) \in \mathcal{A} \cap M$ which is a contradiction since $\mathcal{A} \cap M = \emptyset$. Hence, the result is proved. □

In Theorem 4.17, we prove the existence of the discrete global attractor $\hat{\mathcal{A}}$ and we also relate this attractor with $\tilde{\mathcal{A}} \cap \hat{X}$.

Theorem 4.17 *Assume that $\hat{X}$ is a nonempty closed set. Then $(\hat{X}, g)$ has a discrete global attractor $\hat{\mathcal{A}}$. In addition, if $\mathcal{A} \cap M = \emptyset$ then $\hat{\mathcal{A}} = \tilde{\mathcal{A}} \cap \hat{X}$.*

Proof First, let us prove that $(\hat{X}, g)$ is dissipative. Indeed, since (X, π, M, I) is dissipative, there exists an absorbing set $\mathcal{B}_0$. Let $\hat{B} \in \mathcal{B}(\hat{X})$. There exists $t_{\hat{B}} \geq 0$ such that $\tilde{\pi}(t)\hat{B} \subset \mathcal{B}_0$ for all $t \geq t_{\hat{B}}$. Since $\hat{B} \subset \hat{X} \subset I(M)$ and condition (H1) holds (hence, $\phi(w) \geq \xi$ for all $w \in I(M)$), one can obtain $k_0 \in \mathbb{N}$ such that $\phi(x) + \phi(x_1^+) + \ldots + \phi(x_{k_0}^+) \geq t_{\hat{B}}$ for all $x \in \hat{B}$. Thus,

$$g^n(x) = \tilde{\pi}\left(\sum_{j=0}^{n-1} \phi(x_j^+)\right) x \in \mathcal{B}_0 \cap \hat{X} \quad \text{for all} \quad n \geq k_0 + 1 \quad \text{and} \quad x \in \hat{B}.$$

Therefore, $g^n(\hat{B}) \subset \mathcal{B}_0 \cap \hat{X}$ for every $n \geq k_0 + 1$.

Now, let us prove that $(\hat{X}, g)$ is asymptotically compact. Let $\{x_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $\hat{X}$ and $n_k \stackrel{k\to\infty}{\longrightarrow} \infty$. Note that

$$g^{n_k}(x_k) = \tilde{\pi}\left(\sum_{j=0}^{n_k-1} \phi((x_k)_j^+)\right) x_k, \quad k \in \mathbb{N}.$$

Since $\phi(w) \geq \xi$ for all $w \in I(M)$ (as condition (H1) holds) and $n_k \stackrel{k\to\infty}{\longrightarrow} \infty$, we have $T(x_k) = \sum_{j=0}^{n_k-1} \phi((x_k)_j^+) \stackrel{k\to\infty}{\longrightarrow} \infty$. By the asymptotic compactness of (X, π, M, I), we conclude that $\{\tilde{\pi}(T(x_k))x_k\}_{k\in\mathbb{N}}$ has a convergent subsequence. Therefore, $\{g^{n_k}(x_k)\}_{k\in\mathbb{N}}$ has a convergent subsequence with limit in $\hat{X}$ because it is closed.

By Theorem 3.11, $(\hat{X}, g)$ has a global attractor $\hat{\mathcal{A}}$.

Now, assume that $\mathcal{A} \cap M = \emptyset$. By Proposition 4.7, we have $\hat{\mathcal{A}} \subset \tilde{\mathcal{A}} \cap \hat{X}$. On the other hand, let $x \in \tilde{\mathcal{A}} \cap \hat{X}$. By Theorem 2.15, there exists a bounded global solution $\psi : \mathbb{R} \longrightarrow X$ of $\tilde{\pi}$ through x. Since $x \in \hat{X}$, we have $\phi(x_j^+) < \infty$ for every $j \in \mathbb{N}_0$. Let $t_1 = \phi(x)$ and $t_{n+1} = t_n + \phi(x_n^+)$, $n \in \mathbb{N}$. By Lemma 4.16, there exists a sequence of times $\{t_{-n}\}_{n\in\mathbb{N}}$ with $t_{-n} \to -\infty$ as $n \to \infty$ such that t_{-n} are jump times of ψ and $\psi(-t_{-1}) = x$. Set $t_0 = 0$. By construction, we obtain

$$\psi(t) = \pi(t - t_n)\psi(t_n)$$

for $t \in [t_n, t_{n+1})$ and $n \in \mathbb{Z}$. Thus, $\hat{\psi} : \mathbb{R} \to \hat{X}$ given by $\hat{\psi}(n) = \psi(t_n)$ is a discrete global solution of g through x. Hence, by Theorem 3.14, $x \in \hat{\mathcal{A}}$ and we conclude that $\hat{\mathcal{A}} = \tilde{\mathcal{A}} \cap \hat{X}$. □

5 Applications

5.1 An Integrate-and-Fire Neuron Model

Integrate-and-fire neuron models describe the behavior of a membrane potential $u = u(t)$ (leaky and current-clamped membrane) along with a dissipation constant γ and an applied stimulus $S = S(t)$. Such models can be represented by the following ordinary differential equation

$$u'(t) = -\gamma u(t) + S(t) \tag{5.1}$$

with the additional condition

$$\text{if } \ u(t) = \theta \ \text{ then } \ u(t) \ \text{ is reset to value } \ u_r < \theta. \tag{5.2}$$

The membrane potential $u(t)$ is charged through the excitation, $S(t)$, and when it reaches the threshold value θ, the neuron fires and it is reset to the rest potential u_r, see [14]. Based on [4], we assume that the excitation $S > 0$ is constant with $S \neq \gamma\theta$, $\gamma > 0$ and $\theta > 0$. Define

$$M = \{\theta\} \quad \text{and} \quad I(\theta) = u_r.$$

If $[0, \infty) \ni t \mapsto \pi(t)u_0 \in \mathbb{R}$ is the solution of (5.1) with initial condition u_0 at $t = 0$, then $(\mathbb{R}, \pi, M, I)$ defines an impulsive dynamical system which describes the trajectories of the system (5.1)-(5.2). Note that M is an impulsive set satisfying $I(M) \cap M = \emptyset$. Since $u_r < \theta$ the condition **(H)** is satisfied. Moreover, it is not difficult to see that condition **(T)** also holds.

On one hand, the semiflow $(\mathbb{R}, \pi)$ without condition (5.2) admits a global attractor given by $\mathcal{A} = \left\{\frac{S}{\gamma}\right\}$.

On the other hand, if the excitation S is small, less than the threshold value $\theta\gamma$, it follows that the membrane potential u stabilizes to the value $\frac{S}{\gamma}$, i.e., the global attractor of $(\mathbb{R}, \pi, M, I)$ is given by

$$\tilde{\mathcal{A}} = \left\{\frac{S}{\gamma}\right\} \quad \text{if} \quad \theta\gamma > S.$$

However, if the excitation S is sufficiently larger than the threshold value $\theta\gamma$, then the structure of the attractor undergoes a significant change, meaning that the neuron is now capable of producing action potentials. Indeed, note that $\phi((u_r)_k^+) < \infty$ for all $k \in \mathbb{N}_0$, which implies that

$$\hat{\mathcal{A}} = \{u_r\}.$$

The set

$$B_0 = [u^r, \theta] \cup \left[\frac{S}{\gamma} - 1, \frac{S}{\gamma} + 1\right]$$

is an absorbing set, which implies that $(\mathbb{R}, \pi, M, I)$ is dissipative. By Theorem 4.14, the global attractor $\tilde{\mathcal{A}}$ of $(\mathbb{R}, \pi, M, I)$ is given by

$$\tilde{\mathcal{A}} = [u_r, \theta) \cup \left\{\frac{S}{\gamma}\right\} \quad \text{if} \quad \theta\gamma < S.$$

Remark 5.1 Assume that the integrate-and-fire neuron model is consider under several threshold values $\theta_1, \ldots, \theta_k$, with $\theta_1 < \theta_2 < \ldots < \theta_k$, such that

$$\text{if } \ u(t) = \theta_j \ \text{ for some } \ j \in \{1, \ldots, k\} \ \text{ then } \ u(t) \ \text{ is reset to value } \ u_r^j < \theta_j.$$

If $u_r^1 < \theta_1 < u_r^2 < \theta_2 < \ldots < u_r^p < \theta_p < \dfrac{S}{\gamma} < u_r^{p+1} < \theta_{p+1} < \ldots < u_r^k < \theta_k$ then

$$\tilde{\mathcal{A}} = \left(\bigcup_{j=1}^{p} [u_r^j, \theta_j)\right) \cup \left\{\frac{S}{\gamma}\right\}.$$

5.2 A Nonlinear Reaction–Diffusion Initial Boundary Value Problem

Consider the nonlinear reaction-diffusion initial boundary value problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{for } (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{for } (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & \text{for } x \in \Omega, \end{cases} \tag{5.3}$$

where Ω is a bounded smooth domain of $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary and Δ is the Laplace operator in Ω. The operator $-\Delta$ with the Dirichlet boundary conditions admits an orthonormal complete sequence of eigenfunctions $\{e_i\}_{i=1}^{\infty}$ in $L^2(\Omega)$ with corresponding eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ satisfying $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots, \lambda_n \overset{n\to\infty}{\longrightarrow} \infty$. The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfies the conditions:

(*a*) $|f(t) - f(s)| \leq c|t - s|$, for all $t, s \in \mathbb{R}$, where $c > 0$;
(*b*) $\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1$.

For each $u_0 \in L^2(\Omega)$, there exists a unique solution u of (5.3) with $u \in C([0,\infty), L^2(\Omega))$ such that the map $u_0 \mapsto u(t)$ is continuous in $L^2(\Omega)$. Thus, the map $\pi(t) \colon L^2(\Omega) \to L^2(\Omega)$ given by

$$\pi(t)u_0 = u(t)$$

defines a semidynamical system $(L^2(\Omega), \pi)$ on $L^2(\Omega)$. Also, $\pi(t) \colon L^2(\Omega) \to L^2(\Omega)$ is a compact operator for each $t > 0$. Let us consider the usual norm $\|\cdot\|_2$ and the usual inner product $(\cdot,\cdot)$ in $L^2(\Omega)$. The reader may see [1, 2, 7, 16] for more details.

According to condition (*b*), there exist $\epsilon_0 > 0$ and $R > 0$ such that $\frac{f(s)}{s} \leq \lambda_1 - \epsilon_0$ whenever $|s| > R$. Thus, $sf(s) \leq (\lambda_1 - \epsilon_0)s^2$ provided $|s| > R$. Also, there exists $C > 0$ such that $|sf(s)| \leq C$ for all $s \in [-R, R]$. Hence,

$$sf(s) \leq (\lambda_1 - \epsilon_0)s^2 + C \quad \text{for all} \quad s \in \mathbb{R}.$$

According to the proof of [5, Lemma 4.14], we have

$$\|\pi(t)u_0\|_2^2 \leq \|u_0\|_2^2 e^{-2\epsilon_0 t} + \frac{C|\Omega|}{\epsilon_0}(1 - e^{-2\epsilon_0 t}), \quad \text{for all } t \geq 0. \tag{5.4}$$

By (5.4), the semidynamical system $(L^2(\Omega), \pi)$ is dissipative with absorbing set

$$B_0 = \left\{ v \in L^2(\Omega) : \|v\|_2 \leq \frac{\rho_0 C|\Omega|}{\epsilon_0} \right\}, \quad \rho_0 > 1.$$

Since $\pi(t) \colon L^2(\Omega) \to L^2(\Omega)$ is also compact, it follows by Lemma 2.10 and Remark 2.16 that $(L^2(\Omega), \pi)$ is asymptotically compact. Now, according to Theorem 2.13 and Remark 2.16, we may state the following result.

Lemma 5.2 *The semidynamical system* $(L^2(\Omega), \pi)$ *admits a global attractor* $\mathcal{A}$.

Let $r_0 > \max\left\{1, \frac{2\rho_0 C|\Omega|}{\epsilon_0}\right\}$. Consider the set $M = \{v \in L^2(\Omega) : \ \|v\|_2 = r_0\}$ and the function $I \colon M \to I(M)$ given by

$$I(v) = v + 3r_0 e_1 \quad \text{for all} \quad v \in M.$$

We recall that if $u \in L^2(\Omega)$ then

$$u = \sum_{i=1}^{\infty} \alpha_i(u) e_i,$$

where $\alpha_i(u) = (u, e_i)$ is, for each i, the Fourier coefficient. The solution of the problem (5.3) is given explicitly by the formula

$$\pi(t)u_0 = u(t) = \sum_{i=1}^{\infty} \alpha_i(t) e_i,$$

with $\alpha_i(t) = \alpha_i(u_i(t))$ satisfying the ODE $\alpha_i'(t) + \lambda_i \alpha_i(t) = (f(u(t)), e_i)$, $i \in \mathbb{N}$.

Lemma 5.3 *M is an impulsive set, $I(M)$ is bounded and $I(M) \cap M = \emptyset$.*

Proof Clearly M is a closed set in $L^2(\Omega)$. Let us verify that condition (2.1) holds. Let $\Gamma = \left\{ v \in L^2(\Omega) : \|v\|_2 > \dfrac{C|\Omega|}{\epsilon_0} \right\}$. According to (5.4), for every $u \in \Gamma$, there exists $t_u \geq 0$ such that $\pi(t)u \in B_0$ for all $t \geq t_u$. Also, the map $t \mapsto \|\pi(t)u\|_2$ is strictly decreasing for every $t \geq 0$ such that $\pi(t)u \in \Gamma$. Thus, if $u \in M$ then $\pi((0, \infty))u \cap M = \emptyset$. Hence, M is an impulsive set.

On the other hand, let $w \in I(M)$. Then there exists $v \in M$ such that $w = I(v) = (\alpha_1(v) + 3r_0)e_1 + \sum_{j=2}^{\infty} \alpha_j(v) e_j$. Thus,

$$\|w\|_2^2 = (\alpha_1(v) + 3r_0)^2 + \sum_{j=2}^{\infty} \alpha_j^2(v) = \|v\|_2^2 + 9r_0^2 + 6r_0\alpha_1(v).$$

Since $\|v\|_2 = r_0$, we obtain $|\alpha_1(v)| \leq r_0$. Then

$$4r_0^2 \leq \|w\|_2^2 \leq 16r_0^2. \tag{5.5}$$

This implies that $I(M)$ is bounded and $I(M) \cap M = \emptyset$. □

Lemma 5.4 *There exists $K = K(|f(0)|, |\Omega|, c, C, \epsilon_0) > 0$ such that $(f(\pi(s)w), e_j) \leq K(1 + r_0)$, whenever $w \in I(M)$, $s \geq 0$ and $j \in \mathbb{N}$.*

Proof Let $w \in I(M)$, $j \in \mathbb{N}$ and $s \geq 0$ be arbitrary. By (5.5), we have $\|w\|_2 \leq 4r_0$. Then

$$\begin{aligned}
(f(\pi(s)w), e_j) &\leq \|f(\pi(s)w)\|_2 \sqrt{|\Omega|} \leq \sqrt{|\Omega|} \left(\int_\Omega (|f(0)| + c|\pi(s)w|)^2 dx \right)^{\frac{1}{2}} \\
&\leq |f(0)|\sqrt{2}|\Omega| + \sqrt{2|\Omega|}c\|\pi(s)w\|_2 \\
&\overset{(5.4)}{\leq} |f(0)|\sqrt{2}|\Omega| + \sqrt{2|\Omega|}c \left(\|w\|_2^2 + \frac{C|\Omega|}{\epsilon_0} \right)^{\frac{1}{2}} \\
&\leq K(1 + r_0),
\end{aligned}$$

for some constant $K > 0$. □

Lemma 5.5 *The impulsive dynamical system $(L^2(\Omega), \pi, M, I)$ satisfies conditions* **(H)** *and* **(T)**.

Proof At first, let us show that condition **(H)** holds. Let $w \in I(M)$. Then

$$w = I(v) = (\alpha_1(v) + 3r_0)e_1 + \sum_{j=2}^{\infty} \alpha_j(v)e_j,$$

for some $v \in M$. Let $t \geq 0$ be such that $\pi(t)w \in M$. Then

$$r_0^2 = \|\pi(t)w\|_2^2 = \left((\alpha_1(v) + 3r_0)e^{-\lambda_1 t} + \int_0^t (f(\pi(s)w), e_1)e^{\lambda_1(s-t)}ds\right)^2 + \sum_{j=2}^{\infty}\left(\alpha_j(v)e^{-\lambda_j t} + \int_0^t (f(\pi(s)w), e_j)e^{\lambda_j(s-t)}ds\right)^2.$$

Thus, using Lemma 5.4, we obtain

$$r_0 \geq |\alpha_1(v) + 3r_0|e^{-\lambda_1 t} - \left|\int_0^t (f(\pi(s)w), e_1)e^{\lambda_1(s-t)}ds\right| \geq 2r_0 e^{-\lambda_1 t} - K(1 + r_0)\left(\frac{1 - e^{-\lambda_1 t}}{\lambda_1}\right),$$

that is, $t \geq \frac{1}{\lambda_1}\ln\left(1 + \frac{r_0}{r_0 + K\lambda_1^{-1}(1+r_0)}\right)$. Hence, $\phi(w) \geq \frac{1}{\lambda_1}\ln\left(1 + \frac{r_0}{r_0 + K\lambda_1^{-1}(1+r_0)}\right)$ for all $w \in I(M)$.

Now, let us show that condition **(T)** holds. Let $v \in M$, $\{w_n\}_{n\in\mathbb{N}}$ be a convergent sequence in $L^2(\Omega)$ with limit w, and $t > 0$ be such that $\|\pi(t)w_n - v\|_2 \overset{n\to\infty}{\longrightarrow} 0$. Let $K = \max_{n\in\mathbb{N}} \|w_n\|_2$. Take $\tau > t$ such that $e^{-2\epsilon_0\tau}K^2 < \frac{(\rho_0-1)C|\Omega|}{\epsilon_0}$. By using (5.4), we have the estimate

$$\|\pi(\tau)w_n\|_2^2 \leq \|w_n\|_2^2 e^{-2\epsilon_0\tau} + \frac{C|\Omega|}{\epsilon_0} \leq K^2 e^{-2\epsilon_0\tau} + \frac{C|\Omega|}{\epsilon_0} < r_0^2, \quad n \in \mathbb{N}.$$

On the other hand, we have $v = \pi(t)w$, consequently,

$$r_0^2 = \|v\|_2^2 = \|\pi(t)w\|_2^2 \leq \|w\|_2^2 + \frac{C|\Omega|}{\epsilon_0} < \|w\|_2^2 + \frac{r_0^2}{2\rho_0}.$$

Since $\rho_0 > 1$, there exists $n_0 \in \mathbb{N}$ such that $r_0 < \|w_n\|_2$ for all $n \geq n_0$.

Now, define the function $\Theta_n : [0, \tau] \to \mathbb{R}$ by $\Theta_n(s) = \|\pi(s)w_n\|_2 - r_0$, $s \in [0, \tau]$ and $n \in \mathbb{N}$. Note that

$$\Theta_n(0) > 0 > \Theta_n(\tau) \quad \text{for all} \quad n \geq n_0.$$

By continuity of Θ_n, there exists $r_n \in [0, \tau]$ such that $\|\pi(r_n)w_n\|_2 = r_0$, i.e., $\pi(r_n)w_n \in M$ whenever $n \geq n_0$. We may assume that $r_n \overset{n\to\infty}{\longrightarrow} r \in [0, \tau]$. Thus, $\pi(r)w \in M$ and, hence, $r = t$ as $\|w\|_2 > r_0$ (by Lemma 5.3, the trajectory $\pi^+(w) = \{\pi(t)w : t \geq 0\}$ cross the impulsive set at most once). Taking $\alpha_n = r_n - t$, $n \geq n_0$, we get $\alpha_n \overset{n\to\infty}{\longrightarrow} 0$ and $\pi(t + \alpha_n)w_n = \pi(r_n)w_n \in M$. □

Theorem 5.6 *The impulsive dynamical system $(L^2(\Omega), \pi, M, I)$ is dissipative and asymptotically compact. In addition, $(L^2(\Omega), \pi, M, I)$ admits a global attractor $\tilde{\mathcal{A}}$.*

Proof Since $(L^2(\Omega), \pi)$ is dissipative and $r_0 < \|\tilde{\pi}(t)w\|_2 \leq \|w\|_2 \leq 4r_0$ for all $w \in I(M)$ and all $t \geq 0$ (see (5.5)), it follows that $(L^2(\Omega), \pi, M, I)$ is dissipative with absorbing set $\mathcal{B}_0 = \{v \in L^2(\Omega) : \|v\|_2 \leq 4r_0\}$. By Theorem 2.10, $(L^2(\Omega), \pi, M, I)$ is asymptotically compact. Theorem 2.13 ensures the existence of the global attractor of $(L^2(\Omega), \pi, M, I)$. □

Since $\phi(v_k^+) < \infty$ for all $v \in I(M)$ and all $k \in \mathbb{N}_0$, we have $\hat{L}^2(\Omega) = I(M)$. The set $I(M) = \{v + 3r_0e_1 : v \in M\}$ is closed in $L^2(\Omega)$ as M is closed. Also, $\mathcal{A} \cap M = \emptyset$. In this way, according to Theorem 4.17, $(\hat{L}^2(\Omega), g)$ admits a discrete global attractor $\hat{\mathcal{A}}$. This result is stated next.

Theorem 5.7 *The discrete dynamical system $(\hat{L}^2(\Omega), g)$ admits a discrete global attractor $\hat{\mathcal{A}}$ which satisfies $\hat{\mathcal{A}} = \tilde{\mathcal{A}} \cap \hat{X}$.*

In the last result, we relate the global attractor $\tilde{\mathcal{A}}$ with the attractors $\mathcal{A}$ and $\hat{\mathcal{A}}$.

Theorem 5.8 *There holds*

$$\tilde{\mathcal{A}} = \mathcal{A} \cup \left(\bigcup_{a \in \hat{\mathcal{A}}} \pi([0, \phi(a)))a \right).$$

Proof It is a consequence of Theorem 4.14. □

Acknowledgements The authors would like to thank the anonymous referee for the suggestions that have enhanced this article.

Author Contributions Not applicable.

Funding The first author was partially supported by FAPESP grant 2020/14075-6 and CNPq grant 310540/2019-4. The second author was partially supported by BES-2017-082334.

Availability of Data and Materials Not applicable.

Declarations

Conflict of interest The authors state that there is no conflict of interest.

Ethical Approval Not applicable.

References

1. Arrieta, J.M., Carvalho, A.N.: Abstract parabolic problems with critical nonlinearities and applications to Navier–Stokes and heat equations. Trans. Am. Math. Soc. **352**, 281–310 (2000)
2. Arrieta, J.M., Carvalho, A.N., Rodríguez-Bernal, A.: Attractors of parabolic problems with nonlinear boundary conditions uniform bounds. Comm. Partial Differ. Equ. **25**, 1–37 (2000)
3. Bhatia, N.P., Hajek, O.: Local Semi-Dynamical Systems (Lecture Notes in Mathematics), vol. 90. Springer, Berlin-Heidelbeg-New York (1969)
4. Bonotto, E.M., Kalita, P.: On Attractors of Generalized Semiflows with Impulses. J. Geom. Anal. **30**, 1412–1449 (2020)
5. Bonotto, E.M., Bortolan, M.C., Carvalho, A.N., Czaja, R.: Global attractors for impulsive dynamical systems—a precompact approach. J. Differ. Equ. **259**, 2602–2625 (2015)
6. Bonotto, E. M., Kalita, P.: Long-time behavior for impulsive generalized semiflows. Submitted
7. Chepyzhov, V.V., Vishik, M.I.: Attractors for Equations of Mathematical Physics, Providence, AMS Colloquium Publications v. 49, American Mathematical Society, (2002)
8. Ciesielski, K.: On semicontinuity in impulsive dynamical systems. Bull. Pol. Acad. Sci. Math. **52**, 71–80 (2004)
9. Ciesielski, K.: On stability in impulsive dynamical systems. Bull. Pol. Acad. Sci. Math. **52**, 81–91 (2004)
10. Hale, J. K.: Asymptotic behavior of dissipative dynamical systems, AMS (1988)
11. Kapustyan, O V., Romanyuk, I.V.: Global attractor of an impulsive dynamical system generated by the wave equation, Nelīnīīnī Koliv. 20(3), 361-372 (2017); translation in J. Math. Sci. (N.Y.) 236 (2019), no. 3, 300–312

12. Kaul, S.K.: On impulsive semidynamical systems. J. Math. Anal. Appl. **150**(1), 120–128 (1990)
13. Kaul, S.K.: On impulsive semidynamical systems - II recursive properties. Nonlinear Anal. **16**, 635–645 (1991)
14. Keener, J.P., Hoppensteadt, F.C., Rinzel, J.: Integrate-and-fire models of nerve membrane response to oscillatory input. SIAM J. Appl. Math. **41**, 503–517 (1981)
15. Lakshmikantham, V., Bainov, D.D., Simeonov, P.S.: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
16. Pazy, A.: Applied Mathematical Sciences. Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York (1983)
17. Wan, H., Jiang, H.: Dynamical behaviors of a predator-prey system with prey impulsive diffusion and dispersal delay between two patches. Adv. Differ. Equ. **191**, 1–11 (2019)