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Threshold Dynamics for Infection Age-Structured Epidemic Models with Spatial Diffusion and Degenerate Diffusion

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Abstract

This paper is devoted to studying the threshold dynamics for infection age-structured epidemic models with non-degenerate diffusion and degenerate diffusion. For general infection age-structured epidemic models with non-degenerate diffusion, we establish the basic reproduction number R_0 by using non-densely defined operators and prove that R_0 equals the spectral radius of $-\mathscr{F}\mathscr{A}^{-1}$. For a class of infection age-structured epidemic models with non-degenerate diffusion or degenerate diffusion, we give a general method to prove that R_0 plays the role of the threshold for the extinction or persistence of the disease. Finally, we apply our methods to the infection age-structured SIR, SEIR epidemic models and obtain the threshold results on their global dynamics. Our results on R_0 for the general infection age-structured epidemic models extend the cases of ODE and reaction–diffusion epidemic models. In addition, our method in this paper improves some previous results and is applicable to the Neumann, Dirichlet, and Robin boundary conditions.

Keywords Basic reproduction number · Infection age-structured · Degenerate diffusion · Uniform persistence · Compact attractors

Mathematics Subject Classification 35K20 · 92D30

1 Introduction

The epidemic model is one of the most classical models which has been researched for many years. It was firstly proposed by Kermack and McKendrick in 1927 and is an infection-age-dependent outbreak model [25]. Due to the age-structured effects, the infection age-structured epidemic models are more complicated than models in the form of ordinary differential

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equations. The basic reproduction number R_0 for ODE models can be seen as a threshold for extinction or uniform persistence of the disease, also as a criterion for the global asymptotic stability of the disease-free steady state or endemic steady state [10, 54, 56, 58, 64]. Therefore, the basic reproduction number is of great concern in age-structured models. In the case of infection age-structured epidemic models without diffusion term, Magal et al. studied the SIR model in 2010 and proved that R_0 plays the role of the threshold by using integrated semigroup theory [36]. In addition, many authors have extensively studied age-structured epidemic models [4, 9, 12–15, 26, 42, 60, 61]. Naturally, a question arises whether the basic reproduction number R_0 can be used as a threshold to decide the extinction and uniform persistence of the disease in the infection age-structured models with spatial diffusion.

For ODE and reaction–diffusion epidemic models, R_0 can be defined as the spectral radius of $-FV^{-1}$, where F is the input rate of newly infected individuals and V is the internal evolution of individuals in the infectious compartments. However, the definition of the basic reproduction number in infection age-structured epidemic models is always defined by the spectral radius of the next generation operator instead of $-FV^{-1}$. Therefore, we guess that the basic reproduction number R_0 for infection age-structured models can be defined as the form of $r(-FV^{-1})$. In Sect. 3, inspired by the ideas of Thieme [54], we give an affirmative answer to this conjecture and prove that R_0 for general infection age-structured epidemic models with non-degenerate diffusion also can be defined as the spectral radius of $-\mathscr{F}\mathscr{A}^{-1}$, where $\mathscr{F}$, $\mathscr{A}$ are non-densely defined operators. This result extends the basic reproduction number for ODE and reaction–diffusion epidemic models (such as Theorem 2 in [56], Theorems 3.1, 3.3 and 3.4 in [58] and Corollary 2.1 in [64]). In Sect. 3, we also prove that the spectral bound of $\mathscr{A}+\mathscr{F}$ has the same sign as $R_0 - 1$. Moreover, if $\mathscr{A}$, $\mathscr{F}$ are defined in suitable spaces, the exponential growth bound of $T_{\mathscr{A}_0+\mathscr{F}_0}$ also has the same sign as R_0-1, where $T_{\mathscr{A}_0+\mathscr{F}_0}$ is a C_0-semigroup generated by $\mathscr{A}_0+\mathscr{F}_0$. These results also extend the basic reproduction numbers for many kinds of ODE epidemic models and reaction–diffusion epidemic models (Theorem 3.8 in [44], Theorem 3.1 in [58] and Theorem 2.1 in [64]). In addition, in Sect. 3, we compare our results on R_0 for the infection age-structured epidemic model with Wang and Zhaos' work on reaction–diffusion epidemic models [58].

There have been many pieces of research on the infection age-structured epidemic model with non-degenerate diffusion. However, to the best of our knowledge, almost all research only focused on low-dimensional models, spatially homogeneous environments, and the Neumann boundary conditions. Chekroun and Kuniya studied the infection age-structured SIR model with spatial diffusion under the Neumann and Dirichlet boundary conditions [5–7]. In their work, they only proved the attractiveness of the disease-free steady state when $R_0 < 1$ and the initial value belongs to a subset of phase space instead of the whole phase space. Yet, this only partially showed that R_0 plays a role of the threshold to decide the extinction or uniform persistence of the disease, as happened in some other literature [57, 62]. Especially, if the boundary condition is not Neumann boundary condition or epidemic models with spatial heterogeneity, it becomes extremely difficult to prove the global attractiveness of the disease-free steady state. The major obstruction to use the traditional strategy of constructing Lyapunov functional, is the fact that the expression of the disease-free steady state is not constant. In Sect. 4, we study a class of infection age-structured epidemic model with non-degenerate diffusion and spatial heterogeneity in the high-dimensional case, and give a general method to overcome this obstacle. We prove the global attractiveness of the disease-free steady state if $R_0 < 1$ without restrictions on the initial value condition. A comparison between our and Chekroun's results on the infection age-structured SIR epidemic model [5–7] is presented in Remarks 6.5 and 6.7. Our method can prove the global attractiveness of the disease-free steady state with no limitation on the initial value condition.

There are also many works on the high-dimensional age structure epidemic models without spatial diffusion. In 2013, Magal et al. considered nosocomial infection and established a two-group infection age-structured epidemic model [39]. Kuniya et al. studied the multi-group SIR and SEIR epidemic model with age structure, and applied them to the chlamydia epidemic in Japan [27, 28, 59]. All of these show that it is of great practical significance to study the high-dimensional infection age-structured epidemic model. Nevertheless, due to the effects caused by diffusion terms, almost all the work about the infection age-structured epidemic models with spatial diffusion only focuses on the low-dimensional models. Additionally, many methods for the low-dimensional models with spatial diffusion are no longer suitable for the high-dimensional models. For example, in the case of the SIR model under the Dirichlet boundary condition, Chekroun and Kuniya used the Feynman–Kac formula and the Krasnoselskii's fixed point theorem to prove the existence of an endemic steady state [5]. Their method needs to calculate the Fréchet derivative of the operator defined by the boundary condition. Due to the complexity of operators, this is almost impossible in high-dimensional cases. In Sect. 4, we consider a class of infection age-structured epidemic model with non-degenerate diffusion in a high-dimensional situation. To make up for the absence of Fréchet derivative of some operators, we follow the idea of compact operators theory [18, 38, 49, 65] instead and give another method to prove the uniform persistence of disease and the existence of an endemic steady state.

We would like to mention that the method in Sect. 4 is different from the methods used in infection age-structured models with spatial diffusion in the past. In Sect. 4, based on the approach developed in [11, 39, 45], we give a general method to the class of infection age-structured epidemic model with non-degenerate diffusion and spatial heterogeneity. We overcome some problems left in the past literature (such as the global attractiveness of the disease-free steady state, and the existence of the endemic steady state). This method completely solves the threshold problem for the infection age-structured epidemic model with spatial diffusion. Due to the limitations of these methods, we need to add a condition that there exists a maximum infection age, and this condition is reasonable in age-structured models. To our knowledge, most of the previous studies on spatially diffusive epidemic models in spatially bounded domains assumed the homogeneous Neumann (zero-flux) boundary condition. The advantage of this approach based on operator semigroup theory is that it allows us to treat Neumann, Dirichlet, and Robin boundary conditions when the assumptions hold.

Epidemic models with degenerate diffusion have also been studied by many authors [23, 32, 55, 58, 63]. As far as we are concerned, the infection age-structured model with degenerate diffusion has not been studied in the literature. One of the technical challenges is that the solution semigroup of the degenerate reaction–diffusion equations is not compact. This factor directly prevents us from dealing with this degenerate diffusion model using the above methods. By using a generalized Krein-Rutman Theorem, we prove that the basic reproduction number is still the principal eigenvalue of the next generation operator, under some assumptions. In Sect. 5, we consider a class of infection age-structured epidemic model with degenerate diffusion and spatial heterogeneity. We prove that the basic reproduction number R_0 also plays a role of the threshold to decide extinction or weakly uniform persistence of the disease.

This paper is organized as follows. Our approach is sketched here for readers' convenience. In Sect. 2, we introduce the general infection age-structured epidemic model with spatial diffusion and use the method of characteristic line to transform the age-structured model into Volterra integral equations. In Sect. 3, we prove the existence of the solution and define the basic reproduction number R_0 for the infection age-structured epidemic model with non-degenerate diffusion by using non-densely defined operators. We prove that R_0 can be defined as the spectral radius of the operator $-\mathscr{F}\mathscr{A}^{-1}$ and the spectral bound of $\mathscr{A}+\mathscr{F}$ has the same sign as R_0-1. Moreover, the exponential growth bound of $T_{\mathscr{A}_0+\mathscr{F}_0}$ (solution map) also has the same sign as R_0-1 in some suitable spaces. In addition, we compare our results with Wang and Zhaos' work on reaction–diffusion epidemic models. In Sect. 4, we study a class of infection age-structured epidemic model with non-degenerate diffusion and spatial heterogeneity. We prove that R_0 is the threshold for extinction and persistence of the disease. If $R_0<1$, we prove that the disease-free steady state is globally attractive for the whole phase space by using the comparison principle for the age-structured equation and renewal theorem. If $R_0>1$, we prove the uniform persistence of disease and the existence of an endemic steady state by using the theory of compact attractors. In Sect. 5, we consider the epidemic model under degenerate diffusion and spatial heterogeneity. We follow the definition of the basic reproduction number R_0 in Sect. 3. We prove that $R_0<1$ means the extinction of disease, and $R_0>1$ means weakly uniform persistence of the disease. In Sect. 6, we apply our results to the infection age-structured SIR and SEIR epidemic models and obtain threshold results on its global dynamics. Finally, a brief discussion section completes the paper.

2 Preliminaries

In this paper, we consider the dynamical threshold for the general infection age-structured epidemic models with spatial diffusion and degenerate diffusion. Here, we assume that the number of infected compartments is n and the number of remaining compartments which includes susceptible, removal, and other compartments is m (for simplicity, we mark them as S). We construct the general infection age-structured epidemic model, for $1\le i\le n$, $1\le j\le m$, $t>t_0$, $a>0$ and $x\in\Omega$,

$$\begin{cases} \frac{\partial}{\partial t}S_j(t,x)=b_j(x)\Delta S_j(t,x)+M_j(x,S_1(t,x),\ldots,S_m(t,x),I_1(t,\cdot,x),\ldots,I_n(t,\cdot,x)),\\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)I_i(t,a,x)=d_i(x)\Delta I_i(t,a,x)-V_i(a,x,I_1(t,a,x),\ldots,I_n(t,a,x)),\\ I_i(t,0,x)=F_i(x,S_1(t,x),\ldots,S_m(t,x),\int_0^{+\infty}\beta_{i1}(a,x)I_1(t,a,x)da,\ldots,\\ \int_0^{+\infty}\beta_{in}(a,x)I_n(t,a,x)da), \end{cases} \tag{2.1}$$

under the Neumann boundary condition

$$\frac{\partial S_j}{\partial\nu}=0,\ \frac{\partial I_i}{\partial\nu}=0,\ x\in\partial\Omega, \tag{2.2}$$

with initial value condition

$$S_j(t_0,\cdot)=S_{j0}(\cdot)\in C(\overline{\Omega}),\ I_i(t_0,\cdot,\cdot)=I_{i0}(\cdot,\cdot)\in L^1(\mathbb{R}_+,C(\overline{\Omega})), \tag{2.3}$$

where $d_i(x)$ denotes the diffusion coefficient for the group i of infected compartments and $b_j(x)$ denotes the diffusion coefficient for the group j of remaining groups. Fix $n\in\mathbb{N}_+$. Let $\Omega\subset\mathbb{R}^n$ be a bounded, open, and connected set (domain) with smooth boundary $\partial\Omega$. In system (2.1), functions F_i are the newly infected individuals in the ith infected compartment, V_i is the rate of transfer of individuals between infected compartments, and M_j is a sum of the rate of the birth, out and transfer of remaining compartments.

Following the general setting of ODE and infection age-structured epidemic models, we make the following assumption.

Assumption 2.1 For system (2.1), assume that

(i) for each i and j, $\beta_{ij}(\cdot,\cdot) \in L^\infty_+(\mathbb{R}_+, C(\overline{\Omega})) \cap L^1_+(\mathbb{R}_+, C(\overline{\Omega}))$. Moreover, there exists at least one interval (a_*, a^*) such that $\boldsymbol{B}(a,x) := (\beta_{ij}(a,x))_{1\le i,j\le n}$ is an irreducible and positive matrix for all $a \in (a_*, a^*)$ and $x \in \Omega$;

(ii) $V_i(a, x, I_1(t,a,x), ..., I_n(t,a,x))$ is non-negative and continuous for all $a \in (0, +\infty)$, $x \in \Omega$, and continuously differential with respect to I_k, $\forall 1 \le k \le n$;

(iii) for each $j = 1, 2, \cdots, m$ and each $i = 1, 2, \cdots, n$, diffusion coefficients $b_j(\cdot), d_i(\cdot)$ are continuous functions on $C(\Omega)$ and $d_i(x) \ge 0$, $b_j(x) > 0$, $\forall x \in \Omega$;

(iv) $F_i(x, S_1(t,x), \cdots, S_m(t,x), \int_0^{+\infty} \beta_{i1}(a,x) I_1(t,a,x)da, \cdots, \int_0^{+\infty} \beta_{in}(a,x)$ $I_n(t,a,x)da)$ is a non-negative and continuous function for all $x \in \Omega$, and continuously differential with respect to $\sum\limits_{k=1}^{n} \int_0^{+\infty} \beta_{ik}(a,x) I_k(t,a,x)da$.

For simplicity, we rewrite the system (2.1) into a more compact form. Let $\boldsymbol{S}(t,x)$ and $\boldsymbol{I}(t,a,x)$ be defined as follows

$$\begin{aligned} &\boldsymbol{S}(t,x) := (S_1(t,x), S_2(t,x), \cdots, S_m(t,x))^T, \\ &\boldsymbol{I}(t,a,x) := (I_1(t,a,x), I_2(t,a,x), \cdots, I_n(t,a,x))^T. \end{aligned} \tag{2.4}$$

In addition, we define $\boldsymbol{\Lambda}$ and $\boldsymbol{L}$ as follows

$$\boldsymbol{\Lambda}(x)\boldsymbol{S} := (b_1(x)\Delta S_1, \cdots, b_m(x)\Delta S_m)^T, \boldsymbol{L}(x)\boldsymbol{I} := (d_1(x)\Delta I_1, \cdots, d_n(x)\Delta I_n)^T.$$

In order to study the dynamical threshold for system (2.1), we need to consider the linearization equations of the infected compartments at the disease-free steady state. So we assume that system has a disease-free steady state $(\boldsymbol{S}^0(x), \boldsymbol{I}^0(a,x)) = (\boldsymbol{S}^0(x), \boldsymbol{0}) = (S_1^0(x), S_2^0(x), \cdots, S_m^0(x), \underbrace{0, 0, \cdots, 0}_{n})^T$ (a clearer assumption on the disease-free steady state is in Assumption 2.1). Therefore we consider the following system (the linearization equations of the infected compartments at the disease-free steady state), for $t > t_0$, $a > 0$ and $x \in \Omega$,

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \boldsymbol{I}(t,a,x) = \boldsymbol{L}(x)\boldsymbol{I}(t,a,x) - \boldsymbol{V}^0(a,x)\boldsymbol{I}(t,a,x), \\ \boldsymbol{I}(t,0,x) = \boldsymbol{F}^0(x) \int_0^{+\infty} \boldsymbol{B}(a,x)\boldsymbol{I}(t,a,x)da, \end{cases} \tag{2.5}$$

where $\boldsymbol{B}(a,x) := (\beta_{ij}(a,x))_{1\le i,j\le n}$ and $\boldsymbol{F}^0(x)$, $\boldsymbol{V}^0(x)$ are defined by

$$\begin{aligned} \boldsymbol{F}^0(x) &:= \left(\frac{\partial F_i(x, S_1^0(x), \cdots, S_m^0(x), \int_0^{+\infty} \beta_{i1}(a) I_1^0(a,x)da, \cdots, \int_0^{+\infty} \beta_{in}(a) I_n^0(a,x)da)}{\partial \sum\limits_{j=1}^{n} \int_0^{+\infty} \beta_{ij}(a,x) I_j da} \right)_{1\le, i,j\le n}, \\ \boldsymbol{V}^0(a,x) &:= \left(\frac{\partial V_i(a,x,I_1^0(a,x), \cdots, I_n^0(a,x))}{\partial I_j} \right)_{1\le, i,j\le n}, \end{aligned}$$

where $\boldsymbol{I}^0(a,x) = (I_1^0(a,x), \cdots, I_n^0(a,x)) = (\underbrace{0, 0, \cdots, 0}_{n})$.

Here, we assume that systems (2.1) and (2.5) admit unique mild solutions (we will prove it in Sect. 3). The operators $\boldsymbol{L}(\cdot) - \boldsymbol{V}^0(a,\cdot)$ are associated with an evolutionary system $\mathcal{W}^0 := \{W^0(t,s); 0 \le s \le t \le +\infty\}$ of positive operators on $C(\overline{\Omega}, \mathbb{R}^n)$,

$$\boldsymbol{L}(\cdot) - \boldsymbol{V}^0(a,\cdot) = \lim_{h\to 0+} \frac{1}{h}(W^0(a+h,a)\phi - \phi), \quad \phi \in D(\boldsymbol{L}(\cdot) - \boldsymbol{V}^0(a,\cdot)),$$

where $D(\boldsymbol{L}(\cdot) - \boldsymbol{V}^0(a,\cdot))$ is the set of points for which the limit exists, and the norm of space $C(\overline{\Omega}, \mathbb{R}^n)$ is the usual supremum norm.

The solution of system (2.5) can be abstractly rewritten as

$$\begin{aligned} \boldsymbol{I}(t,a,x) &= \begin{cases} W^0(a,0)\boldsymbol{I}(t-a,0,x), & t-t_0>a, \\ W^0(a,a+t_0-t)\boldsymbol{I}_0(a+t_0-t,x), & t-t_0\le a, \end{cases} \\ \boldsymbol{I}(t,0,x) &= \int_0^{+\infty} \boldsymbol{F}^0(x)\boldsymbol{B}(a,x)\boldsymbol{I}(t,a,x)da, \quad t>t_0, x\in\Omega. \end{aligned} \tag{2.6}$$

We define the exponential growth bound of evolution family $W^0(t,s)$ as

$$\omega(W^0)=\inf\{\hat{\omega}: \exists M\ge 1: \forall s\in\mathbb{R}, t\ge 0: \|W^0(t+s,s)\|\le Me^{\hat{\omega}t}\}.$$

To ensure that the disease-free steady state $(\boldsymbol{S}^0(x),\boldsymbol{0}) = (S_1^0(x), S_2^0(x),\cdots,S_m^0(x),\underbrace{0,0,\cdots,0}_{n})^T$ is stable and note that the internal evolution of individuals in infected compartments is dissipative and exponential decay(such as natural mortalities and disease-induced mortalities), we make the following assumption.

Assumption 2.2 For system (2.1), assume that

(i) $-\boldsymbol{V}^0(a,x)$ is a cooperative and irreducible matrix function for all $x\in\overline{\Omega}$ and $a\in[0,+\infty)$. In addition, $\omega(W^0)<0$, where $\omega(\cdot)$ represents the exponential growth bound;

(ii) the following reaction–diffusion equations under the Neumann boundary condition

$$\frac{d\boldsymbol{S}(t,x)}{dt}=\boldsymbol{\Lambda}(x)\boldsymbol{S}(t,x)+\boldsymbol{M}(\boldsymbol{x},\boldsymbol{S}(\boldsymbol{t},\boldsymbol{x}),\boldsymbol{0}), \quad x\in\Omega \tag{2.7}$$

admits a globally attractive unique positive steady state $\boldsymbol{S}^0(x)$, where $\boldsymbol{M}:=(M_j)_{1\le j\le m}$.

In order to reformulate system (2.5) into Volterra integral equations, we define $z_0(t,t_0;\boldsymbol{I}_0,x)$ as follows

$$z_0(t,t_0;\boldsymbol{I}_0,x):=\boldsymbol{I}(t,0,x)=\boldsymbol{F}^0(x)\int_0^{+\infty}\boldsymbol{B}(a,x)\boldsymbol{I}(t,a,x)da, \quad t\ge t_0, x\in\Omega. \tag{2.8}$$

By (2.8), we have, for $t\ge t_0, x\in\Omega$,

$$\begin{aligned} z_0(t,t_0;\boldsymbol{I}_0,x) &= \boldsymbol{F}^0(x)\int_0^{+\infty}\boldsymbol{B}(a,x)\boldsymbol{I}(t,a,x)da \\ &= \boldsymbol{F}^0(x)\int_0^{t}\boldsymbol{B}(a,x)\boldsymbol{I}(t,a,x)da+\boldsymbol{F}^0(x)\int_t^{+\infty}\boldsymbol{B}(a,x)\boldsymbol{I}(t,a,x)da \\ &= \boldsymbol{F}^0(x)\int_0^{t}\boldsymbol{\Phi}^0(a,x)z_0(t-a,t_0;\boldsymbol{I}_0,x)da+\boldsymbol{F}^0(x)\boldsymbol{H}^0(t,t_0;\boldsymbol{I}_0,x), \end{aligned} \tag{2.9}$$

where $\boldsymbol{\Phi}^0$ and $\boldsymbol{H}^0$ are defined by

$$\begin{aligned} \boldsymbol{\Phi}^0(a,x) &= \boldsymbol{B}(a,x)W^0(a,0), \\ \boldsymbol{H}^0(t,t_0;\boldsymbol{I}_0,x) &= \int_t^{+\infty}\boldsymbol{B}(a,x)W^0(a,a+t_0-t)\boldsymbol{I}_0(a+t_0-t,x)da. \end{aligned}$$

Remark 2.3 System (2.9) can be abstractly seen as Volterra integral equations. Because the methods in this paper are based on the theory of operator semigroup, our results are valid for the Neumann, Dirichlet, and Robin boundary conditions. In this paper, we mainly consider the infection age-structured epidemic model under the Neumann boundary condition.

3 Basic Reproduction Number for General Infection Age-Structured Epidemic Model with Non-degenerate Diffusion

In this section, we consider the infection age-structured epidemic model with non-degenerate diffusion. In Sect. 3.1, we prove the existence of integral solutions of systems (2.1) and (2.5). In Sect. 3.2, we give the definition of the basic reproduction number R_0 and prove that R_0-1 has the same sign as the spectral bound of $\mathscr{A}+\mathscr{F}$, where $\mathscr{A}$, $\mathscr{F}$ are non-densely defined operators. Moreover, if $\mathscr{A}$, $\mathscr{F}$ are defined in suitable spaces, $\omega(T_{\mathscr{A}_0+\mathscr{F}_0})$ also has the same sign as R_0-1. Without loss of generality, we always set $t_0=0$.

Assumption 3.1 There exists a positive constant d_0 such that $d_i(x)\geq d_0$ for each $i=1,2,\cdots,n$ and $x\in\Omega$.

Theorem 3.2 *Let Assumptions 2.1, 2.2, and 3.1 be satisfied. Then the evolution family $W^0(t,s), t\geq s$ is compact.*

Proof By the definition of $W^0(t,s)$, we know that $W^0(t,s)$ is the solution map of the following reaction–diffusion equations under the Neumann boundary condition

$$\frac{d\boldsymbol{u}(t,x)}{dt}=\boldsymbol{L}(x)\boldsymbol{u}(t,x)-\boldsymbol{V}^0(t,x)\boldsymbol{u}(t,x),\quad t\geq 0, x\in\Omega.$$

Note that $\boldsymbol{L}$ is the Laplace operator with Neumann boundary condition, it follows that $W^0(t,s), t\geq s$ is compact. □

3.1 Non-densely Defined Operators and the Well-Posedness

In this subsection, we use the method of the non-densely defined operator to prove the existence of the solutions of systems (2.1) and (2.5).

Set $Y:=C(\overline{\Omega},\mathbb{R}^n)$, equipped with the usual supremum norm. Recall that $\boldsymbol{L}$ is the Laplace operator with the Neumann boundary condition. Then

$$D(\boldsymbol{L}):=\{\boldsymbol{\phi}\in C^2(\Omega,\mathbb{R}^n)\cap C^1(\overline{\Omega},\mathbb{R}^n):\boldsymbol{L\phi}\in C(\Omega,\mathbb{R}^n),\ \frac{\partial\boldsymbol{\phi}}{\partial\nu}=0 \text{ for } x\in\partial\Omega\}.$$

By Chapter 7 of [48], we know that $\boldsymbol{L}$ generates an analytic semigroup of bounded linear operators $T(t)$ on Y.

Let $X:=L^1((0,+\infty),Y)$ and the norm of space X be given by

$$\|\boldsymbol{\varphi}\|_X:=\int_0^{+\infty}\|\boldsymbol{\varphi}(a,\cdot)\|_Y da,\quad \varphi\in X.$$

Let us introduce a new extended space $\mathbb{X}$ and its closed subspace $\mathbb{X}_0$ by

$$\mathbb{X}:=Y\times X,\quad \mathbb{X}_0=\boldsymbol{0}\times X,$$

where $\boldsymbol{0}:=(\underbrace{0,0,...,0}_{n})^T$. For any $(\boldsymbol{\phi},\boldsymbol{\varphi})\in\mathbb{X}$, the norm is defined by

$$\|(\boldsymbol{\phi},\boldsymbol{\varphi})\|_{\mathbb{X}}:=\|\boldsymbol{\phi}\|_Y+\|\boldsymbol{\varphi}\|_X,\quad (\boldsymbol{\phi},\boldsymbol{\varphi})\in\mathbb{X}.$$

Then we consider the family of bounded linear operators $\{R_\lambda\}_{\lambda>0}$ on $\mathbb{X}$, defined by

$$R_\lambda\begin{pmatrix}\boldsymbol{\phi}\\ \boldsymbol{\varphi}\end{pmatrix}=\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{\psi}\end{pmatrix}\Leftrightarrow \boldsymbol{\psi}(a)=e^{-\int_0^a\lambda ds}T(a)\boldsymbol{\phi}+\int_0^a e^{-\int_s^a\lambda dl}T(a-s)\boldsymbol{\varphi}(s)ds.$$

Observe that $\{R_\lambda\}_{\lambda>0}$ is a pseudo-resolvent on $\mathbb{X}$. That is to say that

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu, \quad \forall \lambda, \mu > 0.$$

Moreover, we have

$$R_\lambda x = 0, x \in \mathbb{X} \Rightarrow x \in \mathbb{X}_0$$

and

$$\lim_{\lambda \to +\infty} \lambda R_\lambda x = x, \quad \forall x \in \mathbb{X}_0.$$

Similar to Sect. 1.9 of [43], we deduce that there exists a unique closed linear operator $\mathcal{A}$ which satisfies

$$\mathcal{A} : D(\mathcal{A}) \subset \mathbb{X} \to \mathbb{X}, \quad \overline{D(\mathcal{A})} = \mathbb{X}_0,$$

and

$$R_\lambda = (\lambda I - \mathcal{A})^{-1}, \quad \forall \lambda > 0.$$

Denote by $\mathbb{X}_0^+ := \mathbf{0} \times X_+$ the positive cone of $\mathbb{X}_0$. In addition, we define an operator $\mathcal{F} : \mathbb{X}_0^+ \to \mathbb{X}$ by

$$\mathcal{F}\begin{pmatrix} \mathbf{0} \\ \boldsymbol{\varphi} \end{pmatrix} := \begin{pmatrix} \sum\limits_{i=1}^{n} f_{1i}^0(\cdot) \int_0^{+\infty} \sum\limits_{j=1}^{n} \beta_{ij}(a,\cdot)\varphi_j(a,\cdot)da \\ \vdots \\ \sum\limits_{i=1}^{n} f_{ni}^0(\cdot) \int_0^{+\infty} \sum\limits_{j=1}^{n} \beta_{ij}(a,\cdot)\varphi_j(a,\cdot)da \\ -\sum\limits_{j=1}^{n} v_{1j}^0(a,\cdot)\varphi_j(a,\cdot) \\ \vdots \\ -\sum\limits_{j=1}^{n} v_{nj}^0(a,\cdot)\varphi_j(a,\cdot) \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\varphi} \end{pmatrix} \in \mathbb{X}_0^+,$$

where $\boldsymbol{F}^0 = (f_{ij}^0)_{1\le i,j\le n}$, $\boldsymbol{V}^0 = (v_{ij}^0)_{1\le i,j\le n}$. According to the above definition, we can transform (2.5) into the following semi-linear Cauchy problem in a non-densely defined domain

$$\begin{aligned} \frac{du(t)}{dt} &= \mathcal{A}u(t) + \mathcal{F}u(t), \\ u(0) &= \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\varphi} \end{pmatrix} \in \mathbb{X}_0^+. \end{aligned} \tag{3.1}$$

Similar to [4], we consider the Cauchy problem (3.1) with the following equivalent form (3.2)

$$\begin{aligned} \frac{du(t)}{dt} &= (\mathcal{A} - \tfrac{1}{\varepsilon} I)u(t) + \tfrac{1}{\varepsilon}(I + \varepsilon\mathcal{F})u(t), \\ u(0) &= \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\varphi} \end{pmatrix} \in \mathbb{X}_0^+, \end{aligned} \tag{3.2}$$

where ε is small enough that $I + \varepsilon\mathcal{F}$ map the $\mathbb{X}_0^+$ to the positive cone $\mathbb{X}_+$ of $\mathbb{X}$.

Let $\mathcal{A}_\varepsilon := \mathcal{A} - \frac{1}{\varepsilon}I$ and $\mathcal{F}_\varepsilon := \frac{1}{\varepsilon}(I + \varepsilon\mathcal{F})$. Since the operators are defined on the non-densely defined domain, the classical semigroup theory is not suitable. We use the method of Lipschitz perturbations of the non-densely defined operators [33, 52].

Lemma 3.3 *Let Assumptions 2.1, 2.2, and 3.1 be satisfied. Then $\mathcal{A}_\varepsilon$ satisfies the Hille-Yosida condition.*

Proof We consider the resolvent of $\mathcal{A}_\varepsilon$. Then we have

$$(\lambda I - \mathcal{A}_\varepsilon)\begin{pmatrix} \mathbf{0} \\ \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\varphi} \end{pmatrix} \in \mathbb{X}. \tag{3.3}$$

By $\mathcal{A}_\varepsilon := \mathcal{A} - \frac{1}{\varepsilon}I$ and $R_\lambda = (\lambda I - \mathcal{A})^{-1}$, we have

$$\begin{aligned}\boldsymbol{\psi}(a) &= e^{-\int_0^a \lambda + \frac{1}{\varepsilon} ds} T(a)\boldsymbol{\phi} + \int_0^a e^{-\int_s^a \lambda + \frac{1}{\varepsilon} dl} T(a-s)\boldsymbol{\varphi}(s)ds \\ &= e^{-(\lambda + \frac{1}{\varepsilon})a} T(a)\boldsymbol{\phi} + \int_0^a e^{-(\lambda + \frac{1}{\varepsilon})(a-s)} T(a-s)\boldsymbol{\varphi}(s)ds.\end{aligned}$$

Now, we give the estimate of $(\lambda I - \mathcal{A}_\varepsilon)^{-1}$:

$$\begin{aligned}&\left\| (\lambda I - \mathcal{A}_\varepsilon)^{-1} \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\varphi} \end{pmatrix} \right\|_{\mathbb{X}} = \|\boldsymbol{\psi}\|_X \\ &\le \int_0^{+\infty} e^{-(\lambda + \frac{1}{\varepsilon})a} \|T(a)\boldsymbol{\phi}\| da + \int_0^{+\infty} \int_0^a e^{-(\lambda + \frac{1}{\varepsilon})(a-s)} \|T(a-s)\boldsymbol{\varphi}(s)\| dsda \\ &\le M \int_0^{+\infty} e^{-(\lambda + \frac{1}{\varepsilon})a} e^{-\lambda_0 a} da \, \|\boldsymbol{\phi}\| + M \int_0^{+\infty} \int_0^{+\infty} e^{-(\lambda + \frac{1}{\varepsilon} + \lambda_0)(a-s)} \|\boldsymbol{\varphi}(s)\| dsda \\ &\le \frac{M}{\lambda + \frac{1}{\varepsilon} + \lambda_0} \left\| \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\varphi} \end{pmatrix} \right\|,\end{aligned}$$

where λ_0 is the principal eigenvalue of the following eigenvalue problem:

$$\boldsymbol{L}(x)u(x) + \lambda u(x) = 0.$$

The above eigenvalue problem admits a unique principal eigenvalue λ_0 [3, 20], with the solution semigroup $T(t)$ satisfying $\|T(t)\| \le Me^{-\lambda_0 t}$, where $M \ge 1$. Thus, $\mathcal{A}_\varepsilon$ satisfies the Hille-Yosida estimate. □

Definition 3.4 A continuous function $u : [0, +\infty) \to \mathbb{X}$ is called an integral solution to (3.2) if

$$u(t) = u(t_0) + \mathcal{A}_\varepsilon \int_{t_0}^t u(s)ds + \int_{t_0}^t \mathcal{F}_\varepsilon u(s)ds. \tag{3.4}$$

Remark 3.5 (3.4) implies that $\int_{t_0}^t u(s)ds \in D(\mathcal{A}_\varepsilon)$.

Define the part $\mathcal{A}_{\varepsilon 0}$ of $\mathcal{A}_\varepsilon$ on $\mathbb{X}_0 = \overline{D(\mathcal{A}_\varepsilon)}$:

$$\mathcal{A}_{\varepsilon 0} = \mathcal{A}_\varepsilon \text{ on } D(\mathcal{A}_{\varepsilon 0}) = \{\varphi \in D(\mathcal{A}_\varepsilon); \mathcal{A}_\varepsilon \varphi \in \mathbb{X}_0\}. \tag{3.5}$$

The following Lemmas 3.6 and 3.7 can be found in [33, 52].

Lemma 3.6 *The part $\mathcal{A}_{\varepsilon 0}$ of $\mathcal{A}_\varepsilon$ on $\mathbb{X}_0$ generates a C_0-semigroup $\{T_{\mathcal{A}_{\varepsilon 0}}(t)\}_{t \ge 0}$ on space $\mathbb{X}_0$.*

Lemma 3.7 *The unique continuous solution to (3.2) can be given by (3.6),*

$$u(t) = T_{\mathcal{A}_{\varepsilon 0}}(t - t_0)u(t_0) + \lim_{\lambda \to \infty} \int_{t_0}^t T_{\mathcal{A}_{\varepsilon 0}}(t-s)\lambda(\lambda - \mathcal{A}_\varepsilon)^{-1}\mathcal{F}_\varepsilon u(s)ds. \tag{3.6}$$

and it takes value in $\mathbb{X}_0$.

In fact, Lemma 3.7 follows from the C_0-semigroup theory, integrated semigroup theory, and variation of constants formula. If ε is chosen small enough, $\mathcal{F}_\varepsilon$ can be seen as a local Lipschitz continuous, positive, and bounded perturbation. If we follow the ideas of the Banach fixed point theorem in [21], we can also prove the existence of the local positive solution. Above all, the solution of the system (3.1) has been proved.

Theorem 3.8 *Let Assumptions 2.1, 2.2, and 3.1 be satisfied. Then system (2.5) with initial value $\boldsymbol{I}_0 \in X_+$ has a non-negative solution defined on $C([0,\tau), X)$, $\tau > 0$.*

Remark 3.9 In fact, if we set $\mathcal{Y} = L^2(\Omega, \mathbb{R}^n)$ or other suitable spaces, then Theorem 3.8 still holds. The reason is that the Laplace operator also generates an analytic and compact semigroup on space $L^2(\Omega, \mathbb{R}^n)$. Moreover, we would like to mention that Magal and Ruan's work [34] tells us that we can define $\mathcal{X} := L^p((0,+\infty), \mathcal{Y})$ with $p \geq 1$ and system (2.5) with initial value $\boldsymbol{I}_0 \in \mathcal{X}_+$ has a non-negative solution defined in $C([0,\tau), \mathcal{X})$, $\tau > 0$.

Next, we prove the existence of the solution of the system (2.1). System (2.1) can be abstractly seen as the following form.

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{S}(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Lambda S}(t) + \overline{\boldsymbol{V}}(\boldsymbol{S}(t), u(t)) \\ \mathcal{A}u(t) + \widehat{\boldsymbol{V}}(a, \boldsymbol{S}(t), u(t)) \end{pmatrix}, \tag{3.7}$$

where $\boldsymbol{S}(t,\cdot) = (S_1(t,\cdot), \cdots, S_m(t,\cdot))^T$ and

$$\overline{\mathbf{V}}(\mathbf{S}(t), u(t))(\cdot) = \begin{pmatrix} M_1(\cdot, S_1(t,\cdot), \cdots, S_m(t,\cdot), u_1(t,\cdot,\cdot), \cdots, u_n(t,\cdot,\cdot)) \\ \vdots \\ M_m(\cdot, S_1(t,\cdot), \cdots, S_m(t,\cdot), u_1(t,\cdot,\cdot), \cdots, u_n(t,\cdot,\cdot)) \end{pmatrix},$$

$$\widehat{\boldsymbol{V}}(a, \boldsymbol{S}(t), u(t))(\cdot) \\ = \begin{pmatrix} -V_1(a,\cdot,u_1(t,a,\cdot), \cdots, u_n(t,a,\cdot)) \\ \vdots \\ -V_n(a,\cdot,u_1(t,a,\cdot), \cdots, u_n(t,a,\cdot)) \\ F_1(\cdot, S_1(t,\cdot), \cdots, S_m(t,\cdot), \int_0^{+\infty} \beta_{11}(a,\cdot)u_1(t,a,\cdot)da, \cdots, \int_0^{+\infty} \beta_{1n}(a,\cdot)u_n(t,a,\cdot)da) \\ \vdots \\ F_n(\cdot, S_1(t,\cdot), \cdots, S_m(t,\cdot), \int_0^{+\infty} \beta_{n1}(a,\cdot)u_1(t,a,\cdot)da, \cdots, \int_0^{+\infty} \beta_{nn}(a,\cdot)u_n(t,a,\cdot)da) \end{pmatrix}.$$

Cauchy problem (3.7) can be seen abstractly as

$$\frac{d}{dt}P(t) = \overline{\mathcal{A}}P(t) - (\overline{\mathcal{F}}P)(t),$$

where $P = \begin{pmatrix} \boldsymbol{S} \\ u \end{pmatrix}$, $\overline{\mathcal{A}} = \begin{pmatrix} \boldsymbol{\Lambda} & \boldsymbol{0} \\ \boldsymbol{0} & \mathcal{A} \end{pmatrix}$, $(\overline{\mathcal{F}}P)(t) = \begin{pmatrix} \overline{\boldsymbol{V}}(\boldsymbol{S}(t), u(t)) \\ \widehat{\boldsymbol{V}}(a, \boldsymbol{S}(t), u(t)) \end{pmatrix}$.

Because $\boldsymbol{\Lambda}$ is the Laplace operator with the Neumann boundary condition, the eigenvalues of $\boldsymbol{\Lambda}$ are smaller or equal to 0. It is well known that $\boldsymbol{\Lambda}$ satisfies the Hille-Yosida condition. Combining with Lemma 3.3, we can deduce that $\overline{\mathcal{A}}$ satisfies the Hille-Yosida condition. Similar to Lemmas 3.6, 3.7, and Theorem 3.8, we have the following theorem about the existence of the solution of the system (2.1).

Theorem 3.10 *Let Assumptions 2.1, 2.2, and 3.1 be satisfied. If $\overline{\boldsymbol{V}}$ and $\widehat{\boldsymbol{V}}$ are Lipschitz continuous, then system (2.1) with initial value $(\boldsymbol{S}_0, \boldsymbol{I}_0) \in C_+(\overline{\Omega}, \mathbb{R}^m) \times X_+$ has a mild solution defined on $C([0,\tau), C(\overline{\Omega}, \mathbb{R}^m)) \times C([0,\tau), X)$, $\tau > 0$.*

Remark 3.11 If we define $\mathcal{Y} := L^2(\Omega, \mathbb{R}^n)$ and $\mathcal{X} := L^p((0, +\infty), \mathcal{Y})$, then Theorem 3.10 still holds if we replace phase space $C([0, \tau), C(\overline{\Omega}, \mathbb{R}^m)) \times C([0, \tau), X)$ with $C([0, \tau), L^2(\Omega, \mathbb{R}^m)) \times C([0, \tau), \mathcal{X})$. It is worth mentioning that the results of this section are valid in the spaces $\mathcal{Y} := L^2(\Omega, \mathbb{R}^n)$ and $\mathcal{X} := L^p((0, +\infty), \mathcal{Y})$, $p \geq 1$.

3.2 Basic Reproduction Number

For simplicity, we always set $t_0 = 0$ and spaces $Y := C(\overline{\Omega}, \mathbb{R}^n)$ and $X := L^1((0, +\infty), Y)$ without additional assumptions.

From Sect. 2, we know that there exists an evolution family $W^0(t, s), t \geq s$ on Y for system (2.5) as follows

$$\boldsymbol{I}(t, a, x) = \begin{cases} W^0(a, 0)\boldsymbol{I}(t-a, 0, x), & t-a > 0, \\ W^0(a, a-t)\boldsymbol{I}_0(a-t, x), & t-a \leq 0. \end{cases} \tag{3.8}$$

Based on the boundary condition of (2.5), we have

$$\boldsymbol{I}(t, 0, x) = \boldsymbol{F}^0(x) \int_0^{+\infty} \boldsymbol{B}(a, x)\boldsymbol{I}(t, a, x)da, \quad t \geq 0, x \in \Omega. \tag{3.9}$$

According to the classical theory of the basic reproduction number for the age-structured epidemic models [10, 22], we give the next generation operator Ψ that maps Y into itself as follows,

$$\begin{aligned} \Psi(\boldsymbol{\varphi})(x) :&= \boldsymbol{F}^0(x) \int_0^{+\infty} \Phi^0(a)\boldsymbol{\varphi}(x)da \\ &= \boldsymbol{F}^0(x) \int_0^{+\infty} \boldsymbol{B}(a, x)W^0(a, 0)\boldsymbol{\varphi}(x)da \ \text{ for } \ x \in \overline{\Omega}. \end{aligned} \tag{3.10}$$

Similar to the argument in [10], we define the basic reproduction number R_0 by

$$R_0 := r(\Psi), \tag{3.11}$$

where $r(\cdot)$ is the spectral radius.

Lemma 3.12 *Let Assumptions 2.1, 2.2, and 3.1 be satisfied. Then Ψ is a compact operator and $r(\Psi)$ is the principal eigenvalue of Ψ with a strongly positive eigenvector ψ_*. Moreover, there is no other eigenvalue of Ψ with a positive eigenvector.*

Proof From Theorem 3.2, we know that evolution family $\{W(t, s)\}_{t \geq s}$ is compact. Operator Ψ is a compact operator since it is a composition of a bounded operator and a compact operator (Theorem 4.18 in [46]). Then the rest part of Lemma 3.12 is a direct result of the Krein-Rutman Theorem. □

In Sect. 6 of [54], Thieme used non-densely defined operators to give the threshold operator of a one-dimensional age-structured population model. Inspired by this idea of Thieme, we extend it to the n-dimensional cooperative age-structured epidemic models which are linearized around the disease-free steady state (n-dimensional cooperative non-densely defined Cauchy problem, abstractly). Then we can give another opinion to character the basic reproduction number R_0 of the infection age-structured epidemic model. Define the non-densely defined operators $\mathscr{A}$ and $\mathscr{F}$ on $\mathbb{X}_0$ as follows (the precise definition of $\mathscr{A}$ is given by its resolvent later)

$$\mathscr{A}\begin{pmatrix}\mathbf{0}\\ \boldsymbol{\varphi}\end{pmatrix}=\begin{pmatrix}-\varphi_1(0)\\ \vdots\\ -\varphi_n(0)\\ -\frac{\partial}{\partial a}\varphi_1+d_1(\cdot)\frac{\partial^2\varphi_1}{\partial x^2}-\sum\limits_{j=1}^{n}v_{1j}^0(a,\cdot)\varphi_j\\ \vdots\\ -\frac{\partial}{\partial a}\varphi_n+d_n(\cdot)\frac{\partial^2\varphi_n}{\partial x^2}-\sum\limits_{j=1}^{n}v_{nj}^0(a,\cdot)\varphi_j\end{pmatrix},$$

$$\mathscr{F}\begin{pmatrix}\mathbf{0}\\ \boldsymbol{\varphi}\end{pmatrix}=\begin{pmatrix}\sum\limits_{i=1}^{n}f_{1i}^0(\cdot)\int_0^{+\infty}\sum\limits_{j=1}^{n}\beta_{ij}(a,\cdot)\varphi_j(t,a,\cdot)da\\ \vdots\\ \sum\limits_{i=1}^{n}f_{ni}^0(\cdot)\int_0^{+\infty}\sum\limits_{j=1}^{n}\beta_{ij}(a,\cdot)\varphi_j(t,a,\cdot)da\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

Therefore, system (2.5) is equivalent to

$$\frac{du(t)}{dt}=\mathscr{A}u(t)+\mathscr{F}u(t). \tag{3.12}$$

In order to make the definition of $\mathscr{A}$ precise, we determine its resolvent

$$(\lambda-\mathscr{A})^{-1}(\boldsymbol{\phi},\boldsymbol{\varphi})=(\mathbf{0},\boldsymbol{\psi}),\quad (\boldsymbol{\phi},\boldsymbol{\varphi})\in\mathbb{X}. \tag{3.13}$$

That is to say,

$$\begin{cases}\frac{d}{da}\boldsymbol{\psi}(a)=\boldsymbol{L}\boldsymbol{\psi}(a)-\lambda\boldsymbol{\psi}(a)-\boldsymbol{V}^0(a)\boldsymbol{\psi}(a)+\boldsymbol{\varphi}(a),\\ \boldsymbol{\psi}(0)=\boldsymbol{\phi}.\end{cases}$$

By variation of constants formula, we have

$$\boldsymbol{\psi}(a)=e^{-\lambda a}W^0(a,0)\boldsymbol{\phi}+\int_0^a e^{-\lambda(a-s)}W^0(a,s)\boldsymbol{\varphi}(s)ds. \tag{3.14}$$

Define $\overline{R}_\lambda(\boldsymbol{\phi},\boldsymbol{\varphi})^T=(\mathbf{0},\boldsymbol{\psi})^T$ with $\boldsymbol{\psi}$ given by (3.14). It is easy to see that $\overline{R}_\lambda(\boldsymbol{\phi},\boldsymbol{\varphi})^T=(\mathbf{0},\boldsymbol{\psi})^T$ defines a pseudo-resolvent with zero null-space and this means that there exists an operator $\mathscr{A}$ such that $\overline{R}_\lambda=(\lambda I-\mathscr{A})^{-1}$. Note that W^0 is a positive evolutionary system on space X, we can deduce that $\mathscr{A}$ is a resolvent-positive operator.

Remark 3.13 It is worth mentioning that the method to define non-densely defined operator $\mathscr{A}$ is the same to $\mathcal{A}$ which is defined in Sect. 3.1. More precisely, $\mathscr{A}$ and $\mathcal{A}$ are both defined by their resolvents.

Notice that

$$-\mathscr{F}\mathscr{A}^{-1}(\boldsymbol{\phi},\boldsymbol{\varphi})=(\boldsymbol{F}^0\int_0^{+\infty}\boldsymbol{B}(a)W(a,0)\boldsymbol{\phi}da+Q\boldsymbol{\varphi},\mathbf{0}),\quad (\boldsymbol{\phi},\boldsymbol{\varphi})\in\mathbb{X}$$

with an appropriate operator Q. It follows from Gelfand's formula that $-\mathscr{F}\mathscr{A}^{-1}$ has the same spectral radius on $Y \times \{0\}$ and $\mathbb{X}$. Therefore, R_0 defined by (3.11) equals the spectral radius of $-\mathscr{F}\mathscr{A}^{-1}$.

Remark 3.14 From above, the basic reproduction number R_0 for the infection age-structured epidemic models defined by classical theory (i.e., defined by (3.11)), equals the spectral radius of $-\mathscr{F}\mathscr{A}^{-1}$. This means that we can directly use $r(-\mathscr{F}\mathscr{A}^{-1})$ to define the basic reproduction number. It extends the basic reproduction number for ODE and reaction-diffusion epidemic models (e.g. Theorem 2 in [56], Theorems 3.1, 3.3 and 3.4 in [58] and Corollary 2.1 in [64]).

Theorem 3.15 *Let Assumptions 2.1, 2.2, and 3.1 be satisfied. Then $s(\mathscr{F}+\mathscr{A})$ has the same sign as $R_0 - 1$, where $s(\cdot)$ represents the spectral bound.*

Proof Let $g(t) \in X, \forall t \geq 0$, define

$$[\mathbb{P}(t)g](s) = \begin{cases} W^0(s, s-t)g(s-t), & s > t, \\ 0, & s < t. \end{cases} \tag{3.15}$$

Recall that $W^0(t,s)$ is an evolution family on X. By [8, Proposition 3.11], we know that $\mathbb{P}$ is the evolution semigroup associated with evolution family W^0 on $L^1(\mathbb{R}_+, X)$. Next, we define an evolution family $\widehat{W^0}$ on $\mathbb{X}_0$ and an evolution semigroup $\widehat{\mathbb{P}}$ on $L^1(\mathbb{R}_+, \mathbb{X}_0)$ respectively by

$$\widehat{W^0}(t,s)\begin{pmatrix} 0 \\ \phi \end{pmatrix} := \begin{pmatrix} 0 \\ W^0(t,s)\phi \end{pmatrix} \text{ and } \left[\widehat{\mathbb{P}}(t)\begin{pmatrix} 0 \\ g \end{pmatrix}\right](s) := \begin{pmatrix} 0 \\ W^0(s, s-t)g(s-t) \end{pmatrix}$$

Recall that the exponential growth bound of semigroup $\mathbb{P}(t)$ is

$$\omega(\mathbb{P}) = \inf\{\hat{\omega} \in \mathbb{R}; \exists M \geq 1 : t \geq 0 : \|\mathbb{P}(t)\| \leq Me^{\hat{\omega}t}\}.$$

From (3.13) and (3.14), the restriction of $(\lambda - \mathscr{A})^{-1}$ to $\mathbb{X}_0$ is given as follows

$$(\lambda - \mathscr{A})^{-1}(\mathbf{0}, \boldsymbol{\varphi}) = (\mathbf{0}, \boldsymbol{\psi}).$$

Then we have

$$\begin{aligned} \boldsymbol{\psi}(a) &= \int_0^a e^{-\lambda(a-s)} W^0(a,s)\boldsymbol{\varphi}(s)ds \\ &= \int_0^a e^{-\lambda(a-s)}[\mathbb{P}(a-s)\boldsymbol{\varphi}](a)ds \overset{t=a-s}{=} \int_0^{+\infty} e^{-\lambda t}[\mathbb{P}(t)\boldsymbol{\varphi}](a)dt. \end{aligned}$$

From above, we can see that the restriction of $(\lambda - \mathscr{A})^{-1}$ to $\mathbb{X}_0$ is given by the Laplace transform of semigroup $\widehat{\mathbb{P}}$. It follows from the theory of semigroup that the generator of evolution semigroup $\widehat{\mathbb{P}}$ is $\mathscr{A}_0$ (the part of $\mathscr{A}$ in space $\mathbb{X}_0$). Thus, by [8, Theorem 3.22], we have $\sigma(\widehat{\mathbb{P}}(t))\backslash\{0\} = \exp(t\sigma(\mathscr{A}_0))$ for $t > 0$. Moreover, by [8, Theorem 3.23], we obtain $s(\mathscr{A}_0) = \omega(\widehat{\mathbb{P}}) = \omega(\widehat{W^0})$. Based on the definition of evolution families $W^0(t,s)$, $\widehat{W^0}$ and evolution semigroups $\mathbb{P}(t)$, $\widehat{\mathbb{P}}(t)$, we can find that $\omega(W^0) = \omega(\widehat{W^0})$ and $\omega(\mathbb{P}) = \omega(\widehat{\mathbb{P}})$. This implies that $\omega(\mathbb{P}) = \omega(\widehat{\mathbb{P}}) = \omega(\widehat{W^0}) = \omega(W^0) < 0$. Thus, we have $s(\mathscr{A}_0) < 0$. Note that $\mathscr{A}_0$ and $\mathscr{A}$ has the same resolvent set, $s(\mathscr{A}) = s(\mathscr{A}_0) < 0$. According to Theorem 3.5 in [54], $s(\mathscr{F}+\mathscr{A})$ has the same sign as $r(-\mathscr{F}\mathscr{A}^{-1}) - 1$. Therefore, $s(\mathscr{F}+\mathscr{A})$ has the same sign as $R_0 - 1$. ◻

Similar to Lemma 3.6, $\mathscr{A}_0 + \mathscr{F}_0$ generates a strongly continuous semigroup $T_{\mathscr{A}_0+\mathscr{F}_0}$ on $\mathbb{X}_0$.

Theorem 3.16 *Let Assumptions 2.1, 2.2, and 3.1 be satisfied. Let* $\boldsymbol{\psi} : \mathbb{R}_+ \to C(\mathbb{R}_+, X)$ *and* $(\mathbf{0}, \boldsymbol{\psi}(t)) = T_{\mathscr{A}_0+\mathscr{F}_0}(\mathbf{0}, \boldsymbol{\phi}_0)$ *for all* $t \geq 0$. *Then* $\boldsymbol{\phi}(t, a, x) = \boldsymbol{\psi}(t)(a, x)$ *solves* (3.8) *and* (3.9).

Proof Define $\boldsymbol{b}(t, \cdot) = \int_0^{+\infty} \boldsymbol{F}^0(\cdot)\boldsymbol{B}(a, \cdot)\boldsymbol{\psi}(t, a, \cdot)da$. It is obvious that $\boldsymbol{b}$ is a continuous function. Let $\hat{\boldsymbol{b}}(\lambda) = \int_0^{+\infty} e^{-\lambda t}\boldsymbol{b}(t)dt$ be the Laplace transform of $\boldsymbol{b}(t)$. Then we have

$$(\hat{\boldsymbol{b}}(\lambda), \mathbf{0})^T = \mathscr{F}(\mathbf{0}, \hat{\boldsymbol{\psi}}(\lambda)),$$

where $\hat{\boldsymbol{\psi}}(\lambda)$ is the Laplace transform of $\boldsymbol{\psi}(t)$. By Theorem 3.12 in [54], we obtain $(\mathbf{0}, \hat{\boldsymbol{\psi}}(\lambda))^T = (\lambda - \mathscr{A} - \mathscr{F})^{-1}(\mathbf{0}, \boldsymbol{\phi}_0)^T$. Thus, we have

$$(\mathbf{0}, \boldsymbol{\phi}_0)^T = (\lambda - \mathscr{A} - \mathscr{F})(\mathbf{0}, \hat{\boldsymbol{\psi}}(\lambda))^T = (\lambda - \mathscr{A})(\mathbf{0}, \hat{\boldsymbol{\psi}}(\lambda))^T - (\hat{\boldsymbol{b}}(\lambda), \mathbf{0})^T.$$

Then we have

$$(\lambda - \mathscr{A})(\mathbf{0}, \hat{\boldsymbol{\psi}}(\lambda))^T = (\hat{\boldsymbol{b}}(\lambda), \boldsymbol{\phi}_0)^T \text{ and } (\mathbf{0}, \hat{\boldsymbol{\psi}}(\lambda))^T = (\lambda - \mathscr{A})^{-1}(\hat{\boldsymbol{b}}(\lambda), \boldsymbol{\phi}_0)^T.$$

Define $\boldsymbol{\phi} = \boldsymbol{I}(\cdot, \cdot, \cdot)$ by (3.8) and $\boldsymbol{\varphi}(t) = \boldsymbol{\phi}(t, \cdot)$. It is clear that $\boldsymbol{\varphi}$ is continuous. From (3.13) and (3.14), we have

$$(\mathbf{0}, \widehat{\boldsymbol{\varphi}}(\lambda))^T = (\lambda - \mathscr{A})^{-1}(\hat{\boldsymbol{b}}(\lambda), \boldsymbol{\phi}_0)^T = (\mathbf{0}, \widehat{\boldsymbol{\psi}}(\lambda))^T.$$

According to Theorem 1.7.3 of [2], $\boldsymbol{\varphi} = \boldsymbol{\psi}$ a.e.. Then $\boldsymbol{\phi}(t, a, x) = \boldsymbol{\psi}(t)(a, x)$ solves (3.8) and (3.9). □

Remark 3.17 From Theorem 3.16, we can find that C_0-semigroup $T_{\mathscr{A}_0+\mathscr{F}_0}(t)$ can be seen as the solution map for the infection age-structured epidemic model (2.5). Furthermore, we deduce that $s(\mathscr{A} + \mathscr{F})$ has the same sign as $R_0 - 1$. This means the following relationship is true.

(i) $R_0 < 1$ if and only if $s(\mathscr{A} + \mathscr{F}) < 1$.
(ii) $R_0 = 1$ if and only if $s(\mathscr{A} + \mathscr{F}) = 1$.
(iii) $R_0 > 1$ if and only if $s(\mathscr{A} + \mathscr{F}) > 1$.

If we define $Y := L^1(\Omega, \mathbb{R}^n)$ and $X := L^1((0, +\infty), Y)$(or $Y := L^2(\Omega, \mathbb{R}^n), X := L^2((0, +\infty), Y)$), it follows from Theorem 3.14 of [54] (spectral mapping theorem) that $s(\mathscr{A}_0 + \mathscr{F}_0) = \omega(T_{\mathscr{A}_0+\mathscr{F}_0})$. Then $R_0 - 1$ has the same sign as $\omega(T_{\mathscr{A}_0+\mathscr{F}_0})$ in some suitable spaces. These results extend the basic reproduction numbers for many kinds of ODE epidemic models and reaction–diffusion epidemic models (Theorem 3.8 in [44], Theorem 3.1 in [58] and Theorem 2.1 in [64]). Moreover, if the model (2.1) is under the Dirichlet or Robin boundary conditions, the results in Sect. 3 are still valid.

3.3 Comparison to Reaction–Diffusion Epidemic Models

In this subsection, we compare our results on the basic reproduction number for the infection age-structured epidemic model with Wang and Zhao's work on reaction–diffusion epidemic models. In [58], Wang and Zhao studied the following reaction–diffusion epidemic model

$$\begin{aligned} &\frac{\partial u_i}{\partial t} = \nabla \cdot (d_i(x)\nabla u_i) + f_i(x, u), \quad 1 \leq i \leq n+m, \quad t > 0, x \in \Omega, \\ &\frac{\partial u_i}{\partial \nu} = 0 \quad \forall 1 \leq i \leq n+m \text{ with } d_i > 0, \quad t > 0, x \in \partial\Omega, \end{aligned} \tag{3.16}$$

and the linearization equations of the infected compartments at the disease-free steady state u^0

$$\begin{aligned}&\frac{\partial u_I}{\partial t}=\nabla\cdot(d_I(x)\nabla u_I)+F(x)u_I-V(x)u_I,\quad t>0, x\in\Omega,\\&\frac{\partial u_i}{\partial \nu}=0\quad \forall m+1\leq i\leq m+n \text{ with } d_i>0, t>0, x\in\partial\Omega,\end{aligned}\tag{3.17}$$

where $u_I := (u_{m+1}, ..., u_{m+n})^T$.

The next generation operator is defined by

$$L(\phi)(x):=\int_0^{+\infty}F(x)P(t)\phi dt=F(x)\int_0^{+\infty}P(t)\phi dt,\quad \phi\in C(\overline{\Omega}),\tag{3.18}$$

where $P(t)$ is the solution semigroup of the following reaction–diffusion equations

$$\begin{aligned}&\frac{\partial u_I}{\partial t}=\nabla\cdot(d_I(x)\nabla u_I)-V(x)u_I,\quad t>0,\quad x\in\Omega\\&\frac{\partial u_i}{\partial \nu}=0\quad \forall m+1\leq i\leq m+n \text{ with } d_i>0,\quad t>0\quad x\in\partial\Omega\end{aligned}\tag{3.19}$$

The basic reproduction number R_0 for system (3.16) is defined by $R_0 = r(L)$. Furthermore, Wang and Zhao obtain the following theorem.

Theorem 3.18 *[58] If $-V(x)$ is cooperative $\forall x \in \overline{\Omega}$ and $s(\nabla \cdot (d_I(x)) - V(x)) < 0$, then $L = -FB^{-1}$ and $R_0 - 1$ has the same sign as $s(B + F)$, where B is the generator of semigroup P.*

It is easy to find that Theorem 3.18 is consistent with Theorems 3.15, 3.16, and Remarks 3.14, 3.17. Note that the condition $s(\nabla \cdot (d_I(x)) - V(x)) < 0$ in Theorem 3.18 means $\omega(T) < 0$. It is consistent with Assumption 2.2 (i) in our paper and this assumption is indispensable in almost epidemic models.

Actually, the definition of R_0 of the reaction–diffusion epidemic model relies on the generator B of the operator semigroup $P(t), t \geq 0$. From equation (3.19), we know that operator B is densely defined. However, in the infection age-structured epidemic models, we can not define these densely defined generators, due to the effects by age structure. To make up for the absence of densely defined generators, following the ideas of non-densely defined operators, we overcome this problem. This means that the densely defined generator is not necessary for the definition of the basic reproduction number and we can use non-densely operators to replace it.

4 Infection Age-Structured Epidemic Models with Non-degenerate Diffusion and Spatial Heterogeneity

In this section, based on the approach developed in [11, 39, 45], we give a general method for the following class of infection age-structured epidemic model with non-degenerate diffusion and spatial heterogeneity. For $t > 0$, $a > 0$ and $x \in \Omega$,

$$\begin{cases}\frac{\partial}{\partial t}\boldsymbol{S}(t,x)=\boldsymbol{A}(x)\boldsymbol{S}(t,x)+\boldsymbol{M}(x,\boldsymbol{S}(t,x),\int_0^{+\infty}\boldsymbol{B}(a,x)\boldsymbol{I}(t,a,x)da),\\ \left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\boldsymbol{I}(t,a,x)=\boldsymbol{L}(x)\boldsymbol{I}(t,a,x)-\boldsymbol{V}(a,x)\boldsymbol{I}(t,a,x),\\ \boldsymbol{I}(t,0,x)=\boldsymbol{F}(x,\boldsymbol{S}(t,x))\int_0^{+\infty}\boldsymbol{B}(a,x)\boldsymbol{I}(t,a,x)da,\end{cases}\tag{4.1}$$

under the Neumann boundary condition

$$\frac{\partial S_j}{\partial \nu} = 0, \frac{\partial I_i}{\partial \nu} = 0, \quad x \in \partial \Omega,$$

with initial value condition

$$S_j(0, \cdot) = S_{j0}(\cdot) \in C(\overline{\Omega}), I_i(0, \cdot, \cdot) = I_{i0}(\cdot, \cdot) \in L^1(\mathbb{R}_+, C(\overline{\Omega})).$$

The operators $\boldsymbol{L}(\cdot) - \boldsymbol{V}(a, \cdot)$ are associated with an evolutionary system $\mathcal{W} := \{W(t, s); 0 \leq s \leq t \leq +\infty\}$ of positive operators on $C(\Omega, \mathbb{R}^n)$,

$$\boldsymbol{L}(\cdot) - \boldsymbol{V}(a, \cdot) = \lim_{h \to 0+} \frac{1}{h}(W(a+h, a)\phi - \phi), \phi \in D(\boldsymbol{L}(\cdot) - \boldsymbol{V}(a, \cdot)).$$

Assumption 4.1 For system (4.1), assume that

(i) $-\boldsymbol{V}(a, x) := \{-v_{ij}(a, x)\}$ is a bounded, cooperative, continuous and irreducible matrix function for all $a \in (0, +\infty)$ and $x \in \Omega$.

(ii) The following reaction–diffusion equations under the Neumann boundary condition

$$\frac{d\boldsymbol{S}(t, x)}{dt} = \boldsymbol{\Lambda}(x)\boldsymbol{S}(t, x) + \boldsymbol{M}(x, \boldsymbol{S}(t, x), \boldsymbol{0}), \quad x \in \Omega,$$

admits a globally attractive unique positive steady state $\boldsymbol{S}^0(x)$. In addition, $\boldsymbol{M}(x, \boldsymbol{S}, \int_0^{+\infty} \boldsymbol{B}(a, x)\boldsymbol{I}(a, x)da)$ is monotonically increasing with respect to $\boldsymbol{S}$ and monotonically decreasing with respect to $\boldsymbol{I}$.

(iii) $\boldsymbol{F}(x, \boldsymbol{S}(t, x))$ is a non-negative and continuous function, monotonically increasing with respect to $\boldsymbol{S}$. In addition, $\boldsymbol{F}^0(x)$ is bounded for $x \in \Omega$ and defined by

$$\boldsymbol{F}^0(x) := \boldsymbol{F}(x, \boldsymbol{S}^0(x)).$$

(iv) $\omega(W) < 0$, where $\omega(\cdot)$ represents the exponential growth bound.

(v) For each i and j, $\beta_{ij}(\cdot, \cdot) \in L_+^\infty(\mathbb{R}_+, C(\overline{\Omega})) \cap L_+^1(\mathbb{R}_+, C(\overline{\Omega}))$ and there exists a maximum age of infection denoted by a_+ such that if $a > a_+$ and $x \in \Omega$, $\beta_{ij}(a, x) = 0$. Moreover, there exists at least one interval (a_*, a^*) such that $\boldsymbol{B}(a, x)$ is an irreducible matrix function for $a \in (a_*, a^*)$.

By using the method of characteristic lines stated in Sect. 2, we obtain the following expression of I-equations of the system (4.1),

$$\boldsymbol{I}(t, a, x) = \begin{cases} W(a, 0)\boldsymbol{I}(t-a, 0, x), & t-a > 0, \\ W(a, a-t)\boldsymbol{I}_0(a-t, x), & t-a \leq 0. \end{cases}$$

Therefore, we have, for $t \geq 0, x \in \Omega$,

$$\begin{aligned} z(t, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, x) : &= \boldsymbol{I}(t, 0, x) = \boldsymbol{F}(x, \boldsymbol{S}(t, x)) \int_0^{+\infty} \boldsymbol{B}(a, x)\boldsymbol{I}(t, a, x)da \\ &= \boldsymbol{F}(x, \boldsymbol{S}(t, x)) \int_0^t \Phi(a, x) z(t-a, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, x)da \\ &\quad + \boldsymbol{F}(x, \boldsymbol{S}(t, x))\boldsymbol{H}(t, 0; \boldsymbol{I}_0, x), \end{aligned}$$

where

$$\Phi(a, x) = \boldsymbol{B}(a, x)W(a, 0) \text{ and } \boldsymbol{H}(t, 0; \boldsymbol{I}_0, x) = \int_t^{+\infty} \boldsymbol{B}(a, x)W(a, a-t)\boldsymbol{I}_0(a-t, x)da.$$

4.1 The Perturbed System

In this subsection, we study a perturbed system. Let

$$\widehat{\boldsymbol{F}}^0(x) := \boldsymbol{F}^0(x) + \varepsilon\mathbf{1}, \quad x \in \Omega, \varepsilon \in \mathbb{R}.$$

We consider a perturbed system of (4.1) as follows, for $t > 0, a > 0, x \in \Omega$,

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \boldsymbol{I}_\varepsilon(t,a,x) = \boldsymbol{L}(x)\boldsymbol{I}_\varepsilon(t,a,x) - \boldsymbol{V}(a,x)\boldsymbol{I}_\varepsilon(t,a,x), \\ \boldsymbol{I}_\varepsilon(t,0,x) = (\boldsymbol{F}^0(x) + \varepsilon\mathbf{1}) \int_0^{+\infty} \boldsymbol{B}(a,x)\boldsymbol{I}_\varepsilon(t,a,x)da, \\ \boldsymbol{I}_\varepsilon(0,\cdot,\cdot) = \boldsymbol{I}_0(\cdot,\cdot) \in L^1(\mathbb{R}_+, C(\overline{\Omega}, \mathbb{R}^n)). \end{cases} \tag{4.2}$$

Remark 4.2 If we set $\varepsilon = 0$, system (4.2) coincides with the I-equations of (4.1) around the disease-free steady state.

By using the same method as before, we can obtain the following expression of $\boldsymbol{I}_\varepsilon$ of the system (4.2),

$$\boldsymbol{I}_\varepsilon(t,a,x) = \begin{cases} W(a,0)\boldsymbol{I}_\varepsilon(t-a,0,x), & t-a > 0, \\ W(a,a-t)\boldsymbol{I}_0(a-t,x), & t-a \le 0. \end{cases} \tag{4.3}$$

Thus, we have, for $t \ge 0, x \in \Omega$,

$$\begin{aligned} z_\varepsilon(t,0;\boldsymbol{I}_0,x) :&= \boldsymbol{I}_\varepsilon(t,0,x) = (\boldsymbol{F}^0(x) + \varepsilon\mathbf{1}) \int_0^{+\infty} \boldsymbol{B}(a,x)\boldsymbol{I}_\varepsilon(t,a,x)da \\ &= (\boldsymbol{F}^0(x) + \varepsilon\mathbf{1}) \int_0^t \Phi(a,x) z_\varepsilon(t-a,0;\boldsymbol{I}_0,x)da + (\boldsymbol{F}^0(x) \\ &\quad + \varepsilon\mathbf{1})\boldsymbol{H}(t,0;\boldsymbol{I}_0,x), \end{aligned} \tag{4.4}$$

where

$$\Phi(a,x) = \boldsymbol{B}(a,x)W(a,0),\ \boldsymbol{H}(t,0;\boldsymbol{I}_0,x) = \int_t^{+\infty} \boldsymbol{B}(a,x)W(a,a-t)\boldsymbol{I}_0(a-t,x)da.$$

For $\lambda \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$, we define a linear operator on space Y by

$$\begin{aligned} \Psi_\lambda^\varepsilon(\boldsymbol{\varphi})(x) :&= (\boldsymbol{F}^0(x) + \varepsilon\mathbf{1}) \int_0^{+\infty} e^{-\lambda a}\Phi(a,x)\boldsymbol{\varphi}(x)da \\ &= (\boldsymbol{F}^0(x) + \varepsilon\mathbf{1}) \int_0^{+\infty} e^{-\lambda a}\boldsymbol{B}(a,x)W(a,0)\boldsymbol{\varphi}(x)da. \end{aligned} \tag{4.5}$$

Remark 4.3 If we set $\varepsilon = 0$ and $\lambda = 0$, $\Psi_0^0 = \Psi$. It is easy to see that Ψ is the next generation operator of system (4.1). Following the ideas in Sect. 3, the basic reproduction number is defined by $R_0 = r(\Psi)$.

Similar to Lemma 3.12, we can deduce that operator Ψ_λ^ε is also compact. Let $\Phi_\varepsilon(a,x) := (\boldsymbol{F}^0(x) + \varepsilon\mathbf{1})\Phi(a,x), \forall a \ge 0, x \in \Omega$. From Assumption 4.1 (v), we have $\Phi_\varepsilon(a,x) = 0, \forall a > a_+$. Next, we consider the following Volterra integral equations

$$\boldsymbol{u}(t) = \int_0^t \Phi_\varepsilon(s)\boldsymbol{u}(t-s)ds + \overline{\boldsymbol{u}}(t), \quad t \ge 0, \tag{4.6}$$

where continuous functions $\boldsymbol{u}, \overline{\boldsymbol{u}}$ map $[0,+\infty)$ to Y. Here we hide the spatial variable x by $\Phi_\varepsilon(a,x) = \Phi_\varepsilon(a)(x)$. A family $\Phi_\varepsilon = \{\Phi_\varepsilon(s); s \ge 0\}$ of positive continuous linear operators $\Phi_\varepsilon(s)$ is an operator-value integral kernel on Y. The convolution of a kernel Φ_ε and a function $\boldsymbol{u} \in C([0,+\infty), Y)$ is defined by

$$\Phi_\varepsilon * \boldsymbol{u}(t) = \int_0^t \Phi_\varepsilon(s)\boldsymbol{u}(t-s)ds, \quad t \ge 0. \tag{4.7}$$

Lemma 4.4 *Let Assumptions 3.1 and 4.1 be satisfied. Let* $\boldsymbol{w} \in Y_+ \setminus \{0\}$*. If* $\boldsymbol{u}, \overline{\boldsymbol{u}} \in C([0,+\infty), Y_+)$*,* $\overline{\boldsymbol{u}}(t) \neq 0$ *for some* $t \in [0,\delta]$*, and* $\boldsymbol{u} = \Phi_\varepsilon * \boldsymbol{u} + \overline{\boldsymbol{u}}$*, then there exists* t_0 *such that*

$$\boldsymbol{u}(t) \geq \zeta \boldsymbol{w}, \quad \forall t \in [t_0, t_0+\delta] \tag{4.8}$$

with some $\zeta > 0$ *depending on* $\overline{\boldsymbol{u}}$ *and* $\boldsymbol{w}$*.*

Proof According to the definition of Φ_ε, we have

$$\int_0^t \Phi_\varepsilon(s,x)\boldsymbol{u}(t-s,x)ds = (\boldsymbol{F}^0(x)+\varepsilon\mathbf{1})\int_0^t \boldsymbol{B}(a,x)W(a,0)\boldsymbol{u}(t-a,x)da, \quad \forall t \geq 0, x \in \Omega.$$

By using Assumption 4.1 (v), we know that if $a \in (a_*, a^*)$, then $\boldsymbol{B}(a,x)$ is positive and irreducible for all $x \in \Omega$. Without loss of generality, we assume that there exists $t_1 > 0$ and $i \in 1,2,\cdots,n$ such that $\overline{u}_i(t_1) > 0$. According to cooperation property of $\boldsymbol{B}(a,x)$, we can deduce that $\boldsymbol{u}(t) > \mathbf{0}$, $\forall t \in [t_1 + a_*, +\infty)$. Inequality (4.8) is only considered in a finite time interval. This is clearly true. □

Remark 4.5 If $\boldsymbol{u}$ and $\boldsymbol{w}$ satisfy (4.8), kernel Φ_ε with (4.6) is called a $\boldsymbol{w}$-positive kernel.

Lemma 4.6 *Let Assumptions 3.1 and 4.1 be satisfied. Let* $\boldsymbol{w} \in int(Y_+) \setminus \{0\}$*. Then there exists a constant* c_0 *such that for all* $t \geq 0$*,* $\boldsymbol{v} \in Y$*,*

$$\|\Phi_\varepsilon(t)\boldsymbol{v}\|_{\boldsymbol{w}} \leq c_0\|\boldsymbol{v}\|,$$

where $\|\boldsymbol{v}\|_{\boldsymbol{w}} := \inf\{\|c\| : c \in \mathbb{R}, -c\boldsymbol{w} \leq \boldsymbol{v} \leq c\boldsymbol{w}\}$.

Proof Let

$$\xi_1 := \sup_{x\in\overline{\Omega}} \overline{\beta}(x) \sum_{i=1}^n \sum_{j=1}^n (f_{ij}^0(x)+\varepsilon),$$

where $\overline{\beta}(x) := \max_{i,j=1,2\ldots,n} \sup_{a\in(0,+\infty)} \beta_{ij}(a,x)$. Let $\boldsymbol{v} \neq \mathbf{0}$ is given, we have the following inequality

$$\frac{1}{c_0}\left\|\Phi_\varepsilon(s)\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\right\| \leq \frac{1}{c_0}\left\|(\boldsymbol{F}^0+\varepsilon\mathbf{1})\overline{\beta}W(s,0)\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\right\|, \quad \forall s \geq 0.$$

By Assumption 4.1, we know that $\omega(W) < 0$ and $\|W(t,s)\| \leq Me^{\omega(W)(t-s)}$, $t \geq s$. Therefore, we have the following inequality

$$\frac{1}{c_0}\left\|\Phi_\varepsilon(s)\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\right\| \leq \frac{1}{c_0}\left\|(\boldsymbol{F}^0+\varepsilon\mathbf{1})\overline{\beta}M\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\right\| \leq \frac{M\xi_1}{c_0}, \quad \forall s \geq 0.$$

Let $c_0 > 0$ be large enough such that

$$0 < \frac{M\xi_1}{c_0} < \min_{i=1,2,\cdots,n} w_i(x), \quad \forall x \in \overline{\Omega},$$

where $\boldsymbol{w} = (w_1,\cdots,w_n)^T$. Therefore, we have

$$-\boldsymbol{w} \leq -\frac{M\xi_1}{c_0}\mathbf{1} \leq \frac{1}{c_0\|\boldsymbol{v}\|}\Phi_\varepsilon(s)\boldsymbol{v} \leq \frac{M\xi_1}{c_0}\mathbf{1} \leq \boldsymbol{w}, \quad \forall s \geq 0.$$

It means that

$$-c_0\boldsymbol{w} \leq \frac{1}{\|\boldsymbol{v}\|}\Phi_\varepsilon(s)\boldsymbol{v} \leq c_0\boldsymbol{w}, \quad \forall s \geq 0.$$

Therefore, we have

$$c_0 \geq \left\| \frac{1}{\|\boldsymbol{v}\|} \Phi_\varepsilon(s)\boldsymbol{v} \right\|_{\boldsymbol{w}}, \quad \forall s \in \mathbb{R}_+.$$

□

As a consequence of Lemmas 4.4, 4.6 and Theorems 5.1, 5.2 of [51] or [50], we have the following theorem.

Theorem 4.7 *Let Assumptions 3.1 and 4.1 be satisfied. Then for each $\varepsilon \in \mathbb{R}$, there exists a unique pair $\lambda_\varepsilon \in \mathbb{R}$ and $\widehat{\boldsymbol{\varphi}}_\varepsilon \in C(\overline{\Omega}, \mathbb{R}^n)$ such that the following statements hold*

(i) $\|\widehat{\boldsymbol{\varphi}}_\varepsilon\| = 1$,
(ii) $\Psi^\varepsilon_{\lambda_\varepsilon}(\widehat{\boldsymbol{\varphi}}_\varepsilon) = \widehat{\boldsymbol{\varphi}}_\varepsilon$ *and* $r(\Psi^\varepsilon_{\lambda_\varepsilon}) = 1$,
(iii) *if* $r(\Psi^\varepsilon_{\lambda_1}) < r(\Psi^\varepsilon_{\lambda_\varepsilon}) = 1 < r(\Psi^\varepsilon_{\lambda_2})$*, then* $\lambda_1 < \lambda_\varepsilon < \lambda_2$,
(iv) $\lim_{\varepsilon\to 0} \lambda_\varepsilon = \lambda_0$ *and* $\lim_{\varepsilon\to 0} \widehat{\boldsymbol{\varphi}}_\varepsilon = \widehat{\boldsymbol{\varphi}}_0$,
(v) *if* $\overline{\boldsymbol{u}}_\varepsilon \in C(\mathbb{R}_+, C(\overline{\Omega}, \mathbb{R}^n))$ *with* $\overline{\boldsymbol{u}}_\varepsilon(t) = \boldsymbol{0}$ *for all* $t \geq a_+$ *and* $\boldsymbol{u}_\varepsilon \in C(\mathbb{R}_+, C(\overline{\Omega}, \mathbb{R}^n))$ *satisfies*

$$\boldsymbol{u}_\varepsilon(t) = \int_0^t \Phi_\varepsilon(s)\boldsymbol{u}_\varepsilon(t-s)ds + \overline{\boldsymbol{u}}_\varepsilon(t), \quad \forall t \geq 0,$$

then there exists $\alpha_\varepsilon \geq 0$ *such that*

$$e^{\lambda_\varepsilon t}\boldsymbol{u}_\varepsilon(t) \to \alpha_\varepsilon \widehat{\boldsymbol{\varphi}}_\varepsilon, \quad t \to +\infty.$$

Moreover, if $\overline{\boldsymbol{u}}_\varepsilon \neq \boldsymbol{0}$*, then* $\alpha_\varepsilon > 0$*. If* $\alpha_\varepsilon > 0$*, then*

$$\lim_{t\to+\infty} d_0(e^{\lambda_\varepsilon t}\boldsymbol{u}_\varepsilon(t), \alpha_\varepsilon \widehat{\boldsymbol{\varphi}}_\varepsilon) = 0,$$

where metric d_0 *is defined as* $d_0(\boldsymbol{u}, \boldsymbol{v}) := \inf\{|c| : c \in \mathbb{R}, e^{-c}\boldsymbol{u} \leq \boldsymbol{v} \leq e^c\boldsymbol{u}\}$.

Remark 4.8 Theorem 4.7 (i) and (v) imply that there exists a constant $C := C(\alpha_\varepsilon, \lambda_\varepsilon, \overline{\boldsymbol{u}}_\varepsilon, \widehat{\boldsymbol{\varphi}}_\varepsilon)$ such that $\|\boldsymbol{u}_\varepsilon(t)\| \leq Ce^{-\lambda_\varepsilon t}$.

Remark 4.9 Theorems 5.1 and 5.2 of [51] are called the renewal theorems for Volterra integral equations.

4.2 Extinction and Uniform Persistence of the Disease

In this subsection, we prove that the basic reproduction number R_0 plays a role of the threshold for the extinction and uniform persistence of the disease for the system (4.1). Based on the existence of the solution of system (4.1), we define the solution semiflow $\mathcal{U}(t), t \geq 0$ of the system (4.1) by

$$\mathcal{U}(t)(\boldsymbol{S}_0(\cdot), \boldsymbol{I}_0(\cdot,\cdot)) = (\boldsymbol{S}(t,\cdot), \boldsymbol{I}(t,\cdot,\cdot)), \quad \forall t \geq 0. \tag{4.9}$$

Let sets M, $\overline{M}_0$, M_0, $\partial\overline{M}_0$ and ∂M_0 be defined as follows

$M := C_+(\overline{\Omega}, \mathbb{R}^m) \times X_+$,
$\overline{M}_0 := \{\boldsymbol{\varphi} \in X_+ : \text{there exists } i \in \{1, 2, \cdots, n\} \text{ such that } \int_0^{a_+} \|\varphi_i\| da > 0\}$,
$M_0 := C_+(\overline{\Omega}, \mathbb{R}^m) \times \overline{M}_0$, $\partial\overline{M}_0 := X_+ \setminus \overline{M}_0$, $\partial M_0 := M \setminus M_0 = C_+(\overline{\Omega}, \mathbb{R}^m) \times \partial\overline{M}_0$.

Define a function $\rho : X \to \mathbb{R}_+$ as follows

$$\rho(\boldsymbol{I}_0(a, x)) := \sum_{i=1,2,\cdots,n} \int_0^{a_+} \|I_{0i}(a, \cdot)\| da, \quad \forall \boldsymbol{I}_0(a, x) \in X. \tag{4.10}$$

Combining with the definition of M_0 and ∂M_0, we have

$$\begin{aligned} M_0 &:= \{(\boldsymbol{S}_0(x), \boldsymbol{I}_0(a, x)) \in C_+(\overline{\Omega}, \mathbb{R}^m) \times X_+ : \rho(\boldsymbol{I}_0(a, x)) > 0\}, \\ \partial M_0 &:= \{(\boldsymbol{S}_0(x), \boldsymbol{I}_0(a, x)) \in C_+(\overline{\Omega}, \mathbb{R}^m) \times X_+ : \rho(\boldsymbol{I}_0(a, x)) = 0\}. \end{aligned}$$

Remark 4.10 The set M can be seen as the state space of the susceptible and infectious compartments. $\overline{M}_0$ is the state space of the infectious compartments and the disease exists in the system. M_0 is the state space of the susceptible and infectious compartments with disease exists. $\partial \overline{M}_0$ is the state space of the infectious compartments with no disease. ∂M_0 is the state space of the susceptible and infectious compartments with no disease in the system.

From Assumption 4.1 and Remark 4.10, we directly have the following lemma.

Lemma 4.11 *Let Assumptions 3.1 and 4.1 be satisfied. If* $(\boldsymbol{S}_0, \boldsymbol{I}_0) \in \partial M_0$, *then*

$$(\boldsymbol{S}(t, \cdot), \boldsymbol{I}(t, \cdot, \cdot)) \in \partial M_0, \forall t \geq 0.$$

Lemma 4.12 *Let Assumptions 3.1 and 4.1 be satisfied. Let initial value* $(\boldsymbol{S}_0, \boldsymbol{I}_0) \in C_+(\overline{\Omega}, \mathbb{R}^m) \times X_+$ *be given and* $(\boldsymbol{S}(t, \cdot), \boldsymbol{I}(t, \cdot, \cdot))$ *be the solution of system (4.1) with initial value* $(\boldsymbol{S}_0, \boldsymbol{I}_0)$. *Then for any* $\varepsilon > 0$, *there exists a time* $T_1 \geq 0$ *such that*

$$\boldsymbol{S}(t, x) \leq \boldsymbol{S}^0(x) + \varepsilon \mathbf{1}, \quad \forall t \geq T_1, x \in \Omega, \tag{4.11}$$

where $\mathbf{1} := \underbrace{(1, 1, ..., 1)}_{n}{}^T$.

Proof By the equations of $\boldsymbol{S}$ in system (4.1), we have

$$\frac{d\boldsymbol{S}(t, x)}{dt} \leq \boldsymbol{\Lambda}(x)\boldsymbol{S}(t, x) + \boldsymbol{M}(x, \boldsymbol{S}(t, x), \mathbf{0}), \quad x \in \Omega.$$

From Assumption 4.1, we know that $\boldsymbol{S}^0(t, x) = \boldsymbol{S}^0(x)$ is the global attractive unique steady state of the following equation

$$\frac{d\boldsymbol{S}^0(t, x)}{dt} = \boldsymbol{\Lambda}(x)\boldsymbol{S}^0(t, x) + \boldsymbol{M}(x, \boldsymbol{S}^0(t, x), \mathbf{0}), \quad x \in \Omega.$$

By using the comparison principle for reaction–diffusion equations, we have $\limsup_{t \to +\infty} \boldsymbol{S}(t, x) \leq \boldsymbol{S}^0(x)$. □

In the rest of this subsection, we show that R_0 plays a role in the threshold for extinction or persistence of the disease, even the threshold for global attractiveness of the disease-free steady state or the existence of the endemic steady state. In order to use the comparison principle for age-structured epidemic models, chain transitivity, and the theory of compact attractors, we make the following assumption.

Assumption 4.13 For system (4.1), assume that

(i) for any $\boldsymbol{S}_0 \in C_+(\overline{\Omega}, \mathbb{R}^m)$ and $\boldsymbol{I}_0 \in X_+$, there exists $T_1 > 0$ and a constant $\zeta > 0$ (ζ is independent of initial value) such that

$$\|\mathcal{U}(t)(\boldsymbol{S}_0(\cdot), \boldsymbol{I}_0(\cdot, \cdot))\| \leq \zeta, \quad \forall t \geq T_1.$$

(ii) For any positive element $\boldsymbol{I}_* \in X_+$, the solution $\boldsymbol{S}(t,x)$ of the following system

$$\frac{\partial}{\partial t}\boldsymbol{S}(t,x) = \boldsymbol{\Lambda}(x)\boldsymbol{S}(t,x) + \boldsymbol{M}(x, \boldsymbol{S}(t,x), \int_0^{+\infty} \boldsymbol{B}(a,x)\boldsymbol{I}_*(a,x)da), \quad x \in \Omega,$$

satisfies that there exists a constant $\varepsilon > 0$ such that $\lim_{t\to+\infty} \boldsymbol{S}(t,x) \geq \boldsymbol{S}^0(x) - \varepsilon\mathbf{1} \gg \mathbf{0}, \forall x \in \Omega$.

Theorem 4.14 *Let Assumptions 3.1, 4.1 and 4.13 be satisfied. Then semiflow $\mathcal{U}(t)$ admits a global attractor $A_0 \subset Y_+ \times X_+$.*

Proof According to Assumption 4.13, we obtain that semiflow $\mathcal{U}(t)$ is point dissipative and eventually bounded. Due to the compactness of evolution family $W(t,s)$, $\mathcal{U}(t)$ is also compact. It is well known that compact operators are $\kappa-$condensing operators. By Lemma 2.3.5 of [18], it is asymptotically smooth. According to theorem 2.4.6 of [18] or Theorems 3.1, 3.4 of [19], $\mathcal{U}(t)$ admits a compact attractor A_0 of bounded sets. □

Theorem 4.15 *Let Assumptions 3.1, 4.1 and 4.13 be satisfied. If $R_0 < 1$, then the disease-free steady state $(\boldsymbol{S}^0, \mathbf{0})$ is globally attractive.*

Proof By Theorem 4.7 (ii), we obtain that if $\varepsilon > 0$ is fixed and small, then there exists λ_ε such that $r(\Psi^{\varepsilon}_{\lambda_\varepsilon}) = 1$. By Remark 4.3, we know that $R_0 = r(\Psi^0_0) < 1$. By Theorem 4.7 (ii), there exists a constant λ_0 such that $r(\Psi^0_{\lambda_0}) = 1$. Thus, $r(\Psi^0_0) < r(\Psi^0_{\lambda_0})$. According to Theorem 4.7 (iii), we deduce that $\lambda_0 > 0$. It follows from Theorem 4.7 (iv) that $\lim_{\varepsilon\to0} \lambda_\varepsilon = \lambda_0$. Therefore, if ε is small enough, we have that $\lambda_\varepsilon > 0$.

Define $\widehat{\boldsymbol{F}}^0(x) := \boldsymbol{F}^0(x) + \varepsilon\mathbf{1}$. By Assumption 4.1 (iii), we know that there exists a constant $\delta > 0$ such that

$$\widehat{\boldsymbol{F}}^0(x) \leq \boldsymbol{F}(x, \boldsymbol{S}^0(x) + \delta\mathbf{1}), \quad x \in \Omega.$$

By Lemma 4.12, we obtain that there exists $T_1 > 0$ such that $\boldsymbol{S}(t,x) \leq \boldsymbol{S}^0(x) + \delta\mathbf{1}$, $\forall t \geq T_1$. Therefore, we have the following system, for $t \geq T_1$, $a \geq 0$ and $x \in \Omega$,

$$\begin{cases} \frac{d}{dt}\boldsymbol{S}(t,x) \leq \boldsymbol{\Lambda}(x)\boldsymbol{S}(t,x) + \boldsymbol{M}(x, \boldsymbol{S}(t,x), \mathbf{0}), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)\boldsymbol{I}(t,a,x) = \boldsymbol{L}(x)\boldsymbol{I}(t,a,x) - \boldsymbol{V}(a,x)\boldsymbol{I}(t,a,x), \\ \boldsymbol{I}(t,a,x) \leq (\boldsymbol{F}^0(x) + \varepsilon\mathbf{1}) \int_0^{+\infty} \boldsymbol{B}(a,x)\boldsymbol{I}(t,a,x)da. \end{cases} \tag{4.12}$$

By using the comparison principle in [37] for (4.12), we have

$$\mathbf{0} \leq \boldsymbol{I}(t,a,x) \leq \boldsymbol{I}_\varepsilon(t,a,x), \quad \forall t > T_1, a \geq 0, x \in \overline{\Omega}, \tag{4.13}$$

where $\boldsymbol{I}_\varepsilon$ is the solution of (4.2).

Next, we show

$$\lim_{t\to+\infty} \|\boldsymbol{I}_\varepsilon(t,\cdot,\cdot)\|_X = 0. \tag{4.14}$$

From (4.4) and Remark 4.8, we deduce that there exists a constant C_* such that

$$\|z_\varepsilon(t,0;\boldsymbol{I}_0,x)\| \leq C_* e^{-\lambda_\varepsilon t}, \quad \forall t \geq T_2, x \in \Omega.$$

Note that $\omega(W) < 0$ and $\|W(t,s)\| \leq Me^{\omega(W)(t-s)}$, then we have, for $t \geq T_2$, $a \geq 0$ and $x \in \Omega$,

$$\begin{aligned} \|\boldsymbol{I}_\varepsilon(t,a,x)\|_X &\leq \int_0^t \|W(a,0)\boldsymbol{I}_\varepsilon(t-a,0,x)\|_Y da + \int_t^{+\infty} \|W(a,a-t)\boldsymbol{I}_0(a-t,x)\|_Y da \\ &\leq \int_0^t Me^{\omega(W)a} C_* e^{-\lambda_\varepsilon(t-a)} da + \int_t^{+\infty} \|\boldsymbol{I}_0(a-t,x)\| da. \end{aligned}$$

Therefore, (4.14) holds. By (4.13) and (4.14), we have

$$0 \le \lim_{t\to+\infty} \|\boldsymbol{I}(t,\cdot,\cdot)\|_X \le \lim_{t\to+\infty} \|\boldsymbol{I}_\varepsilon(t,\cdot,\cdot)\|_X = 0. \tag{4.15}$$

Finally, we show that the disease-free steady state $(\boldsymbol{S}^0, \boldsymbol{0})$ is globally attractive.

By (4.15), it remains to prove

$$\lim_{t\to+\infty} \|\boldsymbol{S}(t,x) - \boldsymbol{S}^0(x)\| = 0, \quad x \in \Omega. \tag{4.16}$$

Due to $\lim_{t\to+\infty} \boldsymbol{I}(t,a,x) = \boldsymbol{0}$ uniformly for $a \in [0,+\infty)$ and $x \in \overline{\Omega}$, the equation for $\boldsymbol{S}$ is asymptotic to the following reaction–diffusion equation with Neumann boundary condition

$$\frac{d\boldsymbol{S}(t,x)}{dt} = \boldsymbol{\Lambda}(x)\boldsymbol{S}(t,x) + \boldsymbol{M}(x,\boldsymbol{S}(t,x),\boldsymbol{0}), \quad x \in \Omega.$$

By the theory for asymptotically autonomous semiflows (Corollary 4.3 of [53]) and Assumption 4.1 (ii), we have

$$\lim_{t\to+\infty} \boldsymbol{S}(t,x) = \boldsymbol{S}^0(x)$$

uniformly for $x \in \overline{\Omega}$. □

Remark 4.16 In our proof of Theorem 4.15, we can find that the global attractiveness of the disease-free steady state needs Assumption 4.1 (ii). However, the extinction of the disease only needs (4.11), instead of Assumption 4.1 (ii). Therefore, we can weaken Assumption 4.1 (ii) to (4.11) when we prove the extinction of the disease.

Proposition 4.17 *Let Assumptions 3.1, 4.1 and 4.13 be satisfied. If $\boldsymbol{I}_0 \in \overline{M}_0$, then there exists $T_1 \ge 0$ such that*

$$z_\varepsilon(t,0;\boldsymbol{I}_0,\cdot) \ge \boldsymbol{0} \text{ and } \boldsymbol{I}_\varepsilon(t,\cdot,\cdot) \in \overline{M}_0, \quad \forall t \ge T_1. \tag{4.17}$$

Moreover, if $\lambda_\varepsilon < 0$, then

$$\lim_{t\to+\infty} \|\boldsymbol{I}_\varepsilon(t,\cdot,\cdot)\|_X = +\infty, \quad \forall \boldsymbol{I}_0 \in \overline{M}_0. \tag{4.18}$$

Proof From above, we know that

$$\boldsymbol{H}(t,0;\boldsymbol{I}_0,x) = \int_t^{+\infty} \boldsymbol{B}(a,x)W(a,a-t)\boldsymbol{I}_0(a-t,x)da, \quad \forall t \ge 0, x \in \Omega.$$

It is clear that $\boldsymbol{H}(t,0;\boldsymbol{I}_0,x) = \boldsymbol{0}$, $\forall t \ge a_+, x \in \Omega$. Because $\boldsymbol{I}_0(\cdot,\cdot) \in \overline{M}_0$ and $\boldsymbol{B}(a,x)$ is positive for $a \in (a_*, a^*)$ and $x \in \Omega$, $\boldsymbol{H}(a_*, t_0; \boldsymbol{I}_0, x) > 0$ for all $x \in \Omega$. Therefore, there exists $T_1 > 0$ such that

$$\|z_\varepsilon(t)\| > 0, \quad \forall t \ge T_1.$$

Thus, (4.17) is a direct result. Because $z_\varepsilon(t)$ is strongly positive on space Y, we can directly assume that $z_\varepsilon(t)$ is strongly positive at $t = 0$, i.e., $z_\varepsilon(0) \in int(Y_+)$. In order to show (4.18), we begin with the following claim.

Claim: For any $z_\varepsilon(0) \in int(Y_+)$, there exists two constants c and η such that

$$W(t,s)z_\varepsilon(0) \ge ce^{\eta(t-s)}z_\varepsilon(0), \quad t \ge s.$$

Next, we prove the above claim. Recall that evolution family $W(t,s)$ is the solution operator of the following reaction–diffusion equation under the Neumann boundary condition

$$\frac{du}{dt}(t,x) = \boldsymbol{L}(x)u(t,x) - \boldsymbol{V}(t,x)u(t,x), \quad x \in \Omega. \tag{4.19}$$

Let $\overline{v}_{ij}(x) = \inf_{a\in[0,+\infty)} v_{ij}(a,x)$ for $1 \le i,j \le n$, where $\boldsymbol{V}(a,x) = (v_{ij}(a,x))_{1\le i,j\le n}$. Then we consider the following equation under the Neumann boundary condition

$$\frac{dv}{dt}(t,x) = \boldsymbol{L}(x)v(t,x) - \overline{\boldsymbol{V}}(x)v(t,x), \quad x \in \Omega, \tag{4.20}$$

where $\overline{\boldsymbol{V}}(x) := (\overline{v}_{ij}(x))_{1\le i,j\le n}$. Note that (4.20) is a parabolic equation, (4.20) has a strongly positive solution semigroup $T(t)$. It is well-known that the following eigenvalue problem admits a principal eigenvalue $\eta < 0$ and its corresponding strongly positive eigenvector $v_*(x)$,

$$\boldsymbol{L}(x)v_*(x) - \overline{\boldsymbol{V}}(x)v_*(x) = \eta v_*(x).$$

Moreover, $T(t)v_* = e^{\eta t}v_*, \forall t \ge 0$. For any $z_\varepsilon(0) \in int(Y_+)$, there exists two positive numbers k_1, k_2 such that $k_1 v_*(x) \le z_\varepsilon(0,x) \le k_2 v_*(x)$. Then we have

$$T(t)z_\varepsilon(0) \ge k_1 T(t)v_* = k_1 e^{\eta t}v_* \ge \frac{k_1}{k_2}e^{\eta t}z_\varepsilon(0), \quad \forall t \ge 0.$$

According to the comparison principle for systems (4.19) and (4.20), we have

$$W(a,0)z_\varepsilon(0) \ge T(a)z_\varepsilon(0) \ge \frac{k_1}{k_2}e^{\eta a}z_\varepsilon(0), \quad \forall a \ge 0.$$

This means that **Claim** is true.

Finally, we prove (4.18). By Theorem 4.7 (v), there exists a constant α_ε such that

$$\lim_{t\to+\infty} d_0(e^{\lambda_\varepsilon t}z_\varepsilon(t,0;\boldsymbol{I}_0,x), \alpha_\varepsilon\widehat{\boldsymbol{\varphi}}_\varepsilon) = 0, \quad \forall x \in \Omega.$$

Let δ be given. Then there exists $T_2 \ge 0$ such that

$$0 \le d_0(e^{\lambda_\varepsilon t}z_\varepsilon(t,0;\boldsymbol{\varphi},x), \alpha_\varepsilon\widehat{\boldsymbol{\varphi}}_\varepsilon(x)) < \delta, \quad \forall t \ge T_2, x \in \Omega.$$

This means

$$e^{-\delta}e^{\lambda_\varepsilon t}z_\varepsilon(t,0;\boldsymbol{\varphi},x) \le \alpha_\varepsilon\widehat{\boldsymbol{\varphi}}_\varepsilon(x) \le e^{\delta}e^{\lambda_\varepsilon t}z_\varepsilon(t,0;\boldsymbol{\varphi},x), \quad \forall t \ge T_2, x \in \Omega.$$

For $t \ge T_2$, $a \ge 0$ and $x \in \Omega$, we have

$$\begin{aligned}\|\boldsymbol{I}_\varepsilon(t,a,x)\| &\ge \int_0^t \|W(a,0)z_\varepsilon(t-a,0;\boldsymbol{I}_0,x)\|da \ge \int_0^{t-T_2}\|ce^{\eta a}z_\varepsilon(t-a,0;\boldsymbol{I}_0,x)\|da\\ &\ge \int_0^{t-T_2}\left\|ce^{\eta a}e^{-\delta}e^{-\lambda_\varepsilon(t-a)}\alpha_\varepsilon\left\|\widehat{\boldsymbol{\varphi}}_\varepsilon\right\|\right\|da \ge e^{-\lambda_\varepsilon t}\int_0^{t-T_2}\left\|e^{-\delta}e^{(\lambda_\varepsilon+\eta)a}\alpha_\varepsilon\right\|da.\end{aligned}$$

Due to $\lambda_\varepsilon < 0$, we can find

$$\lim_{t\to+\infty}\|\boldsymbol{I}_\varepsilon(t,\cdot,\cdot)\|_X = +\infty, \quad \forall \boldsymbol{I}_0 \in \overline{M}_0.$$

□

Lemma 4.18 *Let Assumptions 3.1, 4.1 and 4.13 be satisfied. If* $(\boldsymbol{S}_0, \boldsymbol{I}_0) \in M_0$, *then* $(\boldsymbol{S}(t,\cdot), \boldsymbol{I}(t,\cdot,\cdot)) \in M_0, \forall t \ge 0$.

Proof By using the similar method in Proposition 4.17, we deduce that $\|\boldsymbol{H}(a_*, 0; \boldsymbol{I}_0, x)\| > 0, \forall x \in \Omega$. Because $\boldsymbol{B}(a, x)$ is positive for $a \in (a_*, a^*)$ and $x \in \Omega$, $\boldsymbol{z}(t, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, x) > \boldsymbol{0}$ for $t \geq a_*$ and $x \in \Omega$. Suppose, by contradiction, that there exists $T_1 \geq 0$ such that $(\boldsymbol{S}(T_1, \cdot), \boldsymbol{I}(T_1, \cdot, \cdot)) \in \partial M_0$. By Lemma 4.11, we have $(\boldsymbol{S}(t, \cdot), \boldsymbol{I}(t, \cdot, \cdot)) \in \partial M_0, \forall t \geq T_1$. This leads to a contradiction. □

Definition 4.19 The set $\partial \overline{M}_0$ is said to be ρ-ejective for $\mathcal{U}(t)$, if there exists $\varepsilon > 0$ such that for every $x \in \overline{M}_0$ with $0 < \rho(x) < \varepsilon$, there is $T \geq 0$ such that $\rho(\mathcal{U}(T)x) \geq \varepsilon$.

Theorem 4.20 *Let Assumptions 3.1, 4.1 and 4.13 be satisfied. If $R_0 > 1$, then semiflow $\mathcal{U}(t)$ is uniformly persistent, that is, there exists $\delta > 0$ such that*

$$\lim_{t \to +\infty} \rho(\boldsymbol{I}(t, \cdot, \cdot)) \geq \delta, \quad \forall (\boldsymbol{S}_0, \boldsymbol{I}_0) \in M_0. \tag{4.21}$$

Moreover, system (4.1) has an endemic steady state.

Proof First, we prove the following two claims.

Claim 1if $\boldsymbol{0} < \int_0^{+\infty} \boldsymbol{B}(a, x)\boldsymbol{I}(t, a, x)da \leq \vartheta \boldsymbol{1}$ for all $t \geq 0$ and $x \in \Omega$, then $\lim\limits_{t \to +\infty} \|\boldsymbol{I}(t, \cdot, \cdot)\|_X = +\infty$.

According to Assumption 4.1 (iv), we have

$$\frac{\partial}{\partial t}\boldsymbol{S}(t, x) \geq \boldsymbol{\Lambda}(x)\boldsymbol{S}(t, x) + \boldsymbol{M}(x, \boldsymbol{S}(t, x), \vartheta \boldsymbol{1}), \quad t \geq 0.$$

According to Assumption 4.13 (ii), there exists a large enough $T_1 > 0$ such that for some $\hat{\delta} > 0$,

$$\boldsymbol{S}(t, x) \geq \boldsymbol{S}^0(x) - \hat{\delta}\boldsymbol{1}, \quad \forall t \geq T_1, x \in \Omega.$$

Let $\delta := \max\limits_{x \in \Omega}(\boldsymbol{F}^0(x) - \boldsymbol{F}(x, \boldsymbol{S}^0(x) - \hat{\delta}\boldsymbol{1}))$. By using the comparison principle in [37], we have

$$\boldsymbol{I}(t, \cdot, \cdot) \geq \boldsymbol{I}_{-\delta}(t, \cdot, \cdot), \quad \forall t \geq T_1. \tag{4.22}$$

According to Theorem 4.7 (iii), we have

$$r(L^0_{\lambda_0}) = 1 < R_0 = r(L^0_0).$$

This means that $\lambda_0 < 0$. By Theorem 4.7 (iv), we have $\lambda_{-\delta} < 0$ and $r(L^{-\delta}_{\lambda_{-\delta}}) = 1$. By Proposition 4.17, we have $\lim\limits_{t \to +\infty} \|\boldsymbol{I}_{-\delta}(t, \cdot, \cdot)\|_X = +\infty$. Moreover, we have

$$\lim_{t \to +\infty} \|\boldsymbol{I}(t, \cdot, \cdot)\|_X = +\infty. \tag{4.23}$$

Claim 2 $\partial \overline{M}_0$ is said to be ρ-ejective for semiflow $\mathcal{U}(t)$.

Assume, by contradiction, that $\partial \overline{M}_0$ is not ρ-ejective for semiflow $\mathcal{U}(t)$. Because semiflow $\mathcal{U}(t)$ is point dissipative, there exists $T_2 > 0$ and $\zeta > 0$ such that $\|\mathcal{U}(t)(\boldsymbol{S}_0(\cdot), \boldsymbol{I}_0(\cdot, \cdot))\| \leq \zeta, \forall t \geq T_2$. Then we have

$$\rho(\boldsymbol{I}(t, \cdot, \cdot)) \leq \zeta, \quad \forall t \geq T_2. \tag{4.24}$$

By Assumption 4.13 (ii) and Lemma 4.12, we deduce that there exist $T_3 > T_2$ and a constant $C_2 > 0$ such that

$$\|\boldsymbol{F}(x, \boldsymbol{S}(t, x))\| \leq C_2, \quad \forall t \geq T_3, x \in \Omega.$$

From previous contents, we have, for $t \geq T_3$,

$$\|\boldsymbol{z}(t, T_3; \boldsymbol{S}_{T_3}, \boldsymbol{I}_{T_3}, \cdot)\| = \|\boldsymbol{F}(\cdot, \boldsymbol{S}(t, \cdot)) \int_0^{+\infty} \boldsymbol{B}(a, \cdot)\boldsymbol{I}(t, a, \cdot)da\| \leq a_+ C_2 \|\boldsymbol{B}(a, \cdot)\| \|\boldsymbol{I}(t, \cdot, \cdot)\| .$$

Since $\partial \overline{M}_0$ is not ρ-ejective for semiflow $\mathcal{U}(t)$, we see that there exists a initial value $(\boldsymbol{S}_0, \boldsymbol{I}_0) \in M_0$ such that solution $(\boldsymbol{S}(t, \cdot), \boldsymbol{I}(t, \cdot, \cdot))$ satisfies $\rho(\boldsymbol{I}(t, \cdot, \cdot)) < \varepsilon, \forall t \geq 0$. Let ε be given and satisfy

$$a_+ C_2 \|\boldsymbol{B}(a, x)\| \varepsilon \leq \vartheta.$$

Thus, we have

$$\mathbf{0} < \int_0^{+\infty} \boldsymbol{B}(a, x)\boldsymbol{I}(t, a, x)da \leq \vartheta \mathbf{1}, \quad \forall t \geq T_3, x \in \Omega. \tag{4.25}$$

By **Claim 1**, we have $\lim_{t \to +\infty} \|\boldsymbol{I}(t, \cdot, \cdot)\|_X = +\infty$. This contradicts to (4.24). Therefore, **Claim 2** is true.

By **Claim 2**, we know that $\partial \overline{M}_0$ is said to be ρ-ejective for semiflow $\mathcal{U}(t)$. By the proof of Theorem 4.14, we can see that semiflow $\mathcal{U}(t)$ is point dissipative and asymptotically smooth. According to Proposition 3.2 of [38], we can deduce that $\mathcal{U}(t)$ is uniformly persistent. Therefore, global attractor A_0 belongs to M_0, instead of M. Due to the compactness of semiflow $\mathcal{U}(t)$, $\mathcal{U}(t)$ is κ-condensing. According to Theorem 4.5 of [38], semiflow $\mathcal{U}(t)$ has a fixed point in global attractor $A_0 \in M_0$. Thus, the fixed point is an endemic steady state of the system (4.1). □

Remark 4.21 Since our approach is based on operator semigroup theory, it allows us to treat model (4.1) with Neumann, Dirichlet, or Robin boundary conditions. The key to our method is the compactness of the solution map $W(t, s)$. In addition, the solution map $W(t, s)$ of the reaction–diffusion equation with Neumann, Dirichlet, or Robin boundary conditions are all compact. Therefore, if Assumptions 4.1 and 4.13 still hold when the model (4.1) is under Dirichlet or Robin boundary conditions, then the results in Sect. 4 are still valid.

5 Infection Age-Structured Epidemic Model with Degenerate Diffusion and Spatial Heterogeneity

In this section, we consider the following class of infection age-structured epidemic model with degenerate diffusion and spatial heterogeneity. Thus, we give Assumption 5.1 (i). Due to the complexity of the degenerate diffusion, we also consider the following infection age-structured epidemic model which has the same form as (4.1). For $t > 0$, $a > 0$ and $x \in \Omega$,

$$\begin{cases} \frac{\partial}{\partial t}\boldsymbol{S}(t, x) = \boldsymbol{A}(x)\boldsymbol{S}(t, x) + \boldsymbol{M}(x, \boldsymbol{S}(t, x), \int_0^{+\infty} \boldsymbol{B}(a, x)\boldsymbol{I}(t, a, x)da), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \boldsymbol{I}(t, a, x) = \mathbb{L}(x)\boldsymbol{I}(t, a, x) - \boldsymbol{V}(a, x)\boldsymbol{I}(t, a, x), \\ \boldsymbol{I}(t, 0, x) = \boldsymbol{F}(x, \boldsymbol{S}(t, x)) \int_0^{+\infty} \boldsymbol{B}(a, x)\boldsymbol{I}(t, a, x)da, \end{cases} \tag{5.1}$$

under the Neumann boundary condition

$$\frac{\partial S_j}{\partial \nu} = 0, \frac{\partial I_i}{\partial \nu} = 0, \quad x \in \partial\Omega,$$

with initial value condition

$$S_j(0, \cdot) = S_{j0}(\cdot) \in C(\overline{\Omega}), I_i(0, \cdot, \cdot) = I_{i0}(\cdot, \cdot) \in L^1(\mathbb{R}_+, C(\overline{\Omega})),$$

where $\mathbb{L}$ is defined by

$$\mathbb{L}(x)\boldsymbol{u} := (d_1(x)\Delta u_1, \cdots, d_k(x)\Delta u_k, 0, \cdots, 0)^T, \boldsymbol{u} = (u_1, u_2, \cdots, u_n).$$

Assumption 5.1 For system (5.1), assume that

(i) For infected groups, diffusion coefficients d_i satisfy that there exists a positive constant d_0 such that $d_i(x) \geq d_0$ for each $i = 1, 2, ..., k$, $x \in \Omega$ and $d_i(x) = 0$ for $i = k+1, \cdots, n$;
(ii) $-\boldsymbol{V}(a, x) := (-v_{ij}(a, x))_{1 \leq i, j \leq n}$ is a bounded, cooperative, continuous and irreducible matrix function for all $a \in (0, +\infty)$ and $x \in \Omega$.
(iii) The following reaction–diffusion equations under the Neumann boundary condition

$$\frac{d\boldsymbol{S}(t, x)}{dt} = \boldsymbol{\Lambda}(x)\boldsymbol{S}(t, x) + \boldsymbol{M}(x, \boldsymbol{S}(t, x), \boldsymbol{0}), \quad x \in \Omega,$$

admits a globally attractive unique positive steady state $\boldsymbol{S}^0(x)$. In addition, Lipschitz continuous function $\boldsymbol{M}(x, \boldsymbol{S}, \int_0^{+\infty} \boldsymbol{B}(a, x)\boldsymbol{I}(a, x)da)$ is monotonically increasing with respect to $\boldsymbol{S}$ and monotonically decreasing with respect to $\boldsymbol{I}$.
(iv) $\boldsymbol{F}(x, \boldsymbol{S}(t, x))$ is a non-negative, continuous function and monotonically increasing with respect to $\boldsymbol{S}$. In addition, $\boldsymbol{F}^0(x)$ is defined by

$$\boldsymbol{F}^0(x) := \boldsymbol{F}(x, \boldsymbol{S}^0(x)).$$

(v) $\omega(\mathbb{W}) < 0$, where $\omega(\cdot)$ represents the exponential growth bound.
(vi) For each i and j, $\beta_{ij}(\cdot, \cdot) \in L_+^\infty(\mathbb{R}_+, C(\overline{\Omega})) \cap L_+^1(\mathbb{R}_+, C(\overline{\Omega}))$ and there exists a maximum age of infection denoted by a_+ such that if $a > a_+$ and $x \in \Omega$, $\beta_{ij}(a, x) = 0$. Moreover, there exists at least one interval (a_*, a^*) such that $\boldsymbol{B}(a, x)$ is an irreducible matrix function for $a \in (a_*, a^*)$.
(vii) For every positive element $\boldsymbol{I}_* \in X_+$, the solution $\boldsymbol{S}(t, x)$ of the following system

$$\frac{\partial}{\partial t}\boldsymbol{S}(t, x) = \boldsymbol{\Lambda}(x)\boldsymbol{S}(t, x) + \boldsymbol{M}(x, \boldsymbol{S}(t, x), \int_0^{+\infty} \boldsymbol{B}(a, x)\boldsymbol{I}_*(a, x)da), \quad x \in \Omega$$

satisfies that there exists a constant $C > 0$ such that $\lim_{t \to +\infty} \boldsymbol{S}(t, x) \geq \boldsymbol{S}^0(x) - C\boldsymbol{1} \gg \boldsymbol{0}, \forall x \in \Omega$.

Remark 5.2 During the spread of the disease, some infected compartments may not be able to spread in space, such as isolation. Since the equations of infected compartments can be viewed as a cooperative system abstractly, after transformation, the infected compartments with diffusion coefficient 0 can be marked as $k+1, \cdots, n$.

The operators $\mathbb{L}(\cdot) - \boldsymbol{V}(a, \cdot)$ are associated with an evolutionary system $\mathcal{W} := \{\mathbb{W}(t, s); 0 \leq s \leq t \leq +\infty\}$ of positive operators on $C(\overline{\Omega}, \mathbb{R}^n)$,

$$\mathbb{L}(\cdot) - \boldsymbol{V}(a, \cdot) = \lim_{h \to 0+} \frac{1}{h}(\mathbb{W}(a+h, a)\phi - \phi), \quad \phi \in D(\mathbb{L}(\cdot) - \boldsymbol{V}(a, \cdot)).$$

Let $\mathcal{T}(t)$ be the solution semigroup of the following equation under the Neumann boundary condition

$$\frac{d\boldsymbol{I}(t, x)}{dt} = \mathbb{L}(x)\boldsymbol{I}(t, x), \quad t \geq 0, x \in \Omega.$$

In the case where some diffusion coefficients in the system (5.1) are zero, the semigroup $\mathcal{T}(t)$ of the above equation loses compactness. It means that $\mathbb{W}(t,s)$ also loses compactness. Therefore, Theorem 4.7 is not hold under Assumption 5.1. It causes the loss of the compactness of integral kernel Φ_ε defined in (4.7). Thus, we can't use the renewal theorem for Volterra integral equations. Therefore, we can't follow the methods which stated in subsection 4.3 to prove the extinction and uniform persistence of the disease. It also causes a problem that whether R_0 is the principal eigenvalue of the next generation operator or not.

In the following, we give the linearization equations of the infected compartments at the disease-free steady state, for $t > 0$, $a > 0$ and $x \in \Omega$,

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \boldsymbol{I}(t,a,x) = \mathbb{L}(x)\boldsymbol{I}(t,a,x) - \boldsymbol{V}(a,x)\boldsymbol{I}(t,a,x), \\ \boldsymbol{I}(t,0,x) = \boldsymbol{F}^0(x)\int_0^{+\infty} \boldsymbol{B}(a,x)\boldsymbol{I}(t,a,x)da. \end{cases} \tag{5.2}$$

By grouping the infected compartments by the diffusion coefficients (0 or positive), we can divide infected compartments $\boldsymbol{I}$ into two groups of

$$\boldsymbol{I}_1(t,a,x) := (I_1(t,a,x), I_2(t,a,x), ..., I_k(t,a,x))^T, \boldsymbol{I}_2(t,a,x) := (I_{k+1}(t,a,x), \cdots, I_n(t,a,x))^T.$$

Based on $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$, we define $\mathbb{L}_1$ by

$$\mathbb{L}_1(x)\boldsymbol{I}_1 = (d_1(x)\Delta I_1, d_2(x)\Delta I_2, \cdots, d_k(x)\Delta I_k)^T.$$

In addition, we split $\boldsymbol{V}(a,x)$ and $\boldsymbol{F}^0\boldsymbol{B}(a,x) := \boldsymbol{F}^0(x) \times \boldsymbol{B}(a,x)$ into

$$\boldsymbol{V}(a,x) = \begin{pmatrix} \boldsymbol{V}_{11}(a,x) & \boldsymbol{V}_{12}(a,x) \\ \boldsymbol{V}_{21}(a,x) & \boldsymbol{V}_{22}(a,x) \end{pmatrix}, \boldsymbol{F}^0\boldsymbol{B} = \begin{pmatrix} (\boldsymbol{F}^0\boldsymbol{B})_{11} & (\boldsymbol{F}^0\boldsymbol{B})_{12} \\ (\boldsymbol{F}^0\boldsymbol{B})_{21} & (\boldsymbol{F}^0\boldsymbol{B})_{22} \end{pmatrix},$$

where $\boldsymbol{V}_{11}$ and $(\boldsymbol{F}^0\boldsymbol{B})_{11}$ are $k \times k$ matrix functions, $\boldsymbol{V}_{22}$ and $(\boldsymbol{F}^0\boldsymbol{B})_{22}$ are $(n-k) \times (n-k)$ matrix functions.

By using the above notations, we can rewrite system (5.2) into the following form, for $t > 0$, $a > 0$ and $x \in \Omega$,

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \boldsymbol{I}_1(t,a,x) = \mathbb{L}_1(x)\boldsymbol{I}_1(t,a,x) - \boldsymbol{V}_{11}(a,x)\boldsymbol{I}_1(t,a,x) - \boldsymbol{V}_{12}(a,x)\boldsymbol{I}_2(t,a,x), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) \boldsymbol{I}_2(t,a,x) = -\boldsymbol{V}_{21}(a,x)\boldsymbol{I}_1(t,a,x) - \boldsymbol{V}_{22}(a,x)\boldsymbol{I}_2(t,a,x), \\ \boldsymbol{I}_1(t,0,x) = \int_0^{+\infty} (\boldsymbol{F}^0\boldsymbol{B})_{11}(a,x)\boldsymbol{I}_1(t,a,x) + (\boldsymbol{F}^0\boldsymbol{B})_{12}(a,x)\boldsymbol{I}_2(t,a,x)da, \\ \boldsymbol{I}_2(t,0,x) = \int_0^{+\infty} (\boldsymbol{F}^0\boldsymbol{B})_{21}(a,x)\boldsymbol{I}_1(t,a,x) + (\boldsymbol{F}^0\boldsymbol{B})_{22}(a,x)\boldsymbol{I}_2(t,a,x)da. \end{cases} \tag{5.3}$$

However, not all of the results in Sects. 3 and 4 do not hold. We still can follow the ideas in Sects. 3 and 4 to prove the existence of the solution of systems (5.1) and (5.3).

5.1 The Well-Posedness

In this subsection, we follow the ideas in Sect. 3 to prove the existence of the solutions of the systems (5.1) and (5.3).

Firstly, we consider the system (5.3). Let $Y_1 := C(\overline{\Omega}, \mathbb{R}^k)$ with the usual supremum norm. Let $X_1 := L^1((0,+\infty), Y_1)$ and norm of space X_1 be given by

$$\|\varphi\|_{X_1} := \int_0^{+\infty} \|\varphi(a,\cdot)\|_{Y_1} da, \quad \varphi \in X_1.$$

Let us introduce a new extended space $\mathbb{X}_1$ and its closed subspace $\mathbb{X}_{10}$ by

$$\mathbb{X}_1 := Y_1 \times X_1, \mathbb{X}_{10} = \{0_{Y_1}\} \times X_1.$$

Then we consider the family of bounded linear operators $\{\mathcal{R}_\lambda\}_{\lambda>0}$ on $\mathbb{X}_1$, defined by

$$\mathcal{R}_\lambda \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\varphi} \end{pmatrix} = \begin{pmatrix} 0_{Y_1} \\ \boldsymbol{\psi} \end{pmatrix} \Leftrightarrow \boldsymbol{\psi}(a) = e^{-\int_0^a \lambda ds}\mathcal{T}_1(a)\boldsymbol{\phi} + \int_0^a e^{-\int_s^a \lambda dl}\mathcal{T}_1(a-s)\boldsymbol{\varphi}(s)ds.$$

where $\mathcal{T}_1(a), a \geq 0$ is the solution semigroup of the following reaction–diffusion equations under the Neumann boundary condition

$$\frac{d}{dt}u(t,x) = \mathbb{L}_1 u(t,x), \quad u_0(x) \in Y_1.$$

It is clear that $\mathcal{T}_1(a)$ maps Y_1 into itself and is compact. Observe that $\{\mathcal{R}_\lambda\}_{\lambda>0}$ is a pseudo-resolvent on $\mathbb{X}_1$. Moreover, we have

$$\mathcal{R}_\lambda x = 0, x \in \mathbb{X}_1 \Rightarrow x \in \mathbb{X}_{10} \quad \text{and} \quad \lim_{\lambda \to +\infty} \lambda \mathcal{R}_\lambda x = x, \quad \forall x \in \mathbb{X}_{10}.$$

By Sect. 1.9 of [43], we can deduce that there exists a unique closed linear operator $\mathcal{B}_1$ that satisfies

$$\mathcal{B}_1 : D(\mathcal{B}_1) \subset \mathbb{X}_1 \to \mathbb{X}_1, \overline{D(\mathcal{B}_1)} = \mathbb{X}_{10}, \mathcal{R}_\lambda = (\lambda I - \mathcal{B}_1)^{-1}, \quad \forall \lambda > 0.$$

Next, we define $Y_2 := C(\overline{\Omega}, \mathbb{R}^{n-k})$, $X_2 := L^1((0,+\infty), Y_2)$, $\mathbb{X}_2 := Y_2 \times X_2$ and $\mathbb{X}_{20} := \{0_{Y_2}\} \times X_2$. In addition, we introduce an operator $\mathcal{B}_2$ as follows

$$\mathcal{B}_2 \begin{pmatrix} 0_{Y_2} \\ f \end{pmatrix} := \begin{pmatrix} -f(0,\cdot) \\ -\frac{df}{da} \end{pmatrix}, \quad \begin{pmatrix} 0_{Y_2} \\ f \end{pmatrix} \in \{0_{Y_2}\} \times W^{1,1}((0,+\infty), Y_2).$$

We define $\mathcal{B}$ and $\mathcal{G}$ on $\mathbb{X}_1 \times \mathbb{X}_2$ by

$$\mathcal{B} := \begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{pmatrix}, \quad \mathcal{G}\begin{pmatrix} 0_{Y_1} \\ u_1 \\ 0_{Y_2} \\ u_2 \end{pmatrix} := \begin{pmatrix} \int_0^{+\infty} (\boldsymbol{F}^0\boldsymbol{B})_{11}u_1 + (\boldsymbol{F}^0\boldsymbol{B})_{12}u_2 da \\ -\boldsymbol{V}_{11}u_1 - \boldsymbol{V}_{12}u_2 \\ \int_0^{+\infty} (\boldsymbol{F}^0\boldsymbol{B})_{21}u_1 + (\boldsymbol{F}^0\boldsymbol{B})_{22}u_2 da \\ -\boldsymbol{V}_{21}u_1 - \boldsymbol{V}_{22}u_2 \end{pmatrix},$$

where $(0_{Y_1}, u_1) \in \mathbb{X}_1$ and $(0_{Y_2}, u_2) \in \mathbb{X}_2$. Then we can rewrite system (5.3) into the following abstract Cauchy problem

$$\frac{du(t)}{dt} = \mathcal{B}u(t) + \mathcal{G}u(t). \tag{5.4}$$

In order to obtain the existence of the positive solution, we consider the following equivalent system

$$\frac{du(t)}{dt} = \mathcal{B}_\varepsilon u(t) + \mathcal{G}_\varepsilon u(t), \tag{5.5}$$

where $\mathcal{B}_\varepsilon := \mathcal{B} - \frac{1}{\varepsilon}I$ and $\mathcal{G}_\varepsilon := \frac{1}{\varepsilon}(I + \varepsilon\mathcal{G})$.

Remark 5.3 If we define $\overline{\mathcal{B}}$ and $\overline{\mathcal{G}}$ as the following form

$$\overline{\mathcal{B}}\begin{pmatrix} 0_{Y_1} \\ 0_{Y_2} \\ u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} \mathcal{B}_{11}u_1 \\ -u_2(0,\cdot) \\ \mathcal{B}_{12}u_1 \\ -\frac{du_2}{da} \end{pmatrix}, \overline{\mathcal{G}}\begin{pmatrix} 0_{Y_1} \\ 0_{Y_2} \\ u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} \int_0^{+\infty} (\boldsymbol{F}^0\boldsymbol{B})_{11}u_1 + (\boldsymbol{F}^0\boldsymbol{B})_{12}u_2 da \\ \int_0^{+\infty} (\boldsymbol{F}^0\boldsymbol{B})_{21}u_1 + (\boldsymbol{F}^0\boldsymbol{B})_{22}u_2 da \\ -\boldsymbol{V}_{11}u_1 - \boldsymbol{V}_{12}u_2 \\ -\boldsymbol{V}_{21}u_1 - \boldsymbol{V}_{22}u_2 \end{pmatrix}.$$

where $\mathcal{B}_1 := (\mathcal{B}_{11}, \mathcal{B}_{12})^T$. Therefore, system (5.4) can be abstractly seen as the following Cauchy problem on space $\mathbb{X}$,

$$\frac{du(t)}{dt} = \overline{\mathcal{B}}u(t) + \overline{\mathcal{G}}u(t). \tag{5.6}$$

Therefore, it has the same form as the system (3.1) (infection age-structured epidemic model with non-degenerate diffusion). Of course, we can obtain the equivalent system

$$\frac{du(t)}{dt} = \overline{\mathcal{B}}_\varepsilon u(t) + \overline{\mathcal{G}}_\varepsilon u(t), \tag{5.7}$$

where $\overline{\mathcal{B}}_\varepsilon := \overline{\mathcal{B}} - \frac{1}{\varepsilon}I$ and $\overline{\mathcal{G}}_\varepsilon := \frac{1}{\varepsilon}(I + \varepsilon\overline{\mathcal{G}})$.

Lemma 5.4 *Let Assumption 5.1 be satisfied. Then $\mathcal{B}_\varepsilon$ satisfies the Hille-Yosida condition.*

Proof Similar to the proof of Lemma 3.3, we can deduce that $\mathcal{B}_{1\varepsilon} := \mathcal{B}_1 - \frac{1}{\varepsilon}I$ satisfies the Hille-Yosida condition, i.e.,

$$\left\|(\lambda I - \mathcal{B}_{1\varepsilon})^{-1}\begin{pmatrix}\phi_1\\ \varphi_1\end{pmatrix}\right\|_{\mathbb{X}_1} \le \frac{M_1}{\lambda + \lambda_0 + \frac{1}{\varepsilon}}\left\|\begin{pmatrix}\phi_1\\ \varphi_1\end{pmatrix}\right\|_{\mathbb{X}_1},$$

where $M_1 \ge 1$ is a positive constant and $\lambda_0 \le 0$. It means that we only need to prove $\mathcal{B}_{2\varepsilon} := \mathcal{B}_2 - \frac{1}{\varepsilon}I$ also satisfies the Hille-Yosida condition. By following the ideas in Lemma 3.3, it is easy to find that $\mathcal{B}_{2\varepsilon} := \mathcal{B}_2 - \frac{1}{\varepsilon}I$ also satisfies the Hille-Yosida condition, i.e.,

$$\left\|(\lambda I - \mathcal{B}_{2\varepsilon})^{-1}\begin{pmatrix}\phi_2\\ \varphi_2\end{pmatrix}\right\|_{\mathbb{X}_2} \le \frac{M_2}{\lambda + \frac{1}{\varepsilon}}\left\|\begin{pmatrix}\phi_2\\ \varphi_2\end{pmatrix}\right\|_{\mathbb{X}_2},$$

where $M_2 \ge 1$ is a positive constant. Thus, we have

$$\begin{aligned}\left\|(\lambda I - \mathcal{B}_\varepsilon)^{-1}(\phi_1, \varphi_1, \phi_2, \varphi_2)^T\right\|_{\mathbb{X}_1\times\mathbb{X}_2} &\le \frac{M_1}{\lambda + \lambda_0 + \frac{1}{\varepsilon}}\left\|\begin{pmatrix}\phi_1\\ \varphi_1\end{pmatrix}\right\|_{\mathbb{X}_1} + \frac{M_2}{\lambda + \frac{1}{\varepsilon}}\left\|\begin{pmatrix}\phi_2\\ \varphi_2\end{pmatrix}\right\|_{\mathbb{X}_2}\\ &\le \frac{\max\{M_1, M_2\}}{\lambda + \lambda_0 + \frac{1}{\varepsilon}}\left(\left\|\begin{pmatrix}\phi_1\\ \varphi_1\end{pmatrix}\right\|_{\mathbb{X}_1} + \left\|\begin{pmatrix}\phi_2\\ \varphi_2\end{pmatrix}\right\|_{\mathbb{X}_2}\right)\\ &= \frac{\max\{M_1, M_2\}}{\lambda + \lambda_0 + \frac{1}{\varepsilon}}\left\|(\phi_1, \varphi_1, \phi_2, \varphi_2)^T\right\|_{\mathbb{X}_1\times\mathbb{X}_2}\end{aligned}$$

□

Similar to Lemmas 3.6, 3.7, and Theorem 3.8, we have the following results.

Lemma 5.5 *The part $\mathcal{B}_{\varepsilon 0}$ of $\mathcal{B}_\varepsilon$ in $\mathbb{X}_{10} \times \mathbb{X}_{20}$ generates a C_0-semigroup $\{T_{\mathcal{B}_{\varepsilon 0}}(t)\}_{t\ge 0}$ on space $\mathbb{X}_{10} \times \mathbb{X}_{20}$.*

Lemma 5.6 *The unique continuous solution to (5.5) can be given by (5.8), and it take values in $\mathbb{X}_{10} \times \mathbb{X}_{20}$.*

$$u(t) = T_{\mathcal{B}_{\varepsilon 0}}(t)u(0) + \lim_{\lambda\to\infty}\int_0^t T_{\mathcal{B}_{\varepsilon 0}}(t-s)\lambda(\lambda - \mathcal{B}_\varepsilon)^{-1}\mathcal{G}_\varepsilon u(s)ds. \tag{5.8}$$

Remark 5.7 If we consider system (5.7), the part $\overline{\mathcal{B}}_{\varepsilon 0}$ of $\overline{\mathcal{B}}_\varepsilon$ in $\mathbb{X}_0$ generates a C_0-semigroup $\{T_{\overline{\mathcal{B}}_{\varepsilon 0}}(t)\}_{t\ge 0}$ on space $\mathbb{X}_0$. Then the unique continuous solution to (5.7) can be given by (5.9), and it takes values in $\mathbb{X}_0$,

$$u(t) = T_{\overline{\mathcal{B}}_{\varepsilon 0}}(t)u(0) + \lim_{\lambda\to\infty}\int_0^t T_{\overline{\mathcal{B}}_{\varepsilon 0}}(t-s)\lambda(\lambda - \overline{\mathcal{B}}_\varepsilon)^{-1}\overline{\mathcal{G}}_\varepsilon u(s)ds. \tag{5.9}$$

Theorem 5.8 *Let Assumption 5.1 be satisfied. Then the non-negative solution of system (5.2) defined in $C([0,\tau), X)$, $\tau > 0$ exists and is unique.*

Similar to Theorem 3.10, we can obtain the existence of the solution of the system (5.1).

Theorem 5.9 *Let Assumption 5.1 be satisfied. Then the mild solution of system (5.1) defined in $C([0,\tau), C(\overline{\Omega}, \mathbb{R}^m)) \times C([0,\tau), X)$ exists and is unique.*

Remark 5.10 Similar to Remarks 3.9 and 3.11, we also can define $\mathcal{Y} := L^2(\Omega, \mathbb{R}^n)$ and $\mathcal{X} := L^p((0,+\infty), Y)$, or other suitable spaces. Then Theorem 5.9 still holds.

5.2 The Principal Eigenvalue of the Next Generation Operator with Degenerate Diffusion

Following the ideas of the basic reproduction number R_0 stated in Sect. 3, we define the next generation operator Ψ that maps $C(\overline{\Omega}, \mathbb{R}^n)$ into itself as follows

$$\begin{aligned} \Psi(\boldsymbol{\varphi})(x) : &= \boldsymbol{F}^0(x) \int_0^{+\infty} \mathbf{B}(a,x)\mathbb{W}(a,0)\boldsymbol{\varphi}(x)da \\ &= \int_0^{+\infty} (\boldsymbol{F}^0\boldsymbol{B})(a,x)\mathbb{W}(a,0)\boldsymbol{\varphi}(x)da. \end{aligned} \tag{5.10}$$

Similar to the previous section, we define the basic reproduction number R_0 by

$$R_0 = r(\Psi).$$

In the case of the infection age-structured epidemic models with non-degenerate diffusion, Ψ is compact. From Lemma 3.7, we know that $r(\Psi)$ is the principal eigenvalue of Ψ with a strongly positive eigenvector ψ_*. Moreover, there is no other eigenvalue of Ψ with positive eigenvector. However, in the case of the models with degenerate diffusion, Ψ is not compact. It causes the conclusions of Lemma 3.7 cannot be obtained directly by Krein–Rutman theorem. In the following, we prove that $r(\Psi)$ is still the principal eigenvalue of Ψ by using a generalized Krien–Rutman theorem, under the following assumptions.

Assumption 5.11 For system (5.1), assume that

(i) $(\boldsymbol{F}^0\boldsymbol{B})_{12}(a,x) = 0$ for all $a \in (0,\infty)$, $x \in \overline{\Omega}$,
(ii) $\boldsymbol{V}_{21}(a,x) = 0$ for all $a \in (0,\infty)$, $x \in \overline{\Omega}$.

Remark 5.12 Assumption 5.11 does not lead to contradictions in the model. There is a special case that $(\boldsymbol{F}^0\boldsymbol{B})_{12}(a,x) \equiv 0$ and $(\boldsymbol{F}^0\boldsymbol{B})_{22}(a,x) \equiv 0$. In this case, Ψ is compact. Then the principal eigenvalue of Ψ is $r(\Psi)$ by Krein-Rutman theorem. In the following, we mainly consider the case that Ψ is not compact and under Assumption 5.11.

In order to use the generalized Krien–Rutman theorem [29, 41], we give some definitions and theorems. From Sect. 7.5 of [47], the definition of the essential spectrum of a positive bounded operator Ψ is given as follows

$$\sigma_e(\Psi) := \{\lambda \in \sigma(\Psi) : \lambda I - \Psi \text{ is not a Fredholm operator with } ind(\lambda I - \Psi) = 0\},$$

where $ind(\Psi)$ is the Fredholm index defined by $ind(\Psi) = \dim \mathcal{N}(\Psi) - \text{codim} R(\Psi)$, in which $\mathcal{N}(\Psi)$ and $R(\Psi)$ denote the null space and range, respectively, of Ψ. Ψ is said to be a Fredholm operator if $R(\Psi)$ is closed and both of $\dim \mathcal{N}(\Psi)$ and $\text{codim} R(\Psi)$ are finite.

Theorem 5.13 *[29, 41] Let X be a Banach space having a total cone $X_+ \subset X$ and Ψ is a bounded positive operator. If $r(\Psi) > r_e(\Psi)$, then there exists $x \in X_+$ such that $\Psi x = r(\Psi)x$, where $r_e(\Psi)$ denotes the essential spectral radius of Ψ.*

Next, we want to study the properties of the non-compact operator Ψ. Thus, we need a more explicit expression of Ψ. We reconsider the solution map $\mathbb{W}(t,s), t \geq s$. Note that $\mathbb{W}(t,s)$ is the solution map of the following reaction–diffusion equations under the Neumann boundary condition

$$\frac{d\boldsymbol{u}(t,x)}{dt} = \mathbb{L}(x)\boldsymbol{u}(t,x) - \boldsymbol{V}(t,x)\boldsymbol{u}(t,x), \quad x \in \Omega. \tag{5.11}$$

Under Assumption 5.11, (5.11) is equivalent to

$$\frac{d}{da}\begin{pmatrix} \boldsymbol{u}_1(a,x) \\ \boldsymbol{u}_2(a,x) \end{pmatrix} = \begin{pmatrix} \mathbb{L}_1(x)\boldsymbol{u}_1(a,x) \\ 0 \end{pmatrix} - \begin{pmatrix} \boldsymbol{V}_{11}(a,x) & \boldsymbol{V}_{12}(a,x) \\ \boldsymbol{0} & \boldsymbol{V}_{22}(a,x) \end{pmatrix}\begin{pmatrix} \boldsymbol{u}_1(a,x) \\ \boldsymbol{u}_2(a,x) \end{pmatrix}, \tag{5.12}$$

where $\boldsymbol{u}(a,x) := (\boldsymbol{u}_1(a,x), \boldsymbol{u}_2(a,x))^T$. Therefore, we have, for $a \geq 0, x \in \Omega$,

$$\mathbb{W}(a,0)\begin{pmatrix} \boldsymbol{u}_1(0,x) \\ \boldsymbol{u}_2(0,x) \end{pmatrix} := \begin{pmatrix} \mathbb{W}_1(a,0)\boldsymbol{u}_1(0,x) - \int_0^a \mathbb{W}_1(a,s)\boldsymbol{V}_{12}(s)\boldsymbol{u}_2(s,x)ds \\ \mathbb{W}_2(a,0)\boldsymbol{u}_2(0,x) \end{pmatrix}, \tag{5.13}$$

where $\mathbb{W}_1$ is the solution map of $\frac{d\boldsymbol{u}_1(t,x)}{dt} = \mathbb{L}_1(x)\boldsymbol{u}_1(t,x) - \boldsymbol{V}_{11}(t,x)\boldsymbol{u}_1(t,x), x \in \Omega$ and $\mathbb{W}_2$ is the solution map of $\frac{d\boldsymbol{u}_2(t,x)}{dt} = -\boldsymbol{V}_{22}(t)\boldsymbol{u}_2(t,x), x \in \Omega$. By above, we can write Ψ into a more explicit form by

$$\Psi\begin{pmatrix} \boldsymbol{\varphi}_{10}(x) \\ \boldsymbol{\varphi}_{20}(x) \end{pmatrix} = \begin{pmatrix} \Psi_1(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)(x) \\ \Psi_2(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)(x) \end{pmatrix}, \quad x \in \Omega \tag{5.14}$$

where

$$\begin{aligned} \Psi_1(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)(x) &= \int_0^{+\infty}(\boldsymbol{F}^0\boldsymbol{B})_{11}(a,x)(\mathbb{W}_1(a,0)\boldsymbol{\varphi}_1(0,x) - \int_0^a \mathbb{W}_1(a,s)\boldsymbol{V}_{12}(s,x)\boldsymbol{\varphi}_2(s,x)ds)da, \\ \Psi_2(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)(x) &= \int_0^{+\infty}(\boldsymbol{F}^0\boldsymbol{B})_{21}(a,x)(\mathbb{W}_1(a,0)\boldsymbol{\varphi}_1(0,x) - \int_0^a \mathbb{W}_1(a,s)\boldsymbol{V}_{12}(s,x)\boldsymbol{\varphi}_2(s,x)ds)da \\ &\quad + \int_0^{+\infty}(\boldsymbol{F}^0\boldsymbol{B})_{22}(a,x)\mathbb{W}_2(a,0)\boldsymbol{\varphi}_2(0,x)da, \end{aligned} \tag{5.15}$$

where $\boldsymbol{\varphi}(a,x) := (\boldsymbol{\varphi}_1(a,x), \boldsymbol{\varphi}_2(a,x))$ is the solution of the following equations with initial value $\boldsymbol{\varphi}(0,x) = (\boldsymbol{\varphi}_1(0,x), \boldsymbol{\varphi}_2(0,x)) = (\boldsymbol{\varphi}_{10}(x), \boldsymbol{\varphi}_{20}(x))$,

$$\frac{d\boldsymbol{\varphi}(t,x)}{dt} = \mathbb{L}(x)\boldsymbol{\varphi}(t,x) - \boldsymbol{V}(t,x)\boldsymbol{\varphi}(t,x), \quad x \in \Omega. \tag{5.16}$$

In addition, we define an operator $\widehat{\Psi}_2$ on space Y_2 by

$$\widehat{\Psi}_2(\boldsymbol{\varphi}_2)(x) := \int_0^{\infty}(\boldsymbol{F}^0\boldsymbol{B})_{22}(a,x)\mathbb{W}_2(a,0)\boldsymbol{\varphi}_2(x)da, \quad \boldsymbol{\varphi}_2 \in Y_2. \tag{5.17}$$

Lemma 5.14 *Let Assumptions 5.1 and 5.11 be satisfied. Then* $\sigma_e(\Psi) = \sigma_e(\widehat{\Psi}_2)\bigcup\{0\}$.

Proof Note that $\mathcal{T}_1(t)$ is compact, then Ψ_1 is also compact. Moreover, $\Psi_2(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)(x) - \widehat{\Psi}_2(\boldsymbol{\varphi}_2)(x)$ is also compact. We define $\widehat{\Psi}$ by

$$\widehat{\Psi}(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)(x) = (0_{Y_1}, \widehat{\Psi}_2(\boldsymbol{\varphi}_2)(x))^T.$$

Due to Theorem 7.27 of [47] and the fact that $\Psi - \widehat{\Psi}$ is compact, $\sigma_e(\Psi) = \sigma_e(\widehat{\Psi})$. Inspired by the ideas in [29], we divide our proof into the following two steps.

Step 1: $\sigma(\widehat{\Psi}) = \sigma(\widehat{\Psi}_2)\bigcup\{0\}$. It is easy to see that $0 \in \sigma(\widehat{\Psi})$ and $\sigma(\widehat{\Psi}_2) \subset \sigma(\widehat{\Psi})$. Now we prove that $\mu \in \sigma(\widehat{\Psi})$ implies that $\mu \in \sigma(\widehat{\Psi}_2)\bigcup\{0\}$. In fact, we only need to prove the following three claims for any $\mu \neq 0$:

Claim 1: If $\mathcal{N}(\mu I-\widehat{\Psi})\neq\{0\}$, then $\mathcal{N}(\mu I-\widehat{\Psi}_2)\neq\{0\}$.

Claim 2: If $R(\mu I-\widehat{\Psi})$ is not closed, then $R(\mu I-\widehat{\Psi}_2)$ is not either.

Claim 3: If $\overline{R(\mu I-\widehat{\Psi})}\neq Y$, then $\overline{R(\mu I-\widehat{\Psi}_2)}\neq Y_2$.

Firstly, we show **Claim 1**. If $\mathcal{N}(\mu I-\widehat{\Psi})\neq\{0\}$, then there exists some $\boldsymbol{\varphi}=(\boldsymbol{\varphi}_1,\boldsymbol{\varphi}_2)^T\in Y\backslash\{\mathbf{0}\}$ such that

$$(\mu I-\widehat{\Psi})\boldsymbol{\varphi}=(\mu I\boldsymbol{\varphi}_1,(\mu I-\widehat{\Psi}_2)\boldsymbol{\varphi}_2)^T.$$

Moreover, we have $\boldsymbol{\varphi}_1=0_{Y_1}$ and $(\mu I-\widehat{\Psi}_2)\boldsymbol{\varphi}_2=0_{Y_2}$. Furthermore, we can deduce that $\mathcal{N}(\mu I-\widehat{\Psi}_2)\neq\{0_{Y_2}\}$, otherwise $\boldsymbol{\varphi}_2=0_{Y_2}$ and hence $\boldsymbol{\varphi}=\mathbf{0}$.

In the follows, we prove **Claims 2** and **3**. If $R(\mu I-\widehat{\Psi})\neq Y$, for a given $\boldsymbol{\phi}=(\boldsymbol{\phi}_1,\boldsymbol{\phi}_2)^T\notin R(\mu I-\widehat{\Psi})$, we first show that $\boldsymbol{\phi}_2\notin R(\mu I-\widehat{\Psi}_2)$. It is clear that we have

$$(\mu I-\widehat{\Psi})\boldsymbol{\psi}=(\mu I-\widehat{\Psi})(\boldsymbol{\psi}_1,\boldsymbol{\psi}_2)^T=(\mu\boldsymbol{\psi}_1,(\mu I-\widehat{\Psi}_2)\boldsymbol{\psi}_2)^T,\forall(\boldsymbol{\psi}_1,\boldsymbol{\psi}_2)^T\in Y.$$

Therefore, either there exists no $\boldsymbol{\varphi}_1\in Y_1$ such that $\mu\boldsymbol{\varphi}_1=\boldsymbol{\phi}_1$, or there exists no $\boldsymbol{\varphi}_2\in Y_2$ such that $(\mu I-\widehat{\Psi}_2)\boldsymbol{\varphi}_2=\boldsymbol{\phi}_2$. It follows that $\boldsymbol{\varphi}_1=\frac{1}{\mu}\boldsymbol{\phi}_1$. Therefore, there is no $\boldsymbol{\varphi}_2\in Y_2$ such that $(\mu I-\widehat{\Psi})\boldsymbol{\varphi}_2=\boldsymbol{\phi}_2$, i.e., $\boldsymbol{\phi}_2\notin R(\mu I-\widehat{\Psi}_2)$.

For **Claim 2**, if $R(\mu I-\widehat{\Psi})$ is not closed, we choose $\boldsymbol{\phi}^0=(\boldsymbol{\phi}_1^0,\boldsymbol{\phi}_2^0)\in\overline{R(\mu I-\widehat{\Psi})}$ but $\boldsymbol{\phi}^0=(\boldsymbol{\phi}_1^0,\boldsymbol{\phi}_2^0)\notin R(\mu I-\widehat{\Psi})$. By using the above arguments, we can deduce $\boldsymbol{\phi}_2^0\notin R(\mu I-\widehat{\Psi}_2)$. It suffices to prove $\boldsymbol{\phi}_2^0\in\overline{R(\mu I-\widehat{\Psi}_2)}$. Taking a sequence $\boldsymbol{\phi}^{0,n}=(\boldsymbol{\phi}_1^{0,n},\boldsymbol{\phi}_2^{0,n})^T\in R(\mu I-\widehat{\Psi})$ which converges to $\boldsymbol{\phi}^0$ on Y as $n\to+\infty$, we can choose $\boldsymbol{\varphi}^{0,n}=(\boldsymbol{\varphi}_1^{0,n},\boldsymbol{\varphi}_2^{0,n})^T$ such that $(\mu I-\widehat{\Psi})\boldsymbol{\varphi}^{0,n}=\boldsymbol{\phi}^{0,n}$. Then we obtain that $(\mu I-\widehat{\Psi}_2)\boldsymbol{\varphi}_2^{0,n}=\boldsymbol{\phi}_2^{0,n}$. Furthermore, $\boldsymbol{\phi}_2^{0,n}$ converges to $\boldsymbol{\phi}_2^0$ on Y_2 as $n\to+\infty$. This means that $\boldsymbol{\phi}_2^0\in\overline{R(\mu I-\widehat{\Psi}_2)}$.

For **Claim 3**, if $\overline{R(\mu I-\widehat{\Psi})}\neq Y$, we set $\boldsymbol{\phi}^0=(\boldsymbol{\phi}_1^0,\boldsymbol{\phi}_2^0)\in Y$ but $\boldsymbol{\phi}^0=(\boldsymbol{\phi}_1^0,\boldsymbol{\phi}_2^0)\notin\overline{R(\mu I-\widehat{\Psi})}$. Since $\boldsymbol{\phi}_2^0\notin R(\mu I-\widehat{\Psi}_2)$, it suffices to prove $\boldsymbol{\phi}_2^0\notin\overline{R(\mu I-\widehat{\Psi}_2)}$. Suppose, by contradiction, that there is a sequence $\boldsymbol{\phi}_2^{0,n}\in R(\mu I-\widehat{\Psi}_2)$ which converges to $\boldsymbol{\phi}_2^0$ on space Y_2 as $n\to+\infty$. Then we can choose $\boldsymbol{\varphi}_2^{0,n}$ such that $(\mu I-\widehat{\Psi}_2)\boldsymbol{\varphi}_2^{0,n}=\boldsymbol{\phi}_2^{0,n}$. Let $\boldsymbol{\varphi}_1^{0,n}=\frac{1}{\mu}\boldsymbol{\phi}_1^0$, $\boldsymbol{\varphi}^{0,n}=(\boldsymbol{\varphi}_1^{0,n},\boldsymbol{\varphi}_2^{0,n})$ and $\boldsymbol{\phi}^{0,n}=(\mu I-\widehat{\Psi})\boldsymbol{\varphi}^{0,n}=(\boldsymbol{\phi}_1^0,\boldsymbol{\phi}_2^{0,n})^T$. It is easy to find that $\boldsymbol{\phi}^{0,n}$ converges to $\boldsymbol{\phi}^0$ as $n\to+\infty$. Therefore, $\boldsymbol{\phi}^0\in\overline{R(\mu I-\widehat{\Psi})}$. This causes a contradiction. So we finish Step 1.

Step 2: $\sigma_e(\widehat{\Psi})=\sigma_e(\widehat{\Psi}_2)\bigcup\{0\}$. It is easy to see $0\in\sigma_e(\widehat{\Psi})$. According to the definition of the essential spectrum, we only need to prove the following three claims for any $\mu\neq0$.

Claim 4: $\dim\mathcal{N}(\mu I-\widehat{\Psi})=\dim\mathcal{N}(\mu I-\widehat{\Psi}_2)$.

Claim 5: $R(\mu I-\widehat{\Psi})$ is not closed $\Leftrightarrow R(\mu I-\widehat{\Psi}_2)$ is not closed.

Claim 6: If $R(\mu I-\widehat{\Psi})$ is closed, $\operatorname{codim}R(\mu I-\widehat{\Psi})=\operatorname{codim}R(\mu I-\widehat{\Psi}_2)$.

We begin to prove **Claim 4**. We assume that there are some $\boldsymbol{\varphi}^0=(\boldsymbol{\varphi}_1^0,\boldsymbol{\varphi}_2^0)\in Y\setminus\{0\}$ such that

$$(\mu I-\widehat{\Psi})\boldsymbol{\varphi}^0=(\mu I-\widehat{\Psi})(\boldsymbol{\varphi}_1^0,\boldsymbol{\varphi}_2^0)^T=\mathbf{0}.$$

Thus, $\boldsymbol{\varphi}_1^0=0_{Y_1}$ and $(\mu I-\widehat{\Psi})\boldsymbol{\varphi}_2^0=0_{Y_2}$ with $\boldsymbol{\varphi}_2^0\neq0$. Moreover, we have $\dim\mathcal{N}(\mu I-\widehat{\Psi})\leq\dim\mathcal{N}(\mu I-\widehat{\Psi}_2)$. If there exists $\boldsymbol{\varphi}_2^0\in Y_2\setminus\{0_{Y_2}\}$ such that $(\mu I-\widehat{\Psi}_2)\boldsymbol{\varphi}_2^0=0_{Y_2}$, then $(\mu I-\widehat{\Psi})(0_{Y_1},\boldsymbol{\varphi}_2^0)^T=0_Y$. Thus, $\dim\mathcal{N}(\mu I-\widehat{\Psi})\geq\dim\mathcal{N}(\mu I-\widehat{\Psi}_2)$.

Next, we prove **Claim 5**. If $R(\mu I-\widehat{\Psi}_2)$ is not closed, we can choose $\boldsymbol{\phi}_2^0\in\overline{R(\mu I-\widehat{\Psi}_2)}$ but $\boldsymbol{\phi}_2^0\notin R(\mu I-\widehat{\Psi}_2)$. By arguments similar to those in **Claim 2**, we can deduce that $(0_{Y_1},\boldsymbol{\phi}_2^0)^T\in\overline{R(\mu I-\widehat{\Psi})}$ but $(0_{Y_1},\boldsymbol{\phi}_2^0)^T\notin R(\mu I-\widehat{\Psi})$. It follows that $R(\mu I-\widehat{\Psi})$ is not closed. In addition, the converse has been shown in **Claim 2**.

Finally, we prove **Claim 6**. Since $\boldsymbol{\phi}_2^0 \in Y_2/R(\mu I - \widehat{\Psi}_2)$ implies $(0_{Y_1}, \boldsymbol{\phi}_2^0)^T \in Y/R(\mu I - \widehat{\Psi})$, we can deduce that $\operatorname{codim} R(\mu I - \widehat{\Psi}) \geq \operatorname{codim} R(\mu I - \widehat{\Psi}_2)$. To prove the opposite inequality, we choose $\boldsymbol{\phi}^0 = (\boldsymbol{\phi}_1^0, \boldsymbol{\phi}_2^0)^T \in Y/R(\mu I - \widehat{\Psi})$ with $\boldsymbol{\phi}^0 \neq 0_Y$. By the fact $(Y_1, 0_{Y_2})^T \subset R(\mu I - \widehat{\Psi})$, it follows that $\boldsymbol{\phi}_1^0$ can be chosen as 0_{Y_1}. Then $\boldsymbol{\phi}_2^0 \in Y_2/R(\mu I - \widehat{\Psi}_2)$, and hence $\operatorname{codim} R(\mu I - \widehat{\Psi}) \leq \operatorname{codim} R(\mu I - \widehat{\Psi}_2)$. Therefore, we finish Step 2. □

Note that $\widehat{\Psi}_2$ is a positive multiplication operator in X_2, we can determine the spectral radius $r(\widehat{\Psi}_2)$ of $\widehat{\Psi}_2$ by $r(\widehat{\Psi}_2) = \left\| \int_0^\infty (\boldsymbol{F}^0 \boldsymbol{B})_{22}(a, x) \mathbb{W}_2(a, 0) da \right\|_{C(\overline{\Omega}, \mathbb{R}^{n-k})}$. By Proposition 2.7 of [30], we obtain the following proposition.

Proposition 5.15 *Let Assumptions 5.1 and 5.11 be satisfied. Then* $\sigma_e(\widehat{\Psi}_2) = \sigma(\widehat{\Psi}_2) = \bigcup_{x \in \overline{\Omega}} \sigma(\widehat{\Psi}_2(x))$.

Assumption 5.16 For operators Ψ and $\widehat{\Psi}$, assume that $r(\Psi) > r(\widehat{\Psi})$.

Theorem 5.17 *Let Assumptions 5.1, 5.11, and 5.16 be satisfied. Then $r(\Psi)$ is the principal eigenvalue of Ψ with a positive eigenvector $\boldsymbol{\psi}_*$.*

Proof By Lemma 5.14, we know that $r_e(\Psi) = r_e(\widehat{\Psi}_2)$ and $r(\widehat{\Psi}) = r(\widehat{\Psi}_2)$. According to Proposition 5.15, we have $r(\widehat{\Psi}_2) = r_e(\widehat{\Psi}_2)$. Therefore, we obtain $r_e(\Psi) = r(\widehat{\Psi}_2)$. By Assumption 5.16, we see that $r(\Psi) > r(\widehat{\Psi}) = r(\widehat{\Psi}_2) = r_e(\Psi)$. So Theorem 5.17 is a direct result of Theorem 5.13. □

5.3 Extinction and Uniform Persistence of the Disease

For arbitrarily large positive number ξ, we consider the system (5.1) with initial value belongs to the following set

$$B_\xi := \left\{ (\boldsymbol{S}_0(\cdot), \boldsymbol{I}_0(\cdot,\cdot)) \in C_+(\overline{\Omega}, \mathbb{R}^m) \times L_+^1(\mathbb{R}_+, C(\overline{\Omega}, \mathbb{R}^n)) : \boldsymbol{S}_0(\cdot) \leq \boldsymbol{S}^0(\cdot), \boldsymbol{I}_0(a,\cdot) \leq \xi \mathbb{W}(a,0) \boldsymbol{\psi}_*(\cdot) \right\}, \tag{5.18}$$

where $\boldsymbol{\psi}_*$ is the eigenvector of operator Ψ corresponding to $r(\Psi)$ and $\boldsymbol{S}^0$ is the disease-free steady state.

By using a similar approach from Sect. 2, we obtain the following expression of I-equations of the system (5.1),

$$\boldsymbol{I}(t, a, x) = \begin{cases} \mathbb{W}(a, 0) \boldsymbol{I}(t - a, 0, x), & t - a > 0, \\ \mathbb{W}(a, a - t) \boldsymbol{I}_0(a - t, x), & t - a \leq 0. \end{cases}$$

Therefore, we have, for $t \geq 0, x \in \Omega$,

$$\begin{aligned} z(t, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, x) :&= \boldsymbol{I}(t, 0, x) = \boldsymbol{F}(x, \boldsymbol{S}(t, x)) \int_0^{+\infty} \boldsymbol{B}(a, x) \boldsymbol{I}(t, a, x) da \\ &= \boldsymbol{F}(x, \boldsymbol{S}(t, x)) \int_0^t \Phi(a, x) z(t - a, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, x) da \\ &\quad + \boldsymbol{F}(x, \boldsymbol{S}(t, x)) \boldsymbol{H}(t, 0; \boldsymbol{I}_0, x), \end{aligned}$$

where

$$\Phi(a, x) = \boldsymbol{B}(a, x) \mathbb{W}(a, 0), \quad \boldsymbol{H}(t, 0; \boldsymbol{I}_0, x) = \int_t^{+\infty} \boldsymbol{B}(a, x) \mathbb{W}(a, a - t) \boldsymbol{I}_0(a - t, x) da.$$

Lemma 5.18 *Let Assumptions 5.1, 5.11, and 5.16 be satisfied. If $R_0 < 1$ and $(\boldsymbol{S}_0(\cdot), \boldsymbol{I}_0(\cdot,\cdot)) \in B_\xi$, then $\boldsymbol{0} \leq \boldsymbol{S}(t, x) \leq \boldsymbol{S}_0(x)$ and $\boldsymbol{0} \leq z(t, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, x) \leq \xi \boldsymbol{\psi}_*$ for all $t \geq 0$ and $x \in \overline{\Omega}$.*

Proof From the first equation of system (5.1), we have

$$\begin{cases} \frac{dS(t,x)}{dt} \leq \boldsymbol{\Lambda}(x)\boldsymbol{S}(t,x) + \boldsymbol{M}(x, \boldsymbol{S}(t,x), \boldsymbol{0}), & x \in \Omega, \\ \boldsymbol{S}(0,x) \leq \boldsymbol{S}_0(x), & x \in \overline{\Omega}, \\ \partial_\nu \boldsymbol{S}(t,x) = \boldsymbol{0}, & x \in \partial\Omega. \end{cases} \tag{5.19}$$

By using the comparison principle for reaction–diffusion equations, we can deduce that $\boldsymbol{0} \leq \boldsymbol{S}(t,x) \leq \boldsymbol{S}^0(x)$.

By the definition of B_ξ and expression of z, we have, for $t \geq 0, x \in \Omega$,

$$\begin{aligned} z(0,0;\boldsymbol{S}_0,\boldsymbol{I}_0,x) &= \boldsymbol{F}(x,\boldsymbol{S}_0(x)) \int_0^{+\infty} \boldsymbol{B}(a,x)\boldsymbol{I}_0(a,x)da \\ &\leq \boldsymbol{F}^0(x) \int_0^{+\infty} \boldsymbol{B}(a,x)\xi\mathbb{W}(a,0)\boldsymbol{\psi}_*(x)da \\ &\leq \xi\Psi\boldsymbol{\psi}_*(x) = \xi R_0\boldsymbol{\psi}_*(x) < \xi\boldsymbol{\psi}_*(x). \end{aligned}$$

Assume, by contradiction, that there exists $T_1 \geq 0$ and $x_1 \in \Omega$ such that $z(t,0;\boldsymbol{S}_0,\boldsymbol{I}_0,x_1) < \xi\boldsymbol{\psi}_*(x_1)$ for $t \in [0,T_1]$ and $z(T_1+\varepsilon,0;\boldsymbol{S}_0,\boldsymbol{I}_0,x_1) > \xi\boldsymbol{\psi}_*(x_1)$ for some small ε. Therefore, we have

$$\begin{aligned} z(T_1+\varepsilon,0;\boldsymbol{S}_0,\boldsymbol{I}_0,x_1) &\leq \boldsymbol{F}^0(x_1) \int_0^{T_1+\varepsilon} \boldsymbol{B}(a,x_1)\mathbb{W}(a,0)z(T_1+\varepsilon-a,0;\boldsymbol{S}_0,\boldsymbol{I}_0,x_1)da \\ &\quad + \boldsymbol{F}^0(x_1) \int_{T_1+\varepsilon}^{+\infty} \boldsymbol{B}(a,x_1)\mathbb{W}(a,a-T_1-\varepsilon)\boldsymbol{I}_0(a-T_1-\varepsilon,x_1)da \\ &\leq \boldsymbol{F}^0(x_1)\Big(\int_0^{T_1+\varepsilon} \boldsymbol{B}(a,x_1)\mathbb{W}(a,0) \\ &\quad \xi\boldsymbol{\psi}_*(x_1)da + \int_{T_1+\varepsilon}^{+\infty} \boldsymbol{B}(a,x_1)\xi\mathbb{W}(a,0)\boldsymbol{\psi}_*(x_1)da\Big) \\ &= \boldsymbol{F}^0(x_1) \int_0^{+\infty} \boldsymbol{B}(a)\mathbb{W}(a,0)\xi\boldsymbol{\psi}_*(x_1)da \\ &= \xi\Psi\boldsymbol{\psi}_*(x_1) = R_0\xi\boldsymbol{\psi}_*(x_1) < \xi\boldsymbol{\psi}_*(x_1). \end{aligned}$$

This leads to a contradiction. □

Theorem 5.19 *Let Assumptions 5.1, 5.11, and 5.16 be satisfied. If $R_0 < 1$ and $(\boldsymbol{S}_0(\cdot), \boldsymbol{I}_0(\cdot,\cdot)) \in B_\xi$, then the disease-free steady state $(\boldsymbol{S}^0(x), \boldsymbol{0})$ is global attractive.*

Proof By Lemma 5.18, we have $z(t,0;\boldsymbol{S}_0,\boldsymbol{I}_0,\cdot) < \xi\boldsymbol{\psi}_*(\cdot)$. Therefore, we have, for $t \geq 0$,

$$\begin{aligned} \lim_{t\to+\infty} z(t,0;\boldsymbol{S}_0,\boldsymbol{I}_0,\cdot) &= \lim_{t\to+\infty} \Big(\boldsymbol{F}(\cdot,\boldsymbol{S}(t,\cdot)) \int_0^t \boldsymbol{B}(a,\cdot)\mathbb{W}(a,0)z(t-a,0;\boldsymbol{S}_0,\boldsymbol{I}_0,\cdot)da \\ &\quad + \boldsymbol{F}(\cdot,\boldsymbol{S}(t,\cdot)) \int_t^{+\infty} \boldsymbol{B}(a,\cdot)\mathbb{W}(a,a-t)\boldsymbol{I}_0(a-t,\cdot)da\Big) \\ &\leq \boldsymbol{F}^0(\cdot) \int_0^{+\infty} \boldsymbol{B}(a,\cdot)\mathbb{W}(a,0) \lim_{t\to+\infty} z(t-a,0;\boldsymbol{S}_0,\boldsymbol{I}_0,\cdot)da \\ &\leq \boldsymbol{F}^0(\cdot) \int_0^{+\infty} \boldsymbol{B}(a,\cdot)\mathbb{W}(a,0)\xi\boldsymbol{\psi}(\cdot)da \\ &= R_0\xi\boldsymbol{\psi}(\cdot). \end{aligned}$$

After many iterations, we have $\lim_{t\to+\infty} z(t,0;\boldsymbol{S}_0,\boldsymbol{I}_0,\cdot) \leq R_0^n\xi\boldsymbol{\psi}(\cdot)$. This means $\lim_{t\to+\infty} z(t,0;\boldsymbol{S}_0,\boldsymbol{I}_0,\cdot) = 0$. Therefore, we have $\lim_{t\to+\infty} \|\boldsymbol{I}(t,\cdot,\cdot)\|_X = 0$. Moreover, by Assumption 5.1, we can deduce the global attractiveness of the disease-free steady state. □

Define $\mathcal{U}(t), t \geq 0$ as the solution semiflow of the system (5.1) by

$$\mathcal{U}(t)(\boldsymbol{S}_0(\cdot),\boldsymbol{I}_0(\cdot,\cdot)) = (\boldsymbol{S}(t,\cdot),\boldsymbol{I}(t,\cdot,\cdot)), \quad t \geq 0. \tag{5.20}$$

Theorem 5.20 *Let Assumptions 5.1, 5.11, and 5.16 be satisfied. If $R_0 > 1$, then there exists a positive number ε such that*

$$\lim_{t\to+\infty} \sup \|z(t,0;\boldsymbol{S}_0,\boldsymbol{I}_0,\cdot)\| > \varepsilon, \quad \forall(\boldsymbol{S}_0,\boldsymbol{I}_0) \in M_0, \tag{5.21}$$

where set M_0 is defined in Sect. 4.

Proof Assume, by contradiction, that (5.21) is not hold. Then there exists $T_1 \geq 0$ such that $z(t, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, x) \leq \varepsilon \mathbf{1}$ for all $t > T_1$ and $x \in \Omega$. Therefore, we have $\boldsymbol{I}(t, a, x) \leq \varepsilon \mathbf{1}$ for $t > T_1$ and $x \in \Omega$. By Assumption 5.1 (iii) and (vii), we know that there exist constants $\overline{\delta} > 0$ and $T_2 (T_2 > T_1)$ such that

$$\boldsymbol{S}(t, \cdot) \geq \boldsymbol{S}^0(\cdot) - \overline{\delta}\mathbf{1}, \quad t \geq T_2.$$

For $t \geq T_2$ and $x \in \Omega$, we have

$$z(t, T_2; \boldsymbol{S}_{T_2}, \boldsymbol{I}_{T_2}, x) \geq (\boldsymbol{F}^0(x) - \hat{\delta}\mathbf{1}) \int_{T_2}^{t} \boldsymbol{B}(a, x)\mathbb{W}(a, 0) z(t - a, T_2; \boldsymbol{S}_{T_2}, \boldsymbol{I}_{T_2}, x) da,$$

where $\hat{\delta} := \|\boldsymbol{F}^0(x) - \boldsymbol{F}(x, \boldsymbol{S}^0(x) - \overline{\delta}\mathbf{1})\|$. In order to use the Laplace transform of z, we define $L(\lambda)(x) := \int_0^{+\infty} e^{-\lambda t} z(t, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, x) dt$. Therefore, we have, for $t \geq 0$ and $x \in \Omega$,

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda t} z(t, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, x) dt &\geq \int_0^{+\infty} e^{-\lambda t} (\boldsymbol{F}^0(x) - \hat{\delta}\mathbf{1}) \int_0^t \boldsymbol{B}(a, x)\mathbb{W}(a, 0) z(t - a, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, x) da dt \\ &\geq \int_0^{+\infty} (\boldsymbol{F}^0(x) - \hat{\delta}\mathbf{1}) \boldsymbol{B}(a, x) e^{-\int_0^a \lambda ds} \mathbb{W}(a, 0) L(\lambda)(x) da.\end{aligned} \tag{5.22}$$

Next, we define a operator $\Psi_{\hat{\delta},\lambda}$ on $C(\overline{\Omega}, \mathbb{R}^n)$ as follows

$$\Psi_{\hat{\delta},\lambda}(\boldsymbol{\varphi})(x) := \int_0^{+\infty} (\boldsymbol{F}^0(x) - \hat{\delta}\mathbf{1}) \boldsymbol{B}(a) \mathbb{W}(a, 0) e^{-\int_0^a \lambda ds} \boldsymbol{\varphi}(x) da, \quad \boldsymbol{\varphi} \in C(\overline{\Omega}, \mathbb{R}^n).$$

If we set $\hat{\delta} \to 0, \lambda \to 0$ and use the perturbation theory of linear operator [24], we then obtain that $r(\Psi_{\hat{\delta},\lambda})$ is the eigenvalue of $\Psi_{\hat{\delta},\lambda}$ with a positive eigenvector $\boldsymbol{\psi}_*^{\hat{\delta},\lambda}$ which satisfies $\lim_{\hat{\delta} \to 0, \lambda \to 0} r(\Psi_{\hat{\delta},\lambda}) = r(\Psi)$. Let $\boldsymbol{\psi}_*^{\hat{\delta},\lambda}(\cdot) := (\psi_{*,1}^{\hat{\delta},\lambda}(\cdot), ..., \psi_{*,n}^{\hat{\delta},\lambda}(\cdot))^T$. Therefore, we have

$$\begin{aligned}&diag(\psi_{*,1}^{\hat{\delta},\lambda}(\cdot), ..., \psi_{*,n}^{\hat{\delta},\lambda}(\cdot)) L(\lambda)(\cdot) = diag(\psi_{*,1}^{\hat{\delta},\lambda}(\cdot), ..., \psi_{*,n}^{\hat{\delta},\lambda}(\cdot)) \int_0^{+\infty} e^{-\lambda t} z(t, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, \cdot) dt \\ &\geq diag(\psi_{*,1}^{\hat{\delta},\lambda}(\cdot), ..., \psi_{*,n}^{\hat{\delta},\lambda}(\cdot)) \int_0^{+\infty} (\boldsymbol{F}^0(\cdot) - \hat{\delta}\mathbf{1}) \boldsymbol{B}(a, \cdot) e^{-\int_0^a \lambda ds} \mathbb{W}(a, 0) L(\lambda)(\cdot) da \\ &= \int_0^{+\infty} (\boldsymbol{F}^0(\cdot) - \hat{\delta}\mathbf{1}) \boldsymbol{B}(a, \cdot) e^{-\int_0^a \lambda ds} \mathbb{W}(a, 0) \boldsymbol{\psi}_*^{\hat{\delta},\lambda}(\cdot) \times diag(L(\lambda))(\cdot) da \\ &= r(\Psi_{\hat{\delta},\lambda}) diag(\psi_{*,1}^{\hat{\delta},\lambda}(\cdot), ..., \psi_{*,n}^{\hat{\delta},\lambda}(\cdot)) L(\lambda)(\cdot).\end{aligned}$$

where $diag(L(\lambda))(\cdot) := diag(\int_0^{+\infty} e^{-\lambda t} z_1(t, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, \cdot) dt, ..., \int_0^{+\infty} e^{-\lambda t} z_n(t, 0; \boldsymbol{S}_0, \boldsymbol{I}_0, \cdot) dt)$. Due to $r(\Psi_{\hat{\delta},\lambda}) > 1$ and $diag(\boldsymbol{\psi}_*^{\varepsilon,\lambda}(\cdot)) L(\lambda)(\cdot) > \mathbf{0}$, this causes a contradiction. □

Remark 5.21 Semiflow $\mathcal{U}(t)$ with (5.21) means the weakly uniform persistence of the disease. Since our approach based on operator theory, it allows us to treat model (5.1) with Neumann, Dirichlet or Robin boundary conditions. Therefore, if Assumption 5.1 still holds when the model (5.1) is under Dirichlet or Robin boundary conditions, then the results are still valid. In Remark 6.15, we also consider the SEIR model under the Dirichlet boundary condition.

6 Application to Infection Age-Structured SIR and SEIR Epidemic Models

In this section, we apply the methods stated above to the SIR and SEIR epidemic models. In the SIR model, we compare our results with Chekroun and Kuniya's work [5–7]. The SEIR epidemic model can be seen as an application to high-dimensional and degenerate diffusion situations.

6.1 Infection Age-Structured SIR Epidemic Model

In this subsection, we consider the SIR epidemic model under the Neumann boundary condition. The model is constructed as follows, for $t > 0, a > 0, x \in \Omega$

$$\begin{cases} \frac{\partial S(t,x)}{\partial t} = b\Delta S(t,x) + \gamma - S(t,x)\int_0^{+\infty}\beta(a)I(t,a,x)da - \mu S(t,x), \\ \frac{\partial I(t,a,x)}{\partial t} + \frac{\partial I(t,a,x)}{\partial a} = d\Delta I(t,a,x) - [\mu + \eta(a)]I(t,a,x), \\ I(t,0,x) = S(t,x)\int_0^{+\infty}\beta(a)I(t,a,x)da \\ \frac{\partial R(t,x)}{\partial t} = c\Delta R(t,x) + \int_0^{+\infty}\eta(a)I(t,a,x)da - \mu R(t,x), \end{cases} \tag{6.1}$$

with initial value condition

$$S(0,x) = S_0(x),\ I(0,a,x) = I_0(a,x),\ R(0,x) = R_0(x),$$

under the Neumann boundary condition

$$\frac{\partial S}{\partial \nu} = 0,\ \frac{\partial I}{\partial \nu} = 0,\ \frac{\partial R}{\partial \nu} = 0, \quad x \in \partial\Omega.$$

Following the setting of general infection age-structured epidemic models, we make the following assumption.

Assumption 6.1 For system (6.1), assume that

(i) $\gamma > 0, \mu > 0$ and diffusion coefficients $b, c, d > 0$,
(ii) $\beta(\cdot) \in L_+^\infty(\mathbb{R}_+) \cap L_+^1(\mathbb{R}_+)$ and there exists a maximum age of infection denoted by a_+ such that if $a > a_+$, $\beta(a) = 0$. Moreover, there exist positive numbers a_*, a^* such that $\beta(a) > 0, \forall a \in (a_*, a^*)$.
(iii) $\eta(\cdot) \in L_+^\infty(\mathbb{R}_+)$.

It is obvious that Assumption 6.1 is consistent with Assumptions 4.1 (i, iii, iv, v). Therefore, we only need to prove Assumption 4.1 (ii). The disease-free steady state $(S^0(x), 0)$ satisfies the following equations

$$\begin{cases} 0 = b\Delta S^0(x) + \gamma - \mu S^0(x), & x \in \Omega, \\ \partial_\nu S^0(x) = 0, & x \in \partial\Omega. \end{cases} \tag{6.2}$$

It follows from Lemma 2.1 in [31] that we have the following lemma.

Lemma 6.2 *Let Assumption 6.1 be satisfied. Then system (6.1) admits a unique globally attractive disease-free steady state $(S^0(x), 0)$ and $S^0(x) > 0$ for all $x \in \overline{\Omega}$.*

Following the ideas in Sects. 3 and 4, we define the next generation operator L on space $C(\overline{\Omega})$ by

$$L\varphi(x) = S^0(x)\int_0^{+\infty}\beta(a)e^{\int_0^a \mu(s)ds}T(a)\varphi(x)da, \quad \varphi \in C(\overline{\Omega}),$$

where $T(t)$ is the C_0-semigroup generated by $d\Delta$ with Neumann boundary condition. Thus we can define the basic reproduction number R_0 by

$$R_0 = r(L).$$

Next, we show that Assumption 4.13 (i) and (ii) are held. Firstly we prove Assumption 4.13 (ii). For any $I(t,a,x) \geq \zeta$ with a positive constant ζ, we consider

$$\frac{\partial S(t,x)}{\partial t} = b\Delta S(t,x) + \gamma - (a_+\underline{\beta}\zeta - \mu)S(t,x), \quad x \in \Omega, \tag{6.3}$$

where $\underline{\beta} := \inf_{a \in (0,a_+)} \beta(a)$. Similar to Lemma 6.2, we know the system (6.3) under the Neumann boundary condition admits a globally attractive steady state $S^{0,\zeta}$. By comparison principle for the first equation of (6.2), we know that $S^{0,\zeta}(x) < S^0(x)$. Then we have the solution $S(t,x)$ of the system (6.3) satisfies that there exists a large enough time T_1 such that

$$S(t,x) \geq S^{0,\zeta}(x) \geq S^0(x) - \xi, \quad t \geq T_1 x, \in \Omega,$$

where $\xi := \sup_{x \in \overline{\Omega}} (S^0(x) - S^{0,\zeta}(x))$. Therefore, Assumption 4.13 (ii) holds. In the next lemma, we show Assumption 4.13 (i).

Let

$$\widetilde{I}(t,x) := \int_0^{+\infty} I(t,a,x)da \text{ and } \widetilde{I}_0(x) := \int_0^{+\infty} I_0(a,x)da, \quad t \geq 0, x \in \Omega.$$

Lemma 6.3 *Let Assumption 6.1 be satisfied. Let* $(S_0, I_0) \in C(\overline{\Omega}) \times L^1(\mathbb{R}_+, C(\overline{\Omega}))$, $(S(t,\cdot), I(t,\cdot,\cdot))$ *be the solution of system (6.1) with the initial value* (S_0, I_0). *Then there exists a positive constant M (independent of initial value) such that the following inequality holds*

$$\lim_{t \to +\infty} (\|S(t,\cdot)\| + \|\widetilde{I}(t,\cdot)\|) \leq M. \tag{6.4}$$

Proof Note that

$$\int_0^{+\infty} \frac{dI(t,a,x)}{da} da = I(t,+\infty,x) - I(t,0,x), \quad x \in \Omega.$$

It is easy to find that $\lim_{a \to +\infty} I(t,a,x) = 0$ for $t \geq 0$ and $x \in \Omega$. By the boundary condition of (6.1), we have

$$\int_0^{+\infty} \frac{dI(t,a,x)}{da} da = -I(t,0,x) = -S(t,x) \int_0^{+\infty} \beta(a) I(t,a,x) da, \quad t \geq 0, x \in \Omega.$$

Therefore, by integrating both sides of the second equation of system (6.1) on age a, we have, for $t \geq 0, x \in \Omega$,

$$\begin{cases} \frac{dS(t,x)}{dt} = b\Delta S(t,x) + \gamma - \mu S(t,x) - S(t,x) \int_0^{+\infty} \beta(a) I(t,a,x) da, \\ \frac{d\widetilde{I}(t,x)}{dt} = S(t,x) \int_0^{+\infty} \beta(a) I(t,a,x) da + d\Delta \widetilde{I}(t,x) - \mu \widetilde{I}(t,x) - \int_0^{+\infty} \eta(a) I(t,a,x) da, \\ S_0(\cdot) \in C(\overline{\Omega}), \widetilde{I}_0(\cdot) \in C(\overline{\Omega}). \end{cases} \tag{6.5}$$

From Theorem 6.4, we can deduce that $\lim_{t \to +\infty} S(t,x) \leq S^0(x), x \in \Omega$.

Next, we prove this lemma by proving the following 4 claims, step by step.

Claim 1. There exists a positive constant M_1, independent of initial value conditions, such that

$$\limsup_{t \to +\infty} (\|S(t,\cdot)\|_{L^1(\Omega)} + \|\widetilde{I}(t,\cdot)\|_{L^1(\Omega)}) \leq M_1.$$

To prove this claim, we integrate both sides of the first two equations of (6.5) and add up to obtain

$$\frac{\partial}{\partial t} \int_\Omega (S + \widetilde{I}) dx \leq \int_\Omega \gamma dx - \int_\Omega \mu (S + \widetilde{I}) dx \leq \|\Omega\| \gamma - \int_\Omega \mu (S + \widetilde{I}) dx.$$

It is clear that **Claim 1** holds with $M_1 = \frac{\|\Omega\|\gamma}{\mu}$.

Claim 2: For any $k \geq 0$, there exists a positive constant M_{2^k}, independent of initial conditions, such that

$$\limsup_{t\to+\infty}(\|S(t,\cdot)\|^{2^k}_{L^{2^k}(\Omega)} + \|\widetilde{I}(t,\cdot)\|^{2^k}_{L^{2^k}(\Omega)}) \leq M_{2^k}. \tag{6.6}$$

In the following, this claim will be proved by induction. The case $k = 0$ has been proved in **Claim 1**. Then we assume the **Claim 2** holds for $k-1$. By Multiplying both sides of the second equation of (6.5) by $\widetilde{I}^{2^k-1}$ and integrating over Ω [1], we obtain, for a large enough t,

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega \widetilde{I}^{2^k}dx \leq -\frac{2^k-1}{2^{2k-2}}d\int_\Omega |\nabla \widetilde{I}^{2^{k-1}}|^2 dx + \int_\Omega \|S^0\|\overline{\beta}\widetilde{I}^{2^k}dx - \int_\Omega (\mu+\underline{\eta})\widetilde{I}^{2^k}dx,$$

where $\overline{\beta} := \sup_{a\in(0,+\infty)}\beta(a)$ and $\underline{\eta} := \inf_{a\in(0,+\infty)}\eta(a)$. We now recall the interpolation inequality: for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|\xi\|^2_{L^2(\Omega)} \leq \varepsilon\|\nabla\xi\|^2_{L^2(\Omega)} + C_\varepsilon\|\xi\|^2_{L^1(\Omega)}, \text{ for any } \xi \in W^{1,2}(\Omega).$$

Applying the above interpolation inequality with $\varepsilon = \frac{d(2^k-1)}{2^{2k-1}\|S^0\|\overline{\beta}}$, we can obtain

$$\frac{1}{2^k}\frac{\partial}{\partial t}\int_\Omega \widetilde{I}^{2^k}dx \leq -\int_\Omega \|S^0\|\overline{\beta}\widetilde{I}^{2^k}dx - \int_\Omega(\mu+\underline{\eta})\widetilde{I}^{2^k}dx + C_\varepsilon(\int_\Omega \widetilde{I}^{2^{k-1}}da)^2. \tag{6.7}$$

By assumption, we know that (6.6) holds for $k-1$. It follows that

$$\limsup_{t\to+\infty}\int_\Omega \widetilde{I}^{2^{k-1}}dx \leq M^{2^{k-1}}_{2^{k-1}}.$$

Together with (6.7), we can deduce that (6.6) holds for k. And then **Claim 2** is true.

Claim 3: For any $p \geq 1$, there exists a positive constant M_p, independent of initial conditions, such that

$$\limsup_{t\to+\infty}(\|S(t,\cdot)\|^p_{L^p(\Omega)} + \|\widetilde{I}(t,\cdot)\|^p_{L^p(\Omega)}) \leq M_p.$$

In view of **Claim 2** and the continuous embedding $L^q(\Omega) \subset L^p(\Omega), q \geq p \geq 1$, **Claim 3** is a direct result.

Claim 4: There exists a positive M_∞, independent of initial conditions, such that

$$\limsup_{t\to+\infty}\|\widetilde{I}(t,\cdot)\| \leq M_\infty.$$

Let $T_2(t)$ denote the analytic and compact semigroup generated by operator $A := d\Delta - \mu - \overline{\eta}$ in space $Z := L^p(\Omega)$. Let $Z_\alpha, 0 \leq \alpha \leq 1$, be the fractional power space with graph norm. According to the embedding theorem, we can choose $p > \frac{n}{2}$ and $\alpha \geq \frac{n}{2p}$ such that $Z_\alpha \subset C(\overline{\Omega})$. It is well known that there exists $M_\alpha > 0$ such that $\|A^\alpha T_2(t)\| \leq \frac{M_\alpha}{t^\alpha}$ for all $t > 0$. It follows from **Claim 3** that there exists $t_\infty > 1$ such that

$$\|S(t,\cdot)\|_{L^p(\Omega)} \leq M_0 + 1, \|\widetilde{I}(t,\cdot)\|_{L^p(\Omega)} \leq (M_p+1)^{\frac{1}{p}}, \quad \forall t \geq t_\infty - 1.$$

By the second equation of (6.5), for all $t \geq T_\infty - 1$, we have

$$\widetilde{I}(t) \leq T_2(1)\widetilde{I}(t-1) + \int_{t-1}^t T_2(t-s)\|S^0\|\overline{\beta}\widetilde{I}(s)ds.$$

For all $t \geq t_\infty - 1$, we have

$$\begin{aligned}\|A^\alpha \widetilde{I}(t,\cdot)\|_{L^p(\Omega)} &\leq \|A^\alpha T_2(1)\widetilde{I}(t-1)\|_{L^p(\Omega)} + \int_{t-1}^{t} \|A^\alpha T_2(t-s)\|\|S^0\|\overline{\beta}|\widetilde{I}(s)\|_{L^p(\Omega)}ds \\ &\leq M_\alpha \|\widetilde{I}(t-1,\cdot)\|_{L^p(\Omega)} + \|S^0\|\overline{\beta}(M_p+1)^{\frac{1}{p}} \int_t^{t-1} \frac{M_\alpha}{(t-s)^\alpha} ds \\ &\leq M_\alpha (M_p+1)^{\frac{1}{p}} + \frac{\|S^0\|\overline{\beta} M_\alpha (M_p+1)^{\frac{1}{p}}}{1-\alpha}.\end{aligned}$$

Then **Claim 4** follows from the embedding $Z_\alpha \subset C(\overline{\Omega})$. Together with $\lim\limits_{t\to+\infty} S(t,x) \leq S^0(x), \forall x \in \Omega$, Lemma 6.3 holds. □

The above results are sufficient to prove the global attractiveness of the disease-free steady state by Theorem 4.15.

Theorem 6.4 *Let Assumption 6.1 be satisfied. If $R_0 < 1$, then the disease-free steady state $(S^0, 0)$ is globally attractive.*

Remark 6.5 Compare to Chekroun and Kuniya's works on infection age-structured SIR epidemic model under the Neumann and Dirichlet boundary conditions [5–7], we improve the results on the global attractiveness of the disease-free steady state. In Chekroun's work, they only prove the global attractiveness of the disease-free steady state with the initial value belonging to a subset of phase space (Theorem 5.1 in [5], Theorem 4.4 in [6], Theorem 6.2 in [7]). By using our methods, we can overcome this problem. We can prove the global attractiveness of the disease-free steady state with no limitation on the initial value condition. However, due to the limitation of our method, we need to assume that $\beta(a) = 0, \forall a \geq a_+$.

By Theorem 4.20, we have the following theorem.

Theorem 6.6 *Let Assumption 6.1 be satisfied. If $R_0 > 1$, semiflow $\mathcal{U}(t)$ is uniformly persistent and admits a fixed point (i.e. endemic steady state).*

Remark 6.7 These results are consistent with Chekroun and Kuniya's results (Theorems 6.1 and 7.2 in [5], Proposition 5.3 and Theorem 6.1 in [7]).

6.2 Infection Age-Structured SEIR Epidemic Model

In this subsection, we consider an infection age-structured SEIR epidemic model under the Neumann boundary condition. The model is constructed as follows, for $t > 0, a > 0, x \in \Omega$,

$$\begin{cases} \frac{\partial S(t,x)}{\partial t} = b\Delta S(t,x) + \gamma - S(t,x)\int_0^{+\infty} (\beta_2(a)I(t,a,x) + \beta_1(a)E(t,a,x))da - \mu S(t,x), \\ \frac{\partial E(t,a,x)}{\partial t} + \frac{\partial E(t,a,x)}{\partial a} = -\mu E(t,a,x) - \theta_1 E(t,a,x), \\ \frac{\partial I(t,a,x)}{\partial t} + \frac{\partial I(t,a,x)}{\partial a} = d\Delta I(t,a,x) - [\mu + \eta(a)]I(t,a,x) + \theta_1 E(t,a,x), \\ E(t,0,x) = S(t,x)\int_0^{+\infty} \beta_2(a)I(t,a,x)da + S(t,x)\int_0^{+\infty} \beta_1(a)E(t,a,x)da \\ \frac{\partial R(t,x)}{\partial t} = c\Delta R(t,x) + \int_0^{+\infty} \eta(a)I(t,a,x)da - \mu R(t,x), \end{cases} \tag{6.8}$$

with initial value condition

$$S(0,x) = S_0(x), E(0,a,x) = E_0(a,x), I(0,a,x) = I_0(a,x), R(0,x) = R_0(x),$$

under the Neumann boundary condition

$$\frac{\partial S}{\partial \nu} = 0, \frac{\partial E}{\partial \nu} = 0, \frac{\partial I}{\partial \nu} = 0, \frac{\partial R}{\partial \nu} = 0, \quad x \in \partial\Omega.$$

Following the setting of general infection age-structured epidemic models, we make the following assumption.

Assumption 6.8 For system (6.8), assume that

(i) $\gamma > 0, \mu > 0, \theta_1 > 0$ and diffusion coefficients $b, c, d > 0$,
(ii) For each $i = 1, 2$, $\beta_i(\cdot) \in L^\infty_+(\mathbb{R}_+) \cap L^1_+(\mathbb{R}_+)$ and there exists a maximum age of infection denoted by a_+ such that if $a > a_+$, $\beta_i(a) = 0$. Moreover, there exists positive numbers a_*, a^* such that $\beta_i(a) > 0, \forall a \in (a_*, a^*)$.
(iii) $\eta(\cdot) \in L^\infty_+(\mathbb{R}_+)$.

In the SEIR model, we assume that compartment E also has effects on the spread of disease and is with zero diffusion coefficient. Therefore, this SEIR model can be seen as an application in an epidemic model with degenerate diffusion. By Remark 5.10, we consider phase space $Y := L^2(\Omega, \mathbb{R}^2)$ and $X := L^1((0, +\infty), Y)$, instead of $C(\overline{\Omega}, \mathbb{R}^2)$ and $L^1((0, +\infty), C(\overline{\Omega}, \mathbb{R}^2))$. In order to use the method stated before, we introduce some notations as follows

$$\boldsymbol{I}(t,a,x) := \begin{pmatrix} I(t,a,x) \\ E(t,a,x) \end{pmatrix}, \boldsymbol{\eta}(a) := \begin{pmatrix} \eta(a) & 0 \\ 0 & 0 \end{pmatrix}, \boldsymbol{\mu} := \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, \mathbb{L}(x)\boldsymbol{u} := \begin{pmatrix} d\Delta u_1 \\ 0 \end{pmatrix},$$
$$\boldsymbol{S}(t,x) := \begin{pmatrix} 0 & 0 \\ 0 & S(t,x) \end{pmatrix}, \boldsymbol{\theta} := \begin{pmatrix} 0 & \theta_1 \\ 0 & -\theta_1 \end{pmatrix}, \boldsymbol{B}(a) := \begin{pmatrix} 0 & 0 \\ \beta_2(a) & \beta_1(a) \end{pmatrix}.$$

Therefore, we can rewrite system (6.8) into the following form, for $t > 0$, $a > 0$ and $x \in \Omega$,

$$\begin{cases} \frac{\partial S(t,x)}{\partial t} = b\Delta S(t,x) + \gamma - \mu S(t,x) - S(t,x)\int_0^{+\infty}(\beta_2(a)I(t,a,x) + \beta_1(a)E(t,a,x))da, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)\boldsymbol{I}(t,a,x) = \mathbb{L}(x)\boldsymbol{I}(t,a,x) - (\boldsymbol{\mu} + \boldsymbol{\eta}(a))\boldsymbol{I}(t,a,x) + \boldsymbol{\theta}\boldsymbol{I}(t,a,x), \\ \boldsymbol{I}(t,0,x) = \boldsymbol{S}(t,x) \times \int_0^{+\infty}\boldsymbol{B}(a,x)\boldsymbol{I}(t,a,x)da \end{cases} \tag{6.9}$$

As the same with Lemma 6.2, we directly have the following lemma.

Lemma 6.9 *Let Assumption 6.8 be satisfied. Then system (6.8) admits the unique disease-free steady state $(S^0(x), 0)$ and $S^0(x) > 0$ for all $x \in \overline{\Omega}$.*

By using the method in Sect. 5, Ψ and $\widehat{\Psi}$ are defined on space $L^2(\Omega, \mathbb{R}^2)$ as follows

$$\Psi\begin{pmatrix} \varphi_{10}(x) \\ \varphi_{20}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi_2(\varphi_1, \varphi_2)(x) \end{pmatrix}, \widehat{\Psi}\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \widehat{\Psi}_2(\varphi_2)(x) \end{pmatrix}, \quad x \in \Omega,$$

where

$$\begin{aligned}\Psi_2(\varphi_1, \varphi_2)(x) = \int_0^{+\infty} & S^0(x)\beta_2(a)(\mathcal{T}_1(a)e^{-\int_0^a \eta(s)+\mu ds}\varphi_1(0,x) \\ & + \int_0^a \mathcal{T}_1(a-s)e^{-\int_s^a \eta(l)+\mu dl}\theta_1\varphi_2(s,x)ds)da \\ & + \int_0^{+\infty} S^0(x)\beta_2(a)e^{-\int_0^a \mu+\theta_1 ds}\varphi_2(0,x)da,\end{aligned}$$

and

$$\widehat{\Psi}_2(\varphi_{20})(x) = \int_0^{+\infty} S^0(x)\beta_2(a)e^{-\int_0^a \mu+\theta_1 ds}\varphi_{20}(x)da, \quad x \in \Omega.$$

In the above, $\mathcal{T}_1(t)$ is the solution map of the following reaction–diffusion equation under the Neumann boundary condition

$$\frac{d\varphi_1(t,x)}{dt} = d\Delta\varphi_1(t,x), \quad t > 0, x \in \Omega.$$

and $\varphi(a,x) := (\varphi_1(a,x), \varphi_2(a,x))$ is the solution of the following equations with initial value $(\varphi_1(0,x), \varphi_2(0,x)) = (\varphi_{10}(x), \varphi_{20}(x))$,

$$\frac{d\varphi(a,x)}{da} = \mathbb{L}(x)\varphi(a,x) - V(a)\varphi(a,x),$$

where $V(a) := \boldsymbol{\eta}(a) + \boldsymbol{\theta} + \boldsymbol{\mu}$. Then we have, for $a > 0$ and $x \in \Omega$,

$$\begin{pmatrix} \varphi_1(a,x) \\ \varphi_2(a,x) \end{pmatrix} := \begin{pmatrix} \mathcal{T}_1(a)e^{-\int_0^a \mu+\eta(s)ds}\varphi_{10}(x) + \int_0^a \mathcal{T}_1(a-s)e^{-\int_s^a \mu+\eta(l)dl}\theta_1\varphi_2(s,x)ds \\ e^{-(\mu+\theta_1)a}\varphi_{20}(x) \end{pmatrix}.$$

Following the ideas of Sect. 5, we define the basic reproduction number R_0 by

$$R_0 := r(\Psi).$$

It is easy to see that SEIR model is consistent with Assumptions 5.1 and 5.11. It remains to prove Assumption 5.16. By the expression of Ψ and $\widehat{\Psi}$, we have $\Psi \geq \widehat{\Psi}$. Thus, $r(\Psi) \geq r(\widehat{\Psi})$ by Theorem 4.2 of [40]. Therefore, we need to prove the strict monotonicity. Before proof, we introduce some definitions and theorems.

Let X is a Banach space with a cone X_+, a positive operator $A \in \mathcal{L}(X)$ is called $y-$bounded if there exists numbers $\alpha_1, \alpha_2 : X \to \mathbb{R}_+$ such that

$$\alpha_1(x)y \leq Ax \leq \alpha_2(x)y, \quad \forall x \in X_+,$$

where $y \in X_+ \setminus \{0\}$. A linear operator B is called monotonically compact if B is positive, and the relation

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots \geq z$$

implies the convergence of the sequence Bx_n. In particular, if the cone X_+ is regular, every positive linear operator is monotonically compact.

Theorem 6.10 *[16] Let X is a real Banach space with a cone X_+ and positive operators $A, B \in \mathcal{L}(X)$ which satisfy $A \leq B$ and $A \neq B$. If (A.1)-(A.4) are satisfied,*

(A.1) X_+ is normal, minhedral and reproducing cone;
(A.2) B is monotonically compact and $u-$bounded;
(A.3) A is irreducible or $u-$bounded;
(A.4) $(B-A)^2 \neq 0$; then $r(A) < r(B)$.

Remark 6.11 Theorem 6.10 is the results from Theorem 3.5 and Lemmas 3.6, 3.7, 3.8 of [16]. More methods for the strict monotonicity of spectral radius of positive operators can be found in [17, 40].

Define $\underline{\Psi}_2$ on $L^2(\Omega)$ by

$$\begin{aligned} \underline{\Psi}_2(\varphi_{20})(x) &= \int_0^{+\infty} S^0(x)\beta_2(a)\int_0^a \mathcal{T}_1(a-s)e^{-\int_s^a \eta(l)+\mu dl}\theta_1\varphi_2(s,x)dsda \\ &+ \int_0^{+\infty} S^0(x)\beta_2(a)e^{-\int_0^a \mu+\theta_1 ds}\varphi_2(0,x)da \\ &= \int_0^{+\infty} S^0(x)\beta_2(a)\int_0^a \mathcal{T}_1(a-s)e^{-\int_s^a \eta(l)+\mu dl}\theta_1 e^{-(\mu+\theta_1)s}\varphi_2(0,x)dsda \\ &+ \int_0^{+\infty} S^0(x)\beta_2(a)e^{-\int_0^a \mu+\theta_1 ds}\varphi_2(0,x)da, \end{aligned}$$

where $\varphi_{20}(x) = \varphi_2(0,x)$.

Theorem 6.12 *Let Assumption 6.8 be satisfied. Then $r(\Psi) > r(\widehat{\Psi})$.*

Proof Note that $r(\underline{\Psi}_2) > r(\widehat{\Psi}_2)$ means $r(\Psi) > r(\widehat{\Psi})$. By Theorem 6.10, we only need to prove the following four claims.
Claim 1: X_+ is normal, minhedral and reproducing cone;
Claim 2: $\underline{\Psi}_2$ is monotonically compact and $u-$bounded;
Claim 3: $\widehat{\Psi}_2$ is irreducible or $u-$bounded;
Claim 4: $(\underline{\Psi}_2 - \widehat{\Psi}_2)^2 \neq 0$.

We begin to prove **Claim 2**. By the definition of $\underline{\Psi}_2$ and Assumption 6.8, we have, for $x \in \Omega$,

$$\begin{aligned}\underline{\Psi}_2(\varphi_{20})(x) &\le \int_0^{+\infty} S^0(x)\beta_2(a)\varphi_2(0,x)da + \int_0^{+\infty} S^0(x)\beta_2(a)\int_0^a \theta_1\varphi_2(0,x)dsda \\ &\le a_+\overline{\beta}\|S^0\|(1+\theta_1 a_+)\varphi_2(0,x)\end{aligned}$$

where $\overline{\beta} = \sup_{a\in[0,a_+]}\beta_2(a)$. Similar to above, we can also obtain, for $x \in \Omega$,

$$\begin{aligned}\underline{\Psi}_2(\varphi_{20})(x) &\ge \int_0^{+\infty} S^0(x)\beta_2(a)e^{-\int_0^a \mu+\theta_1 ds}\varphi_2(0,x)da \\ &\ge \underline{S}^0\int_{a_*}^{a^*}\beta(a)dae^{(-\mu-\theta_1)a_+}\varphi_2(0,x),\end{aligned}$$

where $\underline{S}^0 = \inf_{x\in\overline{\Omega}} S^0(x)$. Define $h_1 := \underline{S}^0\int_{a_*}^{a^*}\beta(a)dae^{-(\mu+\theta_1)a_+}$ and $h_2 := a_+\overline{\beta}\|S^0\|(1+\theta_1 a_+)$. For any $\varphi_{20}^1, \varphi_{20}^2 \in L_+^2(\Omega)$, there exists two positive constants C_1, C_2 such that $C_1\varphi_{20}^1 \le \varphi_{20}^2 \le C_2\varphi_{20}^1$. Moreover, we have

$$C_1h_1\varphi_{20}^1 \le \underline{\Psi}_2\varphi_{20}^2 \le C_2h_2\varphi_{20}^1.$$

Therefore, $\underline{\Psi}_2$ is φ_{20}^1-bounded. Moreover, $\underline{\Psi}_2$ is defined in $L^p(\Omega)$, it follows that $\underline{\Psi}_2$ is monotonically compact.

It is clear that **Claim 1** and **4** hold. It remains to prove **Claim 3**. It is easy to find that $\widehat{\Psi}_2$ is irreducible and the property of $u-$bounded can be proved by a similar way in **Claim 2**. Therefore, we have $r(\underline{\Psi}_2) > r(\widehat{\Psi}_2)$ and then $r(\Psi) > r(\widehat{\Psi})$. □

Similar to the proof of Theorems 5.19 and 6.4, we obtain the following theorem.

Theorem 6.13 *Let Assumption 6.8 be satisfied. If $R_0 < 1$ and $(S_0, \boldsymbol{I}_0) \in B_\xi$, then the disease-free steady state $(S^0, \mathbf{0})$ is globally attractive, where B_ξ is defined in (5.18).*

Let $\mathcal{U}(t)$ be the solution semiflow of the system (6.8), that is,

$$\mathcal{U}(t)(S_0(\cdot), \boldsymbol{I}_0(\cdot,\cdot)) = (S(t,\cdot), \boldsymbol{I}(t,\cdot,\cdot)), \quad t \ge 0. \tag{6.10}$$

By Theorem 5.20, we have the following theorem.

Theorem 6.14 *Let Assumption 6.8 be satisfied. If $R_0 > 1$, semiflow $\mathcal{U}(t)$ is weakly uniform persistent.*

Remark 6.15 If the model (6.8) is under the Dirichlet boundary condition, we can also prove Theorems 6.13 and 6.14 by a similar method.

7 Discussion

In this paper, we study the dynamical threshold for infection age-structured epidemic model with spatial diffusion and degenerate diffusion. We prove that R_0 can be defined as the spectral radius of operator $-\mathscr{F}\mathscr{A}^{-1}$ and the spectral bound of $\mathscr{A}+\mathscr{F}$ has the same sign as $R_0 - 1$, where $\mathscr{F}, \mathscr{A}$ are non-densely operators. This result extends the basic reproduction numbers for many kinds of ODE and reaction–diffusion epidemic models.

Due to infection age-structured effects, it becomes more difficult to consider the global stability of steady state than epidemic models in the form of ordinary differential equations. When considering the infection age-structured epidemic model with spatial diffusion, almost all work in the literature only concerned the global attractiveness of the disease-free steady state in a subset of phase space instead of the whole phase space or constructing Lyapunov functional. In Sect. 4, we study a class of high-dimensional infection age-structured epidemic models with non-degenerate diffusion and spatial heterogeneity. We overcome this problem by using the comparison principle for the age-structured equation and renewal theorem. We give a general method to prove the global attractiveness of the disease-free steady state without restrictions on the initial values. Of course, the fewer restrictions the better. By the theory of compact attractors, we also give another approach to prove the uniform persistence and the existence of the endemic steady state. It is worth mentioning that the methods used in this paper are suitable to the Neumann, Dirichlet, and Robin boundary conditions. However, due to the limitation caused by our method, we assume that there exists a maximum infection age. In addition, this assumption is reasonable in age-structured models.

In Sect. 5, we consider a class of high-dimensional infection age-structured epidemic models with degenerate diffusion and spatial heterogeneity. Degenerate diffusion leads to compactness loss of solution semigroup. Thus, we cannot follow the methods stated in Sect. 4 to prove the extinction or uniform persistence of disease. Under some assumptions, we can still prove that R_0 is the principal eigenvalue of the next generation operator by a generalized Krein-Rutman Theorem. Moreover, by a Laplace transform, we prove that R_0 also plays a role in the threshold for the extinction and weakly uniform persistence of the disease.

In Sect. 6, we apply our method to infection age-structured SIR and SEIR epidemic models. In the case of the SIR model, we improve some results on the global attractiveness of the disease-free steady state and give another proof for the uniform persistence of semiflow and the existence of the endemic steady state. In addition, we compare our results on the SIR model with those in the literature. In the case of the SEIR model, we consider the SEIR model with degenerate diffusion. We use the method in Sect. 5 and obtain the threshold results on its global dynamics.

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