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On *m*-Minimal Partially Hyperbolic Diffeomorphisms

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Abstract
We discuss the denseness of the strong stable and unstable manifolds of partially hyperbolic diffeomorphisms. To this end, we introduce the concept of m-minimality, wich means that m-almost every point in M has its strong stable and unstable manifolds dense in M. We show that this property has both topological and ergodic consequences. Also, we prove the abundance of m-minimal partially hyperbolic diffeomorphisms in the volume preserving and symplectic scenario.

Keywords Partially hyperbolic diffeomorphisms · m-minimality · Minimality · Stable and unstable foliation · Symplectic · Ergodic theory

1 Introduction

Its well known that, as in the Anosov case, the stable and unstable bundles of a partially hyperbolic diffeomorphism integrate to foliations of the ambient manifold. These foliations are called *strong stable foliation* and *strong unstable foliation* and are denoted by $\mathcal{F}^s$ and $\mathcal{F}^u$, respectively. The structure of such foliations translates to some topological and ergodic properties of the dynamic, as well as the level of recurrence in the dynamics imposes conditions on these foliations.

[1] Recall that a diffeomorphism $f : M \to M$ is *topologically transitive* if there is a point whose forward orbit by f is dense on M. Also, f is *topologically mixing* if given open sets U and V of M, there exists a positive integer n such that $f^j(U)$ intersects V for any $j \geq n$.

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For instance, when one of these foliation is *minimal*, which means that all its leaves are dense in the manifold, it implies that the system is topologically mixing. On the other hand, if an Anosov diffeomorphism is transitive[1] both of its invariant foliation are minimal.

For partially hyperbolic diffeomorphisms, [7] shows that there is an open and dense subset of robustly transitive[2] partially hyperbolic diffeomorphisms in dimension 3 that either $\mathcal{F}^s$ or $\mathcal{F}^u$ is minimal. This result was extended to higher dimensions in [15], when the central bundle is one dimensional (See also [19], where a similar result is obtained for attractors).

However, for partially hyperbolic diffeomorphisms with higher center dimension little is known about the minimality of the strong invariant foliations. For a C^1-generic transitive partially hyperbolic diffemorphism $f : M \to M$, there is a residual subset of M for which the strong stable and unstable leaves are dense in M, see [22]. Moreover, allied with some source of hyperbolicity, the minimality of the invariant foliation can produce robust transitivity, as Pujals and Sambarino showed in [21].

One could also try to understand the relations between the recurrence of the strong foliations and the recurrence of the dynamics itself. On one hand, we can apply Hayashi's Connecting Lemma [13] to show that, C^1-generically, if a partially hyperbolic diffeomorphisms is transitive then each of the strong invariant foliations has a dense leaf. On the other hand, it was shown by Hammerlindl-Potrie [16], that, on three-dimensional nilmanifolds, the strong foliations of any partially hyperbolic diffeomorphism has a dense leaf. However, Yi Shi [27] constructed some such examples with exactly one attractor and one repeller thus not transitive.

In this work, we propose a weaker form of minimality requiring the denseness of the leaves only in a full Lebesgue set of points. We obtain some topological and ergodic consequences, and shows that this condition is abundant in the volume preserving and symplect scenario.

It is worth mentioning that another geometric property related to the invariant foliations is the *accessibility property*,[3] which has been used as a key feature to prove ergodicity (in fact, only *essencial acessibility* is enough). So it is a natural question if this new notion of minimality has any relation with accessibility. We remark that the strong invariant manifolds of a linear Anosov in dimension 3 is minimal (actually the center manifold is also minimal). However, a linear Anosov is not essentially accessible, since the strong stable and unstable directions are jointly integrable, see [23] for more details on other automorphisms of the torus. So we pose the following question.

Question. Does acessibility implies m-minimality?

In the following, we will give the precise definitions and statements of our results.

Let (M, g) be a compact, connected, boundaryless, Riemannian manifold. The Lebesgue measure on M is denoted by m. Any submanifold will be endowed with a metric, which is the restriction of g.

Given a diffeomorphism f on M, we say that a Df-invariant splitting $TM = E \oplus F$ is *dominated* if there exists a positive integer n such that, for every $x \in M$,

$$\|Df_x^n(u)\| \leq \frac{1}{2}\|Df_x^n(v)\|, \text{ for every unitary vectors } u \in E \text{ and } v \in F.$$

[2] Recall that a diffemorphism is *robustly transitive* if every diffeomorphism sufficiently close to it, in the C^1-topology is transitive.

[3] A partially hyperbolic diffeomorphism is accessible if any two points can be joined by a concatenation of curves belonging to unstable or stable manifolds. Actually, this property splits the manifold in accessibility classes. We say that a system is essentially accessible if any union of accessibility classes has zero or full measure.

A diffeomorphism f on M is called *partially hyperbolic* if there exists a continuous Df-invariant dominated splitting $TM = E^s \oplus E^c \oplus E^u$, with non trivial extremal sub-bundles E^s and E^u, and there exists $n \in \mathbb{N}$ such that E^s and E^u are uniformly contracted by Df^n and Df^{-n}, respectively.

If the center bundle E^c is trivial, then f is called *Anosov*. For convenience, given a partially hyperbolic diffeomorphism f, we consider its partially hyperbolic splitting $TM = E^s \oplus E^c \oplus E^u$, such that the extremal bundles contains all the Df-invariant sub-bundles of TM which are contracted or expanded for some iterate of Df. In particular, for us, a partially hyperbolic diffeomorphism with non-trivial center bundle is not Anosov.

As we mentioned before, the strong bundles E^s and E^u integrate to the strong stable and strong unstable foliations $\mathcal{F}^s$ and $\mathcal{F}^u$, which are tangent to E^s and E^u, respectively.

We say that a partially hyperbolic diffeomorphism f is *s-minimal* (*resp. u-minimal*) if its strong stable (resp. strong unstable) foliation $\mathcal{F}^s$ (resp. $\mathcal{F}^u$) is *minimal.*

Now, we introduce the weaker notion of minimality that plays the central role in this paper.

We will denote by $\mathcal{X}^s(f)$ the set of points $x \in M$ such that $\mathcal{F}^s(x)$ is dense. Analogously, we define $\mathcal{X}^u(f)$ for the foliation $\mathcal{F}^u$.

Definition 1.1 Let f be a partially hyperbolic diffeomorphism. We say that f is *ms-minimal* if $m(\mathcal{X}^s(f)) = 1$. We say that f is mu-minimal if $m(\mathcal{X}^u(f)) = 1$. Finally, f is m-minimal if it is both ms and mu-minimal.

In Sect. 2 we will prove many basic properties satisfied by ms and mu-minimal diffeomorphisms. In particular, we prove that m-minimality is a G_δ property in the volume preserving scenario.

In Sect. 3, we prove two main consequences of m-minimallity given in the following Theorem, that gives information about the complexity of the dynamics at the topologic and ergodic level. Recall that a diffeomorphism is *weakly ergodic* if the orbit of m-almost every point x is dense on M.

Theorem 1.2 *Let f be a C^1-partially hyperbolic diffeomorphism preserving the Lebesgue measure m. If f is ms-minimal or mu-minimal then f is topologically mixing. Moreover, if f is also $C^{1+\alpha}$ then f is weakly ergodic.*

Nevertheless, there are even ergodic maps that are not ms or mu-minimal. Consider, for instance, the volume preserving map f on T^3 obtained by a linear Anosov diffeomorphism on the torus $\mathbb{T}^2$ times an irrational rotation of the circle, for which the Lebesgue measure on T^3 is known to be ergodic. Since the irrational rotation is not topologically mixing, neither is the map f itself. Now, the first part of Theorem 1.2 assures that f is not ms or mu-minimal as well.

In Sect. 3, we also introduce the SH property of Pujals and Sambarino, [21], and we use it to obtain a kind of robustness of the m-minimality, see Proposition 3.9.

Our next task is to show the abundance of ms and/or mu-minimal diffeomorphisms in two natural scenarios: volume preserving diffeomorphisms and symplectic diffeomorphisms.

We start with the volume preserving scenario. We denote by $\mathrm{Diff}^1_m(M)$ the set formed by diffeomorphisms f on M that preserve the volume form m, i.e., $f^*m = m$.

We say that a diffemorphism f exhibit a homoclinic tangency if there exists a non empty and non transversal intersection between the stable and unstable manifold of a hyperbolic periodic point of f. Hence, denoting by $\mathcal{HT}$ the subset of C^1 diffeomorphisms exhibiting a homoclincic tangency, we know there exists an open and dense subset in $\mathrm{Diff}^1_m(M)\backslash cl(\mathcal{HT})$ formed by partially hyperbolic diffeomorphisms. This was proved by Crovisier, Sambarino and Yang in [11]. See also [3]. Moreover, such diffemorphisms were such that the center bundle admits a sub splitting in one dimensional sub bundles.

Our next result says that in the volume preserving scenario, far from homoclinic tangency, the presence of m-minimality is abundant. The hypothesis of being far from tangencies is a technical requirement that allow us to make use of important features. We do not know if this hypothesis can actually be dropped.

Theorem 1.3 *There exists an open and dense subset $\mathcal{G} \subset \mathrm{Diff}^1_m(M) \setminus cl(\mathcal{HT})$, such that any C^2-diffeomorphism $f \in \mathcal{G}$ is a partially hyperbolic diffeomorphism which is m-minimal.*

Now, since m-minimality is a G_δ property, Proposition 2.5, from the previous theorem we also have the following corollary.

Corollary 1.4 *There exists a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_m(M) \backslash cl(\mathcal{HT})$, such that any $f \in \mathcal{R}$ is a partially hyperbolic diffeomorphism which is m-minimal.*

We consider now the symplectic scenario.

Let (M, ω) be a symplectic manifold, been ω the two symplectic form on M. In this case, m will denote the volume form in M induced by ω. The set of symplectic diffemorphisms will be denoted by $\mathrm{Diff}^1_\omega(M)$. Recall that f is a symplectic diffeomorphism if $f^*\omega = \omega$.

In this setting, the symplectic structure allow us to prove that generically any partially hyperbolic diffeomorphism is m-minimal. It is worth to point out that in the symplectic setting all non Anosov diffeomorphisms are approximated by diffeomorphisms exhibiting homoclinic tangency, see Newhouse [18].

Theorem 1.5 *Let (M^{2d}, ω) be a symplectic manifold, and consider $m = \omega^d$ a volume form in M. There exists a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M)$, such that if $f \in \mathcal{R}$ is a partially hyperbolic diffeomorphism then f is m-minimal.*

The paper is organized in the following way: In Sect. 2, we give basic properties of m-minimality, and in Sect. 3, we prove some dynamical consequences of m-minimality. Finally, in Sect. 4 we prove the abundance of m-minimal partially hyperbolic diffeomorphisms, i.e., we prove Theorems 1.3 and 1.5.

2 Basic Properties of *m*-Minimality

In this section we list some basic properties related to the m-minimality.

We first remark that if $\nu << m$ and f is ms-minimal then f is νs-minimal.

We now examinate the invariance by iterations.

Proposition 2.1 *Let $n > 0$, f is ms-minimal if, and only if, f^n is ms-minimal. Let $n < 0$ then f is ms-minimal if, and only if, f^n is mu-minimal.*

Proof We only need to check that $\mathcal{F}^s(x, f) = \mathcal{F}^s(x, f^n)$ if $n > 0$ and $\mathcal{F}^s(x, f) = \mathcal{F}^u(x, f^n)$ if $n < 0$. □

Now, we study the behaviour of the property under products.

Proposition 2.2 *f is ms-minimal if, and only if, $f \times f$ is $(m \times m)s$-minimal.*

Proof We begin noticing that if $(a, b) \in \mathcal{F}^s((x, y), f \times f)$, then

$$d((f^n(x), f^n(y)), (f^n(a), f^n(b))) \to 0 \text{ as } n \to \infty.$$

So, this implies that $a \in \mathcal{F}^s(x)$ and $b \in \mathcal{F}^s(y)$.

Now, we show that $\mathcal{X}^s(f \times f) \subset \mathcal{X}^s(f) \times \mathcal{X}^s(f)$. Indeed, let $(x, y) \in \mathcal{X}^s(f \times f)$. Since $\mathcal{F}^s((x, y), f \times f) \cap (U \times V) \neq \emptyset$, for any open sets U and V, we have that $x \in \mathcal{X}^s(f)$ and $y \in \mathcal{X}^s(f)$.

Reciprocally, let $(x, y) \in \mathcal{X}^s(f) \times \mathcal{X}^s(f)$. Any open set of $M \times M$ contains an open set like $U \times V$. Hence, there exists $a \in U \cap \mathcal{F}^s(x, f)$ and $b \in V \cap \mathcal{F}^s(y, f)$.

For any $\varepsilon > 0$, there exists N such that $n \geq N$ implies $d(f^n(x), f^n(a)) < \varepsilon$ and $d(f^n(y), f^n(b)) < \varepsilon$. So

$$d((f \times f)^n((x, y)), (f \times f)^n((a, b))) < \varepsilon \text{ if } n \geq N.$$

Hence, $(a, b) \in \mathcal{F}^s((x, y), f \times f)$.

Thus $\mathcal{X}^s(f \times f) = \mathcal{X}^s(f) \times \mathcal{X}^s(f)$, and the result follows. □

Next result shows that it is possible to preserve m-minimality under some conjugacies.

Proposition 2.3 *Let f and g be two partially hyperbolic diffeomorphisms. Let h be a m-regular homeomorphism (i.e. preserves sets of null m-measure), such that $h(\mathcal{F}^s(x)) = \mathcal{F}^s(h(x))$. Then f is ms-minimal if, and only if, g is ms-minimal.*

Proof Just notice that $h(\mathcal{X}^s(f)) = \mathcal{X}^s(g)$. □

Remark 2.4 Regarding the hypothesis of the previous statement, we would like to point out that there are partially hyperbolic system which are conjugated but the conjugacy does not send strong leaves over strong leaves, see for instance [26]. Even so, it is a question if both foliations can be ms-minimal. Another related question is the following: if f and g are two partially hyperbolic diffeomorphisms central conjugated (see [14]) such that f is ms-minimal, is it true that g is also ms-minimal?

Now, we will show that m-minimality is a G_δ property, for any borelian probability m. We denote by $\mathcal{PH}^1_m(M)$ the set of partially hyperbolic diffeomorphisms which preserve m, i.e. $f^*m = m$, endowed with the C^1 topology.

For any partially hyperbolic diffeomorphism f, we denote by $\mathcal{X}^s_\delta(f)$ (resp. $\mathcal{X}^u_\delta(f)$) the subset of points x in M such that $\mathcal{F}^s(x)$ (resp. $\mathcal{F}^u(x)$) is δ-dense in M. Recall that a subset $A \subset M$ is *δ-dense* if A intersects any open ball with diameter larger than δ. Also, we denote by $\mathcal{F}^{s(u)}_K(x)$ a compact disc with radius K and centre x inside the leaf $\mathcal{F}^{s(u)}(x)$. The sets $\mathcal{F}^{s(u)}_K(x)$ varies continuously with respect to the diffeomorphism f and with respect to x. In particular, we can conclude that $\mathcal{X}^s_\delta(f)$ (resp. $\mathcal{X}^u_\delta(f)$) is an open subset of M.

Proposition 2.5 *The set of m-minimal diffeomorphisms is a countable intersection of open sets of $\mathcal{PH}^1_m(M)$.*

Proof For any $\varepsilon, \delta > 0$ we define:

$$\mathcal{B}^s_m(\varepsilon, \delta) = \{f \in \mathcal{PH}^1_m(M)/ \quad m(\mathcal{X}^s_\delta(f)) > 1 - \varepsilon\} \text{ and} \tag{1}$$
$$\mathcal{B}^u_m(\varepsilon, \delta) = \{f \in \mathcal{PH}^1_m(M)/ \quad m(\mathcal{X}^u_\delta(f)) > 1 - \varepsilon\}. \tag{2}$$

Observe that if a partially hyperbolic diffemorphism f is ms-minimal (resp. mu-minimal), then f belongs to $\mathcal{B}^s(\varepsilon, \delta)$ (resp. $\mathcal{B}^u(\varepsilon, \delta)$) for every positive ε and δ. In particular, the rest of the proof is a directly consequence of the next lemma, which implies that m-minimality is a G_δ property. □

Lemma 2.6 *The subsets $\mathcal{B}^s(\varepsilon, \delta)$ and $\mathcal{B}^u(\varepsilon, \delta)$ are open subsets of $\mathcal{PH}^1_m(M)$.*

Proof of Lemma 2.6 Since f belongs to $\mathcal{B}^s(\varepsilon, \delta)$ if, and only if, f^{-1} belongs to $\mathcal{B}^u(\varepsilon, \delta)$, it is enough to prove that $\mathcal{B}^s(\varepsilon, \delta)$ is an open subset of $\mathcal{PH}^1_m(M)$.

Let $f \in \mathcal{B}^s(\varepsilon, \delta)$ be a partially hyperbolic diffeomorphism with decomposition $TM = E^s \oplus E^c \oplus E^u$.

By continuity of the partially hyperbolic splitting, any diffeomorphism g close enough to f is also partially hyperbolic. We suppose first that every diffeomorphism g close to f has a partially hyperbolic splitting with stable and unstable bundle dimensions equal to the dimensions of the respectively sub-bundles in the partially hyperbolic splitting of f.

Now, given $x \in \mathcal{X}^s_\delta(f)$, there exists $K_x > 0$ such that $\mathcal{F}^s_{K_x}(f, x)$ is δ-dense in M, since M is a compact manifold. Thus since the strong stable manifolds varies continuously with respect to the diffeomorphism in compact parts, there exists a neighborhood $\mathcal{V}_x$ of f and a neighborhood U_x of x, such that:

$$\mathcal{F}^s_{K_x}(g, y) \text{ is } \delta\text{-dense in } M \text{ for every } y \in U_x \text{ and } g \in \mathcal{V}_x.$$

In particular, note that $U_x \subset \mathcal{X}^s_\delta(g)$ for every $g \in \mathcal{V}_x$.

These open sets U_x give a natural open cover of $\mathcal{X}^s_\delta(f)$, and since $m(\mathcal{X}^s_\delta(f)) > 1 - \varepsilon$, we can use Vitalli's Theorem, to obtain $x_1, \ldots, x_k \in \mathcal{X}^s_\delta(f)$ such that

$$m\left(\bigcup_{i=1}^{n} U_{x_i}\right) > 1 - \varepsilon.$$

Hence, considering $\mathcal{V} = \cap_{1 \le i \le n} \mathcal{V}_{x_i}$, we have that $m(\mathcal{X}^s_\delta(g)) > 1 - \varepsilon$ for every $g \in \mathcal{V}$, which implies $\mathcal{V} \subset \mathcal{B}^s(\varepsilon, \delta)$.

Now, if there is g close to f with different strong sub bundles dimension from f, then we note that g has a partially hyperbolic splitting $\tilde{E}^{ss} \oplus \tilde{E}^{cs} \oplus \tilde{E}^c \oplus \tilde{E}^{cu} \oplus \tilde{E}^{uu}$ of TM to g, such that the stable (unstable) bundle of g is $\tilde{E}^{ss(uu)} \oplus \tilde{E}^{cs(cu)}$, with $\tilde{E}^{ss(uu)}(g)$ being a sub bundle close to $E^{s(u)}(f)$. Hence, by [17], there are invariant sub manifolds integrating $\tilde{E}^{ss(uu)}(g)$ contained in the strong stable (resp. unstable) leaf of g which are close to the strong manifolds of f, and thus the above arguments can also be used in this situation to conclude the proof. □

3 Dynamical Consequences of *m*-Minimality

In this section, we give three consequences of m-minimality.

3.1 Topological Mixing

In this subsection we prove the following:

Proposition 3.1 *Let f be a partially hyperbolic diffeomorphism preserving a volume form m. If f is ms or mu-minimal, then f is topologically mixing.*

Proof :Since f is ms-minimal if, and only if, f^{-1} is mu-minimal, without loss of generality we can suppose f is mu-minimal to prove the proposition.

Let U and V be two arbitrary open sets of M. We choose $\varepsilon > 0$ and an open ball $B \subset U$ of diameter 2ε, such that the compact disc $\mathcal{F}^s_\varepsilon(x)$ with radius ε and centre x inside the strong stable leaf of x, is contained in U for every $x \in B$. Now, let $\delta > 0$ be such that there is an open ball of diameter $\delta > 0$ inside V. In particular, any δ-dense subset of M should intersect V. We also denote $b = m(B)$.

Now, using that f is mu-minimal, i.e. $m(\mathcal{X}^u(f)) = 1$, and the continuity of the strong unstable manifold, we can repeat the arguments in the proof of Lemma 2.6 to find an open set $W \subset M$ and $K > 0$ such that $\mathcal{F}^{uu}_K(x)$ is δ-dense for every $x \in W$ and $m(W) \geq 1 - b$.

Now, by the partial hyperbolicity of f there exists $N_0 > 0$ such that for any $n \geq N_0$ and any $x \in M$ if $D \supset \mathcal{F}^{uu}_\varepsilon(x)$ then $f^n(D)$ contains $\mathcal{F}^{uu}_K(f^n(x))$. Using this information, we will prove that $f^n(U) \cap V \neq \emptyset$, for any $n \geq N_0$, which implies f is topologically mixing, since U and V were taken arbitrary.

Given $n \geq N_0$, since f preserves the Lebesgue measure m, $m(f^{-n}(W)) = m(W)$ which is bigger than $1 - b$. Hence, since $b = m(B)$, there exists $x \in f^{-n}(W) \cap B$. By choice of B we can consider a disk $D \subset \mathcal{F}^{uu}(x) \cap U$ with centre x and radius $\varepsilon > 0$. Thus, $f^n(D)$ contains $\mathcal{F}^{uu}_K(x)$, since $n \geq N_0$. Therefore, provided that $f^n(x) \in W$, $f^n(D)$ is δ-dense in M which implies $f^n(D) \cap V \neq \emptyset$. □

3.2 Weak Ergodicity

In this sub-section we prove the following theorem:

Theorem 3.2 *Let f be a $C^{1+\alpha}$-partially hyperbolic diffeomorphism preserving the Lebesgue measure m. If f is ms-minimal or mu-minimal then f is weakly ergodic.*

We give two proofs. The first one is direct and uses ideas from Pesin [20]. The second one is more indirect, using a result due to Zhang [29]. However, it produces other results that can be useful.

Proof We fix an open set U, by Poincaré's Recurrence Theorem, we have a subset $R \subset U$ with $m(U - R) = 0$ formed by recurrent points. Hence, if $z \in R$ and $w \in \mathcal{F}^s(z)$ there exists $n_k \to \infty$ such that $f^{n_k}(w) \in U$. □

For any $x \in \mathcal{X}^s$, we know that there exists $y \in \mathcal{F}^s(x) \cap U$. Moreover, there exists an open set $V \subset U$ containing y such that $\bigcup_{z \in V} \mathcal{F}^s(z)$ is a neighborhood of x.

But, by absolute continuity, we have that $W_x = \bigcup_{z \in R \cap V} \mathcal{F}^s(z)$ has full measure in $\bigcup_{z \in U} \mathcal{F}^s(z)$ and the orbit of every point in W meets U.

Hence, using Lebesgue density points, we have that $W = \bigcup_{x \in \mathcal{X}^s} W_x$ is a full measure set. Moreover, the orbit of every point in W meets U. Using a countable basis of neighborhoods we obtain the weak ergodicity. □

Remark 3.3 It is important to remark that the arguments used in the Proof 1 can also be used together with accessibility property to obtain weakly ergodicity, for $C^{1+\alpha}$ volume preserving partially hyperbolic diffeomorphisms. See [5] for such proof.

Second proof: The second proof is based in the following result due to Zhang. We recall that an acip is an invariant probability which is absolutely continuous with respect to Lebesgue.

Theorem 3.4 *(Zhang [29]) Let $f \in \mathrm{Diff}^r(M)$ for some $r > 1$ and Λ be a strongly partially hyperbolic set supporting some acip μ. Then Λ is bi-saturated, that is, for each point $p \in \Lambda$, the global stable manifolds and the global unstable manifolds over p lies on Λ.*

We remark that Zhang used this result to prove that essential accessibility implies weak ergodicity, when the diffeomorphism supports some acip.

Lemma 3.5 *Let f be a $C^{1+\alpha}$, ms-minimal partially hyperbolic diffeomorphism. If $\Lambda \subset M$ is a compact f-invariant set with $m(\Lambda) > 0$, then $\Lambda = M$.*

Proof Let Λ be a compact invariant set with positive Lebesgue measure. By Theorem 3.4, Λ is bi-saturated. Since f is ms-minimal, we have that $m(\Lambda \cap \mathcal{X}^s) = m(\Lambda) > 0$. In particular, $\Lambda \cap \mathcal{X}^s$ is non-empty. For any $x \in \Lambda \cap \mathcal{X}^s$, we have that $\mathcal{F}^s(x) \subset \Lambda$ and $cl(\mathcal{F}^s(x)) = M$. Since Λ is closed, we get that $\Lambda = M$. □

As a consequence we obtain a criterion to obtain minimality.

Proposition 3.6 *Let f be a $C^{1+\alpha}$, ms-minimal partially hyperbolic diffeomorphism. If $\mathcal{X}^s$ admits some compact invariant subset Λ with positive measure, then f is s-minimal.*

Proof By Lemma 3.5, we have that $\Lambda = M$, and since $\Lambda \subset \mathcal{X}^s$, we conclude that $\mathcal{X}^s = M$, which means that f is s-minimal. □

Finally, we deal with weak ergodicity.

Theorem 3.7 *Every $C^{1+\alpha}$ partially hyperbolic diffeomorphism f that is ms-minimal is weakly ergodic.*

Proof Let $\{U_n\}_{n\in\mathbb{N}}$ be a base of the topology of M. For a fixed $k \in \mathbb{N}$, consider the set $A_k = \{x \in M \mid \mathcal{O}(x) \cap U_k = \emptyset\}$. The sets A_k's are closed and f-invariant. Clearly, $m(A_k) < 1$, since $A_k \cap U_k = \emptyset$. By Lemma 3.5, we conclude that $m(A_k) = 0$ for every $k \in \mathbb{N}$. Hence the set $\bigcap_{k\in\mathbb{N}} A_k^c$ has full measure. By construction, the orbit of every point in this set passes trough every U_n, so it is a dense orbit. □

Corollary 3.6 and Theorem 3.7 have analogous versions for the mu-minimal case.

3.3 The SH Property

Pujals and Sambarino introduced in [21] a property for partially hyperbolic diffeomorphisms which they call by SH. Roughly, this property says that there are points in any unstable large disks where the dynamics f behaves as a hyperbolic one. Moreover, they proved that SH is a robust property. An amazing consequence of such property is that implies robustness of minimality of the strong foliation.

We could ask if SH would also imply the robustness of m-minimality. We do not have an answer for that, yet. However, we can prove that if a ms-minimal partially hyperbolic diffeomorphism has SH property, then the set $\mathcal{X}^s(g)$ still has measure close to one for any C^1-diffemorphism g near f. In particular $\mathcal{X}^s$ is robustly a large set. This kind of result goes in the same spirit of the result of Tahzibi in [25]. Also, it is important to point out that there are examples of minimal foliations having foliations arbitrary close having no dense leaf in the ambient manifold. For instance, the foliation given by the irrational rotation in the torus.

Definition 3.8 (Property SH) Let f be a partial hyperbolic diffeomorphism. We say that f exhibits the property SH (or has the property SH) if there exist $\lambda > 1$ and $C > 0$ such that, for any $x \in M$, there exists $y^u(x) \in \mathcal{F}_1^{uu}(x))$ (the ball of radius 1 in $\mathcal{F}^{uu}(x)$ centered at x) satisfying

$$m\{Df^n_{|E^c(f^l(y^u(x)))}\} > C\lambda^n \text{ for any } n > 0,\ l > 0.$$

In this definition, $m(.)$ is the co-norm of the linear map.

Proposition 3.9 *Let f be a volume preserving partially hyperbolic diffeomorphism ms-minimal having the SH property. Then given $\varepsilon > 0$, there exists a neighborhood $\mathcal{V}$ of f such that $m(\mathcal{X}^s(g)) > 1 - \varepsilon$ for every $g \in \mathcal{V}$.*

Proof Since f is ms-minimal, we can choose a small neighborhood $\mathcal{V}$ of f inside $\mathcal{B}(\varepsilon, \delta)$, for $\delta > 0$ arbitrary small. See (1) in the proof of Proposition 2.5 to recall the definition of $\mathcal{B}(\varepsilon, \delta)$. Moreover, we can suppose that every diffeomorphisms in $\mathcal{U}$ has SH property, by robustness of such property. Hence, if $g \in \mathcal{U}$ and $\delta > 0$ is small enough, given any open set U, we can use property SH as in [21] to prove that $\mathcal{F}^s(g, g^{-k}(x))$ intersects U for any large positive integer k, and every $x \in \mathcal{X}^s_\delta(g)$.

Now, we have, by Poincaré Recurrence Theorem, that almost every point in $\mathcal{X}^s_\delta(g)$ is recurrent. Also, since $\mathcal{X}^s_\delta(g)$ is an open set of M, for almost every point $x \in \mathcal{X}^s_\delta(g)$ there is arbitrary large positive integer n_{k_x} such that $f^{n_{k_x}}(x) \in \mathcal{X}^s_\delta(g)$. Thus, using the first part of the proof we have $\mathcal{F}^s(x)$ must intersects U. Since this open set was taken arbitrary, we have just proved that almost every point x in $\mathcal{X}^s_\delta(g)$ also belongs to $\mathcal{X}^s(g)$. Which implies $m(\mathcal{X}^s(g)) > 1 - \varepsilon$, since $g \in \mathcal{B}(\varepsilon, \delta)$. ◻

4 The Abundance of *m*-Minimallity

4.1 A Criterion to Obtain Density of the Strong Leaves

In this section we obtain information about the strong stable and unstable leaves of points in the manifold containing hyperbolic periodic points in their ω-limit or α-limit sets. Recall that the *ω-limit (resp. α-limit) set of x, $\omega(x)$ (resp. $\alpha(x)$)*, is the set of points y in M such that there exists a sequence of forward (resp. backward) iterates of x converging to y.

What follows is our criterion to show δ-density of strong leaves of a partially hyperbolic diffeomorphisms. In the next results, we address only the case of strong stable leaves. However, there are similar results for the strong unstable leaves, which can be obtained by considering f^{-1}.

Proposition 4.1 *Let f be a C^1 partially hyperbolic diffeomorphism with splitting $TM = E^s \oplus E^c \oplus E^u$, having a periodic point p with period $\tau(p)$. Given $\delta > 0$, if:*

a) *$\mathcal{F}^s(f^j(p))$ is δ-dense in M, for any $0 \leq j < \tau(p)$;*
b) *For any small enough neighborhood V of p, there exist a submanifold $D \subset V$ containing p which integrates $E^c \oplus E^u$, i.e. $T_D M = E^c \oplus E^u$, such that $f^{-\tau(p)}(D) \subset D$;*

Then, for any $x \in M$ such that $p \in \omega(x)$ we have $\mathcal{F}^s(x)$ is also δ-dense in M.

Remark 4.2 Recalling that the *index* of a hyperbolic periodic point is its stable bundle dimension, we remark that every hyperbolic periodic point p of a partially hyperbolic diffeomorphism f having index equal to the dimension of the strong stable bundle of f, satisfies condition (b) of the previous proposition.

Before we prove Proposition 4.1, we use it and the previous remark to obtain a criterion to see density of strong stable leaves.

Proposition 4.3 *Let f be a C^1 partially hyperbolic diffeomorphism with splitting $TM = E^s \oplus E^c \oplus E^u$, having a hyperbolic periodic point p with index s, where $s = dim\ E^s$, such that $\mathcal{F}^s(p)$ is dense in M. Thus, if $x \in M$ is such that $p \in \omega(x)$ then $\mathcal{F}^s(x)$ is dense in M.*

Proof Since f is a diffeomorphism implies that $f^j(p)$ is dense in M for any integer j. In particular, item (a) of Proposition 4.1 is satisfied for any $\delta > 0$. Therefore, since item (b) of such proposition is also true by Remark 4.2, we have that $\mathcal{F}^s(x)$ is dense in M. □

Let we prove now Proposition 4.1.

Proof of Proposition 4.1 Let p be the periodic point of f satisfying items (a) and (b) in the hypothesis.. And let $x \in M$ different of p, such that $p \in \omega(x)$. Hence, there exist positive integers n_k converging to infinity when k goes to infinity, such that $f^{n_k}(x)$ converges to p. Recalling that $\tau(p)$ is the period of p, it is not difficulty to see that there exists some $0 \leq j < \tau(p)$ such that $n_k + j$ is a multiple of $\tau(p)$ for infinitely many positive integers k. Hence, replacing the sequence $(n_k)_{k\in\mathbb{N}}$ for a subsequence we can assume $n_k + j = m_k\tau(p)$ for any $k \in \mathbb{N}$.

Let $U \subset M$ be an arbitrary open set, containing a disk with diameter larger than δ.

Using item (a) of the properties satisfied for p, there exists $K > 0$ such that the compact part $\mathcal{F}^s_K(f^j(p))$ of the strong stable leaf of $f^j(p)$ intersects U, for every $0 \leq j < \tau(p)$. Moreover, since the strong stable leaves varies continuously in compact parts there exists a neighborhood V of p such that $\mathcal{F}^s_K(y) \cap U \neq \emptyset$ for every $y \in f^j(V)$, $0 \leq j < \tau(p)$.

Taking V smaller, if necessary, let $D \subset V$ the sub manifold given by item (b) in the hypothesis. Moreover, as a consequence of the partial hyperbolicity of f, we can take another small neighborhood $\tilde{V} \subset V$ of p, such that $\mathcal{F}^s_{loc}(y)$ intersects transversally $f^j(D)$ for any $y \in f^j(\tilde{V})$, $0 \leq j < \tau(p)$. In particular, by choice of x, there exists k_0 such that $\mathcal{F}^s(f^{n_{k_0}}(x))$ intersects transversally D in a point z_{k_0}. Now, by choice of D, we have that $f^{-m_{k_0}\tau(p)}(z_{k_0}) \in f^j(D) \subset f^j(V)$. Which implies that $\mathcal{F}^s(f^{-m_{k_0}\tau(p)}(z_{k_0}))$ intersects U. Thus, if we observe that by choice of j we have that $x \in \mathcal{F}^s(f^{-m_{k_0}\tau(p)}(z_{k_0}))$, then we conclude $\mathcal{F}^s(x)$ intersects U. And since U was taken arbitrary we have just finished the proof of proposition. □

4.2 The Existence of Dense Strong Leaves

The next result is a consequence of a standard application of the Hayashi's connecting lemma applied to transitive dynamics.

Proposition 4.4 *Let $f \in \mathrm{Diff}^1_m(M)$ (resp. $f \in \mathrm{Diff}^1_\omega(M)$) be partially hyperbolic, and p be a hyperbolic periodic point (resp. either hyperbolic or m-elliptic periodic point). Given a small enough neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_m(M)$ (resp. in $\mathrm{Diff}^1_\omega(M)$), there exists a dense subset $\mathcal{D} \subset \mathcal{U}$ formed by partially hyperbolic diffeomorphisms such that $\mathcal{F}^s(p(g), g)$ is dense in M for every $g \in \mathcal{D}$. Where $p(g)$ denotes the analytic continuation of p for g.*

Proof We can reduce $\mathcal{U}$, if necessary, such that it is defined an analytic continuation to the hyperbolic periodic point p. If p is a m-elliptic periodic point, then there alson exists an analytic continuation for such point in the symplectic setting. Also, in this last case, it is important to remark that $2m$ is a number smaller than center dimension of f.

From now on in this proof, we assume p is hyperbolic and f is volume preserving. For the other cases, the proof is analogous. Reducing $\mathcal{U}$ again, if necessary, we can also suppose that every diffemorphism in $\mathcal{U}$ is partially hyperbolic.

Let $U_1, \ldots, U_n, \ldots$ be an enumerable basis of opens sets of M. We define $\mathcal{B}^s_m \subset \mathcal{U}$ the subset of diffeomorphisms g such that g is partially hyperbolic and $\mathcal{F}^s(p(g), g)$ intersects U_m. By continuity of the strong stable foliation $\mathcal{B}^s_m$ is an open set inside $\mathcal{U}$.

Let $\mathcal{R} \subset \text{Diff}^1_m(M)$ the residual subset given by [6] formed by transitive diffeomorphisms (in the symplectic setting this is prove in [1]). In particular, $\mathcal{R}$ is dense in $\mathcal{U}$. Let $g \in \mathcal{R} \cap \mathcal{U}$. Since g is transitive, we can use the connecting lemma (see [28]) to perturb g and find a partially hyperbolic diffeomorphism $\tilde{g}$ arbitrary close to g such that $\mathcal{F}^s(p(\tilde{g}), \tilde{g})$ intersects U_m. Thus, we have that $\mathcal{B}^s_m$ is open and dense in $\mathcal{U}$. Since, U_m was taken arbitrary, $\mathcal{D} = \cap \mathcal{B}^s_m$ is a dense subset in $\mathcal{U}$ formed by diffeomorphisms satisfying the thesis of the proposition. □

4.3 *m*-Minimality in the Conservative Setting

In this section we prove Theorem 1.3. Before that, although we know that weakly ergodicity does not implies m-minimality, the following result says that weakly ergodicity implies m-minimality in some setting.

Theorem 4.5 *Let $f \in \text{Diff}^1_m(M)$ be weakly ergodic and partially hyperbolic with decomposition $TM = E^s \oplus E^c \oplus E^u$. If there exists a hyperbolic periodic point p of f with $ind\ p = dim\ E^s$ (resp. $ind\ p = dim\ E^s \oplus E^c$), and such that $\mathcal{F}^s(p)$ (resp. $\mathcal{F}^u(p)$) is dense in M, then f is ms-minimal (resp. mu-minimal).*

Proof This theorem is in fact a directly consequence of our criterion in the Sect. 4.1. In fact, since f is weakly ergodic, then for almost every point x in M the forward orbit of x is dense in M, in particular the hyperbolic periodic point $p \in \omega(x)$, which implies by Proposition 4.3 that $\mathcal{F}^s(x)$ is dense in M, for almost every point x.

Respectively, if $ind\ p = dim\ E^s \oplus E^c$ we can use Proposition 4.3, as before, but now for f^{-1} to conclude that $\mathcal{F}^u(x)$ is dense for almost every point x. □

In the sequence we will use Theorem 4.5 to prove Theorem 1.3.

Another important tool we use in the proof of Theorem 4.5 is the well known blender sets, introduced by Bonatti and Diaz, [6]. What follows is a definition of a blender given in [8].

Definition 4.6 Let $f : M \to M$ be a diffeomorphism and p a hyperbolic periodic point of index i. We say that f has a *blender associated to* p if there is a C^1-neighborhood $\mathcal{U}$ of f and a C^1 open set $\mathcal{D}$ of embeddings of an $(d - i - 1)$-dimensional disk D into M, such for every $g \in \mathcal{U}$, every disk $D \in \mathcal{D}$ intersects the closure of $W^s(p(g))$, where $p(g)$ is the continuation of the periodic point p for g. Moreover, we say that a blender is *acctivated* by a hyperbolic periodic point $\tilde{p}$ of index $i + 1$ if the unstable manifold of $\tilde{p}$ contains a disk of the superposition region.

According to the above definition we have the following result:

Lemma 4.7 *(Lemma 6.12 in [8]) Let $f : M \to M$ be a diffeomorphism having a blender associated to a hyperbolic periodic point p of index i. Suppose that the blender is activated by a hyperbolic periodic point $\tilde{p}$ of index $i + 1$. Then, for every diffeomorphism g in a small enough C^1-neighborhood of f, the closure of $W^s(p(g))$ contains $W^s(\tilde{p}(g))$.*

The next result gives an abundance of Blenders in the conservative setting, (See also [9]).

Theorem 4.8 *(Theorem 1.1 in [24]) Let $f \in \text{Diff}^r_m(M)$ such that f has two hyperbolic periodic points p of index i and $\tilde{p}$ of index $i + 1$. Then there are C^r diffeomorphisms arbitrary C^1-close to f which preserve m and admit a blender associated to the analytic continuation of p.*

What follows is a by-product of a conservative version of results in [11] and [2]:

Theorem 4.9 *There is a residual subset* $\mathcal{R} \subset \mathrm{Diff}^1_m(M)\backslash cl(\mathcal{HT})$ *such that for every* $f \in \mathcal{R}$, f *is partially hyperbolic having non trivial extremal sub bundles with decomposition* $TM = E^s \oplus E^c \oplus E^u$, *and moreover there exists hyperbolic periodic points* $p_0, \dots, p_k$ *of* f, *where* $k = dim\ E^s \oplus E^c$, *such that* $ind\ p_i = dim\ E^s + i$, *for any* $i = 0, \dots, k$.

Finally we can prove Theorem 1.3.

Proof of Theorem 1.3: Let $\mathcal{R}$ be the residual subset given by Theorem 4.9. Hence, we consider $f \in \mathcal{R}$ and $p_0, \dots, p_k$ the hyperbolic periodic points given by Theorem 4.9. Recall $k = dim\ E^s \oplus E^c$, if $TM = E^s \oplus E^c \oplus E^u$ is the partially hyperbolic decomposition given by f. Since the index of a hyperbolic periodic point does not change for its analytic continuation, and since blender sets are robust, we can use Theorem 4.8 to find an open set $\mathcal{U} \subset \mathrm{Diff}^1_m(M)$ arbitrary close to f such that for every $g \in \mathcal{U}$ there exists a blender set $\Lambda_i(g)$ associated to each $p_i(g)$, for any $i = 0, \dots, k$.

Now, using that generic volume preserving diffeomorphisms are transitive (see [6]) and the connecting lemma, we can find $g_1 \in \mathcal{U}$ such that the blender set $\Lambda_1(g_1)$ is activated by $p_2(g_1)$. This implies, by Lemma 4.7, that there exists an open set $\mathcal{U}_1 \subset \mathcal{U}$ such that the closure of $W^s(p_1(g))$ contains $W^s(p_2(g))$ for every $g \in \mathcal{U}_1$. Using the above arguments again, we can find an open set $\mathcal{U}_2 \subset \mathcal{U}_1$ such that $W^s(p_2(g))$ contains $W^s(p_3(g))$ for every $g \in \mathcal{U}_1$. And thus, repeating this process finitely many times we can obtain an open set $\mathcal{V} \subset \mathcal{U}$ such that

$$cl(W^s(p_i(g))) \supset W^s(p_{i+1}(g)), \quad \text{for every } i = 0, \dots, k-1, \text{ and } g \in \mathcal{V}. \tag{3}$$

Reducing the open set $\mathcal{V}$, if necessary, we can use [12] and Remark 3.3 to assume that every C^2-diffeomorphism g in $\mathcal{V}$ is weakly ergodic.

Hence, let $g \in \mathcal{V}$ be a C^2-diffemorphism. Hence, g is topologically transitive, which implies the existence of a dense backward orbit of g, say $\{g^{-n}(x)\}_{n\in\mathbb{N}}$. Since g is partially hyperbolic, the local strong unstable manifolds has uniform length, and thus there exists $n_0 \in \mathbb{N}$ such that $\mathcal{F}^u(f^{-n_0}(x))$ intersects transversally $W^s_{loc}(p_k(g))$. Thus, the accumulation points of $\{f^{-n}(x)\}_{n\in\mathbb{N}}$ is also accumulated by points in $W^s(p_k(g))$, which implies the stable manifold of $p_k(g)$ is dense in M. Then, using (3) we conclude that $W^s(p_0(g)) = \mathcal{F}^s(p_0(g))$ is also dense in M.

We have just proved that g satisfies the hypothesis of Theorem 1.3, which implies g is ms-minimal.

Therefore, since $\mathcal{V}$ is arbitrary close to f, and f is arbitrary in $\mathcal{R}$, by standard topology arguments we can find an open set $\mathcal{A}_s \subset \mathrm{Diff}^1_m(M)\backslash cl(\mathcal{HT})$ such that any $g \in \mathcal{A}_s$ is ms-minimal, and $\mathcal{R}$ is contained in the closure of $\mathcal{A}_s$.

Now, since the partially hyperbolic diffeomorphisms in $\mathcal{R}$ has non trivial extremal sub bundles, the above arguments can also be done for f^{-1}, to find an open set $\mathcal{A}_u \subset \mathrm{Diff}^1_m(M)\backslash cl(\mathcal{HT})$ such that any $g \in \mathcal{A}_u$ is mu-minimal, and $\mathcal{R}$ is also contained in the closure of $\mathcal{A}_u$.

The proof of theorem is finished taking $\mathcal{A} = \mathcal{A}_s \cap \mathcal{A}_u$. □

4.4 *m*-Minimality in the Symplectic Setting

In this section we will prove Theorem 1.5. Here (M, ω) is a symplectic manifold, being ω a symplectic form on M. Also, in this subsection m will denotes the volume form on M induced by the exterior powers of ω.

The next result use m-elliptic periodic points to see density of strong leaves of partially hyperbolic diffemorphisms. Recall that a periodic point p of a symplectic diffeomorphism f with period $\tau(p)$ is called *m-elliptic* if $Df^{\tau(p)}(p)$ has exactly $2m$ modulus one eigenvalues, which must be non real and simple eigenvalues.

Before we state the result, given δ and ε positive, we denote by $\mathcal{B}^s_\omega(\varepsilon,\delta)$ and $\mathcal{B}^u_\omega(\varepsilon,\delta)$ the set formed by symplectic partially hyperbolic diffeomorphisms f such that $m(\mathcal{X}^s_\delta(f)) > 1-\varepsilon$ and $m(\mathcal{X}^u_\delta(f)) > 1-\varepsilon$, respectively.

Proposition 4.10 *Let $f \in \mathrm{Diff}^1_\omega(M)$ be partially hyperbolic having a $2m$-dimensional center bundle, and a m-elliptic periodic point p. Thus, for any neighborhood $\mathcal{U} \subset \mathrm{Diff}^1_\omega(M)$ of f and any ε and $\delta > 0$ there exists a symplectic partially hyperbolic diffeomorphism $g \in \mathcal{U}$ which belongs to $\mathcal{B}^s(\varepsilon,\delta)$ (resp. $\mathcal{B}^u(\varepsilon,\delta)$).*

Proof We prove the existence of such g inside $\mathcal{B}^s(\varepsilon,\delta)$. The other case is a consequence by considering f^{-1} instead of f.

By continuity of the partially hyperbolic splitting and the robustness of m-elliptic periodic points in the symplectic scenario, we can suppose all diffeomorphisms in $\mathcal{U}$ are partially hyperbolic having a partially hyperbolic decomposition with same central bundle, and moreover, every diffeomorphism in $\mathcal{U}$ has a m-elliptic periodic point $p(g)$ which is the analytic continuation of p.

After a perturbation, using Proposition 4.4, we can assume f is such that $\mathcal{F}^s(p)$ is dense in M. Now, let $\varepsilon > 0$ and $\delta > 0$ given arbitrary. Since the strong stable foliation also varies continuously with the diffeomorphism in compact parts, taking a small neighborhood V of p and $\mathcal{U}$ smaller, if necessary, we can suppose $\mathcal{F}^s(x,g)$ is δ-dense in M for every diffeomorphism $g \in \mathcal{U}$ and $x \in V$.

By Zender [30], there is a C^2-diffeomorphism $f_1 \in \mathcal{U}$. Moreover, as in the conservative setting, we can use accessibility and Remark 3.3 to assume that there exists a neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ such that every C^2-diffeomorphism g in $\mathcal{U}_1$ is weakly ergodic. Recall that acessibility is also true in an open and dense subset among partially hyperbolic symplectic diffeomorphisms.

To simplify the notation we still denote by p the analytic continuation of p for f_1. Now, using Pasting Lemma of Arbieto and Matheus [4], we can perturb f_1 to find a C^2-diffeomorphism $f_2 \in \mathcal{U}_1$ such that p still is a m-elliptic periodic point of f_2, and moreover $f_2 = Df_1(p)$ in a small neighborhood of p, in local coordinates. Hence, replacing V by a small neighborhood of p and looking to V in local coordinates, if we consider E^s and E^c the stable and center bundles of f_2, respectively, we have that $D = (E^c \oplus E^u(p)) \cap V$ is locally $f_2^{-\tau(p)}$-invariant. In fact, we have that $Df_1^{-1}(p)|E^u$ contracts and $Df_1^{-\tau(p)}(p)|E^c$ has norm equal to one.

Now, since f_2 is C^2 and belongs to $\mathcal{U}_1$ it is weakly ergodic. Thus, for almost every point x in M, p belongs to $\omega(x)$, which implies by Proposition 4.1 that $\mathcal{F}^s(x, f_2)$ is δ-dense in M, since $\mathcal{F}^s(p, f_2)$ is. $\square$

In the hypothesis of Proposition 4.10, we have the existence of a m-elliptic periodic point for a partially hyperbolic diffeomorphism with center dimension equal to $2m$. This hypothesis was essential in the proof of such result. However, it is not guaranteed that this point actually exists for an arbitrary symplectic partially hyperbolic diffeomorphism. In fact, this was a question posed by [1]. Fortunately, it was proved in [10] that this is the case for an open and dense subset among partially hyperbolic symplectic diffeomorphisms. More precisely:

Theorem 4.11 *(Theorem A in [10]) There exists an open and dense subset $\mathcal{A} \subset \mathrm{Diff}^1_\omega(M)$, such that if $f \in \mathcal{A}$ is a partially hyperbolic diffeomorphism with $2m$-dimensional center bundle, then f has a m-elliptic periodic point.*

Proof of Theorem 1.5: Let we consider the open and dense subset $\mathcal{A}$ inside the partially hyperbolic symplectic diffeomorphisms given by Theorem 4.11. Hence, given $m, n \in \mathbb{N}$, by Proposition 4.10 we have that $\mathcal{B}^s(1/m, 1/n)$ is dense in $\mathcal{A}$. Since these sets are open in $\mathrm{Diff}^1_\omega(M)$ we have that $\mathcal{R}^s = (\cap \mathcal{B}^s(1/m, 1/n) \cap \mathcal{A}) \cup (cl(\mathcal{A}))^c$ is a residual subset inside $\mathrm{Diff}^1_\omega(M)$ such that every partially hyperbolic diffeomorphism $f \in \mathcal{R}^s$ is ms-minimal.

Considering the maps f^{-1}, we can also find a residual subset $\mathcal{R}^u \subset \mathrm{Diff}^1_\omega(M)$ such that every partially hyperbolic diffeomorphism $f \in \mathcal{R}^u$ is mu-minimal.

Thus the proof is finished taking $\mathcal{R} = \mathcal{R}^s \cap \mathcal{R}^u$. □

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