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# Construction of Multi-solitons for a Generalized Derivative Nonlinear Schrödinger Equation

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**Abstract**
We consider a derivative nonlinear Schrödinger equation with general nonlinearlity:

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0,$$

In Tang and Xu (J Differ Equ 264(6):4094–4135, 2018), the authors prove the stability of two solitary waves in energy space for $\sigma \in (1, 2)$. As a consequence, there exists a solution of the above equation which is close arbitrary to sum of two solitons in energy space when $\sigma \in (1, 2)$. Our goal in this paper is proving the existence of multi-solitons in energy space for $\sigma \geq \frac{3}{2}$. Our proofs proceed by fixed point arguments around the desired profile, using Strichartz estimates.

**Keywords** Multi-solitons · Nonlinear derivative Schrödinger equations · Strichartz estimates · Fixed point method · Grönwall inequality

**Mathematics Subject Classification** 35Q55 · 35C08 · 35Q51

## Contents



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## 1 Introduction

In this paper, we consider the following generalized derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0, \tag{1.1}$$

where $\sigma \in \mathbb{R}^+$ is a given constant and $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$.

The Eq. (1.1) was studied in many works. In the special case $\sigma = 1$, local well-posedness, global well posedness, stability of solitary waves and stability of multi-solitons have been investigated. In [15], Ozawa gave a sufficient condition for global well posedness of (1.1) in the energy space by using a Gauge transformation to remove the derivative terms. In [2], Colin–Ohta showed that the equation has a two parameters family of solitary waves and proved the stability of these particular solutions by using variational methods. In [8], Kwon-Wu gave a result on stability of solitary waves when the parameters are at the threshold between existence and non-existence. In [11], Le Coz–Wu proved stability of multi-solitons in the energy space under some conditions on the parameters of the composing solitons.
In the general case, the local well-posedness and global well- posedness of (1.1) was studied in [6] when the initial data is in the Sobolev space $H_0^1(\Omega)$, where $\Omega$ is any unbounded interval of $\mathbb{R}$. In this work, Hayashi-Ozawa used an approximation argument. In [16], Santos proved the local well-posedness for small size initial data in weighted Sobolev spaces. The arguments used in this work follow parabolic regularization approach introduced by Kato [7].

The Eq. (1.1) has a two parameters family of solitons. The stability of the solitons has attracted the attention of many researchers. In [12], by using the abstract theory of Grillakis–Shatah–Strauss [3, 4], Liu–Simpson–Sulem proved that in the case $\sigma \geq 2$, the solitons of (1.1) are orbitally unstable; in the case $0 < \sigma < 1$, they are orbitally stable and in the case $\sigma \in (1, 2)$ they are orbitally stable if $c < 2z_0\sqrt{\omega}$ and orbitally unstable if $c > 2z_0\sqrt{\omega}$ for some constant $z_0 \in (0, 1)$. In the critical case $c = 2z_0\sqrt{\omega}$, Guo–Ning–Wu [5] proved that solitons are always orbitally unstable. In [1], Bai–Wu–Xue proved that when $\sigma \geq \frac{3}{2}$, the solution is global and scattering when the initial data small in $H^s(\mathbb{R})$, $\frac{1}{2} \leq s \leq 1$. Moreover, the authors showed that when $\sigma < 2$, the scattering may not occur even under smallness conditions on the initial data. Therefore, in this model, the exponent $\sigma \geq 2$ is optimal for small data scattering. In [17], in the case $\sigma \in (1, 2)$, Tang and Xu proved the stability of the sum of two solitary waves in the energy space provided that solitons are stables i.e $c < 2z_0\sqrt{\omega}$, using perturbation arguments, modulational analysis and an energy argument as in [13, 14].

In this paper, we show the existence of multi-solitons in energy space in the case $\sigma \geq \frac{3}{2}$. Before stating the main result, we give some preliminaries on multi-solitons of (1.1).

As mentioned in [12], the Eq. (1.1) admits a two-parameters family of solitary waves solutions given by

$$\psi_{\omega,c}(t,x) = \varphi_{\omega,c}(x-ct)\exp\left(i\left(\omega t + \frac{c}{2}(x-ct) - \frac{1}{2\sigma+2}\int_{-\infty}^{x-ct}\varphi_{\omega,c}^{2\sigma}(\eta)\,d\eta\right)\right), \tag{1.2}$$

where $\omega > \frac{c^2}{4}$ and

$$\varphi_{\omega,c}^{2\sigma}(y) = \frac{(\sigma+1)(4\omega - c^2)}{2\sqrt{\omega}\left(\cosh(\sigma\sqrt{4\omega - c^2}\,y) - \frac{c}{2\sqrt{\omega}}\right)}. \tag{1.3}$$

The profile $\varphi_{\omega,c}$ is a positive solution of

$$-\partial_y^2\varphi_{\omega,c}+\left(\omega-\frac{c^2}{4}\right)\varphi_{\omega,c}+\frac{c}{2}|\varphi_{\omega,c}|^{2\sigma}\varphi_{\omega,c}-\frac{2\sigma+1}{(2\sigma+2)^2}|\varphi_{\omega,c}|^{4\sigma}\varphi_{\omega,c}=0. \quad (1.4)$$

Define

$$\phi_{\omega,c}(y)=\varphi_{\omega,c}(y)e^{i\theta_{\omega,c}(y)}, \quad (1.5)$$

where

$$\theta_{\omega,c}(y)=\frac{c}{2}y-\frac{1}{2\sigma+2}\int_{-\infty}^{y}\varphi_{\omega,c}^{2\sigma}(\eta)\,d\eta. \quad (1.6)$$

Clearly, we have

$$\psi_{\omega,c}(t,x)=e^{i\omega t}\phi_{\omega,c}(x-ct). \quad (1.7)$$

and $\phi_{\omega,c}$ solves

$$-\partial_y^2\phi_{\omega,c}+\omega\phi_{\omega,c}+ic\partial_y\phi_{\omega,c}-i|\phi_{\omega,c}|^{2\sigma}\partial_y\phi_{\omega,c}=0,\quad y\in\mathbb{R}. \quad (1.8)$$

Let $K\in\mathbb{N}$, $K\geq 2$. For each $1\leq j\leq K$, let $(\omega_j,c_j,\theta_j)\in\mathbb{R}^3$ be parameters such that $\omega_j>\frac{c_j^2}{4}$. Define, for each $j=1,\dots,K$

$$R_j(t,x)=e^{i\theta_j}\psi_{\omega_j,c_j}(t,x)$$

and define the multi-soliton profile by

$$R=\sum_{j=1}^{K}R_j. \quad (1.9)$$

For convenience, define $h_j=\sqrt{4\omega_j-c_j^2}$, for each $j=1,\dots,K$. Our main result is the following.

**Theorem 1.1** *Let $\sigma\geq\frac{3}{2}$, $K\in\mathbb{N}$, $K\geq 2$ and for each $1\leq j\leq K$, $(\theta_j,\omega_j,c_j)$ be a sequence of parameters such that $\theta_j\in\mathbb{R}$, $c_j\neq c_k$, for $j\neq k$. The multi-soliton profile $R$ is given as in (1.9). There exists a certain positive constant $C_*$ such that if the parameters $(\omega_j,c_j)$ satisfy*

$$\begin{aligned}C_*&\left(\left(1+\|R\|_{L^\infty L^\infty}^{2(\sigma-1)}\right)\left(1+\|R\|_{L^\infty H^1}^2\right)\left(1+\|\partial_x R\|_{L^\infty L^\infty}+\|R\|_{L^\infty L^\infty}^{2\sigma+1}\right)\right)\\ &\leq v_*=\inf_{j\neq k}h_j|c_j-c_k|,\end{aligned} \quad (1.10)$$

*then there exists a solution $u$ of (1.1) such that*

$$\|u-R\|_{H^1}\leq Ce^{-\lambda t},\quad \forall t\geq T_0,$$

*for positive constants $C,T_0$ depending only on the parameters $\omega_1,\dots,\omega_K,c_1,\dots,c_K$ and $\lambda=\frac{1}{16}v_*$.*

***Remark 1.2*** The condition $\sigma\geq\frac{3}{2}$ is used to prove the existence of solution $\eta$ of (2.14) by using contraction mapping theorem.

The condition (1.10) is an implicit condition on the parameters. Below, we show that for large, negative and enough separated velocities, the condition (1.10) holds.

***Remark 1.3*** We prove that there exist parameters $(\omega_j, c_j, \theta_j)$ for $1 \le j \le K$ such at the condition (1.10) is satisfied for any prescribed $h_j$ and ratio $c_1 : c_2 : \cdots : c_K$ between negative $c_j$. Let $M > 0$, $h_j > 0$, $d_j < 0$, for each $1 \le j \le K$. We chose $(c_j, \omega_j) = \left(Md_j, \frac{1}{4}(h_j^2 + M^2 d_j^2)\right)$. We verify that this choice satisfies the condition (1.10) for $M$ large enough. Indeed, we see that $c_j < 0$ and $h_j \ll |c_j|$ for $M$ large enough. We have

$$\varphi_{\omega_j,c_j}^{2\sigma} \approx \frac{h_j^2}{2\sqrt{\omega_j}\left(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}}\right)}$$

$$\partial_x \varphi_{\omega_j,c_j} \approx \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{-\sinh(\sigma h_j y)}{\left(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}}\right)^{1+\frac{1}{2\sigma}}}.$$

Using $|\sinh(x)| \le |\cosh(x)|$ for all $x \in \mathbb{R}$ we have

$$|\partial_x \varphi_{\omega_j,c_j}| \le \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{1}{(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}})^{\frac{1}{2\sigma}}} \lesssim |\varphi_{\omega_j,c_j}|.$$

Thus,

$$\begin{aligned}
\|R_j\|_{L^\infty L^\infty} &= \|\varphi_{\omega_j,c_j}\|_{L^\infty} \lesssim \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \ll 1 \\
\|\partial_x R_j\|_{L^\infty L^\infty} &= \|\partial_x \phi_{\omega_j,c_j}\|_{L^\infty} \\
&\lesssim \|\partial_x \varphi_{\omega_j,c_j}\|_{L^\infty} + \left\| \frac{c_j}{2}\varphi_{\omega_j,c_j} - \frac{1}{2\sigma+2}\varphi_{\omega_j,c_j}^{2\sigma+1} \right\|_{L^\infty} \\
&\lesssim \|\varphi_{\omega_j,c_j}\|_{L^\infty} + |c_j| \|\varphi_{\omega_j,c_j}\|_{L^\infty} \\
&\lesssim \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|R\|_{L^\infty L^\infty} &\lesssim \sum_j \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \lesssim 1 \\
\|\partial_x R\|_{L^\infty L^\infty} &\lesssim \sum_j \left( \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|R_j\|_{L^\infty H^1}^2 &= \|R_j\|_{L^\infty L^2}^2 + \|\partial_x R_j\|_{L^\infty L^2}^2 = \|\varphi_{\omega_j,c_j}\|_{L^2}^2 + \|\partial_x \varphi_{\omega_j,c_j}\|_{L^2}^2 \\
&\lesssim \|\varphi_{\omega_j,c_j}\|_{L^2}^2 \lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{\sigma}} \left\| \frac{1}{\cosh(\sigma h_j y)^{\frac{1}{2\sigma}}} \right\|_{L^2}^2 \lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{\sigma}} \|e^{-\frac{h_j}{2}|y|}\|_{L^2}^2 \\
&\approx \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{\sigma}} \frac{1}{h_j} \lesssim h_j^{\frac{1}{\sigma}} h_j^{-1} = h_j^{\frac{1}{\sigma}-1},
\end{aligned}$$

where we use $h_j \le 2\sqrt{\omega_j}$. Thus,

$$\|R\|_{L^\infty H^1}^2 \lesssim \sum_j h_j^{\frac{1}{\sigma}-1}.$$

The condition (1.10) satisfies if the following estimate holds:

$$C_*\left(\left(1+\sum_j h_j^{\frac{1}{\sigma}-1}\right)\left(1+\sum_j\left(\sqrt[2\sigma]{\frac{h_j^2}{|c_j|}}+|c_j|\sqrt[2\sigma]{\frac{h_j^2}{|c_j|}}\right)\right)\right)\le \inf_{j\neq k} h_j|c_j-c_k|. \tag{1.11}$$

We see that the left hand side of (1.11) is order $M^{1-\frac{1}{2\sigma}}$ and the right hand side of (1.11) is order $M^1$. Hence, the condition (1.10) satisfies if we choose $M$ large enough.

***Remark 1.4*** The exponent $\sigma = 2$ is the borderline for the existence of stable solitons. Since the example given in Remark 1.3 chooses all $c_j$ negative, by the work of Liu-Simpson-Sulem [12], solitons are stable for $\sigma < 2$ and unstable for $\sigma \ge 2$. This shows that in Theorem 1.1, we can construct multi-solitons from stable solitons or unstable solitons.

Our strategy of the proof of Theorem 1.1 is as follows. First, we define $\varphi, \psi$ based on $u$ in such a way that $\varphi$ and $\psi$ satisfy a system of nonlinear Schrödinger equations without derivatives (see (2.3)). Let $R$ be a multi-soliton profile which satisfies the assumptions of Theorem 1.1. Then $R$ solves (1.1) up to a small perturbation. Let $(h, k)$ be defined in a similar way as $(\varphi, \psi)$ but replace $u$ by $R$. We see that $(h, k)$ solves (2.3) up to small perturbations. Setting $\tilde{\varphi} = \varphi - h$ and $\tilde{\psi} = \psi - k$, we see that if $u$ solves (1.1) then $(\tilde{\varphi}, \tilde{\psi})$ solves a system and a relation between $\tilde{\varphi}$ and $\tilde{\psi}$ holds and vice versa. By using the Banach fixed point theorem, we prove that there exists a solution $(\tilde{\varphi}, \tilde{\psi})$ of this system which decays exponentially in time on $H^1(\mathbb{R})$ for $t$ large. Combining with the assumption (1.10), we can prove a relation between $\tilde{\varphi}$ and $\tilde{\psi}$. Thus, we easily obtain the solution $u$ of (1.1) satisfying the desired property.

This paper is organized as follows. In Sect. 2, we prove the existence of multi-solitons for the Eq. (1.1). In Sect. 3, we prove some technical results which are used in the proof of the main result Theorem 1.1. More precisely, we prove the exponential decay of perturbations in the equations of $h, k$ (Lemma 3.1) and the existence of decaying solutions for the system of equations of $\tilde{\varphi}, \tilde{\psi}$ (Lemma 3.8).

Before proving the main result, we introduce some notation used in this paper.

**Notation** (1) *We denote the Schrödinger operator as follows*

$$L = i\partial_t + \partial_x^2.$$

(2) *Given a time $t \in \mathbb{R}$, the Strichartz space $S([t,\infty))$ is defined via the norm*

$$\|u\|_{S([t,\infty))} = \sup_{(q,r)\ admissible} \|u\|_{L_t^q L_x^r([t,\infty)\times\mathbb{R})}.$$

*We denote the dual space by $N[t,\infty) = S([t,\infty))^*$. Hence for any $(q,r)$ admissible pair we have*

$$\|u\|_{N([t,\infty))} \le \|u\|_{L_t^{q'} L_x^{r'}([t,\infty)\times\mathbb{R})}.$$

(3) *For $a, b \in \mathbb{R}^2$, we denote $|(a,b)| = |a| + |b|$.*

(4) *Let* $a, b > 0$. *We denote* $a \lesssim b$ *if* $a$ *is smaller than* $b$ *up to multiplication by a positive constant and denote* $a \lesssim_c b$ *if* $a$ *is smaller than* $b$ *up to multiplication by a positive constant depending on* $c$. *Moreover, we denote* $a \approx b$ *if* $a$ *equals to* $b$ *up to multiplication by a positive constant.*

## 2 Proof of the Main Result

In this section we give the proof of Theorem 1.1. We use the Banach fixed point theorem and Strichartz estimates. We divide our proof in three steps. **Step 1. Preliminary analysis.** Let $u \in C(I, H^1(\mathbb{R}))$ be a $H^1(\mathbb{R})$ solution of (1.1) on $I$. Consider the following transform:

$$\varphi(t,x) = \exp(i\Lambda)u(t,x), \tag{2.1}$$

$$\psi = \exp(i\Lambda)\partial_x u = \partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi, \tag{2.2}$$

where

$$\Lambda = \frac{1}{2}\int_{-\infty}^{x} |u(t,y)|^{2\sigma}\, dy.$$

As in [6, section 4], using $|u| = |\varphi|$ and $\mathcal{I}m(\overline{u}\partial_x u) = \mathcal{I}m(\overline{\varphi}\psi)$, we have

$$\partial_t\Lambda = -\sigma\mathcal{I}m(|u|^{2(\sigma-1)}\overline{u}\partial_x u) + \sigma\mathcal{I}m\left[\int_{-\infty}^{x} \partial_x(|u|^{2(\sigma-1)}\overline{u})\partial_x u\, dy\right] - \frac{1}{4}|u|^{4\sigma}.$$

Thus, using $|u| = |\varphi|$ and $\mathcal{I}m(\overline{u}\partial_x u) = \mathcal{I}m(\overline{\varphi}\psi)$, we have

$$\begin{aligned}\partial_t\Lambda &= -\sigma|\varphi|^{2(\sigma-1)}\mathcal{I}m(\overline{\varphi}\psi) + \sigma\int_{-\infty}^{x} \partial_x(|u|^{2(\sigma-1)})\mathcal{I}m(\overline{u}\partial_x u)\, dx - \frac{1}{4}|\varphi|^{4\sigma}\\ &= -\sigma|\varphi|^{2(\sigma-1)}\mathcal{I}m(\overline{\varphi}\psi) + \sigma\int_{-\infty}^{x} \partial_x(|\varphi|^{2(\sigma-1)})\mathcal{I}m(\overline{\varphi}\psi)\, dx - \frac{1}{4}|\varphi|^{4\sigma}.\end{aligned}$$

Since $u$ solves (1.1), we have

$$\begin{aligned}L\varphi &= L(\exp(i\Lambda))u + \exp(i\Lambda)Lu + 2\partial_x(\exp(i\Lambda))\partial_x u\\ &= L(\exp(i\Lambda))u + \exp(i\Lambda)(Lu + i|u|^{2\sigma}u)\\ &= L(\exp(i\Lambda))u\\ &= (i\partial_t + \partial_x^2)(\exp(i\Lambda))u,\\ &= \left[-\exp(i\Lambda)\partial_t\Lambda + \partial_x(\exp(i\Lambda)\frac{i}{2}|u|^{2\sigma})\right]u\\ &= -\varphi\partial_t\Lambda + \left[\exp(i\Lambda)\frac{-1}{4}|u|^{2\sigma} + \frac{i}{2}\exp(i\Lambda)\partial_x(|u|^{2\sigma})\right]u\\ &= -\varphi\partial_t\Lambda + \varphi\left[-\frac{1}{4}|\varphi|^{4\sigma} + \frac{i}{2}\partial_x(|\varphi|^{2\sigma})\right]\\ &= \sigma|\varphi|^{2(\sigma-1)}\varphi\mathcal{I}m(\overline{\varphi}\psi) - \sigma\varphi\int_{-\infty}^{x} \partial_x(|\varphi|^{2(\sigma-1)})\mathcal{I}m(\overline{\varphi}\psi)\, dx\\ &\quad + \frac{1}{4}|\varphi|^{4\sigma}\varphi - \frac{1}{4}\varphi|\varphi|^{4\sigma} + i\sigma|\varphi|^{2(\sigma-1)}\varphi\mathcal{R}e(\overline{\varphi}\partial_x\varphi)\\ &= \sigma|\varphi|^{2(\sigma-1)}\varphi(\mathcal{I}m(\overline{\varphi}\psi) + i\mathcal{R}e(\overline{\varphi}\partial_x\varphi))\end{aligned}$$

$$
\begin{aligned}
&-\sigma\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}(\sigma-1)\partial_x(|\varphi|^2)\mathcal{I}m(\overline{\varphi}\psi)\,dx\\
&=\sigma|\varphi|^{2(\sigma-1)}\varphi(\mathcal{I}m(\overline{\varphi}\psi)+i\mathcal{R}e(\overline{\varphi}\psi))\\
&\quad-\sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}2\mathcal{R}e(\overline{\varphi}\psi)\mathcal{I}m(\overline{\varphi}\psi)\,dx\\
&=i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{\psi}-\sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\psi^2\overline{\varphi}^2)\,dy.
\end{aligned}
$$

As in [6, section 4], we have

$$
\begin{aligned}
L\psi &= L(\exp(i\Lambda)\partial_x u)\\
&=\exp(i\Lambda)\Bigg[-\frac{i}{2}\partial_x(|u|^{2\sigma})\partial_x u+\sigma|u|^{2(\sigma-1)}\mathcal{I}m(\overline{u}\partial_x u)\partial_x u\\
&\quad-\sigma\int_{-\infty}^{x}\mathcal{I}m(\partial_x(|u|^{2(\sigma-1)}\overline{u})\partial_x u)\,dy\,\partial_x u\Bigg]\\
&=-\frac{i}{2}\partial_x(|\varphi|^{2\sigma})\psi+\sigma|\varphi|^{2(\sigma-1)}\mathcal{I}m(\overline{\varphi}\psi)\psi-\sigma\int_{-\infty}^{x}\partial_x(|u|^{2(\sigma-1)})\mathcal{I}m(\overline{u}\partial_x u)\,dy\,\psi\\
&=-\frac{i}{2}\partial_x(|\varphi|^{2\sigma})\psi+\sigma|\varphi|^{2(\sigma-1)}\psi\mathcal{I}m(\overline{\varphi}\psi)-\sigma\psi\int_{-\infty}^{x}\partial_x(|\varphi|^{2(\sigma-1)})\mathcal{I}m(\overline{\varphi}\psi)\,dy\\
&=\sigma|\varphi|^{2(\sigma-1)}\psi(\mathcal{I}m(\overline{\varphi}\psi)-i\mathcal{R}e(\overline{\varphi}\partial_x\varphi))\\
&\quad-\sigma\psi\int_{-\infty}^{x}(\sigma-1)|\varphi|^{2(\sigma-1)}2\mathcal{R}e(\overline{\varphi}\partial\varphi)\mathcal{I}m(\overline{\varphi}\psi)\,dy\\
&=\sigma|\varphi|^{2(\sigma-1)}\psi(\mathcal{I}m(\overline{\varphi}\psi)-i\mathcal{R}e(\overline{\varphi}\psi))\\
&\quad-\sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}2\mathcal{R}e(\overline{\varphi}\psi)\mathcal{I}m(\overline{\varphi}\psi)\mathcal{I}m(\overline{\varphi}\psi)\,dy\\
&=-i\sigma|\varphi|^{2(\sigma-1)}\psi^2\overline{\varphi}-\sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\psi^2\overline{\varphi}^2)\,dy.
\end{aligned}
$$

Thus, if $u$ solves (1.1) then $(\varphi, \psi)$ solves

$$
\begin{cases}
L\varphi = i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{\psi}-\sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\psi^2\overline{\varphi}^2)\,dy,\\
L\psi = -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\overline{\varphi}-\sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\psi^2\overline{\varphi}^2)\,dy.
\end{cases}
\tag{2.3}
$$

For convenience, we define

$$
P(\varphi,\psi)=i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{\psi}-\sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\psi^2\overline{\varphi}^2),\tag{2.4}
$$

$$
Q(\varphi,\psi)=-i\sigma|\varphi|^{2(\sigma-1)}\psi^2\overline{\varphi}-\sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\psi^2\overline{\varphi}^2).\tag{2.5}
$$

Let $R$ be the multi-soliton profile satisfying the assumption of Theorem 1.1. Define $h, k$ by

$$
\begin{aligned}
h(t,x)&=\exp\left(\frac{i}{2}\int_{-\infty}^{x}|R(t,x)|^{2\sigma}\,dy\right)R(t,x),\\
k&=\partial_x h-\frac{i}{2}|h|^{2\sigma}h.
\end{aligned}
$$

Since $R_j$ solves (1.1) for each $1 \leq j \leq K$, we have

$$LR + i|R|^{2\sigma} R_x = -\sum_j i|R_j|^{2\sigma} R_{jx} + i|R|^{2\sigma} R_x. \tag{2.6}$$

By Lemma 3.1 for $t \gg T_0$ large enough we have

$$\left\| -\sum_j i|R_j|^{2\sigma} R_{jx} + i|R|^{2\sigma} R_x \right\|_{H^2} \leq e^{-\lambda t}. \tag{2.7}$$

Thus, we rewrite (2.6) as follows:

$$LR + i|R|^{2\sigma} R_x = e^{-\lambda t}\Omega, \tag{2.8}$$

where

$$\Omega = e^{\lambda t}(-\sum_j i|R_j|^{2\sigma} R_{jx} + i|R|^{2\sigma} R_x). \tag{2.9}$$

By an elementary calculation, we have

$$\begin{cases} Lh = i\sigma|h|^{2(\sigma-1)}h^2\overline{k} - \sigma(\sigma-1)h\int_{-\infty}^x |h|^{2(\sigma-2)}\mathcal{I}m(k^2\overline{h}^2)\,dy + e^{-\lambda t}m(t,x), \\ Lk = -i\sigma|h|^{2(\sigma-1)}k^2\overline{h} - \sigma(\sigma-1)k\int_{-\infty}^x |h|^{2(\sigma-2)}\mathcal{I}m(k^2\overline{h}^2)\,dy + e^{-\lambda t}n(t,x). \end{cases} \tag{2.10}$$

where

$$m = \exp\left(\frac{i}{2}\int_{-\infty}^x |R|^{2\sigma}\,dy\right)\Omega - \sigma h\int_{-\infty}^x |R|^{2(\sigma-1)}\mathcal{I}m(\overline{R}\Omega)\,dy, \tag{2.11}$$

$$n = \exp\left(\frac{i}{2}\int_{-\infty}^x |R|^{2\sigma}\,dy\right)e^{-\lambda t}(\partial_x\Omega - \sigma\partial_x R\int_{-\infty}^x |R|^{2(\sigma-1)}\mathcal{I}m(\overline{R}\Omega)\,dy). \tag{2.12}$$

Since $\Omega$ is uniformly bounded in time in $H^2(\mathbb{R})$, we see that $m, n$ are uniformly bounded in time in $H^1(\mathbb{R})$. Let $\tilde{\varphi} = \varphi - h$ and $\tilde{\psi} = \psi - k$. Then $(\tilde{\varphi}, \tilde{\psi})$ solves:

$$\begin{cases} L\tilde{\varphi} = P(\varphi,\psi) - P(h,k) - e^{-\lambda t}m(t,x), \\ L\tilde{\psi} = Q(\varphi,\psi) - Q(h,k) - e^{-\lambda t}n(t,x). \end{cases} \tag{2.13}$$

Set $\eta = (\tilde{\varphi}, \tilde{\psi})$, $W = (h,k)$ and $f(\varphi,\psi) = (P(\varphi,\psi), Q(\varphi,\psi))$ and $-H = e^{-\lambda t}(m,n)$. We will find in Step 2 a solutions of (2.13) in Duhamel form:

$$\eta(t) = i\int_t^\infty S(t-s)[f(W+\eta) - f(W) + H](s)\,ds, \tag{2.14}$$

where $S(t)$ denote the Schrödinger group. Moreover, since $\psi = \partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$, we will prove in Step 3

$$\tilde{\psi} = \partial_x\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi}+h|^{2\sigma}(\tilde{\varphi}+h) - |h|^{2\sigma}h). \tag{2.15}$$

**Step 2 Existence of a solution of the system** From Lemma 3.8, there exists $T_* \gg 1$ such that for $T_0 \geq T_*$ there exists a unique solution $\eta$ of (2.13) defined on $[T_0, \infty)$ such that

$$\|\eta\|_X := \sup_{t>T_0}\left(e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{\lambda t}\|\partial_x\eta\|_{S([t,\infty))\times S([t,\infty))}\right) \leq 1. \tag{2.16}$$

Thus, for all $t \geq T_0$, we have

$$\|\tilde{\varphi}\|_{H^1} + \|\tilde{\psi}\|_{H^1} \lesssim e^{-\lambda t}. \tag{2.17}$$

**Step 3 Existence of a multi-soliton train** We first prove that the solution $\eta = (\tilde{\varphi}, \tilde{\psi})$ of (2.13) satisfies the relation (2.15). Set $\varphi = \tilde{\varphi} + h$, $\psi = \tilde{\psi} + k$ and $v = \partial_x \varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$ and $\tilde{v} = v - k$. Since $(\tilde{\varphi}, \tilde{\psi})$ solves (2.13) and $(h, k)$ solves (2.10), we have $(\varphi, \psi)$ solves (2.3). Furthermore,

$$Lv = \partial_x L\varphi - \frac{i}{2} L(|\varphi|^{2\sigma}\varphi). \tag{2.18}$$

Moreover,

$$\begin{aligned}
&L(|\varphi|^{2\sigma}\varphi)\\
&= (i\partial_t + \partial_x^2)(\varphi^{\sigma+1}\overline{\varphi}^{\sigma}) = i\partial_t(\varphi^{\sigma+1}\overline{\varphi}^{\sigma}) + \partial_x^2(\varphi^{\sigma+1}\overline{\varphi}^{\sigma})\\
&= i(\sigma+1)|\varphi|^{2\sigma}\partial_t\varphi + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_t\overline{\varphi}\\
&\quad + \partial_x((\sigma+1)|\varphi|^{2\sigma}\partial_x\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\overline{\varphi})\\
&= i(\sigma+1)|\varphi|^{2\sigma}\partial_t\varphi + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_t\overline{\varphi} + (\sigma+1)\left[\partial_x^2\varphi|\varphi|^{2\sigma} + \partial_x\varphi\partial_x(|\varphi|^{2\sigma})\right]\\
&\quad + \sigma\left[\partial_x^2\overline{\varphi}|\varphi|^{2(\sigma-1)}\varphi^2 + (\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + (\sigma-1)|\varphi|^{2(\sigma-2)}\varphi^3(\partial_x\overline{\varphi})^2\right]\\
&= (\sigma+1)|\varphi|^{2\sigma}(i\partial_t\varphi + \partial_x^2\varphi) + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(i\partial_t\overline{\varphi} + \partial_x^2\overline{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma})\\
&\quad + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\\
&= (\sigma+1)|\varphi|^{2\sigma}L\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(-\overline{L\varphi} + 2\partial_x^2\overline{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma})\\
&\quad + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3.
\end{aligned}$$

Combining with (2.18) and using (2.3), we have

$$\begin{aligned}
Lv &= \partial_x L\varphi - \frac{i}{2}L(|\varphi|^{2\sigma}\varphi)\\
&= \partial_x L\varphi - \frac{i}{2}\Big[(\sigma+1)|\varphi|^{2\sigma}L\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(-\overline{L\varphi} + 2\partial_x^2\overline{\varphi})\\
&\quad +(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big]\\
&= \partial_x(P(\varphi,\psi) - P(\varphi,v)) + \partial_x P(\varphi,v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\,(P(\varphi,\psi) - P(\varphi,v))\\
&\quad - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi,v) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi,\psi)} - \overline{P(\varphi,v)})\\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi,v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi}\\
&\quad - \frac{i}{2}\Big[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\\
&\quad +\sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big]\\
&= \partial_x(P(\varphi,\psi) - P(\varphi,v)) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\,(P(\varphi,\psi) - P(\varphi,v))\\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi,\psi)} - \overline{P(\varphi,v)}) + G(\varphi,v),
\end{aligned}$$

where $G(\varphi, v)$ contains the remaining ingredients and $G(\varphi, v)$ only depends on $\varphi$ and $v$:

$$
\begin{aligned}
&G(\varphi, v)\\
&= \partial_x P(\varphi, v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma} P(\varphi, v) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi, v)}\\
&\quad - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi} - \frac{i}{2}\Big[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\\
&\quad + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big].
\end{aligned}
\tag{2.19}
$$

As the calculations of $L\psi$ in the step 1, noting that the role of $v$ is similar to the role of $\psi$ in the process of calculation, we have $G(\varphi, v) = Q(\varphi, v)$ (see Lemma 3.7 for a detailed proof). Hence,

$$
\begin{aligned}
L\psi - Lv = Q(\varphi, \psi) - Q(\varphi, v) &- \partial_x(P(\varphi, \psi) - P(\varphi, v))\\
&+ \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\left(P(\varphi, \psi) - P(\varphi, v)\right)\\
&- \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \psi)} - \overline{P(\varphi, v)}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
L\tilde{\psi} - L\tilde{v} &= L\psi - Lv\\
&= Q(\varphi, \tilde{\psi}+k) - Q(\varphi, \tilde{v}+k) - \partial_x(P(\varphi, \tilde{\psi}+k) - P(\varphi, \tilde{v}+k)\\
&\quad + \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi, \tilde{\psi}+k) - P(\varphi, \tilde{v}+k))\\
&\quad - \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \tilde{\psi}+k)} - \overline{P(\varphi, \tilde{v}+k)}).
\end{aligned}
\tag{2.20}
$$

Multiplying both side of (2.20) by $\overline{\tilde{\psi} - \tilde{v}}$, taking imaginary part and integrating over space with integration by parts we obtain

$$
\begin{aligned}
&\frac{1}{2}\partial_t\|\tilde{\psi} - \tilde{v}\|_{L^2}^2\\
&= \mathcal{Im}\int_{\mathbb{R}}(Q(\varphi, \tilde{\psi}+k) - Q(\varphi, \tilde{v}+k))(\overline{\tilde{\psi}} - \overline{\tilde{v}})\,dx &(2.21)\\
&\quad - \mathcal{Im}\int_{\mathbb{R}}\partial_x(P(\varphi, \tilde{\psi}+k) - P(\varphi, \tilde{v}+k))(\overline{\tilde{\psi}} - \overline{\tilde{v}})\,dx &(2.22)\\
&\quad + (\sigma+1)\mathcal{Im}\int_{\mathbb{R}}\frac{i}{2}|\varphi|^{2\sigma}(P(\varphi, \tilde{\psi}+k) - P(\varphi, \tilde{v}+k))(\overline{\tilde{\psi}} - \overline{\tilde{v}})\,dx &(2.23)\\
&\quad - \sigma\mathcal{Im}\int_{\mathbb{R}}\frac{i}{2}|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \tilde{\psi}+k)} - \overline{P(\varphi, \tilde{v}+k)})(\overline{\tilde{\psi}} - \overline{\tilde{v}})\,dx. &(2.24)
\end{aligned}
$$

We denote by $A, B, C, D$ the terms (2.21), (2.22), (2.23) and (2.24) respectively. First, we try to estimate $A, B, C, D$ in term of $R$. We have

$$
\begin{aligned}
|A| &\lesssim \left|\int_{\mathbb{R}}(Q(\varphi, \tilde{\psi}+k) - Q(\varphi, \tilde{v}+k))(\overline{\tilde{\psi}} - \overline{\tilde{v}})\,dx\right|\\
&\lesssim \left|\int_{\mathbb{R}}|\varphi|^{2(\sigma-1)}\overline{\varphi}((\tilde{\psi}+k)^2 - (\tilde{v}+k)^2)(\overline{\tilde{\psi}} - \overline{\tilde{v}})\,dx\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\int_{\mathbb{R}}\left[(\tilde{\psi}+k)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m((\tilde{\psi}+k)^2\overline{\varphi}^2)\,dy\right.\right.\\
&\left.\left.-(\tilde{v}+k)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m((\tilde{v}+k)^2\overline{\varphi}^2)\,dy\right](\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
\lesssim&\left|\int_{\mathbb{R}}|\varphi|^{2(\sigma-1)}\overline{\varphi}((\tilde{\psi}+k)^2-(\tilde{v}+k)^2)(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&+\left|\int_{\mathbb{R}}\left[(\tilde{\psi}-\tilde{v})\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m((\tilde{\psi}+k)^2\overline{\varphi}^2)\,dy\right](\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&+\left|\int_{\mathbb{R}}\left[(\tilde{v}+k)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy\right](\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
\lesssim&\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi\|_{L^\infty}^{2\sigma-1}\|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty}\\
&+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\left\|\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m((\tilde{\psi}+k)^2\overline{\varphi}^2)\,dy\right\|_{L_x^\infty}\\
&+\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\tilde{v}+k\|_{L^2}\left\|\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy\right\|_{L_x^\infty}\\
\lesssim&\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi\|_{L^\infty}^{2\sigma-1}\|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2(\sigma-1)}(\tilde{\psi}+k)^2\|_{L_x^1}\\
&+\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\tilde{v}+k\|_{L^2}\|\varphi^{2(\sigma-1)}((\tilde{\psi}+k)^2-(\tilde{v}+k)^2)\|_{L^1}\\
\lesssim&\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi\|_{L^\infty}^{2\sigma-1}\|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2(\sigma-1)}(\tilde{\psi}+k)^2\|_{L^1}\\
&+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\tilde{v}+k\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}\\
\lesssim&\|\tilde{\psi}-\tilde{v}\|_{L^2}^2K_1,
\end{aligned}
\tag{2.25}
$$

where,

$$
\begin{aligned}
K_1:=&\|\varphi\|_{L^\infty}^{2\sigma-1}\|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty}+\|\varphi^{2(\sigma-1)}(\tilde{\psi}+k)^2\|_{L^1}\\
&+\|\tilde{v}+k\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|B|\lesssim&\left|\int_{\mathbb{R}}\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2(\overline{\tilde{\psi}}-\overline{\tilde{v}}))(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&+\left|\int_{\mathbb{R}}\partial_x\left(\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy\right)(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
\lesssim&\left|\int_{\mathbb{R}}\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)(\overline{\tilde{\psi}}-\overline{\tilde{v}})^2\,dx\right|+\left||\varphi|^{2(\sigma-1)}\varphi^2\frac{1}{2}\partial_x((\tilde{\psi}-\tilde{v})^2)\,dx\right|\\
&+\left|\int_{\mathbb{R}}\partial_x\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))\,dy(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&+\left|\int_{\mathbb{R}}\varphi|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|.
\end{aligned}
\tag{2.26}
$$

By using integration by parts for the second term of (2.26) and using Hölder inequality we have

$$
\begin{aligned}
|B|\lesssim&\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}\\
&+\|\partial_x\varphi\|_{L^2}\|\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))\,dy\|_{L_x^\infty}\|\tilde{\psi}-\tilde{v}\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty}\\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}\\
&\quad+\|\partial_x\varphi\|_{L^2}\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k)\|_{L^1_x} \qquad (2.27)\\
&\quad+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty}\\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}\\
&\quad+\|\partial_x\varphi\|_{L^2}\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty}\\
&=\|\tilde{\psi}-\tilde{v}\|_{L^2}^2K_2, \qquad (2.28)
\end{aligned}
$$

where

$$
\begin{aligned}
K_2:\ =\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}+\|\partial_x\varphi\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}\\
+\tilde{v}+2k)\|_{L^2}+\|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty}.
\end{aligned}
$$

Using (2.4), we have

$$
\begin{aligned}
|C|&\lesssim\left|\int_{\mathbb{R}}|\varphi|^{2\sigma}|\varphi|^{2(\sigma-1)}\varphi^2(\overline{\tilde{\psi}}-\overline{\tilde{v}})^2\,dx\right|\\
&\quad+\left|\int_{\mathbb{R}}|\varphi|^{2\sigma}\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&\lesssim\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty}\\
&\quad+\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2\sigma+1}\|_{L^2}\left\|\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))\,dy\right\|_{L_x^\infty}\\
&\lesssim\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k)\|_{L^1}\\
&\lesssim\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}\\
&=\|\tilde{\psi}-\tilde{v}\|_{L^2}^2K_3, \qquad (2.29)
\end{aligned}
$$

where

$$
K_3:=\|\varphi^{4\sigma}\|_{L^\infty}+\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}.
$$

Now, we give an estimate for $D$. We have

$$
\begin{aligned}
|D|&\lesssim\left|\int_{\mathbb{R}}|\varphi|^{2(\sigma-1)}\varphi^2|\varphi|^{2(\sigma-1)}\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&\quad+\left|\int_{\mathbb{R}}|\varphi|^{2(\sigma-1)}\varphi^2\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&\lesssim\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty}\\
&\quad+\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2\sigma+1}\|_{L^2}\left\|\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy\right\|_{L_x^\infty}\\
&\lesssim\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k)\|_{L^1}\\
&\lesssim\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}\\
&=\|\tilde{\psi}-\tilde{v}\|_{L^2}^2K_3, \qquad (2.30)
\end{aligned}
$$

Combining (2.25), (2.27), (2.29) and (2.30), we have

$$\left|\partial_t \|\tilde{\psi} - \tilde{v}\|_{L^2}^2\right| \lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 (K_1 + K_2 + K_3).$$

Using the Grönwall inequality, we have for any $t < N < \infty$,

$$\begin{aligned}\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 &\lesssim \|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2 \exp\left(\int_t^N (K_1 + K_2 + K_3)\, ds\right) \\ &\leq e^{-2\lambda N} \exp\left(\int_t^N (K_1 + K_2 + K_3)\, ds\right). \end{aligned} \tag{2.31}$$

The second inequality is by (2.17). Now, we try to estimate $K_1 + K_2 + K_3$ in term of $R$. When we have this kind of estimate, we will use the assumption (1.10) to obtain that $\tilde{\psi} = \tilde{v}$. We have

$$\begin{aligned}&\int_t^N (K_1 + K_2 + K_3)\, ds \\ &= \int_t^N \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\varphi^{2(\sigma-1)}(\tilde{\psi} + k)^2\|_{L^1} \\ &\quad + \|\tilde{v} + k\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2}\, ds \qquad (2.32) \\ &\quad + \int_t^N \|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} + \|\partial_x\varphi\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \\ &\quad + \|\varphi^{2\sigma-1}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty}\, ds \qquad (2.33) \\ &\quad + \int_t^N \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2}\, ds. \qquad (2.34)\end{aligned}$$

Using (2.16) and (2.17), we have

$$\|\varphi\|_{L^\infty} \leq \|\tilde{\varphi}\|_{L^\infty} + \|h\|_{L^\infty} \lesssim 1 + \|h\|_{L^\infty} \tag{2.35}$$
$$\|\varphi\|_{L^2} \leq \|\tilde{\varphi}\|_{L^2} + \|h\|_{L^2} \lesssim 1 + \|h\|_{L^2} \tag{2.36}$$
$$\|\psi\|_{L^\infty} \lesssim 1 \tag{2.37}$$

We denote by $Z_1$, $Z_2$, $Z_3$ the terms (2.32), (2.33) and (2.34) respectively. Using (2.35), (2.36), (2.37), (2.16) and (2.17), for $N \gg t$, we have

$$\begin{aligned}|Z_1| &\lesssim \|\varphi\|_{L^4(t,N)L^\infty}^3 \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-2)} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^4(t,N)L^\infty} \\ &\quad + (N-t)\|\varphi\|_{L^\infty L^\infty}^{2(\sigma-1)} (\|\tilde{\psi}\|_{L^\infty L^2} + \|k\|_{L^\infty L^2})^2 \\ &\quad + \|\tilde{v} + k\|_{L^{\frac{4}{3}}(t,N)L^2} \|\varphi\|_{L^\infty L^2} \|\varphi\|_{L^\infty L^\infty}^{2(\sigma-1)} (\|\tilde{\psi} + \tilde{v}\|_{L^4(t,N)L^\infty} + \|k\|_{L^4(t,N)L^\infty}) \\ &\lesssim (N-t)^{\frac{3}{4}} \|\varphi\|_{L^\infty L^\infty}^{2\sigma-1} (1 + \|k\|_{L^\infty L^\infty}(N-t)^{\frac{1}{4}}) \\ &\quad + (N-t)(1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|k\|_{L^\infty L^2}^2) \\ &\quad + (N-t)^{\frac{3}{4}}(1 + \|k\|_{L^\infty L^2})(1 + \|h\|_{L^\infty L^2})(1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + (N-t)^{\frac{1}{4}}\|k\|_{L^\infty L^\infty}) \\ &\lesssim (N-t)\|k\|_{L^\infty L^\infty}(1 + \|h\|_{L^\infty L^\infty}^{2\sigma-1}) + (N-t)(1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|k\|_{L^\infty L^2}^2) \\ &\quad + (N-t)\|k\|_{L^\infty L^\infty}(1 + \|k\|_{L^\infty L^2})(1 + \|h\|_{L^\infty L^2})(1 + \|h\|_{L^\infty L^\infty}^{2(\sigma-1)}) \\ &:= (N-t) W_1(h, k).\end{aligned}$$

Similarly, for $N \gg t$, we have

$$
\begin{aligned}
|Z_2| &\lesssim \|\partial_x \varphi \varphi^{2\sigma-1}\|_{L^1(t,N)L^\infty} + (N-t)\|\partial_x \varphi\|_{L^\infty(t,N)L^2}\|\varphi\|^{2(\sigma-1)}_{L^\infty L^\infty}\|\tilde{\psi}+\tilde{v}+k\|_{L^\infty(t,N)L^2} \\
&\quad + (N-t)^{\frac{3}{4}}\|\varphi\|^{2\sigma-1}_{L^\infty L^\infty}(\|\tilde{\psi}+\tilde{v}\|_{L^4(t,N)L^\infty} + \|k\|_{L^4(t,N)L^\infty}) \\
&\lesssim (N-t)^{\frac{3}{4}}(\|\partial_x \tilde{\varphi}\|_{L^4(t,N)L^\infty} + \|\partial_x h\|_{L^4(t,N)L^\infty})\|\varphi\|^{2\sigma-1}_{L^\infty L^\infty} \\
&\quad + (N-t)(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|k\|_{L^\infty L^2}) \\
&\quad + (N-t)^{\frac{3}{4}}(1+\|h\|^{2\sigma-1}_{L^\infty L^\infty})(1+(N-t)^{\frac{1}{4}}\|k\|_{L^\infty L^\infty}) \\
&\lesssim (N-t)\|\partial_x h\|_{L^\infty L^\infty}(1+\|h\|^{2\sigma-1}_{L^\infty L^\infty}) + (N-t)(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|k\|_{L^\infty L^2}) \\
&\quad + (N-t)\|k\|_{L^\infty L^\infty}(1+\|h\|^{2\sigma-1}_{L^\infty L^\infty}) \\
&:= (N-t)W_2(h,k),
\end{aligned}
$$

and

$$
\begin{aligned}
|Z_3| &\lesssim (N-t)(\|\tilde{\varphi}\|_{L^\infty L^\infty} + \|h\|_{L^\infty L^\infty})^{4\sigma} \\
&\quad + (N-t)\|\varphi\|_{L^\infty L^2}\|\varphi\|^{2\sigma}_{L^\infty L^\infty}\|\varphi\|^{2(\sigma-1)}_{L^\infty L^\infty}(\|\tilde{\psi}+\tilde{v}\|_{L^\infty L^2} + \|k\|_{L^\infty L^2}) \\
&\lesssim (N-t)(1+\|h\|^{4\sigma}_{L^\infty L^\infty}) + (N-t)(1+\|h\|_{L^\infty L^2})(1+\|h\|^{4\sigma-2}_{L^\infty L^\infty})(1+\|k\|_{L^\infty L^2}) \\
&:= (N-t)W_3(h,k).
\end{aligned}
$$

Hence, from (2.31), we have

$$
\begin{aligned}
\|\tilde{\psi}(t)-\tilde{v}(t)\|^2_{L^2} &\lesssim e^{-2\lambda N}\exp\left(\int_t^N (K_1+K_2+K_3)\,ds\right) \\
&\lesssim e^{-2\lambda N}\exp((N-t)(W_1(h,k)+W_2(h,k)+W_3(h,k))) \qquad (2.38)
\end{aligned}
$$

The above estimate is not enough explicit. As said above, we would like to estimate the right hand side of (2.38) in terms of $R$. Noting that $|h| = |R|$ and $|k| = |\partial_x R|$, we have

$$
\begin{aligned}
W_1(h,k) &= \|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|^{2\sigma-1}_{L^\infty L^\infty}) + (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|\partial_x R\|^2_{L^\infty L^2}) \\
&\quad + \|\partial_x R\|_{L^\infty L^\infty}(1+\|\partial_x R\|_{L^\infty L^2})(1+\|R\|_{L^\infty L^2})(1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty}) \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})\big[\|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|_{L^\infty L^\infty}) + (1+\|\partial_x R\|_{L^\infty L^2}) \\
&\quad +\|\partial_x R\|_{L^\infty L^\infty}(1+\|\partial_x R\|_{L^\infty L^2})(1+\|R\|_{L^\infty L^2})\big] \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})\times \\
&\quad \times\big[\|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|_{L^\infty H^1}) + (1+\|R\|^2_{L^\infty H^1}) + \|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|^2_{L^\infty H^1})\big] \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty}).
\end{aligned}
$$

Similarly, by noting that $|\partial_x h| \le |k| + |h|^{2\sigma+1}$, we have

$$
\begin{aligned}
W_2(h,k) &\lesssim (\|k\|_{L^\infty L^\infty} + \|h\|^{2\sigma+1}_{L^\infty L^\infty})(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|h\|_{L^\infty L^\infty}) \\
&\quad + (1+\|h\|^{2(\sigma-1)})(1+\|k\|_{L^\infty L^2}) + \|k\|_{L^\infty L^\infty}(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|h\|_{L^\infty L^\infty}) \\
&\lesssim (1+\|h\|^{2(\sigma-1)})\times \\
&\quad \times\Big[(\|k\|_{L^\infty L^\infty} + \|h\|^{2\sigma+1}_{L^\infty L^\infty})(1+\|h\|_{L^\infty L^\infty}) \\
&\quad +(1+\|k\|_{L^\infty L^2}) + \|k\|_{L^\infty L^\infty}(1+\|h\|_{L^\infty L^\infty})\Big] \\
&\lesssim (1+\|h\|^{2(\sigma-1)})\times
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[(1+\|h\|_{L^\infty L^\infty})(\|k\|_{L^\infty L^\infty}+\|h\|^{2\sigma+1}_{L^\infty L^\infty})+(1+\|k\|_{L^\infty L^2})\right]\\
&=(1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})\times\\
&\quad\times \left[(1+\|R\|_{L^\infty L^\infty})(\|\partial_x R\|_{L^\infty L^\infty}+\|R\|^{2\sigma+1}_{L^\infty L^\infty})+(1+\|\partial_x R\|_{L^\infty L^2})\right]\\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty}+\|R\|^{2\sigma+1}_{L^\infty L^\infty})\\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty}+\|R\|^{2\sigma+1}_{L^\infty L^\infty}),
\end{aligned}
$$

and

$$
\begin{aligned}
W_3(h,k)&=(1+\|R\|^{4\sigma}_{L^\infty L^\infty})+(1+\|R\|_{L^\infty L^2})(1+\|R\|^{4\sigma-2}_{L^\infty L^\infty})(1+\|\partial_x R\|_{L^\infty L^2})\\
&\lesssim (1+\|R\|^{4\sigma-2}_{L^\infty L^\infty})\left[(1+\|R\|^2_{L^\infty L^\infty})+(1+\|R\|_{L^\infty L^2})(1+\|\partial_x R\|_{L^\infty L^2})\right]\\
&\lesssim (1+\|R\|^{4\sigma-2}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1}).
\end{aligned}
$$

Combining the above estimates, we have

$$
\begin{aligned}
&W_1(h,k)+W_2(h,k)+W_3(h,k)\\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})\left(1+\|\partial_x R\|_{L^\infty L^\infty}+\|R\|^{2\sigma+1}_{L^\infty L^\infty}\right)\\
&\quad+(1+\|R\|^{4\sigma-2}_{L^\infty L^\infty})\left(1+\|R\|^2_{L^\infty H^1}\right)\\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})\left(1+\|\partial_x R\|_{L^\infty L^\infty}+\|R\|^{2\sigma+1}_{L^\infty L^\infty}\right)\\
&\quad+(1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^{2\sigma}_{L^\infty L^\infty})\left(1+\|R\|^2_{L^\infty H^1}\right)\\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})\left(1+\|\partial_x R\|_{L^\infty L^\infty}+\|R\|^{2\sigma+1}_{L^\infty L^\infty}\right).
\end{aligned}
$$

Thus, there exists a positive constant $C_0$ such that

$$
\begin{aligned}
&W_1(h,k)+W_2(h,k)+W_3(h,k)\\
&\le C_0\left((1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})\left(1+\|\partial_x R\|_{L^\infty L^\infty}+\|R\|^{2\sigma+1}_{L^\infty L^\infty}\right)\right).
\end{aligned}
$$

Note that the constant $C_0$ in the right side is independent of parameters $\omega_j, c_j$. Let $C_*=16C_0$. Using the assumption (1.10), we have

$$
W_1(h,k)+W_2(h,k)+W_3(h,k)\le \frac{v_*}{16}=\lambda,
$$

for $t$ large enough. Thus, by (2.38), we have

$$
\|\tilde{\psi}(t)-\tilde{v}(t)\|^2_{L^2}\le e^{-2\lambda N+(N-t)\lambda},
$$

for $t$ large enough. Letting $N\to\infty$ in the above estimate, we obtain

$$
\|\tilde{\psi}(t)-\tilde{v}\|^2_{L^2}=0,
$$

for all $t$ large enough. This implies that

$$
\tilde{\psi}=\partial_x\varphi-\frac{i}{2}|\varphi|^{2\sigma}\varphi-k, \tag{2.39}
$$

which is equivalent to (2.15) and then

$$
\psi=\partial_x\varphi-\frac{i}{2}|\varphi|^{2\sigma}\varphi. \tag{2.40}
$$

Moreover, since $(\tilde{\psi}, \tilde{\varphi})$ solves (2.13) we have $(\psi, \varphi)$ solves (2.3). Combining with (2.40), if we set

$$u = \exp\left(-\frac{i}{2}\int_{-\infty}^{x} |\varphi|^{2\sigma}\, dy\right)\varphi \tag{2.41}$$

then $u$ solves (1.1) by Lemma 3.6. Furthermore, by Lemma 3.6, we have

$$\begin{aligned}\|u - R\|_{H^1} &= \left\|\exp\left(-\frac{i}{2}|\varphi|^{2\sigma}\, dy\right)\varphi - \exp\left(\frac{i}{2}|h|^{2\sigma}\, dy\right)h\right\|_{H^1}\\ &\lesssim C(\|\varphi\|_{H^1}, \|h\|_{H^1})\|\varphi - h\|_{H^1} \lesssim \|\tilde{\varphi}\|_{H^1} \lesssim e^{-\lambda t},\end{aligned}$$

Thus for $t$ large enough, we have

$$\|u - R\|_{H^1} \leq Ce^{-\lambda t}, \tag{2.42}$$

for $\lambda = \frac{1}{16}v_*$ and $C = C(\omega_1, \ldots, \omega_K, c_1, \ldots, c_K)$. This completes the proof of Theorem 1.1.

## 3 Some Technical Lemmas

### 3.1 Properties of Solitons

In this section, we give the proof of (2.7). We have the following result.

**Lemma 3.1** *Let $\lambda = \frac{1}{16}v_*$, where $v_*$ is defined by (1.10). There exist $C > 0$ and such that for $t > 0$ large enough, the estimate (2.7) uniformly holds in time.*

***Proof*** First, we need some estimates on the profile. We have

$$\begin{aligned}|R_j(t,x)| &= |\psi_{\omega_j,c_j}(t,x)| = |\phi_{\omega_j,c_j}(x - c_j t)| = |\varphi_{\omega_j,c_j}(x - c_j t)|\\ &\approx \left(\frac{4\omega_j - c_j^2}{2\sqrt{\omega_j}\left(\cosh(\sigma h_j(x - c_j t)) - \frac{c_j}{2\sqrt{\omega_j}}\right)}\right)^{\frac{1}{2\sigma}}\\ &\lesssim \left(\frac{4\omega_j - c_j^2}{2\sqrt{\omega_j}\left(\cosh(\sigma h_j(x - c_j t)) - \frac{|c_j|}{2\sqrt{\omega_j}}\cosh(\sigma h_j(x - c_j t))\right)}\right)^{\frac{1}{2\sigma}}\\ &\lesssim \left(\frac{4\omega_j - c_j^2}{(2\sqrt{\omega_j} - |c_j|)\cosh(\sigma h_j(x - c_j t))}\right)^{\frac{1}{2\sigma}} \lesssim \left(\frac{2\sqrt{\omega_j} + |c_j|}{\cosh(\sigma h_j(x - c_j t))}\right)^{\frac{1}{2\sigma}}\\ &\lesssim_{\omega_j,|c_j|} e^{-\frac{h_j}{2}|x - c_j t|},\end{aligned}$$

Furthermore,

$$\partial_x \varphi_{\omega_j,c_j}(y) \approx \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{-\sinh(\sigma h_j y)}{\left(\cosh(\sigma h_j y) - \frac{c_j}{\sqrt{\omega_j}}\right)^{1+\frac{1}{2\sigma}}}.$$

Thus,

$$|\partial_x \varphi_{\omega_j,c_j}(y)| \lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{|\sinh(\sigma h_j y)|}{\left(1-\frac{|c_j|}{\sqrt{\omega_j}}\right)^{1+\frac{1}{2\sigma}} \cosh(\sigma h_j y)^{1+\frac{1}{2\sigma}}}$$
$$\lesssim_{\omega_j,|c_j|} \frac{1}{\cosh(\sigma h_j y)^{\frac{1}{2\sigma}}} \lesssim_{\omega_j,|c_j|} e^{-\frac{h_j}{2}|y|},$$

Using the above estimates, we have

$$\begin{aligned}
|\partial_x R_j(t,x)| &= |\partial_x \psi_{\omega_j,c_j}(t,x)| = |\partial_x \phi_{\omega_j,c_j}(x-c_jt)| \\
&= |\partial_x \varphi_{\omega_j,c_j}(x-c_jt) + i\varphi_{\omega_j,c_j}(x-c_jt)\partial_x\theta_{\omega_j,c_j}(x-c_jt)| \\
&\lesssim |\partial_x \varphi_{\omega_j,c_j}(x-c_jt)| + |\varphi_{\omega_j,c_j}(x-c_jt)||\partial_x\theta_{\omega_j,c_j}(x-c_jt)| \\
&\lesssim_{\omega_j,|c_j|} |\partial_x \varphi_{\omega_j,c_j}(x-c_jt)| + e^{\frac{-h_j}{2}|x-c_jt|} \\
&\lesssim_{\omega_j,|c_j|} e^{-\frac{h_j}{2}|x-c_jt|}.
\end{aligned}$$

By similar arguments, we have

$$|\partial_x^2 R_j(t,x)| + |\partial_x^3 R_j(t,x)| \lesssim_{\omega_j,|c_j|} e^{\frac{-h_j}{2}|x-c_jt|},$$

For convenience, we set

$$\begin{aligned}
&\chi = -i|R|^{2\sigma}\partial_x R + i\Sigma_j |R_j|^{2\sigma}\partial_x R_j, \\
&f(R,\overline{R},\partial_x R) = i|R|^{2\sigma}\partial_x R, \\
&g(R,\overline{R},\partial_x R,\partial_x\overline{R},\partial_x^2 R) = i\partial_x(|R|^{2\sigma}\partial_x R), \\
&r(R,\partial_x R,\ldots,\partial_x^3 R,\partial_x\overline{R},\partial_x^2\overline{R}) = i\partial_x^2(|R|^{2\sigma}\partial_x R).
\end{aligned}$$

Fix $t > 0$, for each $x \in \mathbb{R}$, choose $m = m(x) \in \{1, 2, \ldots, K\}$ so that

$$|x - c_m t| = \min_j |x - c_j t|.$$

For $j \neq m$ we have

$$|x-c_jt| \geq \frac{1}{2}(|x-c_jt| + |x-c_mt|) \geq \frac{1}{2}|c_jt - c_mt| = \frac{t}{2}|c_j - c_m|.$$

Thus, we have

$$\begin{aligned}
&|(R-R_m)(t,x)| + |\partial_x(R-R_m)(t,x)| + |\partial_x^2(R-R_m)(t,x)| + |\partial_x^3(R-R_m)(t,x)| \\
&\leq \sum_{j\neq m}(|R_j(t,x)| + |\partial_x R_j(t,x)| + |\partial_x^2 R_j(t,x)| + |\partial_x^3 R_j(t,x)|) \\
&\lesssim_{\omega_1,\ldots,\omega_K,|c_1|,\ldots,|c_K|} \delta_m(t,x) := \sum_{j\neq m} e^{\frac{-h_j}{2}|x-c_jt|}.
\end{aligned}$$

Recall that

$$v_* = \inf_{j\neq k} h_j|c_j - c_k|.$$

We have

$$
\begin{aligned}
&|(R-R_m)(t,x)| + |\partial_x(R-R_m)(t,x)| + |\partial_x^2(R-R_m)(t,x)| + |\partial_x^3(R-R_m)(t,x)| \\
&\quad \lesssim \delta_m(t,x) \\
&\quad \lesssim e^{-\frac{1}{4}v_* t}.
\end{aligned}
$$

We see that $f, g, r$ are polynomials in $R$, $\partial_x R$, $\partial_x^2 R$, $\partial_x^3 R$, $\partial_x \overline{R}$ and $\partial_x^2 \overline{R}$. Denote

$$
A = \sup_{|u|+|\partial_x u|+|\partial_x^2 u|+|\partial_x^3 u| \le \sum_j \|R_j\|_{H^4}} (|df| + |dg| + |dr|).
$$

We have

$$
\begin{aligned}
&|\chi| + |\partial_x \chi| + |\partial_x^2 \chi| \\
&\le |f(R,\overline{R},\partial_x R) - f_{R_m,\partial_x \overline{R}_m,R_m}| + |g(R,\overline{R},\partial_x R,..) - g(R_m,\overline{R}_m,\partial_x R_m,..)| \\
&\quad + |r(R,\partial_x R,\dots,\partial_x^3 R,\overline{R},..) - r(R_m,\partial_x R_m,\dots,\partial_x^3 R_m,\overline{R}_m,..)| \\
&\quad + \Sigma_{j\ne m}(f(R_j,\overline{R}_j,\partial_x R_j) + g(R_j,\partial_x R_j,\partial_x^2 R_j,\overline{R}_j,\partial_x \overline{R}_j) + r(R_j,\dots,\partial_x^3 R_j,\overline{R}_j,\dots,\partial_x^2 \overline{R}_j)) \\
&\lesssim A(|R-R_m| + |\partial_x(R-R_m)| + |\partial_x^2(R-R_m)| + |\partial_x^3(R-R_m)|) \\
&\quad + A\Sigma_{j\ne m}(|R_j| + |\partial_x R_j| + |\partial_x^2 R_j| + |\partial_x^3 R_j|) \\
&\lesssim 2A\Sigma_{j\ne m}(|R_j| + |\partial_x R_j| + |\partial_x^2 R_j| + |\partial_x^3 R_j|) \\
&\lesssim 2A\delta_m(t,x).
\end{aligned}
$$

In particular,

$$
\|\chi\|_{W^{2,\infty}} \lesssim e^{-\frac{1}{4}v_* t}. \tag{3.1}
$$

Moreover,

$$
\begin{aligned}
\|\chi\|_{W^{2,1}} &\lesssim \Sigma_j(\||R_j|^{2\sigma}\partial_x R_j\|_{L^1} + \|\partial_x(|R_j|^{2\sigma}\partial_x R_j)\|_{L^1} + \|\partial_x^2(|R_j|^{2\sigma}\partial_x R_j)\|_{L^1}) \\
&\lesssim \Sigma_j(\|R_j\|_{H^1}^( 2\sigma+1) + \|R_j\|_{H^2}^{2\sigma+1} + \|R_j\|_{H^3}^{2\sigma+1}) < \infty.
\end{aligned}
$$

Thus, using Hölder inequality we obtain

$$
\|\chi\|_{H^2} \lesssim_{\omega_1,\dots,\omega_K,|c_1|,\dots,|c_K|} e^{-\frac{1}{8}v_* t}.
$$

It follows that if $t \gg \max\{\omega_1,\dots,\omega_K,|c_1|,\dots,|c_K|\}$ is large enough then

$$
\|\chi\|_{H^2} \le e^{-\frac{1}{16}v_* t} = e^{-\lambda t}.
$$

This implies the desired result. □

### 3.2 Some Useful Estimates

**Lemma 3.2** *Fix $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$. We have*

$$
\left|(w+z)^\alpha(\overline{w}+\overline{z})^\beta - w^\alpha \overline{w}^\beta\right| \lesssim |z|^{\alpha+\beta} + |z||w|^{\alpha+\beta-1}. \tag{3.2}
$$

***Proof*** We may assume that $w \ne 0$. We may assume $w = 1$ be replacing $z$ by $\frac{z}{w}$. Let

$$
f(t) = (1+tz)^\alpha(1+t\overline{z})^\beta.
$$

It suffices to show

$$|f(1) - f(0)| \lesssim |z|^{\alpha+\beta} + |z|. \tag{3.3}$$

When $|z| \geq \frac{1}{2}$, we have $|f(1)-f(0)| \lesssim |z|^{\alpha+\beta}$. When $|z| < \frac{1}{2}$, we have $f(1)-f(0) = f'(t)$ for some $t \in (0, 1)$ by mean value theorem, but $\sup_{0<t<1} |f'(t)| \lesssim |z|$. This shows (3.3) and hence (3.2). □

As a consequence of (3.2), we have the following result.

**Lemma 3.3** *Fix* $\alpha > 0$. *We have*

$$\big||w+z|^{\alpha} - |w|^{\alpha}\big| \lesssim |z|^{\alpha} + |z||w|^{\alpha-1}. \tag{3.4}$$

***Proof*** Using (3.2) for $\alpha = \beta$, we obtain (3.4). □

**Lemma 3.4** *Let* $w_1, w_2, \eta_1, \eta_2 \in \mathbb{C}$ *and* $\sigma \geq \frac{3}{2}$. *Define*

$$J(\eta_1, \eta_2) = |w_1 + \eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2 + \eta_2)^2 (\overline{w}_1 + \overline{\eta}_1)^2).$$

*Then*

$$|J(\eta_1, \eta_2) - J(0, 0)| \lesssim |\eta|(|\eta|^{2\sigma-1} + |W|^{2\sigma-1}),$$

*where* $|\eta| = |\eta_1| + |\eta_2|$ *and* $|W| = |w_1| + |w_2|$.

***Proof*** We have

$$|J(\eta_1, \eta_2) - J(0, 0)| \leq |J(\eta_1, \eta_2) - J(\eta_1, 0)| + |J(\eta_1, 0) - J(0, 0)|.$$

Moreover,

$$\begin{aligned}
|J(\eta_1, \eta_2) - J(\eta_1, 0)| &\lesssim |w_1 + \eta_1|^{2(\sigma-1)} |(w_2 + \eta_2)^2 - w_2^2| \\
&\lesssim |w_1 + \eta_1|^{2(\sigma-1)} |\eta_2|(|\eta_2| + |w_2|) \\
&\lesssim (|W| + |\eta|)^{2\sigma-1} |\eta| \\
&\lesssim (|W|^{2\sigma-1} + |\eta|^{2\sigma-1})|\eta|.
\end{aligned}$$

Thus, it suffices to check the other term. Rewrite

$$\begin{aligned}
J(\eta_1, 0) &= \frac{1}{2i} |w_1 + \eta_1|^{2(\sigma-2)} \left[ w_2^2 (\overline{w}_1 + \overline{\eta}_1)^2 - \overline{w}_2^2 (w_1 + \eta_1)^2 \right] \\
&= \frac{w_2^2}{2i} (w_1 + \eta_1)^{\sigma-2} (\overline{w}_1 + \overline{\eta}_1)^{\sigma} - \frac{\overline{w}_2^2}{2i} (w_1 + \eta_1)^{\sigma} (\overline{w}_1 + \overline{\eta}_1)^{\sigma-2}.
\end{aligned}$$

By (3.2),

$$\begin{aligned}
|J(\eta_1, 0) - J(0, 0)| &\lesssim |w_2|^2 (|\eta_1|^{2\sigma-2} + |\eta_1||w_1|^{2\sigma-3}) \\
&\lesssim |W|^2 |\eta| (|W|^{2\sigma-3} + |\eta|^{2\sigma-3}) \\
&\lesssim |\eta|(|\eta|^{2\sigma-1} + |W|^{2\sigma-1}),
\end{aligned}$$

where in the second estimate, the term $|\eta_1|^{2\sigma-2}$ is superlinear provided $\sigma \geq \frac{3}{2}$ and in the last estimate, we use the Cauchy inequality $|W|^2 |\eta|^{2\sigma-3} \lesssim |W|^{2\sigma-1} + |\eta|^{2\sigma-1}$ provided $\sigma \geq \frac{3}{2}$. This implies the desired result. □

**Lemma 3.5** *Let $w, \eta \in \mathbb{C}$ and $\sigma \geq 1$. We have*

$$|\partial_x(|w+\eta|^{2(\sigma-1)}(w+\eta)^2 - |w|^{2(\sigma-1)}w^2)| \lesssim |\partial_x \eta|(|w|^{2\sigma-1} + |\eta|^{2\sigma-1}) \\ +|\partial_x w||\eta|(|\eta|^{2\sigma-2} + |w|^{2\sigma-2}).$$

***Proof*** We have

$$\begin{aligned}
&|\partial_x(|w+\eta|^{2(\sigma-1)}(w+\eta)^2 - |w|^{2(\sigma-1)}w^2)| \\
&= |\partial_x((w+\eta)^{\sigma+1}(\overline{w}+\overline{\eta})^{\sigma-1} - w^{\sigma+1}\overline{w}^{\sigma-1})| \\
&\lesssim |(\sigma+1)(\partial_x w + \partial_x \eta)(w+\eta)^{\sigma}(\overline{w}+\overline{\eta})^{\sigma-1} - (\sigma+1)\partial_x w w^{\sigma}\overline{w}^{\sigma-1}| \\
&\quad + |(\sigma-1)(w+\eta)^{\sigma+1}(\partial_x\overline{w} + \partial_x\overline{\eta})(\overline{w}+\overline{\eta})^{\sigma-2} - (\sigma-1)w^{\sigma+1}\partial_x\overline{w}\overline{w}^{\sigma-2}|.
\end{aligned}$$

We only need to treat the first term. The second term is similar. Using $\sigma \geq \frac{3}{2}$ and (3.2), we have

$$\begin{aligned}
&|(\sigma+1)(\partial_x w + \partial_x \eta)(w+\eta)^{\sigma}(\overline{w}+\overline{\eta})^{\sigma-1} - (\sigma+1)\partial_x w w^{\sigma}\overline{w}^{\sigma-1}| \\
&\lesssim |\partial_x \eta||w+\eta|^{2\sigma-1} + |\partial_x w||(w+\eta)^{\sigma}(\overline{w}+\overline{\eta})^{\sigma-1} - w^{\sigma}\overline{w}^{\sigma-1}| \\
&\lesssim |\partial_x \eta|(|w|^{2\sigma-1} + |\eta|^{2\sigma-1}) + |\partial_x w|(|\eta|^{2\sigma-1} + |\eta||w|^{2\sigma-2}) \\
&= |\partial_x \eta|(|w|^{2\sigma-1} + |\eta|^{2\sigma-1}) + |\partial_x w||\eta|(|\eta|^{2\sigma-2} + |w|^{2\sigma-2}).
\end{aligned}$$

□

**Lemma 3.6** *Let $u$ be defined as in (2.41). Then $u$ is a solution of (1.1) on $(T_0, \infty)$. Moreover,*

$$\|u\|_{L^\infty(T_0,\infty;H^1(\mathbb{R}))} \lesssim \|\varphi\|_{L^\infty(T_0,\infty;H^1(\mathbb{R}))} + \|\varphi\|^{2\sigma+1}_{L^\infty(T_0,\infty;H^1(\mathbb{R}))}, \tag{3.5}$$

*and*

$$\|u\|_{L^\infty(T_0,\infty;H^1(\mathbb{R}))} \lesssim 1. \tag{3.6}$$

***Proof*** Since $\psi = \partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$ and $(\varphi, \psi)$ solves (2.3), we have

$$L\varphi = P(\varphi, \psi) = P\left(\varphi, \partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi\right). \tag{3.7}$$

We recall that

$$u = \exp\left(\frac{-i}{2}\int_{-\infty}^{x} |\varphi|^{2\sigma}\,dy\right)\varphi. \tag{3.8}$$

Using (3.7) and (3.8), we may prove that $u$ is a solution of (1.1).

Since (3.8), for each $t > 0$,

$$\begin{aligned}
\|u(t)\|_{L^2(\mathbb{R})} &= \|\varphi(t)\|_{L^2(\mathbb{R})} \\
\|\partial_x u(t)\|_{L^2} &\leq \|\partial_x\varphi(t)\|_{L^2(\mathbb{R})} + \frac{1}{2}\||\varphi|^{2\sigma+1}\|_{L^2(\mathbb{R})} \\
&\lesssim \|\varphi(t)\|_{H^1(\mathbb{R})} + \|\varphi(t)\|^{2\sigma+1}_{H^1(\mathbb{R})}.
\end{aligned}$$

This implies (3.5).

Moreover, using (2.17), we have, for all $t \geq T_0$,

$$\|\varphi(t)\|_{H^1(\mathbb{R})}$$

$$
\begin{aligned}
&\leq \|\tilde{\varphi}(t)\|_{H^1(\mathbb{R})} + \|h\|_{L^\infty(\mathbb{R},H^1(R))} \\
&\lesssim e^{-\lambda T_0} + \|R\|_{L^\infty(\mathbb{R},H^1(R))} + \|R\|^{2\sigma+1}_{L^\infty(\mathbb{R},H^1(R))} \\
&\lesssim 1.
\end{aligned}
$$

Combining with (3.5), we obtain (3.6). This completes the proof. □

### 3.3 Proof $G(\varphi, v) = Q(\varphi, v)$

Let $G(\varphi, v)$ be defined as in (2.19) and $Q$ be defined as in (2.5). Then we have the following result.

**Lemma 3.7** *Let $v = \partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$. Then the following equality holds:*

$$G(\varphi, v) = Q(\varphi, v).$$

***Proof*** We have

$$
\begin{aligned}
P(\varphi, v) &= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{v} - \sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy, \\
Q(\varphi, v) &= -i\sigma|\varphi|^{2(\sigma-1)}v^2\overline{\varphi} - \sigma(\sigma-1)v\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy \\
G(\varphi, v) &= \partial_x P(\varphi, v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi, v) \\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi, v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi} \\
&\quad - \frac{i}{2}\Big[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi \\
&\quad + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big].
\end{aligned}
$$

The term contains $\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy$ in the expression of $G(\varphi, v)$ is the following.

$$
\begin{aligned}
&-\sigma(\sigma-1)\partial_x\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy \\
&\quad - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(-1)\sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy \\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(-1)\sigma(\sigma-1)\overline{\varphi}\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy \\
&= -\sigma(\sigma-1)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\left(\partial_x\varphi - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\varphi + \frac{i}{2}\sigma|\varphi|^{2\sigma}\varphi\right) \\
&= -\sigma(\sigma-1)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\left(\partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi\right) \\
&= -\sigma(\sigma-1)v\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy,
\end{aligned}
$$

which equals to the term contains $\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy$ in the expression of $Q(\varphi, v)$. We only need to check the equality of the remaining terms. The remaining terms of $G(\varphi, v)$ is the following.

$$i\sigma\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2\overline{v}) - \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi\mathcal{I}m(v^2\overline{\varphi}^2)$$

$$
\begin{aligned}
&-\frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{v})\\
&+\frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(-i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}^2v)-i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi} \qquad (3.9)\\
&-\frac{i}{2}\Big[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma})+\sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\\
&+\sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big]. \qquad (3.10)
\end{aligned}
$$

Noting that $\partial_x(|\varphi|^2)=2\mathcal{R}e(v\overline{\varphi})$ and $v=\partial_x\varphi-\frac{i}{2}|\varphi|^{2\sigma}\varphi$, we have

$$
\begin{aligned}
&\text{the term (3.9)}\\
&=i\sigma\partial_x(|\varphi|^{2(\sigma-1)})\varphi^2\overline{v}+i\sigma|\varphi|^{2(\sigma-1)}2\varphi\partial_x\varphi\overline{v}+i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\overline{v}\\
&\quad-\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi2\mathcal{R}e(v\overline{\varphi})\mathcal{I}m(v\overline{\varphi})+\frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v}\\
&\quad+\sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{R}e(\varphi\overline{v})-i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi}\\
&=2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\mathcal{R}e(v\overline{\varphi})\varphi^2\overline{v}+2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\overline{v}+i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x(\overline{v}-\overline{\partial_x\varphi})\\
&\quad-2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi\mathcal{R}e(v\overline{\varphi})\mathcal{I}m(v\overline{\varphi})+\frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v}+\sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{R}e(\varphi\overline{v})\\
&=2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\mathcal{R}e(v\overline{\varphi})\varphi(i\varphi\overline{v}-\mathcal{I}m(v\overline{\varphi}))+2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\overline{v}\\
&\quad+i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\left(\frac{i}{2}|\varphi|^{2\sigma}\overline{\varphi}\right)+\frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v}+\sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{R}e(\varphi\overline{v})\\
&=2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{R}e(v\overline{\varphi}))^2+2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v}\\
&\quad-\frac{1}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(2\sigma|\varphi|^{2(\sigma-1)}\mathcal{R}e(v\overline{\varphi})+|\varphi|^{2\sigma}\partial_x\overline{\varphi})\\
&\quad+\frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v}+\sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{R}e(\varphi\overline{v})\\
&=2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{R}e(v\overline{\varphi}))^2+2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v}\\
&\quad-\frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\partial_x\overline{\varphi}+\frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v}\\
&=2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{R}e(v\overline{\varphi}))^2+2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v}+\frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2(\overline{v}-\partial_x\overline{\varphi})\\
&=2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{R}e(v\overline{\varphi}))^2+2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v}+\frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi.
\end{aligned}
$$

Moreover, using $\mathcal{R}e(\partial_x\varphi\overline{\varphi})=\mathcal{R}e(v\overline{\varphi})$ we have

$$
\begin{aligned}
&\text{the term (3.10)}\\
&=\frac{-i}{2}\Big[\sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi+\sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi(\partial_x\overline{\varphi}\varphi+\partial_x\varphi\overline{\varphi})\\
&\quad+\sigma(\sigma-1)(\partial\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\Big]\\
&=\frac{-i}{2}\Big[2\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi+\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\overline{\varphi}\varphi^2(\partial_x\varphi\overline{\varphi}+\partial_x\overline{\varphi}\varphi)\\
&\quad+2\sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\Big]\\
&=\frac{-i}{2}\Big[2\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi+2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\overline{\varphi}\varphi^2\mathcal{R}e(v\overline{\varphi})
\end{aligned}
$$

$$
\begin{aligned}
&\quad +2\sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi \mathcal{R}e(v\overline{\varphi})\Big] \\
&= -i\left[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\overline{\varphi}\varphi^2\mathcal{R}e(v\overline{\varphi})\right. \\
&\quad \left. +\sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\right] \\
&= -i\left[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\mathcal{R}e(v\overline{\varphi})\varphi(\partial_x\overline{\varphi}\varphi + \partial_x\varphi\overline{\varphi})\right. \\
&\quad \left. +2\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\right] \\
&= -i\left[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}(\mathcal{R}e(v\overline{\varphi}))^2\varphi\right] \\
&= -2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{R}e(v\overline{\varphi}))^2 \\
&\quad - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi - 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi}).
\end{aligned}
$$

Combining the above expressions we obtain

$$
\begin{aligned}
&\text{the remaining term of } G(\varphi, v) \\
&= 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v} + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi - 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi}) \\
&= 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(\varphi\overline{v} - \mathcal{R}e(v\overline{\varphi})) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi \\
&= -2\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{I}m(\varphi\overline{v}) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi \\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(2\mathcal{I}m(\varphi\overline{v}) + i\partial_x\overline{\varphi}\varphi) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(2\mathcal{I}m(\varphi\partial_x\overline{\varphi}) + |\varphi|^{2\sigma+2} + i\mathcal{R}e(\varphi\partial_x\overline{\varphi}) - \mathcal{I}m(\varphi\partial_x\overline{\varphi})) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(|\varphi|^{2\sigma+2} + i\overline{\varphi}\partial_x\varphi) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}(\partial_x\varphi)^2 - \sigma|\varphi|^{4\sigma}\partial_x\varphi + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}\left(v + \frac{i}{2}|\varphi|^{2\sigma}\varphi\right)^2 - \sigma|\varphi|^{4\sigma}\left(v + \frac{i}{2}|\varphi|^{2\sigma}\varphi\right) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi \\
&= -i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}v^2.
\end{aligned}
$$

This is exactly the remaining terms of $Q(\varphi, v)$. Thus, $G(\varphi, v) = Q(\varphi, v)$. □

### 3.4 Existence of a Solution of the System

In this section, using similar arguments as in [9, 10], we prove the existence of a solution of (2.13). For convenience, we recall the equation:

$$
\eta(t) = i\int_t^{\infty} S(t-s)[f(W+\eta) - f(W) + H](s)\,ds, \tag{3.11}
$$

where

$$
\begin{aligned}
W &= (h, k), \\
H &= e^{-\lambda t}(m, n),
\end{aligned}
$$

$$f(\varphi,\psi)=(P(\varphi,\psi),Q(\varphi,\psi)).$$

We have the following lemma.

**Lemma 3.8** *Let $H=H(t,x):[0,\infty)\times\mathbb{R}\to\mathbb{C}^2$, $W=W(t,x):[0,\infty)\times\mathbb{R}\to\mathbb{C}^2$ be given vector functions which satisfy for some $C_1>0$, $C_2>0$, $\lambda>0$, $T_0\geq 0$:*

$$\|W(t)\|_{L^\infty(\mathbb{R})\times L^\infty(\mathbb{R})}+e^{\lambda t}\|H(t)\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})}\leq C_1,\quad \forall t\geq T_0, \tag{3.12}$$

$$\|\partial W(t)\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})}+\|\partial W(t)\|_{L^\infty(\mathbb{R})\times L^\infty(\mathbb{R})}+e^{\lambda t}\|\partial H(t)\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})}\leq C_2,\quad \forall t\geq T_0. \tag{3.13}$$

*Consider Eq. (3.11). There exists a constant $\lambda_*$ independent of $C_2$ such that if $\lambda\geq\lambda_*$ then there exists a unique solution $\eta$ of (3.11) on $[T_0,\infty)\times\mathbb{R}$ satisfying*

$$e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))}+e^{\lambda t}\|\partial\eta\|_{S([t,\infty))\times S([t,\infty))}\leq 1,\quad \forall t\geq T_0.$$

***Proof*** We rewrite (3.11) by $\eta=\Phi\eta$. We show that, for $\lambda$ large enough, $\Phi$ is a contraction map in the following ball

$$B=\left\{\eta:\|\eta\|_X:=\sup_{t>T_0}\left(e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))}+e^{\lambda t}\|\partial_x\eta\|_{S([t,\infty))\times S([t,\infty))}\right)\leq 1\right\}.$$

We will use condition $\lambda\gg 1$ in the proof without specifying it.

**Step 1. Proof $\Phi$ maps $B$ into $B$**

Let $t\geq T_0$, $\eta=(\eta_1,\eta_2)\in B$, $W=(w_1,w_2)$ and $H=(h_1,h_2)$. By Strichartz estimates, we have

$$\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))}\lesssim\|f(W+\eta)-f(W)\|_{N([t,\infty))\times N([t,\infty))}, \tag{3.14}$$

$$+\|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}. \tag{3.15}$$

For (3.15), using (3.12), we have

$$\begin{aligned}\|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}&=\|h_1\|_{L^1_\tau L^2_x([t,\infty))}+\|h_2\|_{L^1_\tau L^2_x([t,\infty))}\\&\lesssim\int_t^\infty e^{-\lambda\tau}\,d\tau\leq\frac{1}{\lambda}e^{-\lambda t}<\frac{1}{10}e^{-\lambda t}.\end{aligned} \tag{3.16}$$

For (3.14), we have

$$|P(W+\eta)-P(W)|$$
$$=|P(w_1+\eta_1,w_2+\eta_2)-P(w_1,w_2)|$$
$$\lesssim\left||w_1+\eta_1|^{2\sigma-1)}(w_1+\eta_1)^2\overline{w_2+\eta_2}-|w_1|^{2(\sigma-1)}w_1^2\overline{w_2}\right| \tag{3.17}$$
$$+\left|(w_1+\eta_1)\int_{-\infty}^x|w_1+\eta_1|^{2(\sigma-2)}\mathcal{I}m((w_2+\eta_2)^2(\overline{w_1}+\overline{\eta_1})^2)\right.$$
$$\left.-w_1\int_{-\infty}^x|w_1|^{2(\sigma-2)}\mathcal{I}m(w_2^2\overline{w_1}^2)\right|. \tag{3.18}$$

Using the assumption $\sigma\geq\frac{3}{2}$ and the inequality (3.4), we have

the term (3.17)

$$\lesssim\left|\left||w_1+\eta_1|^{2(\sigma-1)}-|w_1|^{2(\sigma-1)}\right||w_1+\eta_1|^2|w_2+\eta_2|\right|$$
$$+\left||w_1|^{2(\sigma-1)}|(w_1+\eta_1)^2-w_1^2||w_2+\eta_2|\right|+\left||w_1|^{2(\sigma-1)}|w_1|^2|\eta_2|\right|$$

$$
\begin{aligned}
&\lesssim (|\eta_1|^{2(\sigma-1)} + |\eta_1||w_1|^{2(\sigma-1)-1})(|W| + |\eta|)^3 \\
&\quad + |w_1|^{2(\sigma-1)}(|w_1||\eta_1| + |\eta_1|^2)|w_2 + \eta_2| + |w_1|^{2\sigma}|\eta_2| \\
&\lesssim (|\eta|^{2(\sigma-1)} + |\eta||W|^{2(\sigma-1)-1})(|W|^3 + |\eta|^3) \\
&\quad + |W|^{2(\sigma-1)}(|W||\eta| + |\eta|^2)(|W| + |\eta|) + |W|^{2\sigma}|\eta| \\
&\lesssim |\eta|(|\eta|^{2\sigma-3} + |W|^{2\sigma-3})(|\eta|^3 + |W|^3) + |\eta||W|^{2(\sigma-1)}(|W|^2 + |\eta|^2) + |W|^{2\sigma}|\eta| \\
&\lesssim |\eta|(|\eta|^{2\sigma} + |W|^{2\sigma}) + |\eta||W|^{2\sigma} + |\eta|^3|W|^{2(\sigma-1)} + |W|^{2\sigma}|\eta| \\
&\lesssim |\eta|^{2\sigma+1} + |\eta||W|^{2\sigma}.
\end{aligned}
$$

Moreover, using Lemma 3.4, we have

$$
\begin{aligned}
&\text{the term (3.18)} \\
&\lesssim |\eta_1| \int_{-\infty}^{x} |w_1 + \eta_1|^{2(\sigma-2)}|w_2 + \eta_2|^2|w_1 + \eta_1|^2\, dy \\
&\quad + |w_1| \int_{-\infty}^{x} J(\eta_1, \eta_2) - J(0,0) dy \\
&\lesssim |\eta| \int_{-\infty}^{x} |W|^{2\sigma} + |\eta|^{2\sigma}\, dy + |W| \int_{-\infty}^{x} |\eta|(|W|^{2\sigma-1} + |\eta|^{2\sigma-1})\, dy \\
&= |\eta| \int_{-\infty}^{x} |W|^{2\sigma} + |\eta|^{2\sigma}\, dy + |W| \int_{-\infty}^{x} |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma}\, dy.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&|P(W+\eta) - P(W)| \\
&\quad \lesssim |\eta|^{2\sigma+1} + |\eta||W|^{2\sigma} + |\eta| \int_{-\infty}^{x} |W|^{2\sigma} + |\eta|^{2\sigma}\, dy + |W| \int_{-\infty}^{x} |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma}\, dy.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|Q(W+\eta) - Q(W)| \\
&\quad \lesssim |\eta|^{2\sigma+1} + |\eta||W|^{2\sigma} + |\eta| \int_{-\infty}^{x} |W|^{2\sigma} + |\eta|^{2\sigma}\, dy + |W| \int_{-\infty}^{x} |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma}\, dy.
\end{aligned}
$$

Hence, using $\sigma \geq \frac{3}{2}$, we have:

$$
\begin{aligned}
&\|f(W+\eta) - f(W)\|_{N([t,\infty)) \times N([t,\infty))} \\
&\lesssim \|P(W+\eta) - P(W)\|_{L^1_\tau L^2_x([t,\infty))} + \|Q(W+\eta) - Q(W)\|_{L^1_\tau L^2_x([t,\infty))} \\
&\lesssim \||\eta|^{2\sigma+1}\|_{L^1_\tau L^2_x([t,\infty))} + \||\eta| \int_{-\infty}^{x} |W|^{2\sigma} + |\eta|^{2\sigma}\, dy\|_{L^1_\tau L^2_x([t,\infty))} \\
&\quad + \||W| \int_{-\infty}^{x} |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma}\, dy\|_{L^1_\tau L^2_x([t,\infty))} \\
&\lesssim \||\eta|\|_{L^\infty L^2_x([t,\infty))} \||\eta|\|^4_{L^4_\tau L^\infty_x([t,\infty))} \\
&\quad + \||\eta|\|_{L^1_\tau L^2_x([t,\infty))} \left\| \int_{-\infty}^{x} |W|^{2\sigma} + |\eta|^{2\sigma}\, dy \right\|_{L^\infty_\tau L^\infty_x([t,\infty))} \\
&\quad + \||W|\|_{L^\infty_\tau L^2_x([t,\infty))} \|\int_{-\infty}^{x} |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma}\, dy\|_{L^1_\tau L^\infty_x([t,\infty))}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim e^{-5\lambda t} + \| |\eta| \|_{L^1_\tau L^2_x([t,\infty))} \| |W|^{2\sigma} + |\eta|^{2\sigma} \|_{L^\infty_\tau L^1_x} \\
&\quad + \|W\|_{L^\infty_t L^2_x} \|\eta\|_{L^1_\tau L^2_x([t,\infty))} \| |W|^{2\sigma-1} + |\eta|^{2\sigma-1} \|_{L^\infty_\tau L^2_x([t,\infty))} \\
&\lesssim e^{-5\lambda t} + \| |\eta| \|_{L^1_\tau L^2_x([t,\infty))} = e^{-5\lambda t} + \int_t^\infty e^{-\lambda\tau} d\tau \\
&\lesssim e^{-5\lambda t} + \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t},
\end{aligned}
$$

Combining with (3.16) and (3.14), (3.15) we obtain

$$
\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} < \frac{1}{5} e^{-\lambda t}. \tag{3.19}
$$

We have

$$
\begin{aligned}
\|\partial_x \Phi\eta\|_{S([t,\infty))\times S([t,\infty))} &\lesssim \|\partial_x (f(W+\eta) - f(W))\|_{N([t,\infty))\times N([t,\infty))} &(3.20)\\
&\quad + \|\partial_x H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}. &(3.21)
\end{aligned}
$$

For (3.21), using (3.13) we have

$$
\|\partial_x H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))} \lesssim \int_t^\infty e^{-\lambda\tau}\, d\tau = \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t}, \tag{3.22}
$$

For (3.20), we have

$$
\begin{aligned}
\|\partial_x (f(W+\eta) - f(W))\|_{N([t,\infty))\times N([t,\infty))} &= \|\partial_x (P(W+\eta) - P(W))\|_{N([t,\infty))} \\
&\quad + \|\partial_x (Q(W+\eta) - Q(W))\|_{N([t,\infty))}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&|\partial_x (P(W+\eta) - P(W))| \\
&\lesssim |\partial_x (|w_1+\eta_1|^{2(\sigma-1)} (w_1+\eta_1)^2 (\overline{w}_2 + \overline{\eta}_2) - |w_1|^{2(\sigma-1)} w_1^2 \overline{w}_2)| &(3.23)\\
&\quad + \Big| \partial_x (w_1+\eta_1) \int_{-\infty}^x |w_1+\eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2+\eta_2)^2 (\overline{w}_1 + \overline{\eta}_1)^2)\, dy \\
&\quad - \partial_x w_1 \int_{-\infty}^x |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \overline{w}_1^2)\, dy \Big| &(3.24)\\
&\quad + \Big| (w_1+\eta_1)|w_1+\eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2+\eta_2)^2 (\overline{w}_1 + \overline{\eta}_1)^2) \\
&\quad - w_1 |w_1|^{2(\sigma-2)} \mathcal{I}m(w_2^2 \overline{w}_1^2) \Big|. &(3.25)
\end{aligned}
$$

For (3.23), using Lemma 3.5 and (3.2) we have

$$
\begin{aligned}
&\text{the term (3.23)} \\
&\lesssim |\partial_x (|w_1+\eta_1|^{2(\sigma-1)} (w_1+\eta_1)^2 - |w_1|^{2(\sigma-1)} w_1^2) \overline{w}_2| \\
&\quad + |\eta_2| (|\partial_x W| + |\partial_x \eta|)(|W|^{2\sigma-1} + |\eta|^{2\sigma-1}) \\
&\quad + |(|w_1+\eta_1|^{2(\sigma-1)} (w_1+\eta_1)^2 - |w_1|^{2(\sigma-1)} w_1^2) \partial_x \overline{w}_2| \\
&\quad + |\partial_x \eta_2| (|W|^{2\sigma} + |\eta|^{2\sigma}) \\
&\lesssim |\partial_x \eta_1| (|w_1|^{2\sigma-1} + |\eta_1|^{2\sigma-1}) + |\partial_x w_1||\eta_1| (|\eta_1|^{2\sigma-2} + |w_1|^{2\sigma-2}) \\
&\quad + |\eta_2| (|\partial_x W| + |\partial_x \eta|)(|W|^{2\sigma-1} + |\eta|^{2\sigma-1}) \\
&\quad + |\partial_x w_2| (|\eta_1|^{2\sigma} + |\eta_1||w_1|^{2\sigma-1})
\end{aligned}
$$

$$+ |\partial_x \eta_2|(|W|^{2\sigma} + |\eta|^{2\sigma}).$$

Thus,

$$\|\text{the term (3.23)}\|_{L^1_\tau L^2_x([t,\infty))} \lesssim \||\eta| + |\partial\eta|\|_{L^1_\tau L^2_x([t,\infty))} \lesssim \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t}.$$

For (3.24), using the inequality (3.4), we have

$$\begin{aligned}
&\| \text{ the term (3.24)}\|_{L^1_\tau L^2_x([t,\infty))} \\
&\lesssim \|\partial\eta_1\|_{L^1_\tau L^2_x([t,\infty))} \|\int_{-\infty}^{x} |w_1+\eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)\, dy\|_{L^\infty_t L^\infty_x} \\
&\quad + \|\partial_x w_1\|_{L^\infty_t L^2_x} \times \\
&\quad \times \left\| \int_{-\infty}^{x} J(\eta_1,\eta_2) - J(0,0)\, dy \right\|_{L^1_\tau L^\infty_x} \\
&\lesssim \|\partial\eta_1\|_{L^1_\tau L^2_x([t,\infty))} \||w_1+\eta_1|^{2(\sigma-2)} \mathcal{I}m((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)\|_{L^\infty_t L^1_x} \\
&\quad + \|J(\eta_1,\eta_2) - J(0,0)\|_{L^1_\tau L^1_x} \\
&\lesssim \|\partial\eta_1\|_{L^1_\tau L^2_x([t,\infty))} + \||\eta|\|_{L^1_\tau L^2_x([t,\infty))} \le \int_t^\infty e^{-\lambda\tau}\, d\tau \lesssim \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t},
\end{aligned}$$

For (3.25), using the inequality (3.4), we have

$$\begin{aligned}
&\|\text{the term (3.25)}\|_{L^1_\tau L^2_x([t,\infty))} \\
&\lesssim \||\eta|(|W|^{2\sigma} + |\eta|^{2\sigma})\|_{L^1_\tau L^2_x([t,\infty))} + \||W|(J(\eta_1,\eta_2) - J(0,0))\|_{L^1_\tau L^2_x([t,\infty))} \\
&\lesssim \||\eta|(|W|^{2\sigma} + |\eta|^{2\sigma})\|_{L^1_\tau L^2_x([t,\infty))} + \||\eta||W|(|W|^{2\sigma-1} + |\eta|^{2\sigma-1})\|_{L^1_\tau L^2_x([t,\infty))} \\
&\lesssim \||\eta|\|_{L^1_\tau L^2_x([t,\infty))} \\
&\le \int_t^\infty e^{-\lambda\tau}\, d\tau \lesssim \frac{1}{\lambda} e^{-\lambda t} < \frac{1}{10} e^{-\lambda t},
\end{aligned}$$

Combining the above estimates, we obtain

$$\begin{aligned}
&\|\partial_x(P(W+\eta) - P(W))\|_{N([t,\infty))} \\
&\le \|\partial_x(P(W+\eta) - P(W))\|_{L^1_\tau L^2_x([t,\infty))} \le \frac{3}{10} e^{-\lambda t},
\end{aligned} \tag{3.26}$$

Similarly,

$$\|\partial_x(Q(W+\eta) - Q(W))\|_{N([t,\infty))} \le \frac{3}{10} e^{-\lambda t}, \tag{3.27}$$

Combining the estimates (3.20), (3.21), (3.22), (3.26) and (3.27), we have

$$\|\partial_x \Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \le \frac{7}{10} e^{-\lambda t}. \tag{3.28}$$

Combining (3.19) with (3.28), we obtain

$$\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} + \|\partial_x \Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \le \frac{9}{10} e^{-\lambda t}, \tag{3.29}$$

Thus, for $\lambda$ large enough

$$\|\Phi\eta\|_X < 1.$$

This implies that $\Phi$ maps $B$ into $B$.
**Step 2. $\Phi$ is a contraction map on B**
By using (3.12), (3.13) and a similar estimate of (3.29), we can show that, for any $\eta \in B$ and $\kappa \in B$ we have

$$\|\Phi\eta - \Phi\kappa\|_X \leq \frac{1}{2}\|\eta - \kappa\|_X.$$

for λ large enough. From Banach fixed point theorem, there exists a unique solution in $B$ of (3.11) and thus a solution of (2.13). This completes the proof of Lemma 3.8. □

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**Data Availability** All data generated or analysed during this study are included in this published article [and its supplementary information files].

## Declarations

**Conflict of interest** I am the only author of this paper and there is no conflict of interest.

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