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Spreading Dynamics for a Three Species Predator–Prey System with Two Preys in a Shifting Environment

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Abstract
This paper deals with the long time behavior of a reaction–diffusion system modeling the spatio-temporal interaction of two preys and one predator in a shifting environment. Here the prey populations weakly compete, and the environment becomes hostile for the three species with a constant positive speed. We investigate the survival of the species and describe the spreading speeds of different species. The dynamical behavior exhibits various regions composed of different combinations of species, which propagate with different wave speeds.

Keywords Predator-prey model · Spreading · Extinction · Shifting environment · Entire solution

Mathematics Subject Classification 35K45 · 35K57 · 92D25

1 Introduction

Global climate change has led to the shift in the habitat of many ecological species, which causes to influence the survival and spreading of species. Mathematical models with an environmental shift represented by reaction-diffusion equations have attracted a lot of attentions

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in the last decades. A single species model in shifting environment is given by

$$u_t(x,t) = du_{xx}(x,t) + u(x,t)f(x-st, u(x,t)), \;\; x \in \mathbb{R}, \;\; t > 0,$$

where the function f represents the shifting environment with a changing speed s. We refer the reader to [1, 3–5, 17, 18, 22] for single equation models. For multi-species interaction systems in shifting environments, we refer the reader to [16, 30, 31, 34, 35] for 2-species competition systems, [33] for a cooperative model and [10, 11] for predator-prey models. See also [6–8, 21, 23, 26, 36] and references therein for more related works on models in shifting environments.

In this paper, we consider the following diffusive predator-prey model with two preys and one predator:

$$\begin{aligned} u_t(x,t) &= d_1 u_{xx}(x,t) + r_1 u(x,t)[1 + \alpha(x-st) \\ &\quad -(u + hv + aw)(x,t)], \;\; x \in \mathbb{R}, \; t > 0, \end{aligned} \tag{1.1}$$

$$\begin{aligned} v_t(x,t) &= d_2 v_{xx}(x,t) + r_2 v(x,t)[1 + \alpha(x-st) \\ &\quad -(ku + v + aw)(x,t)], \;\; x \in \mathbb{R}, \; t > 0, \end{aligned} \tag{1.2}$$

$$\begin{aligned} w_t(x,t) &= d_3 w_{xx}(x,t) + r_3 w(x,t)[-1 + \alpha(x-st) \\ &\quad +(bu + bv - w)(x,t)], \;\; x \in \mathbb{R}, \; t > 0, \end{aligned} \tag{1.3}$$

where the unknown functions u, v and w, respectively, stand for the population densities of two preys and predator species at position x and time t. Parameters $d_i, r_i, i = 1, 2, 3, h, k, a, b$ are positive and represent the diffusion coefficients, intrinsic growth rates, competing rates, predation rate and conversion rate, respectively. In this work, we assume that the competition between the two preys is weak, i.e., $0 < h, k < 1$.

In system (1.1)–(1.3), the term $\alpha(x - st)$ represents an environmental shift. Here the constant $s > 0$ denotes the shifting speed of the environment while the function $\alpha(\cdot)$ describes the environmental change. We assume that it models a gradual deterioration of the environment, propagating from left to right at the speed s. More precisely, we always assume that α is a continuously differentiable and nondecreasing function in $\mathbb{R}$ such that

$$\alpha(-\infty) \in (-\infty, -1), \;\; \alpha(\infty) = 0. \tag{1.4}$$

Here $\alpha(\infty) = 0$ is imposed so that the maximal carrying capacities of both preys are normalized to be 1. Moreover, due to (1.4), the growth rates of two preys, $r_1(1 + \alpha(x - st))$ and $r_2(1 + \alpha(x - st))$, are positive in the region $x - st \gg 1$ (ahead the climate change) and negative for $x - st \ll -1$. This means that the environment is favorable to the preys ahead of the climate change, then gradually deteriorates until it becomes hostile to the species. Note also that the predator species can feed on both prey populations but cannot survive without any prey due to the negative constant -1 in (1.3). See also Remark 4.1 in §4. As far as the predator is concerned, we assume that $b > 1$, which means that the predator can survive when the total populations of these two preys stay at a certain high level in the favorable region $x - st \gg 1$.

Less work is done for the predator-prey interaction system with climate change because of some mathematical difficulties arising from the lack of a comparison principle. Also, an indirect influence of climate change on predators has been considered so far [10, 11], in which the predators are only affected by the ecological change of their food in shifting environments. Unlike the previous works, this paper considers a situation where climate change directly affects the predator. Such direct effects of climate change on both prey and predator can happen in nature; for example, the habitat of both predator and prey is lost through desertification or rising sea levels due to climate change; the polar bears' habitat shrinks due to the melting sea ice. Our goal is to investigate the propagation of species in such a shifting environment.

To this aim, we investigate the long time behavior of the solutions of (1.1)–(1.3), equipped with suitable initial data, typically compactly supported or with support bounded above, that is mostly concentrated in the unfavorable region $x \ll -1$. Here we study the persistence of the different populations under the forced environmental shift and describe the spreading properties. As far as the spreading property is concerned, homogeneous reaction-diffusion systems in which the comparison principle does not hold have attracted much attention in recent years. We refer the reader to [15, 25, 27, 29] for arguments based on refined estimates of the heat kernel, to [9] for arguments based on local comparison, and to [12–14, 28, 32] for ideas based on uniform persistence like arguments in dynamical systems.

In this paper, we analyze the long time behavior for the solutions of (1.1)–(1.3) by using partial comparison arguments and some ideas based on dynamical system theory. This long time asymptotic behavior will strongly depend upon the constant equilibria of the problem in the favorable region, that when $\alpha = 0$. One may observe that these constant equilibria without the α-term are given by

$$\begin{cases} E_1 = (1, 0, 0),\ E_2 = (0, 1, 0),\ E_u = (u_p, 0, w_p), \\ E_v = (0, v_p, w_p),\ E_c = (u_c, v_c, 0),\ E_* = (u_*, v_*, w_*), \end{cases}$$

where

$$\begin{cases} v_p = u_p = \frac{1+a}{1+ab},\ w_p = \frac{b-1}{1+ab},\ u_c = \frac{1-h}{1-hk},\ v_c = \frac{1-k}{1-hk}, \\ u_* = \frac{(1+a)(1-h)}{1-hk+ab(2-h-k)},\ v_* = \frac{(1+a)(1-k)}{1-hk+ab(2-h-k)},\ w_* = \frac{b(2-h-k)-1+hk}{1-hk+ab(2-h-k)}. \end{cases}$$

As will be seen later, the boundary equilibria will have a strong impact for the dynamical behavior of the problem, yielding various propagation regions.

In order to describe this dynamical behavior, we introduce various linear speeds associated to these boundary equilibria by setting

$$\begin{aligned} &s_1^* := 2\sqrt{d_1 r_1},\ s_2^* := 2\sqrt{d_2 r_2},\ s_3^* := 2\sqrt{d_3 r_3 \beta_1}, \\ &s_1^{**} := 2\sqrt{d_1 r_1 (1-h)},\ s_2^{**} := 2\sqrt{d_2 r_2 (1-k)},\ s_3^{**} := 2\sqrt{d_3 r_3 \beta_2}, \\ &s_1^{***} := 2\sqrt{d_1 r_1 \delta_1},\ s_2^{***} = 2\sqrt{d_2 r_2 \delta_2},\ s_3^{***} = 2\sqrt{d_3 r_3 (b-1)}, \end{aligned}$$

where

$$\beta_1 := 2b - 1,\ \beta_2 := b(u_c + v_c) - 1,\ \delta_1 := 1 - h v_p - a w_p,\ \delta_2 := 1 - k u_p - a w_p.$$

Note that

$$s_i^* > s_i^{**} > s_i^{***},\ i = 1, 2, 3,$$

due to $0 < h, k < 1$ and $b > 1$.

Let (u, v, w) be a solution of system (1.1)–(1.3) with initial data (u_0, v_0, w_0). Set

$$X_K := \{\varphi \in C^0(\mathbb{R}) : 0 \le \varphi \le K \text{ for } x \in \mathbb{R}\}$$

for a positive constant K. It is easy to see that $(u, v, w)(\cdot, t) \in X_1 \times X_1 \times X_{2b-1}$ for $t > 0$, when $(u_0, v_0, w_0) \in X_1 \times X_1 \times X_{2b-1}$.

Now, we state the main results of this paper as follows.

First, we have the following extinction results, when the spreading of the preys cannot follow the environmental shift.

Theorem 1.1 *Assume that* $(u_0, v_0, w_0) \in X_1 \times X_1 \times X_{2b-1}$ *and all supports of* u_0, v_0 *and* w_0 *are bounded above. Then*

$$\lim_{t\to\infty} u(x,t) = 0, \text{ uniformly for } x \in \mathbb{R}, \text{ if } s > s_1^*; \tag{1.5}$$

$$\lim_{t\to\infty} v(x,t) = 0, \text{ uniformly for } x \in \mathbb{R}, \text{ if } s > s_2^*; \tag{1.6}$$

$$\lim_{t\to\infty} w(x,t) = 0, \text{ uniformly for } x \in \mathbb{R}, \text{ if } s > \hat{s} := \max\{s_1^*, s_2^*\}. \tag{1.7}$$

Moreover, if u_0 *and* v_0 *have compact supports and the support of* w_0 *is bounded above, then*

$$\lim_{t\to\infty} w(x,t) = 0, \quad \text{uniformly for } x \in \mathbb{R}, \text{ if } s > s_3^*.$$

Theorem 1.2 *Assume that* $(u_0, v_0, w_0) \in X_1 \times X_1 \times X_{2b-1}$. *The following statements hold.*
(i) *For any small* $\zeta > 0$, *we have*

$$\lim_{t\to\infty} \sup_{x\le(s-\zeta)t} u(x,t) = \lim_{t\to\infty} \sup_{x\le(s-\zeta)t} v(x,t) = \lim_{t\to\infty} \sup_{x\le(s-\zeta)t} w(x,t) = 0.$$

(ii) *If* $u_0(x) = v_0(x) = w_0(x) = 0$ *for* $x \ge K$ *for some constant* K, *then*

$$\lim_{t\to\infty} \sup_{x\ge(s_1^*+\tau)t} u(x,t) = \lim_{t\to\infty} \sup_{x\ge(s_2^*+\tau)t} v(x,t) = \lim_{t\to\infty} \sup_{x\ge(s_3^*+\tau)t} w(x,t) = 0, \tag{1.8}$$

$$\lim_{t\to\infty} \sup_{x\ge(\hat{s}+\tau)t} w(x,t) = 0, \ \hat{s} = \max\{s_1^*, s_2^*\}, \tag{1.9}$$

for any $\tau > 0$.

Next, for the spreading behaviors we have

Theorem 1.3 *Let* $s_2^* < s_1^*$. *Assume that* $(u_0, v_0, w_0) \in X_1 \times X_1 \times X_{2b-1}$, $u_0 \neq 0$, *and the supports of* v_0 *and* w_0 *are bounded above. Suppose that* $s_3^* < s_1^*$. *If* $s \in (\max\{s_2^*, s_3^*\}, s_1^*)$, *then*

$$\lim_{t\to\infty} \left\{ \sup_{(s+\varepsilon)t\le x\le(s_1^*-\varepsilon)t} \Big[|u(x,t)-1| + v(x,t) + w(x,t) \Big] \right\} = 0 \tag{1.10}$$

for all $\varepsilon \in (0, (s_1^* - s)/2)$.

A similar result to Theorem 1.3 holds for E_2, by interchanging the roles of u and v.

Theorem 1.4 *Let* $s_2^* < s_1^*$. *Assume that* $(u_0, v_0, w_0) \in X_1 \times X_1 \times X_{2b-1}$ *and* u_0, v_0 *and* w_0 *have nonempty supports which are bounded above. Suppose that* $s_2^* < s_3^{***}$. *If* $s \in (s_2^*, s_{**})$, $s_{**} := \min\{s_1^*, s_3^{***}\}$, *then*

$$\lim_{t\to\infty} \left\{ \sup_{(s+\varepsilon)t\le x\le(s_{**}-\varepsilon)t} \Big[|u(x,t)-u_p| + v(x,t) + |w(x,t)-w_p| \Big] \right\} = 0 \tag{1.11}$$

for all $\varepsilon \in (0, (s_{**} - s)/2)$

A similar result to Theorem 1.4 holds for E_v, by interchanging the roles of u and v.

Theorem 1.5 *Assume that* $(u_0, v_0, w_0) \in X_1 \times X_1 \times X_{2b-1}$, $u_0 \neq 0$, $v_0 \neq 0$ *and the support of* w_0 *is bounded above. Suppose that* $s_3^* < s_1^{**}$ *and* $s_3^* < s_2^{**}$. *If* $s \in (s_3^*, s^{**})$, $s^{**} := \min\{s_1^{**}, s_2^{**}\}$, *then*

$$\lim_{t\to\infty} \left\{ \sup_{(s+\varepsilon)t\le x\le(s^{**}-\varepsilon)t} \Big[|u(x,t)-u_c| + |v(x,t)-v_c| + w(x,t) \Big] \right\} = 0 \tag{1.12}$$

for all $\varepsilon \in (0, (s^{**} - s)/2)$.

Lastly, for the co-existence case, we have

Theorem 1.6 *Assume that* $(u_0, v_0, w_0) \in X_1 \times X_1 \times X_{2b-1}$ *such that* $u_0 \neq 0$*,* $v_0 \neq 0$ *and* $w_0 \neq 0$*. Set* $\underline{s} := \min\{s_1^{***}, s_2^{***}, s_3^{***}\}$*. If* $s < \underline{s}$*, then*

$$\lim_{t\to\infty}\left\{\sup_{(s+\varepsilon)t\le x\le(\underline{s}-\varepsilon)t}\Big[|u(x,t)-u_*|+|v(x,t)-v_*|+|w(x,t)-w_*|\Big]\right\}=0 \tag{1.13}$$

for all $\varepsilon \in (0, (\underline{s} - s)/2)$.

The above theorems describe the spreading speeds and the spreading region for the different species: a single prey in Theorem 1.3, a single prey and the predator in Theorem 1.4, the two preys without predator in Theorem 1.5 and the three species in Theorem 1.6. As in [13], the long time behavior of three species problem without climate shift is not fully understood. It can be expected that the weak competition between the two preys may produce the spreading speed of invasion which is non-linearly determined (see [19, 24] for the purely competitive case). Roughly speaking, when the climate shift is included, these spreading speeds should be compared with the forced speed for the environmental shift since no species can survive in the hostile region $x - st \ll -1$ in the large times $t \gg 1$.

Consequently, due to the weak competition between the two preys, the spreading regions mentioned in the above results are sometimes not optimal. For instance one may expect that the optimal result for the spreading to E_c in Theorem 1.5 is the range $s \in (s_3^*, \min\{s_1^*, s_2^*\})$. However, due to the best known result for the spreading result of the two weak species competition system (see [30] or Proposition 3.7), we can only derive the spreading to E_c for $s \in (s_3^*, s^{**})$.

In the absence of the slow prey v, i.e., when $s_2^* < s < s_1^*$, the survival of both fast prey u and the predator w can be obtained similarly to that in [10] for a predator-prey system with a single prey. However, when $s < \min\{s_1^*, s_2^*, s_3^*\}$, the description of the spreading speed and region for the predator w is very difficult to obtain. Roughly speaking, for the survival of the predator, its growth rate has to be positive, that is, $u + v > 1/b$ in the favorable region $x - st \gg 1$. While the property $u + v$ uniformly positive in some regions can be achieved, proving that $u + v$ reaches a suitable level turns out to be a complicated issue. We can only derive some sufficient conditions to ensure such a property. As mentioned above, our proofs are mostly based on dynamical systems arguments. One of the difficulties to overcome is to derive some positive lower bounds estimation of solutions. Although the upper bound $\underline{s}$ of shifting speed to the co-existence state E_* is far from optimal, the proof of Theorem 1.6 is quite intricate.

The rest of this paper is organized as follows. In Sect. 2, we prove our extinction results, namely Theorem 1.1 and Theorem 1.2. The proofs are very similar to that in [10]. To be self-contained, we provide the details here for the reader's convenience. One should note that the shifting effect is taken directly in each species in system (1.1)–(1.3), while it was imposed only for the prey species in [10]. Section 3 is devoted to the spreading dynamics. We prove Theorems 1.3, 1.4, 1.5 and 1.6 in Sect. 3. Finally, in §4, we provide some numerical simulations for readers to better understand the spreading dynamics of system (1.1)–(1.3).

2 Extinction

In this section, we show the extinction of species as $t \to +\infty$ when the species cannot keep pace with the environmental shifting speed.

2.1 Complete Extinction: Theorem 1.1

Proof of Theorem 1.1 First we assume that $s > s_1^*$. Then, by the comparison principle for the scalar equation, we get that $u(x,t) \le \overline{u}(x,t)$ for $x \in \mathbb{R}, t > 0$, where $\overline{u}$ is the solution of the initial value problem

$$\begin{cases} \overline{u}_t(x,t) = d_1\overline{u}_{xx}(x,t) + r_1\overline{u}(x,t)[1+\alpha(x-st) - \overline{u}(x,t)], & x \in \mathbb{R},\ t>0, \\ \overline{u}(x,0) = u_0(x) \ge 0, & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

From [23, Theorem 2.1] and $u_0 \le 1$, we have that $\lim_{t\to\infty} \overline{u}(x,t) = 0$ uniformly for $x \in \mathbb{R}$ when $s > s_1^*$. Hence (1.5) follows. The case for v in (1.6) is similar, by comparing v with $\overline{v}$, where $\overline{v}$ is the solution of

$$\begin{cases} \overline{v}_t(x,t) = d_2\overline{v}_{xx}(x,t) + r_2\overline{v}(x,t)[1+\alpha(x-st) - \overline{v}(x,t)], & x \in \mathbb{R},\ t>0, \\ \overline{v}(x,0) = v_0(x) \ge 0, & x \in \mathbb{R}. \end{cases} \tag{2.2}$$

Next, let $s > \hat{s} = \max\{s_1^*, s_2^*\}$. Then, by (1.5) and (1.6), for a given $\varepsilon \in (0, 1/(2b))$ there exists $T_0 > 0$ such that $u(x,t) \le \varepsilon$ and $v(x,t) \le \varepsilon$ for $x \in \mathbb{R}, t > T_0$. Consider the function

$$W(x,t) := (2b-1)e^{-\sigma(t-T_0)},\ \sigma \in (0, r_3(1-2b\varepsilon)).$$

Then

$$\begin{aligned} &W_t - \{d_3W_{xx} + r_3W(-1+2b\varepsilon - W)\} \\ &= -(2b-1)e^{-\sigma(t-T_0)}[\sigma - r_3(1-2b\varepsilon) - r_3(2b-1)e^{-\sigma(t-T_0)}] \ge 0. \end{aligned}$$

By comparison, $w(x,t) \le W(x,t)$ for $x \in \mathbb{R},\ t \ge T_0$, since $W(x,T_0) = 2b-1 \ge w(x,T_0)$. Thus $\lim_{t\to\infty} w(x,t) = 0$ uniformly for $x \in \mathbb{R}$, when $s > \hat{s}$.

Finally, we assume that $s > s_3^*$. Choose $\delta > 0$ so small that $s > 2\sqrt{r_3(2b+2b\delta-1)}$, and consider the scalar equation for $i = 1, 2$ that

$$z_t(x,t) = d_iz_{xx}(x,t) + r_iz(x,t)[\alpha(x-st) + 1 + \delta - z(x,t)],\ x \in \mathbb{R},\ t > 0. \tag{2.3}$$

From [22, Theorem 1.1], (2.1) has a forced traveling wave solution $\phi_{i,\delta}(x-st)$ such that $\phi_{i,\delta}$ is nondecreasing, $\phi_{i,\delta}(-\infty) = 0$ and $\phi_{i,\delta}(\infty) = 1+\delta$. Since u_0 and v_0 are compactly supported with $u_0 \le 1$ and $v_0 \le 1$, we can choose $L > 0$ such $u_0(x) < \phi_{1,\delta}(x+L)$ and $v_0(x) < \phi_{2,\delta}(x+L)$ for $x \in \mathbb{R}$. Then, by comparison principle, $u(x,t) \le \phi_{1,\delta}(x-st+L)$ and $v(x,t) \le \phi_{2,\delta}(x-st+L)$ for $x \in \mathbb{R},\ t > 0$.

Now, let

$$f(x-st) := -1 + \alpha(x-st) + b[\phi_{1,\delta}(x-st+L) + \phi_{2,\delta}(x-st+L)].$$

Then f is nondecreasing with $f(-\infty) = -1+\alpha(-\infty) < 0$ and $f(\infty) = 2b+2b\delta-1 > 0$. Let $\overline{w}$ be the solution of the initial value problem

$$\begin{cases} \overline{w}_t(x,t) = d_3\overline{w}_{xx}(x,t) + r_3\overline{w}(x,t)[f(x-st) - \overline{w}(x,t)], & x \in \mathbb{R},\ t>0, \\ \overline{w}(x,0) = w_0(x) \ge 0, & x \in \mathbb{R}. \end{cases} \tag{2.4}$$

From the comparison principle, $w(x,t) \le \overline{w}(x,t)$ for $x \in \mathbb{R}$, $t > 0$. Then it follows from [23, Theorem 2.1] that $w(x,t)$ converges to 0 uniformly for $x \in \mathbb{R}$ as $t \to \infty$. The theorem is thus proved. □

2.2 Partial Extinction: Theorem 1.2

Next, we give a proof of Theorem 1.2 as follows.

Proof of Theorem 1.2 By a comparison with suitable supersolutions as defined in the proof of Theorem 1.1 (see (2.1), (2.2) and (2.4)), part (i) and (1.8) in part (ii) follow from [23, Theorem 2.2 (i)] and [23, Theorem 2.2 (ii)], respectively.

It remains to prove (1.9) when $s_3^* > \hat{s}$. Without loss of generality we may assume that $\hat{s} = s_1^*$. For this, let $\tau > 0$ be given and let λ_i the smaller positive root of

$$d_i\lambda^2 - (s_i^* + \tau/2)\lambda + r_i = 0$$

for $i = 1, 2$. Then $z_i(x,t) := A_i e^{-\lambda_i[x-(s_i^*+\tau/2)t]}$ is a solution of the following linear equation

$$z_{i,t}(x,t) = d_i z_{i,xx}(x,t) + r_i z_i(x,t),$$

for any positive constant A_i. Since $u_0(x) = v_0(x) = 0$ for $x \ge K$, we can choose A_i large such that

$$u_0(x) \le A_1 e^{-\lambda_1 x}, \quad v_0(x) \le A_2 e^{-\lambda_2 x}$$

for all $x \in \mathbb{R}$. Then, by the comparison principle,

$$0 \le u(x,t) \le A_1 e^{-\lambda_1[x-(s_1^*+\tau/2)t]}, \quad 0 \le v(x,t) \le A_2 e^{-\lambda_2[x-(s_2^*+\tau/2)t]} \tag{2.5}$$

for $x \in \mathbb{R}$, $t \ge 0$.

Now, we consider a function $W(x,t) := \min\left\{2b-1,\, Be^{-\lambda_3[x-(s_1^*+\tau/2)t]}\right\}$, where $\lambda_3 > 0$ is chosen small enough such that

$$d_3\lambda_3^2 - (s_1^* + \tau/2)\lambda_3 - \frac{r_3}{2} < 0,$$

and $B > 0$ will be chosen later. Note that, for (x,t) with $W(x,t) = Be^{-\lambda_3[x-(s_1^*+\tau/2)t]}$,

$$e^{-\lambda_i[x-(s_i^*+\tau/2)t]} = \left[e^{-\lambda_3[x-(s_i^*+\tau/2)t]}\right]^{\lambda_i/\lambda_3} < \left[\frac{2b-1}{B}\right]^{\lambda_i/\lambda_3}, \quad i = 1, 2.$$

Then, for such (x,t), $W(x,t)$ satisfies

$$\begin{aligned}&W_t - d_3W_{xx} - r_3W[-1 + b(A_1e^{-\lambda_1[x-(s_1^*+\tau/2)t]} + A_2e^{-\lambda_2[x-(s_2^*+\tau/2)t]}) - W]\\ &\ge r_3W\left\{\frac{1}{2} - b\Big[A_1\left(\frac{2b-1}{B}\right)^{\lambda_1/\lambda_3} + A_2\left(\frac{2b-1}{B}\right)^{\lambda_2/\lambda_3}\Big]\right\} \ge 0\end{aligned}$$

if we choose B large enough. For (x,t) with $W(x,t) = 2b-1$, it is clear that

$$W_t - d_3W_{xx} - r_3W(-1 + 2b - W) = 0.$$

Hence W is a supersolution of

$$\overline{w}_t = d_3\overline{w}_{xx} + r_3\overline{w}\left[-1 + b\Big(\min\{1, A_1e^{-\lambda_1[x-(s_1^*+\tau/2)t]}\} + \min\{1, A_2e^{-\lambda_2[x-(s_2^*+\tau/2)t]}\}\Big) - \overline{w}\right].$$

Recall from (2.5) that

$$u(x,t) \le \min\{1, A_1 e^{-\lambda_1[x-(s_1^*+\tau/2)t]}\}, \ v(x,t) \le \min\{1, A_2 e^{-\lambda_2[x-(s_2^*+\tau/2)t]}\}.$$

Using $\alpha \le 0$, it follows from the comparison principle that

$$w(x,t) \le Be^{-\lambda_3 \tau t/2}, \quad x \ge (s_1^* + \tau)t, \ t \ge T_0,$$

for some large T_0. This proves (1.9) and so Theorem 1.2 is proved. □

3 Spreading Dynamics

3.1 Spreading to E_1

We assume that $s_2^* < s_1^*$, $s_3^* < s_1^*$ and $s \in (\max\{s_2^*, s_3^*\}, s_1^*)$.

Proof of Theorem 1.3 Let $s_1^*(\delta) := 2\sqrt{d_1 r_1(1-\delta)}$, $\delta \in (0,1)$. Fix $s \in (\max\{s_2^*, s_3^*\}, s_1^*)$ and $\varepsilon \in (0, (s_1^* - s)/2)$. Choose $\delta_0 > 0$ small enough such that

$$s_1^* - \varepsilon \le s_1^*(\delta) - \varepsilon/2, \ s < s_1^*(\delta), \ \varepsilon < (s_1^*(\delta) - s), \ \forall \delta \in (0, \delta_0).$$

Since $s > \max\{s_2^*, s_3^*\}$, by Theorem 1.1, both v and w converge to zero uniformly on $\mathbb{R}$ as $t \to \infty$.

For a $\delta \in (0, \delta_0)$, there is $T_0 \gg 1$ such that $(hv + aw)(x,t) \le \delta$ for all $x \in \mathbb{R}$, $t \ge T_0$. Let $\underline{u}$ and $\overline{u}$ be the solutions of

$$\begin{cases} \underline{u}_t(x,t) = d_2 \underline{u}_{xx}(x,t) + r_2 \underline{u}(x,t)[1 + \alpha(x - st) - \delta - \underline{u}(x,t)], & x \in \mathbb{R}, \ t > T_0, \\ \underline{u}(x,0) = u(x,T_0) \ge 0, & x \in \mathbb{R}, \end{cases}$$

and

$$\begin{cases} \overline{u}_t(x,t) = d_2 \overline{u}_{xx}(x,t) + r_2 \overline{u}(x,t)[1 + \alpha(x - st) - \overline{u}(x,t)], & x \in \mathbb{R}, \ t > T_0, \\ \overline{u}(x,0) = u(x,T_0) \ge 0, & x \in \mathbb{R}, \end{cases}$$

respectively. Then $\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t)$ for $x \in \mathbb{R}$, $t \ge T_0$, by comparison.

On the other hand, from [23, Theorem 2.2 (iii)], we have

$$\lim_{t\to\infty} \left\{ \sup_{(s+\varepsilon/2)t \le x \le (s_1^*(\delta)-\varepsilon/2)t} |\underline{u}(x,t) - (1-\delta)| \right\} = 0,$$

$$\lim_{t\to\infty} \left\{ \sup_{(s+\varepsilon)t \le x \le (s_1^*-\varepsilon)t} |\overline{u}(x,t) - 1| \right\} = 0,$$

since $\varepsilon \in (0, (s_1^*(\delta) - s))$. Hence we deduce that

$$\limsup_{t\to\infty} \left\{ \sup_{(s+\varepsilon)t \le x \le (s_1^*-\varepsilon)t} |u(x,t) - 1| \right\} \le \delta.$$

Letting $\delta \downarrow 0$, we have the desired result. □

3.2 Spreading to E_u

Throughout this section we fix (u, v, w) a solution of (1.1)–(1.3) equipped with an initial data (u_0, v_0, w_0) such that all components have nonempty supports which are bounded above.

For $s < c_1 < c_2$ we define $\omega_{[c_1,c_2]}$ as the set of the functions $(\tilde{u}, \tilde{v}, \tilde{w}) : \mathbb{R}^2 \to \mathbb{R}^3$ such that there exist sequences $\{t_n\} \subset [0, \infty)$ and $\{x_n\} \subset \mathbb{R}$ with $t_n \to \infty$ and $x_n \in [c_1 t_n, c_2 t_n]$ for all $n \geq 0$ such that

$$(\tilde{u}, \tilde{v}, \tilde{w})(x, t) = \lim_{n\to\infty} (u, v, w)(x + x_n, t + t_n) \text{ locally uniformly for } (x, t) \in \mathbb{R}^2.$$

Let us also observe that, since $c_1 > s$, any $(\tilde{u}, \tilde{v}, \tilde{w}) \in \omega_{[c_1,c_2]}$ becomes an entire solution of the homogeneous problem

$$u_t(x, t) = d_1 u_{xx}(x, t) + r_1 u(x, t)[1 - (u + hv + aw)(x, t)], \ x \in \mathbb{R}, \ t \in \mathbb{R}, \tag{3.1}$$

$$v_t(x, t) = d_2 v_{xx}(x, t) + r_2 v(x, t)[1 - (ku + v + aw)(x, t)], \ x \in \mathbb{R}, \ t \in \mathbb{R}, \tag{3.2}$$

$$w_t(x, t) = d_3 w_{xx}(x, t) + r_3 w(x, t)[-1 + (bu + bv - w)(x, t)], \ x \in \mathbb{R}, \ t \in \mathbb{R}, \tag{3.3}$$

We start with the following lemma about a weak pointwise spreading lemma in the fast prey case. We assume $s_2^* < s_1^*$, $s_2^* < s_3^*$ and set $s^* = \min\{s_1^*, s_3^*\}$.

Lemma 3.1 *Suppose that $s \in (s_2^*, s^*)$. Fix any $c_1 < c_2$ such that $s < c_1 < c_2 < s^*$. Then for any $c \in [c_1, c_2]$ there exists $\mu_1(c) \in (0, 1)$ such that the solution (u, v, w) satisfies*

$$\limsup_{t\to\infty} u(ct, t) \geq \mu_1(c), \tag{3.4}$$

and for any $(\tilde{u}, \tilde{v}, \tilde{w}) \in \omega_{[c_1,c_2]}$ with $\tilde{u} \neq 0$

$$\limsup_{t\to\infty} \tilde{u}(ct, t) \geq \mu_1(c). \tag{3.5}$$

Proof Fix $s < c_1, c_2 < s^*$. Here we focus on proving (3.5), that is for solutions in $\omega_{[c_1,c_2]}$. The proof of (3.4), for the solution itself, follows from the same arguments.

First recalling that $s > s_2^*$, Theorem 1.1 ensures as a special case that

$$\sup_{x\in[c_1 t, c_2 t]} v(x, t) \to 0 \text{ as } t \to \infty,$$

this implies that:

$$\text{for any } (\tilde{u}, \tilde{v}, \tilde{w}) \in \omega_{[c_1,c_2]} \text{ one has } \tilde{v} = 0.$$

Now to prove (3.5) we argue by contradiction by assuming that there are $c \in [c_1, c_2]$, sequences $\{t_n\}$ and $\{(\tilde{u}_n, 0, \tilde{w}_n)\} \subset \omega_{[c_1,c_2]}$ with $\tilde{u}_n \neq 0$ for all $n \geq 0$ and $\lim_{n\to\infty} t_n = \infty$ such that

$$\lim_{n\to\infty} \sup_{t\geq t_n} \tilde{u}_n(ct, t) = 0. \tag{3.6}$$

Then we claim that

$$\lim_{n\to\infty} \{ \sup_{|x-ct|\leq R, t\geq t_n} \tilde{u}_n(x, t)\} = 0, \ \forall R > 0. \tag{3.7}$$

Otherwise, there exist sequences $\{x_n\} \subset [-R, R]$ and $\{\tau_n\}$ with $\tau_n \geq t_n$ such that

$$\liminf_{n\to\infty} \tilde{u}_n(x_n + c\tau_n, \tau_n) > 0.$$

Without loss of generality (up to a subsequence) we may assume that $x_n \to x_0$ as $n \to \infty$ for some $x_0 \in [-R, R]$. By the standard parabolic estimates and using $c > s$, up to the extraction of a subsequence, we have

$$(\tilde{u}_n, 0, \tilde{w}_n)(x + c\tau_n, t + \tau_n) \to (u_\infty, 0, w_\infty)(x, t) \ \text{ as } n \to \infty$$

locally uniformly in $\mathbb{R} \times \mathbb{R}$, where (u_∞, w_∞) is a nonnegative bounded entire solution of

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - aw), \ x \in \mathbb{R}, \ t \in \mathbb{R}, \\ w_t = d_3 w_{xx} + r_3 w(-1 + bu - w), \ x \in \mathbb{R}, \ t \in \mathbb{R}. \end{cases} \tag{3.8}$$

Since $u_\infty(0, 0) = 0$ by (3.6), the strong maximum principle implies that $u_\infty \equiv 0$. However, $u_\infty(x_0, 0) > 0$, a contradiction. Hence (3.7) holds.

Next, we derive that

$$\lim_{n\to\infty} \{ \sup_{|x-ct|\le R, t\ge t_n} \tilde{w}_n(x, t)\} = 0, \ \forall R > 0. \tag{3.9}$$

Indeed, taking any sequence $\{x_n\} \subset [-R, R]$ and $\{\tau_n\}$ with $\tau_n \ge t_n$ for all n, set

$$(u_\infty, w_\infty)(x, t) := \lim_{n\to\infty} (\tilde{u}_n, \tilde{w}_n)(x + c\tau_n, t + \tau_n).$$

Then $u_\infty \equiv 0$, by (3.7), and w_∞ is a nonnegative bounded entire solution of

$$(w_\infty)_t = d_3 (w_\infty)_{xx} + r_3 w_\infty(-1 - w_\infty), \ x \in \mathbb{R}, \ t \in \mathbb{R}.$$

This leads to $w_\infty \equiv 0$. Hence (3.9) follows.

Now, let

$$\lambda_R := \frac{c^2}{4d_1} + \frac{d_1\pi^2}{4R^2}, \ \phi(x) := e^{-cx/(2d_1)} \cos\left(\frac{\pi x}{2R}\right).$$

Then (λ_R, ϕ) satisfies

$$-d_1\phi_{xx} - c\phi_x = \lambda_R \phi \ \text{ in } (-R, R); \ \phi(\pm R) = 0.$$

Since $c < s_1^*$, we can find constants $0 < \delta \ll 1$ and $R \gg 1$ such that $c^2/(4d_1) < \lambda_R < r_1(1 - 2\delta)$. Note that $c > s$ and v tends to zero uniformly in $\mathbb{R}$. Hence, by (1.1), (3.7) and (3.9), for large enough n the function $\tilde{u}_n(x, t)$ satisfies

$$(\tilde{u}_n)_t \ge d_1(\tilde{u}_n)_{xx} + r_1(1 - \delta)\tilde{u}_n \ \text{ for } x \in (ct - R, ct + R), t \ge t_n.$$

Then $\hat{u}_n(x, t) := \tilde{u}_n(x + ct, t)$ satisfies

$$(\hat{u}_n)_t \ge d_1(\hat{u}_n)_{xx} + c(\hat{u}_n)_x + r_1(1 - \delta)\hat{u}_n \ \text{ for } x \in (-R, R), t \ge t_n.$$

Note that $\tilde{u}_n > 0$ by the strong maximum principle. We can choose a positive constant γ such that $\hat{u}_n(x, t_n) \ge \gamma e^{r_1\delta t_n}\phi(x)$ for all $x \in [-R, R]$. Then a comparison principle gives that

$$\hat{u}_n(x, t) \ge \gamma e^{r_1\delta t}\phi(x), \ |x| \le R, \ t \ge t_n.$$

This implies that $\tilde{u}_n(ct, t) \to \infty$ as $t \to \infty$, a contradiction. Hence the lemma is proved. □

Next, we derive a pointwise spreading lemma for the fast prey.

Lemma 3.2 *Suppose that* $s \in (s_2^*, s^*)$. *For any* $c \in (s, s^*)$ *there exists* $\mu_2(c) \in (0, 1)$ *such that*

$$\liminf_{t\to\infty} u(ct, t) \geq \mu_2(c), \tag{3.10}$$

where (u, v, w) *is the fixed solution of (1.1)–(1.3).*

Proof Again, proceed by a contradiction by assuming that there are $c \in (s, s^*)$ and a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} u(ct_n, t_n) = 0. \tag{3.11}$$

Fix $s < c_1 < c < c_2 < s^*$. By (3.4), we can choose a sequence $\{t_n'\}$ with $t_n' < t_n$ and $t_n' \to \infty$ such that

$$u(ct_n' t_n') \geq \mu_1(c)/2 \ \text{ for all } \ n.$$

Then we claim that $t_n - \tau_n \to \infty$ as $n \to \infty$, wherein we have set

$$\tau_n := \sup\{t \geq t_n' \mid u(ct, t) \geq \gamma_1(c)\} \text{ with } \gamma_1(c) := \mu_1(c)/2.$$

Indeed, by taking the limit and using $c > s > s_2^*$, we have due to Theorem 1.1 (up to extraction of a subsequence)

$$(u_n, v_n, w_n)(x + c\tau_n, t + \tau_n) \to (u_\infty, 0, w_\infty)(x, t)$$

locally uniformly in $\mathbb{R} \times \mathbb{R}$, where $(u_\infty, 0, w_\infty) \in \omega_{[c_1,c_2]}$.

If the sequence $\{t_n - \tau_n\}$ has a bounded subsequence then, up to a subsequence, one may assume that $t_n - \tau_n \to t_0$ as $n \to \infty$ for some $t_0 \in \mathbb{R}$. Moreover, we have

$$u_\infty(ct_0, t_0) = \lim_{n\to\infty} u(c(t_n - \tau_n) + c\tau_n, (t_n - \tau_n) + \tau_n) = \lim_{n\to\infty} u(ct_n, t_n) = 0,$$

by (3.11). It then follows from the strong maximum principle that $u_\infty \equiv 0$. This is a contradiction to $u_\infty(0, 0) = \gamma_1(c)$, since $u(c\tau_n, \tau_n) = \gamma_1(c)$ for all n. Hence $t_n - \tau_n \to \infty$ as $n \to \infty$.

Furthermore, since

$$u(ct, t) \leq \gamma_1(c), \ \ \forall t \in (\tau_n, t_n), \ \ \forall n \geq 0,$$

or equivalently

$$u(ct + c\tau_n, t + \tau_n) \leq \gamma_1(c), \ \ \forall t \in (0, t_n - \tau_n), \ \ \forall n \geq 0,$$

and since $t_n - \tau_n \to \infty$ as $n \to \infty$ we obtain by letting $n \to \infty$ that

$$u_\infty(ct, t) \leq \gamma_1(c) \ \text{ for all } \ t \geq 0. \tag{3.12}$$

However, as already observed, $(u_\infty, 0, w_\infty) \in \omega_{[c_1,c_2]}$ with $u_\infty \neq 0$ so that (3.12) contradicts (3.5) and this completes the proof of the lemma. □

Now we show the uniform spreading of the fast prey u as follows.

Proposition 3.3 *Suppose that* $s \in (s_2^*, s^*)$. *Then for any* $\varepsilon \in (0, (s^* - s)/2)$ *there is a positive constant* $\kappa_1 = \kappa_1(\varepsilon)$ *such that*

$$\liminf_{t\to\infty} \left\{ \inf_{(s+\varepsilon)t \leq x \leq (s^*-\varepsilon)t} u(x, t) \right\} \geq \kappa_1. \tag{3.13}$$

Proof Fix $\varepsilon \in (0, (s^* - s)/2)$ and set $c_1 = s + \varepsilon$ and $c_2 = s^* - \varepsilon$. Consider the shifted function

$$(\hat{u}, \hat{v}, \hat{w})(x, t) = (u, v, w)(x + c_1 t, t).$$

By contradiction, we assume that there exist $\tilde{c} \in [0, c_2 - c_1)$ and a sequence $\{(c_k, t_k)\}$ with $c_k \in [0, c_2 - c_1)$ such that $c_k \to \tilde{c}$, $t_k \to \infty$ as $k \to \infty$, and

$$\hat{u}(c_k t_k, t_k) = u(c_k t_k + c_1 t_k, t_k) \to 0 \text{ as } k \to \infty. \tag{3.14}$$

Next, define the sequence $\{t_k'\}$ by

$$t_k' := \frac{c_k t_k}{c_2 - c_1},$$

so that we have $t_k' < t_k$. One may observe that $t_k' \to \infty$ as $k \to \infty$. Indeed, suppose that one has (at least along a subsequence) $c_k t_k \to x_\infty$ as $k \to \infty$ for some $x_\infty \in [0, \infty)$. Consider the sequence of functions $(u_k, v_k, w_k)(x, t) = (u, v, w)(x + c_1 t_k, t + t_k)$ that converges (possibly along a subsequence) to $(u_\infty, v_\infty, w_\infty) \in \omega_{[c_1, c_2]}$. Then, due to (3.14),

$$u_\infty(x_\infty, 0) = \lim_{k \to \infty} u_k(c_k t_k, 0) = \lim_{k \to \infty} u(c_k t_k + c_1 t_k, t_k) = 0.$$

The strong maximum principle gives that $u_\infty \equiv 0$. On the other hand, from Lemma 3.2 one has

$$u(c_1 t_k, t_k) \geq \frac{3\mu_2(c_1)}{4}, \ \forall k \gg 1,$$

which implies that $u_\infty(0, 0) \geq \mu_2(c_1)/2 > 0$, a contradiction. This proves that $c_k t_k \to \infty$, or equivalently $t_k' \to \infty$ as $k \to \infty$.

Now we observe from Lemma 3.2 that

$$u(c_2 t_k', t_k') \geq \mu_2(c_2)/2 \ \text{ for all } k \text{large enough.}$$

This rewrites as

$$u(c_1 t_k' + (c_2 - c_1) t_k', t_k') = u(c_1 t_k' + c_k t_k, t_k') = \hat{u}(c_k t_k, t_k') \geq \mu_2(c_2)/2 \ \text{ for all } \ k \gg 1.$$

Then let us introduce

$$\tau_k := \sup\{t \geq t_k' \mid \hat{u}(c_k t_k, t) \geq \gamma_0\} \text{ with } \gamma_0 := \min\{\mu_2(c_2), \mu_1(c_1)\}/2.$$

Note that $\tau_k < t_k$ for all large k. We claim that $t_k - \tau_k \to \infty$ as $k \to \infty$. Indeed, to see this, we assume by contradiction that (at least for a subsequence) $t_k - \tau_k \to t_0$ as $k \to \infty$ for some $t_0 \in \mathbb{R}$. Consider the sequence of functions

$$(u_k, v_k, w_k)(x, t) := (u, v, w)(x + c_1 t_k + c_k t_k, t + t_k) \to (u_\infty, v_\infty, w_\infty)(x, t).$$

Then, by (3.14), $u_\infty(0, 0) = 0$ and so $u_\infty \equiv 0$ due to the strong maximum principle. On the other hand, due to the definition of τ_k (together with (3.14)) one also has

$$u_k(c_1(\tau_k - t_k), \tau_k - t_k) = \hat{u}(c_k t_k, \tau_k) = \gamma_0 \text{ for all } k \gg 1.$$

This yields, letting $k \to \infty$,

$$u_\infty(-c_1 t_0, -t_0) = \gamma_0 > 0,$$

a contradiction with $u_\infty \equiv 0$. This proves that $t_n - \tau_n \to \infty$ as $n \to \infty$.

Finally, to complete the proof of the proposition, note that for all k large enough one has

$$\hat{u}(c_k t_k, t' + \tau_k) \le \hat{u}(c_k t_k, \tau_k) = \gamma_0, \ \forall t' \in [0, t_k - \tau_k]. \tag{3.15}$$

Consider the sequence of functions

$$(u_k, v_k, w_k)(x, t) := (u, v, w)(x + c_1\tau_k + c_k t_k, t' + \tau_k),$$

and possibly along a subsequence, one may assume that

$$(u_k, v_k, w_k)(x, t) \to (u_\infty, v_\infty, w_\infty)(x, t) \text{ locally uniformly for } (x, t) \in \mathbb{R}^2 \text{ as } k \to \infty.$$

Now let us observe that (3.15) coupled with $t_k - \tau_k \to \infty$ as $k \to \infty$ ensures that

$$u_\infty(0, 0) = \gamma_0 \text{ and } u_\infty(c_1 t', t') \le \gamma_0, \ \forall t' \ge 0. \tag{3.16}$$

To reach a contradiction, let us further note that for all k one has

$$c_1\tau_k \le c_1\tau_k + c_k t_k = c_1\tau_k + (c_2 - c_1)t_k' \le c_1\tau_k + (c_2 - c_1)\tau_k \le c_2\tau_k,$$

so that $(u_\infty, v_\infty, w_\infty) \in \omega_{[c_1,c_2]}$ together with $u_\infty \ne 0$. Hence Lemma 3.1 (see (3.5)) ensures that

$$\limsup_{t\to\infty} u_\infty(c_1 t, t) \ge \mu_1(c_1) > \gamma_0,$$

a contradiction with (3.16), that completes the proof of the proposition. □

Remark 3.1 From Proposition 3.3, we obtain an important lower estimate for the functions in $\omega_{[c_1,c_2]}$ with $s_2^* < s < c_1 < c_2 < s^*$. More precisely, fix $c_1 < c_2$ and $\varepsilon > 0$ small enough such that

$$s + \varepsilon < c_1 \text{ and } c_2 < s^* - \varepsilon,$$

then for all $(\tilde{u}, \tilde{v}, \tilde{w}) \in \omega_{[c_1,c_2]}$ one has:

$$\tilde{v} = 0 \text{ and } \tilde{u} \ge \kappa_1(\varepsilon),$$

where $\kappa_1(\varepsilon) > 0$ is the constant provided in Proposition 3.3.

Recall $s_{**} = \min\{s_1^*, s_3^{***}\} < s^*$ and $s_3^{***} = 2\sqrt{d_3 r_3(b-1)}$. Assume now $s_2^* < s_{**}$. For the predator, we first derive the following weak pointwise spreading property.

Lemma 3.4 *Assume that $s \in (s_2^*, s_{**})$ and let $\varepsilon > 0$ small enough such that*

$$s + \varepsilon < s_{**} - \varepsilon.$$

Set $c_1 := s + \varepsilon$ and $c_2 = s_{**} - \varepsilon$. Then for any $c \in [c_1, c_2]$ there is a positive constant $\mu_3(c) = \mu_3^\varepsilon(c)$ such that

$$\limsup_{t\to\infty} w(ct, t) \ge \mu_3(c),$$

and for all $(\tilde{u}, \tilde{v}, \tilde{w}) \in \omega_{[c_1,c_2]}$ with $\tilde{w} \ne 0$

$$\limsup_{t\to\infty} \tilde{w}(ct, t) \ge \mu_3(c).$$

Proof As before, we only prove the second statement of the lemma. The proof for the solution itself follows from the same arguments.

Let $c \in [c_1, c_2]$ be given. By contradiction, we assume that there exist a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ and a sequence $\{(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n)\} \subset \omega_{[c_1,c_2]}$ such that

$$\tilde{w}_n \neq 0,\ \forall n,\ \text{ and } \lim_{n\to+\infty} \sup_{t\geq t_n} \tilde{w}_n(ct, t) = 0.$$

By passing to the limit as $n \to \infty$, and applying the strong maximum principle, we have

$$\limsup_{n\to\infty}\{\sup_{t\geq t_n, |x-ct|\leq R} \tilde{w}_n(x, t)\} = 0 \tag{3.17}$$

for any $R > 0$ (as in the proof of Lemma 3.1). Then we claim

$$\limsup_{n\to\infty}\{\inf_{t\geq t_n, |x-ct|\leq R} \tilde{u}_n(x, t)\} = 1 \tag{3.18}$$

for any $R > 0$.

To this aim, for contradiction, we assume that there is a sequence $\{(x_n, t'_n)\}$ with $t'_n \geq t_n$ and $x_n \in [ct'_n - R, ct'_n + R]$ such that $\limsup_{n\to\infty} \tilde{u}_n(x_n, t'_n) < 1$. Then, by standard parabolic estimates and extracting a subsequence, we have that $(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n)(x + x_n, t + t'_n)$ converges to $(u_\infty, v_\infty, w_\infty)$ as $n \to \infty$. According to Remark 3.1, one has

$$\tilde{v}_n = 0 \text{ and } \tilde{u}_n \geq \kappa_1(\varepsilon/2) \text{ for all } n,$$

so that

$$v_\infty = 0 \text{ and } u_\infty \geq \kappa_1(\varepsilon/2).$$

In addition, since $w_\infty(0, t) = 0$ for all $t > 0$, by the strong maximum principle we see that $w_\infty \equiv 0$. Hence u_∞ satisfies

$$\begin{cases} (u_\infty)_t = d_1(u_\infty)_{xx} + r_1 u_\infty(1 - u_\infty), & (x, t) \in \mathbb{R} \times \mathbb{R}, \\ \inf_{(x,t)\in\mathbb{R}^2} u_\infty(x, t) \geq \kappa_1(\varepsilon/2) > 0. \end{cases}$$

This implies that $u_\infty \equiv 1$, a contradiction to $u_\infty(0, 0) < 1$. Hence (3.18) is proved.

Now, for any small $\delta \in (0, (b-1)/2)$ and large $R > 0$, we have

$$(\tilde{w}_n)_t \geq d_3(\tilde{w}_n)_{xx} + r_3(b - 1 - 2\delta)\tilde{w}_n,\ |x - ct| \leq R,\ t \geq t_n,$$

for n large enough. Proceed as in the proof of Lemma 3.1, using $c < 2\sqrt{d_3 r_3(b-1)}$, we reach a contradiction to (3.17). The lemma is thus proved. □

Next, we derive the pointwise spreading property of the predator.

Lemma 3.5 *Assume that $s \in (s_2^*, s_{**})$. Then for any $c \in (s, s_{**})$ there exists $\mu_4(c) > 0$ such that the solution (u, v, w) satisfies*

$$\liminf_{t\to\infty} w(ct, t) \geq \mu_4(c).$$

Proof Fix $c \in (s, s_{**})$ and let us proceed by contradiction as in the proof of Lemma 3.2. Fix $\varepsilon > 0$ small enough such that $c \in [s + \varepsilon, s_{**} - \varepsilon]$. Then as in the proof of Lemma 3.2 there exists an entire solution $(u_\infty, 0, w_\infty) \in \omega_{[s+\varepsilon, s_{**}-\varepsilon]}$ of (3.1)-(3.3) such that

$$w_\infty(0, 0) = \frac{\mu_3(c)}{2}, \quad w_\infty(ct, t) \leq \frac{\mu_3(c)}{2}\ \forall t \geq 0, \tag{3.19}$$

so that $w_\infty \neq 0$ and the second condition provides a contradiction with Lemma 3.4. Thus the lemma follows. □

The last step is to show the uniform spreading of the predator w.

Proposition 3.6 *Suppose that $s \in (s_2^*, s_{**})$. Then for any $\varepsilon \in (0, (s_{**} - s)/2)$ there is a positive constant $\kappa_2 = \kappa_2(\varepsilon)$ such that the solution (u, v, w) satisfies*

$$\liminf_{t\to+\infty} \left\{ \inf_{(s+\varepsilon)t \le x \le (s_{**}-\varepsilon)t} w(x,t) \right\} \ge \kappa_2. \tag{3.20}$$

Proof The proof is exactly the same as that of Proposition 3.3, using Lemma 3.5 instead of Lemma 3.2. We omit it safely. □

Proof of Theorem 1.4 Note that v tends to zero uniformly on $\mathbb{R}$ as $t \to \infty$, since $s > s_2^*$. Along with Propositions 3.3 and 3.6, the theorem follows by a similar argument to the proof of [10, Theorem 2.7]. We also omit the details here. □

Remark 3.2 From the above discussions, it is easy to see that [10, Theorem 2.7] holds for the system

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u[1 + \alpha(x - st) - u - aw], \\ v_t = d_2 v_{xx} + r_2 v[-1 + \alpha(x - st) + bu - v], \end{cases}$$

in which the predator is directly affected by the shifting environment.

3.3 Spreading to E_c

We assume that $s_3^* < s_1^{**}$, $s_3^* < s_2^{**}$ and $s \in (s_3^*, s^{**})$, where $s^{**} = \min\{s_1^{**}, s_2^{**}\}$.

To prove Theorem 1.5, we first consider the following competition system

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u[1 + \alpha(x - st) - u - hv], \; x \in \mathbb{R}, \; t > 0, \\ v_t = d_2 v_{xx} + r_2 v[1 + \alpha(x - st) - ku - v], \; x \in \mathbb{R}, \; t > 0, \\ u(x, 0) = u_0(x), \; v(x, 0) = v_0(x), \; x \in \mathbb{R}. \end{cases} \tag{3.21}$$

Recall the following spreading theorem on (3.21) from [30] (see also [34, Theorem 2.7] when $r_1 = r_2$).

Proposition 3.7 *Let (u, v) be a solution of (3.21) with initial data $(u_0, v_0) \in X_1 \times X_1$ such that $u_0 \ne 0$ and $v_0 \ne 0$. Then for any $s \in (0, s^{**})$ we have*

$$\lim_{t\to\infty} \left\{ \sup_{(s+\varepsilon)t \le x \le (s^{**}-\varepsilon)t} \left[|u(x,t) - u_c| + |v(x,t) - v_c| \right] \right\} = 0$$

for all $\varepsilon \in (0, (s^{**} - s)/2)$.

Proof of Theorem 1.5 Since $s > s_3^*$, w tends to zero uniformly on $\mathbb{R}$ as $t \to \infty$. Given $\delta \in (0, \min\{1 - h, 1 - k\})$. There exists $T \gg 1$ such that $aw(x, t) \le \delta$ for all $x \in \mathbb{R}$, $t \ge T$. Let $V = 1 - v$. Then (u, V) satisfies

$$\begin{aligned} u_t &= d_1 u_{xx} + r_1 u[1 - h + \alpha(x - st) - u + hV - aw], \; x \in \mathbb{R}, \; t \ge T, \\ V_t &= d_2 V_{xx} + r_2(1 - V)[-\alpha(x - st) + ku - V + aw], \; x \in \mathbb{R}, \; t \ge T. \end{aligned}$$

Hence

$$u_t \le d_1 u_{xx} + r_1 u[1 - h + \alpha(x - st) - u + hV], \; x \in \mathbb{R}, \; t \ge T,$$

$$V_t \le d_2 V_{xx} + r_2(1-V)[\delta - \alpha(x-st) + ku - V],\ x \in \mathbb{R},\ t \ge T,$$

It follows from the comparison principle for cooperative systems that $u(x,t) \le \overline{u}(x,t)$ and $V(x,t) \le \overline{V}(x,t)$ for $x \in \mathbb{R}$, $t \ge T$, where $(\overline{u}, \overline{V})$ is the solution of

$$\begin{cases} \overline{u}_t = d_1 \overline{u}_{xx} + r_1 \overline{u}[1 - h + \alpha(x-st) - \overline{u} + h\overline{V}],\ x \in \mathbb{R},\ t \ge T, \\ \overline{V}_t = d_2 \overline{V}_{xx} + r_2(1-\overline{V})[\delta - \alpha(x-st) + k\overline{u} - \overline{V}],\ x \in \mathbb{R},\ t \ge T, \\ \overline{u}(x,T) = u(x,T),\ \overline{V}(x,T) = 1 - v(x,T),\ x \in \mathbb{R}. \end{cases} \tag{3.22}$$

Set $\underline{v} = 1 - \overline{V}$. Then it follows from (3.22) that $(\overline{u}, \underline{v})$ satisfies

$$\begin{cases} \overline{u}_t = d_1 \overline{u}_{xx} + r_1 \overline{u}[1 + \alpha(x-st) - \overline{u} - h\underline{v}],\ x \in \mathbb{R},\ t \ge T, \\ \underline{v}_t = d_2 \underline{v}_{xx} + r_2 \underline{v}[1 - \delta + \alpha(x-st) - k\overline{u} - \underline{v}],\ x \in \mathbb{R},\ t \ge T, \\ \overline{u}(x,T) = u(x,T),\ \underline{v}(x,T) = v(x,T),\ x \in \mathbb{R}. \end{cases}$$

Since $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$, both $u(x,T)$ and $v(x,T)$ are positive by the strong maximum principle. It follows from Proposition 3.7 that for any $s \in (s_3^*, s_\delta^{**})$,

$$s_\delta^{**} := \min\{s_1^{**}, 2\sqrt{d_2 r_2(1-k-\delta)}\},$$

we have

$$\lim_{t\to\infty} \left\{ \sup_{(s+\varepsilon/2)t \le x \le (s_\delta^{**} - \varepsilon/2)t} \left[|\overline{u}(x,t) - \bar{u}_c| + |\underline{v}(x,t) - \bar{v}_c| \right] \right\} = 0 \tag{3.23}$$

for all $\varepsilon \in (0, (s_\delta^{**} - s))$, where

$$\bar{u}_c := \frac{1 + h\delta - h}{1 - hk},\ \ \bar{v}_c := \frac{1 - k - \delta}{1 - hk}.$$

Similarly, $u(x,t) \ge \underline{u}(x,t)$ and $V(x,t) \ge \underline{V}(x,t)$ for $x \in \mathbb{R}$, $t \ge T$, where $(\underline{u}, \underline{V})$ is the solution of

$$\begin{aligned} &\underline{u}_t = d_1 \underline{u}_{xx} + r_1 \underline{u}[1 - h - \delta + \alpha(x-st) - \underline{u} + h\underline{V}],\ x \in \mathbb{R},\ t \ge T, \\ &\underline{V}_t = d_2 \underline{V}_{xx} + r_2(1-\underline{V})[-\alpha(x-st) + k\underline{u} - \underline{V}],\ x \in \mathbb{R},\ t \ge T, \\ &\underline{u}(x,T) = u(x,T),\ \underline{V}(x,T) = 1 - v(x,T),\ x \in \mathbb{R}. \end{aligned}$$

Set $\overline{v} = 1 - \underline{V}$. Then $(\underline{u}, \overline{v})$ satisfies

$$\begin{cases} \underline{u}_t = d_1 \underline{u}_{xx} + r_1 \underline{u}[1 - \delta + \alpha(x-st) - \underline{u} - h\overline{v}],\ x \in \mathbb{R},\ t \ge T, \\ \overline{v}_t = d_2 \overline{v}_{xx} + r_2 \overline{v}[1 + \alpha(x-st) - k\underline{u} - \overline{v}],\ x \in \mathbb{R},\ t \ge T, \\ \underline{u}(x,T) = u(x,T),\ \overline{v}(x,T) = v(x,T),\ x \in \mathbb{R}. \end{cases}$$

It follows from Proposition 3.7 again that for any $s \in (s_3^*, s_{**}^\delta)$,

$$s_{**}^\delta := \min\{2\sqrt{d_1 r_1(1-h-\delta)}, s_2^{**}\},$$

we have

$$\lim_{t\to\infty} \left\{ \sup_{(s+\varepsilon/2)t \le x \le (s_{**}^\delta - \varepsilon/2)t} \left[|\underline{u}(x,t) - \hat{u}_c| + |\overline{v}(x,t) - \hat{v}_c| \right] \right\} = 0 \tag{3.24}$$

for all $\varepsilon \in (0, (s^{\delta}_{**} - s))$, where

$$\hat{u}_c := \frac{1-h-\delta}{1-hk}, \ \hat{v}_c := \frac{1+k\delta-k}{1-hk}.$$

Finally, using $\underline{u} \le u \le \overline{u}$ and $\underline{v} \le v \le \overline{v}$ in $\mathbb{R} \times [T, \infty)$, it follows from (3.23) and (3.24) that

$$\lim_{t\to\infty} \left\{ \sup_{(s+\varepsilon)t \le x \le (s^{**}-\varepsilon)t} \Big[|u(x,t) - u_c| + |v(x,t) - v_c| \Big] \right\} = 0 \tag{3.25}$$

for all $\varepsilon \in (0, (s^{**} - s)/2)$, by letting $\delta \downarrow 0$, since

$$(\bar{u}_c, \bar{v}_c) \to (u_c, v_c), \ (\hat{u}_c, \hat{v}_c) \to (u_c, v_c), \ s^{**}_{\delta} \to s^{**} \text{ and } s^{\delta}_{**} \to s^{**} \text{ as } \delta \downarrow 0.$$

Hence Theorem 1.5 is proved. □

3.4 Spreading to E_*

In the sequel, we let $s_* := \min\{s_1^*, s_2^*\}$.

First, we prove the following lemma on the weak pointwise persistence of $u + v$.

Lemma 3.8 *Suppose that $s < s_*$. Then for any $c \in (s, s_*)$ there exists $\nu_1(c) > 0$ such that for each $(u_0, v_0, w_0) \in X_1 \times X_1 \times X_{2b-1}$ with $u_0 + v_0 \ne 0$ the solution (u, v, w) of (1.1)-(1.3) satisfies*

$$\limsup_{t\to\infty} (u+v)(ct, t) \ge \nu_1(c). \tag{3.26}$$

Proof We assume for contradiction that there are $c \in (s, s_*)$, sequences

$$\{(u_{0n}, v_{0n}, w_{0n})\} \subset X_1 \times X_1 \times X_{2b-1}$$

with $u_{0n} + v_{0n} \ne 0$ for all $n \ge 0$ and $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$\lim_{n\to\infty} \sup_{t \ge t_n} (u_n + v_n)(ct, t) = 0,$$

wherein (u_n, v_n, w_n) denotes the solution of (1.1)-(1.3) with the initial datum $(u_{0,n}, v_{0,n}, w_{0,n})$. Then we have

$$\lim_{n\to\infty} \sup_{t\ge t_n} u_n(ct, t) = 0, \ \lim_{n\to\infty} \sup_{t\ge t_n} v_n(ct, t) = 0. \tag{3.27}$$

By the same argument as before, we also obtain that

$$\lim_{n\to\infty} \{ \sup_{|x-ct|\le R, t\ge t_n} u_n(x,t) \} = \lim_{n\to\infty} \{ \sup_{|x-ct|\le R, t\ge t_n} v_n(x,t) \} = 0, \ \forall R > 0. \tag{3.28}$$

From (3.28), we can further derive that

$$\lim_{n\to\infty} \{ \sup_{|x-ct|\le R, t\ge t_n} w_n(x,t) \} = 0, \ \forall R > 0. \tag{3.29}$$

Indeed, by the same limiting argument, the limit function w_∞ satisfies (using also $u_\infty = v_\infty \equiv 0$)

$$w_t = d_3 w_{xx} + r_3 w(-1 - w), \ x \in \mathbb{R}, \ t \in \mathbb{R}.$$

Since w_∞ is nonnegative and bounded, it must be identically zero. Hence (3.29) follows.

To conclude we argue as above by setting for $i = 1, 2$ and $R > 0$

$$\lambda_R^i := \frac{c^2}{4d_i} + \frac{d_i \pi^2}{4R^2}, \ \phi^i(x) := e^{-cx/(2d_i)} \cos\left(\frac{\pi x}{2R}\right).$$

Then (λ_R^i, ϕ^i) satisfies

$$-d_i \phi_{xx}^i - c\phi_x^i = \lambda_R^i \phi \ \text{ in } (-R, R); \ \phi^i(\pm R) = 0.$$

Since $c < s_i^*$ for $i = 1, 2$, one can find constants $0 < \delta \ll 1$ and $R \gg 1$ such that

$$c^2/(4d_i) < \lambda_R^i < r_i(1 - 2\delta).$$

With such R, due to (3.28) and (3.29), choose $n \geq 0$ large enough such that

$$\sup_{t \geq t_n} \sup_{|x-ct| \leq R} (u_n + hv_n + aw_n)(x, t) \leq \delta,$$
$$\sup_{t \geq t_n} \sup_{|x-ct| \leq R} (ku_n + v_n + aw_n)(x, t) \leq \delta.$$

Since $u_{0n} + v_{0n} \neq 0$, either $u_{0n} \neq 0$ or $v_{0n} \neq 0$. If $u_{0n} \neq 0$, then for some constant $\kappa_{1,n}$ we have

$$u_n(x + ct, t) \geq \kappa_{1,n} e^{r_1 \delta t} \phi^1(x), \ |x| \leq R, \ t \geq t_n.$$

This implies that $u_n(ct, t) \to \infty$ as $t \to \infty$, a contradiction. Similarly, if $v_{0n} \neq 0$, then for some constant $\kappa_{2,n}$ we have

$$v_n(x + ct, t) \geq \kappa_{2,n} e^{r_2 \delta t} \phi^1(x), \ |x| \leq R, \ t \geq t_n,$$

and we also reach a contradiction. Hence the lemma is proved. □

Remark 3.3 It is easy to see that the same argument as in Lemma 3.8 also leads to the following results: Given any $c \in [0, s_*)$ there exists $\hat{v}_0(c) > 0$ such that for any nontrivial nonnegative solution $(u, v, w) = (u, v, w)(\cdot, t) \in X_1 \times X_1 \times X_{2b-1}$ with $u + v \neq 0$ of the system

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u - hv - aw), \\ v_t = d_2 v_{xx} + r_2 v(1 - ku - v - aw), \\ w_t = d_3 w_{xx} + r_3 w(-1 + bu + bv - w), \end{cases} \tag{3.30}$$

satisfies

$$\limsup_{t \to \infty} (u + v)(ct, t) \geq \hat{v}_0(c).$$

Next, we prove the pointwise persistence for $u + v$.

Lemma 3.9 *Assume that $s < s_*$. Let $(u_0, v_0, w_0) \in X_1 \times X_1 \times X_{2b-1}$ be a given initial data with $u_0 + v_0 \neq 0$. Then for any $c \in (s, s_*)$ there exists $v_2(c) > 0$ such that the solution (u, v, w) of (1.1)-(1.3) starting from (u_0, v_0, w_0) satisfies*

$$\liminf_{t \to \infty} (u + v)(ct, t) \geq v_2(c). \tag{3.31}$$

Proof Again, proceed by a contradiction. Assume that there is a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that the solution (u, v, w) satisfies

$$\lim_{n \to \infty} (u + v)(ct_n, t_n) = 0.$$

This implies that

$$\lim_{n\to\infty} u(ct_n, t_n) = \lim_{n\to\infty} v(ct_n, t_n) = 0. \tag{3.32}$$

Since $u_0 + v_0 \neq 0$, by using (3.26) we can choose a sequence $\{t'_n\}$ with $t'_n < t_n$ and $t'_n \to \infty$ such that

$$(u+v)(ct'_n t'_n) \geq \nu_1(c)/2 \text{ for all } n \geq 0.$$

Next, define

$$\tau_n := \sup\{t \geq t'_n \mid (u+v)(ct, t) \geq \rho_1\}, \text{ with } \rho_1 := \min\{\nu_1(c), \hat{\nu}_0(c)\}/2.$$

Note that for n large enough one has $\tau_n < t_n$ and $(u+v)(c\tau_n, \tau_n) = \rho_1$. It also follows from a limiting argument and the strong maximum principle that $t_n - \tau_n \to \infty$ as $n \to \infty$. Indeed, by taking the limit, we have (up to extraction of a subsequence)

$$(u, v, w)(x + c\tau_n, t + \tau_n) \to (u_\infty, v_\infty, w_\infty)(x, t)$$

locally uniformly in $\mathbb{R} \times \mathbb{R}$, where $(u_\infty, v_\infty, w_\infty)$ is an entire solution of system (3.30). If (up to a subsequence) $t_n - \tau_n \to t_0$ as $n \to \infty$ for some $t_0 \in \mathbb{R}$, then

$$u_\infty(ct_0, t_0) = \lim_{n\to\infty} u(c(t_n - \tau_n) + c\tau_n, (t_n - \tau_n) + \tau_n) = \lim_{n\to\infty} u(ct_n, t_n) = 0,$$

by (3.32). It then follows from the strong maximum principle that $u_\infty \equiv 0$. Similarly, we have $v_\infty \equiv 0$. This is a contradiction to $(u_\infty + v_\infty)(0, 0) = \rho_1$, since $(u+v)(c\tau_n, \tau_n) = \rho_1$ for all n. Hence $t_n - \tau_n \to \infty$ as $n \to \infty$.

Furthermore, since

$$(u+v)(c\tau_n, \tau_n) = \rho_1 \text{ and } (u+v)(ct, t) \leq \rho_1, \ \forall t \in (\tau_n, t_n),$$

we obtain

$$(u_\infty + v_\infty)(0, 0) = \rho_1 \text{ and } (u_\infty + v_\infty)(ct, t) \leq \rho_1 \text{ for all } t \geq 0, \tag{3.33}$$

due to $t_n - \tau_n \to \infty$ as $n \to \infty$. On the other hand note that $(u_\infty, v_\infty, w_\infty)$ is a solution of (3.30) in $X_1 \times X_1 \times X_{2b-1}$ with $u_\infty + v_\infty \neq 0$. Hence the second condition in (3.33) contradicts Remark 3.3. This completes the proof of the lemma. □

With Lemma 3.9, we can show the uniform persistence of $u + v$.

Proposition 3.10 *Assume that $s < s_*$. Let $(u_0, v_0, w_0) \in X_1 \times X_1 \times X_{2b-1}$ be a given initial data with $u_0 + v_0 \neq 0$. Then the solution (u, v, w) of (1.1)-(1.3) starting from (u_0, v_0, w_0) satisfies for any $\varepsilon \in (0, (s_* - s)/2)$ there is a positive constant θ_ε such that*

$$\liminf_{t\to\infty} \left\{ \inf_{(s+\varepsilon)t \leq x \leq (s_*-\varepsilon)t} (u+v)(x, t) \right\} \geq \theta_\varepsilon. \tag{3.34}$$

Proof Again to prove this proposition we argue by contradiction by assuming that for some given $\varepsilon \in (0, (s_* - s)/2)$, there is a sequence $\{(x_n, t_n)\}$ such that $t_n \to \infty$ as $n \to \infty$,

$$x_n \in [(s+\varepsilon)t_n, (s_* - \varepsilon)t_n], \ \forall n \geq 0, \ (u+v)(x_n, t_n) \to 0 \text{ as } n \to \infty. \tag{3.35}$$

First, by Lemma 3.9, we have

$$\liminf_{t\to\infty} (u+v)(c_1 t, t) \geq \nu_2(c_1), \quad \text{with } c_1 := s_* - \varepsilon/2.$$

Then, for the sequence $\{t_n' := x_n/c_1\}$, we have $t_n' < t_n$, $t_n' \to \infty$ as $n \to \infty$ and

$$(u+v)(x_n, t_n') = (u+v)(c_1 t_n', t_n') \geq \nu_2(c_1)/2 \ \text{ for all } \ n \gg 1. \tag{3.36}$$

Now introduce for all n

$$\tau_n := \sup\{t \geq t_n' \mid (u+v)(c_1 t_n', t) \geq \rho_2\},$$

where ρ_2 is given by

$$\rho_2 := \min\{\nu_2(c_1), \hat{v}_0(0)\}/2.$$

Here $\hat{v}_0(0)$ is defined in Remark 3.3. Note that $\tau_n < t_n$ and $(u+v)(c_1 t_n', \tau_n) = \rho_2$ for all large n, due to (3.35).

Claim that $t_n - \tau_n \to \infty$ as $n \to \infty$. For this, we let

$$(u_\infty, v_\infty, w_\infty)(x,t) := \lim_{n\to\infty} (u_n, v_n, w_n)(x + c_1 t_n', t + \tau_n). \tag{3.37}$$

Note that

$$\begin{aligned} x + c_1 t_n' - s(t+\tau_n) &= x - st + x_n - s\tau_n \\ &\geq x - st + (s+\varepsilon)t_n - s\tau_n \geq x - st + \varepsilon t_n \to \infty \text{ as } n \to \infty. \end{aligned}$$

Hence $(u_\infty, v_\infty, w_\infty)$ is a nonnegative bounded entire solution of (3.30). If, for a subsequence, $t_n - \tau_n \to t_0$ as $n \to \infty$ for some $t_0 \in \mathbb{R}$, then

$$(u_\infty + v_\infty)(0, t_0) = \lim_{n\to\infty} (u_n + v_n)(c_1 t_n', (t_n - \tau_n) + \tau_n) = \lim_{n\to\infty} (u_n + v_n)(x_n, t_n) = 0,$$

by (3.35). Hence $u_\infty(0, t_0) = v_\infty(0, t_0) = 0$. It follows from the strong maximum principle that $u_\infty = v_\infty \equiv 0$, a contradiction to

$$(u_\infty + v_\infty)(0,0) = \lim_{n\to\infty} (u+v)(c_1 t_n', \tau_n) = \rho_2 > 0.$$

This shows that $t_n - \tau_n \to \infty$ as $n \to \infty$. Hence we obtain that $(u_\infty, v_\infty, w_\infty)$ is a solution of (3.30) with $(u_\infty + v_\infty)(0,0) = \rho_2 \neq 0$ and

$$(u_\infty + v_\infty)(0,t) \leq \rho_2 \leq \hat{v}_0(0)/2, \ \forall t \geq 0.$$

This contradicts Remark 3.3 and completes the proof of the proposition. □

We now investigate the spreading for the predator w. Here we fix an initial data (u_0, v_0, w_0) in $X_1 \times X_1 \times X_{2b-1}$ with

$$u_0 \neq 0, \ \ v_0 \neq 0 \text{ and } w_0 \neq 0.$$

We denote by (u, v, w) the solution of (1.1)-(1.3) starting at time $t = 0$ from (u_0, v_0, w_0). As before for $s < c_1 < c_2$ we define $\omega_{[c_1,c_2]}$ as the set of the functions $(\tilde{u}, \tilde{v}, \tilde{w}) : \mathbb{R}^2 \to \mathbb{R}^3$ such that there exists $\{(x_n, t_n)\}$ with

$$c_1 t_n \leq x_n \leq c_2 t_n, \ \ \forall n, \ \ \text{and } t_n \to \infty \text{ as } \ n \to \infty,$$

such that

$$(\tilde{u}, \tilde{v}, \tilde{w})(x,t) = \lim_{n\to\infty} (u, v, w)(x + x_n, t + t_n) \text{ locally uniformly for } (x,t) \in \mathbb{R}^2.$$

Moreover, according to Proposition 3.10, when $s < s_*$ for each $\varepsilon \in (0, (s_* - s)/2)$ we have for any $(\tilde{u}, \tilde{v}, \tilde{w}) \in \omega_{[s+\varepsilon, s_*-\varepsilon]}$

$$(\tilde{u} + \tilde{v})(x,t) \geq \theta_\varepsilon, \ \ \forall (x,t) \in \mathbb{R}^2. \tag{3.38}$$

Here the constant θ_ε is provided by Proposition 3.10.

We now derive the following lemma on the weak pointwise persistence of w. Recall $s^{**} = \min\{s_1^{**}, s_2^{**}\} < s_*$.

Lemma 3.11 *Suppose that $s < \bar{s} := \min\{s^{**}, s_3^{***}\}$. Fix $\varepsilon \in (0, \bar{s} - s)/2)$ and let $c_1 = s + \varepsilon$, $c_2 = \bar{s} - \varepsilon$. Then for any $c \in [c_1, c_2]$ there is a constant $\nu_3(c) > 0$ such that*

$$\limsup_{t\to\infty} w(ct, t) \geq \nu_3(c), \tag{3.39}$$

and for any solution $(\tilde{u}, \tilde{v}, \tilde{w}) \in \omega_{[c_1,c_2]}$ with $\tilde{w} \neq 0$ one has

$$\limsup_{t\to\infty} \tilde{w}(ct, t) \geq \nu_3(c). \tag{3.40}$$

Proof Here again we prove (3.40) while the proof of (3.39) for the solution itself is similar.

To prove (3.40) we argue by contradiction and assume that for some $c \in [c_1, c_2]$ there exist a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ and a sequence $\{(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n)\} \in \omega_{[c_1,c_2]}$ with $\tilde{w}_n \neq 0$ such that

$$\lim_{n\to+\infty} \sup_{t\geq t_n} \tilde{w}_n(ct, t) = 0. \tag{3.41}$$

The same argument as before leads

$$\limsup_{n\to\infty}\{\sup_{t\geq t_n, |x-ct|\leq R} \tilde{w}_n(x, t)\} = 0, \;\; \forall R > 0. \tag{3.42}$$

Next, fix $\eta > 0$ small enough and $R > 0$ large enough. Then, using (3.42), for all $n \gg 1$ one has

$$\tilde{w}_n(x, t) \leq \eta, \;\; \forall t \geq t_n, \;\; |x - ct| \leq R.$$

Since $u \leq 1$ and $v \leq 1$, $(\tilde{u}_n, \tilde{v}_n)$ satisfies

$$(\tilde{u}_n)_t(x, t) \geq d_1(\tilde{u}_n)_{xx}(x, t) + r_1\tilde{u}_n(x, t)[1 - (h + a\eta) - \tilde{u}_n(x, t)], \;\; |x - ct| \leq R, \; t > t_n,$$
$$(\tilde{v}_n)_t(x, t) \geq d_2(\tilde{v}_n)_{xx}(x, t) + r_2\tilde{v}_n(x, t)[1 - (k + a\eta) - \tilde{v}_n(x, t)], \;\; |x - ct| \leq R, \; t > t_n.$$

We also have

$$(\tilde{u}_n + \tilde{v}_n)(x, t) \geq \theta_\varepsilon > 0 \text{ for all } (x, t) \in \mathbb{R}^2 \text{ and } n \geq 0.$$

As a consequence, since $c < s_1^{**}$, if $\tilde{u}_n \neq 0$, then using the same arguments as for [14, Lemma 5.2] one has

$$\liminf_{t\to\infty} \tilde{u}_n(x + ct, t) \geq q_{\eta,R}^1(x), \;\; \forall |x| \leq R,$$

where $q_{\eta,R}^1$ denotes a positive solution of the problem

$$d_1(q_{\eta,R}^1)_{xx} + c(q_{\eta,R}^1)_x + r_1 q_{\eta,R}^1[1 - (h + a\eta) - q_{\eta,R}^1] = 0, \;\; x \in (-R, R), \;\; q_{\eta,R}^1(\pm R) = 0.$$

Note that this is true when $R > 0$ is large enough and $\eta > 0$ small enough so that $c < 2\sqrt{d_1 r_1(1 - (h + a\eta))}$. Moreover, we have

$$q_{\eta,R}^1(x) \to 1 - h \text{ as } \eta \to 0, \;\; R \to \infty, \;\; \text{locally uniformly in } \mathbb{R}.$$

As a consequence, one obtains that

$$\liminf_{t\to\infty} \tilde{u}_n(x + ct, t) \geq 1 - h, \;\; \forall x \in \mathbb{R}.$$

Similarly, since $c < s_2^{**}$, either $\tilde{v}_n = 0$ or $\tilde{v}_n \neq 0$ and

$$\liminf_{t\to\infty} \tilde{v}_n(x+ct,t) \geq 1-k, \ \forall x \in \mathbb{R}.$$

As a consequence of [20] (see also [13]), we obtain

(1) If $\tilde{v}_n \neq 0$ and $\tilde{u}_n = 0$, then

$$\tilde{v}_n(x+ct,t) \to 1 \text{ locally uniformly.}$$

(2) If $\tilde{u}_n \neq 0$ and $\tilde{v}_n = 0$, then

$$\tilde{u}_n(x+ct,t) \to 1 \text{ locally uniformly.}$$

(3) If $\tilde{u}_n \neq 0$ and $\tilde{v}_n \neq 0$, then

$$(\tilde{u}_n, \tilde{v}_n)(x+ct,t) \to (u_c, v_c) \text{ locally uniformly.}$$

In any case, for all $R > 0$ there exists a sequence $\{t_n'\}$ with $t_n' \geq t_n$ for all $n \geq 0$ such that

$$\limsup_{n\to\infty} \left\{ \inf_{t\geq t_n'} \inf_{|x-ct|\leq R} (\tilde{u}_n + \tilde{v}_n)(x,t) \right\} \geq \min\{u_c + v_c, 1\} = 1. \tag{3.43}$$

Using (3.43), we complete the proof of the result. Indeed, for any small $\delta > 0$ and large $R > 0$, the function $\tilde{w}_n$ satisfies

$$(\tilde{w}_n)_t \geq d_3(\tilde{w}_n)_{xx} + r_3(b-1-\delta)\tilde{w}_n, \ |x-ct| \leq R, \ t \geq t_n',$$

for any n large enough. As before, since $\tilde{w}_n \neq 0$ and using $s < c < s_3^{***}$, we construct for $R \gg 1$ an unbounded sub-solution and reach a contradiction.

Thereby the lemma is proved. □

With Lemma 3.11, a similar argument to the proof of Lemma 3.2 we can prove the following lemma. We omit its proof.

Lemma 3.12 *Suppose that $s < \bar{s}$. Then for any $c \in (s, \bar{s})$ there is a constant $\nu_4(c) > 0$ such that*

$$\liminf_{t\to\infty} w(ct,t) \geq \nu_4(c). \tag{3.44}$$

Then, similar to that of Proposition 3.3, we have the following uniform persistence of w.

Proposition 3.13 *Suppose that $s < \bar{s}$. Then for any $\varepsilon \in (0, (\bar{s}-s)/2)$ there is a positive constant θ_1^ε such that the solution (u, v, w) satisfies*

$$\liminf_{t\to\infty} \left\{ \inf_{(s+\varepsilon)t\leq x\leq(\bar{s}-\varepsilon)t} w(x,t) \right\} \geq \theta_1^\varepsilon. \tag{3.45}$$

Now, we come to the question of the persistence of the fast prey. Here recall that (u, v, w) is a fixed solution of (1.1)–(1.3) equipped with the initial data (u_0, v_0, w_0) with $u_0 \neq 0$, $v_0 \neq 0$ and $w_0 \neq 0$.

Lemma 3.14 *Set*

$$\underline{s}_1 := \min\{s_1^{***}, \bar{s}\}, \tag{3.46}$$

and suppose that $s < \underline{s}_1$. *Fix* $s < c_1 < c_2 < \underline{s}_1$. *Then for each* $c \in [c_1, c_2]$ *there exists* $\nu_5(c) > 0$ *such that the solution* (u, v, w) *satisfies*

$$\limsup_{t\to\infty} u(ct, t) \geq \nu_5(c), \tag{3.47}$$

and for any $(\tilde{u}, \tilde{v}, \tilde{w}) \in \omega_{[c_1,c_2]}$ *with* $\tilde{u} \neq 0$

$$\limsup_{t\to\infty} \tilde{u}(ct, t) \geq \nu_5(c). \tag{3.48}$$

Proof Fix $\varepsilon > 0$ small such that

$$s + \varepsilon < c_1 < c_2 < \underline{s}_1 - \varepsilon \leq \bar{s} - \varepsilon.$$

First note that Propositions 3.10 and 3.13 ensure that for all $(\tilde{u}, \tilde{v}, \tilde{w}) \in \omega_{[c_1,c_2]}$ one has

$$\tilde{u} + \tilde{v} \geq \theta^{\varepsilon}, \quad \tilde{w} \geq \theta_1^{\varepsilon}. \tag{3.49}$$

Now as before to prove the lemma we only prove (3.48). We argue by contradiction assuming that there are sequences $\{(\tilde{u}_n, \tilde{v}_n, \tilde{w}_n)\} \subset \omega_{[c_1,c_2]}$ with $\tilde{u}_n \neq 0$ for all $n \geq 0$ and $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$\lim_{n\to\infty} \sup_{t\geq t_n} \tilde{u}_n(ct, t) = 0, \tag{3.50}$$

for some given $c \in [c_1, c_2]$. Then, as before, we have

$$\lim_{n\to\infty} \{ \sup_{|x-ct|\leq R, t\geq t_n} \tilde{u}_n(x, t)\} = 0, \ \forall R > 0. \tag{3.51}$$

From (3.51), we can further derive that

$$\lim_{n\to\infty} \{ \sup_{|x-ct|\leq R, t\geq t_n} [|\tilde{v}_n(x, t) - u_p| + |\tilde{w}_n(x, t) - w_p|]\} = 0, \ \forall R > 0. \tag{3.52}$$

Indeed, by a contradiction argument, assume that (3.52) does not hold for some $R > 0$. Then there is a sequence $\{(x_n, \tau_n)\}$ with $\tau_n \geq t_n$ and $|x_n - c\tau_n| \leq R$ for all n such that

$$|\tilde{v}_n(x_n, \tau_n) - u_p| + |\tilde{w}_n(x_n, \tau_n) - w_p| \geq \delta, \ \forall n, \tag{3.53}$$

for some $\delta > 0$. Let

$$(u_\infty, v_\infty, w_\infty)(x, t) := \lim_{n\to\infty} (\tilde{u}_n, \tilde{v}_n, \tilde{w}_n)(x + x_n, t + \tau_n), \ x, t \in \mathbb{R}.$$

Then $u_\infty \equiv 0$ and due to (3.49), (v_∞, w_∞) satisfies

$$v_\infty(x, t) \geq \theta^{\varepsilon}, \quad w_\infty(x, t) \geq \theta_1^{\varepsilon}, \ \forall (x, t) \in \mathbb{R}^2,$$

and it is a nonnegative bounded entire solution of

$$\begin{cases} v_t = d_2 v_{xx} + r_2 v(1 - v - aw), \\ w_t = d_3 w_{xx} + r_3 w(-1 + bv - w). \end{cases} \tag{3.54}$$

Hence [10, Theorem 2.7] ensures that $(v_\infty, w_\infty) \equiv (u_p, w_p)$ a contradiction with (3.53). Hence (3.52) is proved.

Now, it follows from (3.51) and (3.52) that there are positive constants δ small and R large such that the positive function $\tilde{u}_n$ satisfies

$$(\tilde{u}_n)_t \geq d_1(\tilde{u}_n)_{xx} + r_1(1 - hu_p - aw_p - \delta)\tilde{u}_n \ \text{ for } x \in (ct - R, ct + R), t \geq t_n,$$

for large enough n. The same argument as before leads to a contradiction, using $s < c < s_1^{***}$. Hence the lemma is proved. □

With Lemma 3.14, we show the pointwise persistence of u component.

Lemma 3.15 *Recalling (3.46), suppose that* $s < \underline{s}_1$. *Let* $\varepsilon > 0$ *be given such that* $\varepsilon < (\underline{s}_1 - s)/2$. *For any* $c \in [s+\varepsilon, \underline{s}_1 - \varepsilon]$ *there exists* $\nu_6(c) = \nu_6^\varepsilon(c) > 0$ *such that the solution* (u, v, w) *satisfies*

$$\liminf_{t\to\infty} u(ct, t) \geq \nu_6(c).$$

Proof Again, proceed by a contradiction. Assume that there exist $c \in [c_1, c_2]$ with $c_1 = s+\varepsilon$ and $c_2 = \underline{s}_1 - \varepsilon$ and a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} u(ct_n, t_n) = 0. \tag{3.55}$$

By (3.47), we can choose a sequence $\{t_n'\}$ with $t_n' < t_n$ and $t_n' \to \infty$ such that

$$u(ct_n' t_n') \geq \nu_5(c)/2 \;\text{ for all }\; n.$$

Recalling the definition of $\hat{\nu}_0(c)$ in Remark 3.3, we set

$$\tau_n := \sup\{t \geq t_n' \mid u(ct, t) \geq \rho_5\}, \quad \text{with } \rho_5 := \min\{\nu_5(c), \hat{\nu}_0(c)\}/2.$$

As above, it follows from a limiting argument and a strong maximum principle that $t_n - \tau_n \to \infty$ as $n \to \infty$.

Now, let

$$(u_\infty, v_\infty, w_\infty)(x, t) := \lim_{n\to\infty} (u_n, v_n, w_n)(x + c\tau_n, t + \tau_n), \;\; (x, t) \in \mathbb{R}^2.$$

Note that $(u_\infty, v_\infty, w_\infty) \in \omega_{[c_1, c_2]}$ with

$$u_\infty(0, 0) = \rho_5 > 0 \text{ and } u_\infty(ct, t) \leq \rho_5 \;\text{ for all }\; t \geq 0.$$

that contradicts Lemma 3.14 and proves the lemma. □

Then, as before, we obtain the uniform persistence of prey u as follows.

Proposition 3.16 *Suppose that* $s < \underline{s}_1$. *Then for any* $\varepsilon \in (0, (\underline{s}_1 - s)/2)$ *there is a positive constant* θ_2^ε *such that the solution* (u, v, w) *satisfies*

$$\liminf_{t\to\infty} \left\{ \inf_{(s+\varepsilon)t \leq x \leq (\underline{s}_1 - \varepsilon)t} u(x, t) \right\} \geq \theta_2^\varepsilon.$$

Finally, we come to the question of the persistence of the slow prey v. In fact, a similar argument to that for Lemma 3.14 gives

Lemma 3.17 *Set*

$$\underline{s}_2 := \min\{s_2^{***}, \bar{s}\}, \tag{3.56}$$

and suppose that $s < \underline{s}_2$. *Fix* $s < c_1 < c_2 < \underline{s}_2$. *Then for each* $c \in [c_1, c_2]$ *there exist* $\nu_7(c) > 0$ *such that the solution* (u, v, w) *satisfies*

$$\limsup_{t\to\infty} v(ct, t) \geq \nu_7(c),$$

and for any $(\tilde{u}, \tilde{v}, \tilde{w}) \in \omega_{[c_1, c_2]}$ *with* $\tilde{v} \neq 0$ *one has*

$$\limsup_{t\to\infty} \tilde{v}(ct, t) \geq \nu_7(c).$$

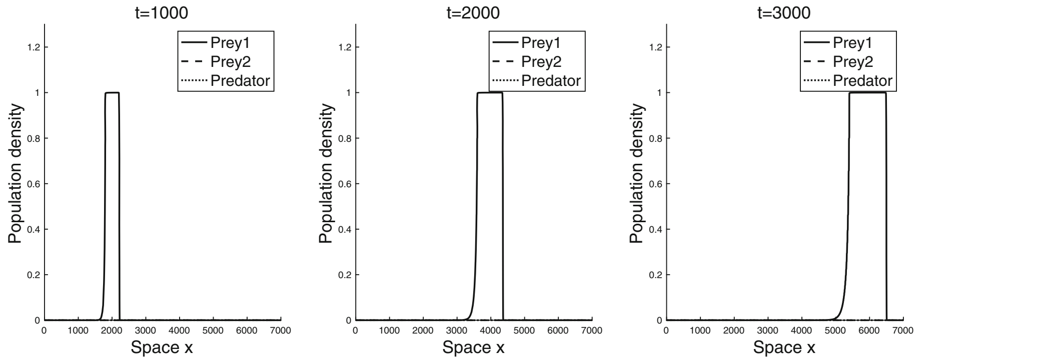

Fig. 1 Spreading of the fast prey for given parameters $\{d_1 = 1.3, d_2 = 0.4, d_3 = 0.3, s = 1.8\}$. The conditions in Theorem 1.3 are satisfied, since $s_1^* \approx 2.28$, $s_2^* \approx 1.26$, $s_3^* \approx 1.78$

Then as before, we obtain the following result.

Lemma 3.18 *Recalling (3.56), assume that* $s < \underline{s}_2$. *Fix* $\varepsilon \in (0, \bar{s} - s)/2)$ *and let* $c_1 = s + \varepsilon$, $c_2 = \bar{s} - \varepsilon$. *Then for any* $c \in [c_1, c_2]$ *there is a constant* $\nu_8(c) > 0$ *such that the solution* (u, v, w) *satisfies*

$$\liminf_{t\to\infty} v(ct, t) \geq \nu_8(c).$$

Hence we obtain the following uniform persistence of prey v.

Proposition 3.19 *Suppose that* $s < \underline{s}_2$. *Then for any* $\varepsilon \in (0, (\underline{s}_2 - s)/2)$ *there is a positive constant* $\theta_3^\varepsilon > 0$ *such that the given solution* (u, v, w) *satisfies*

$$\liminf_{t\to\infty} \left\{ \inf_{(s+\varepsilon)t \leq x \leq (\underline{s}_2 - \varepsilon)t} v(x, t) \right\} \geq \theta_3^\varepsilon.$$

Therefore, Theorem 1.6 follows from Propositions 3.13, 3.16 and 3.19, by a similar proof to that of Theorem 1.4 using a Lyapunov functional approach of [13, Lemma 4.3] and [20, Theorem 1.1].

4 Some Numerical Simulations

In this section, we first present some numerical simulations for the spreading dynamics of system (1.1)-(1.3) described in Theorems 1.3–1.6. The following parameters and functions are used in our numerical simulations.

$$\begin{cases} u_0(x) = v_0(x) = w_0(x) = \begin{cases} 0.5\sin(\pi x), & x \in [0, 1], \\ 0, \text{otherwise}; \end{cases} \\ \alpha(x - st) = \frac{2}{\pi}\arctan 10(x - st) - 1; \\ r_1 = r_2 = 1.0;\ \ r_3 = 1.2;\ \ a = 0.5;\ \ b = 1.6;\ \ h = 0.2;\ \ k = 0.8. \end{cases}$$

The diffusion coefficients will be given for each case in Figs. 1, 2, 3, 4 which represent the spreading of species for Theorem 1.3–1.6, respectively.

As we observe from these figures, both Figs. 1 and 2 have the same spatial ranges in (1.10) and (1.11) as stated in Theorems 1.3 and 1.4, respectively. However, the spatial ranges

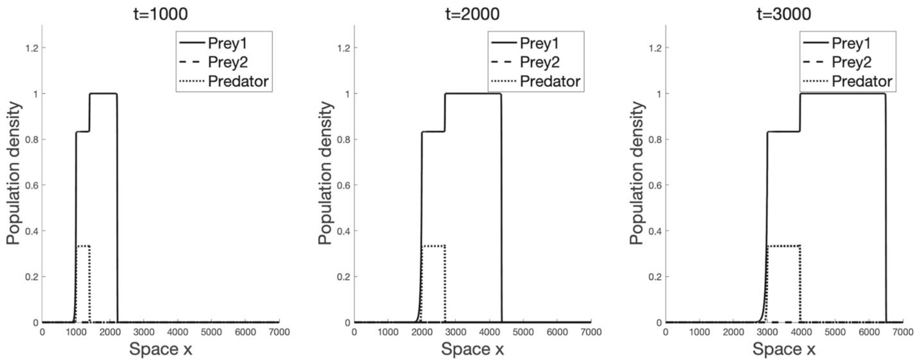

Fig. 2 Spreading of the predator and the fast prey for given parameters $\{d_1 = 1.3, d_2 = 0.1, d_3 = 0.6, s = 1.0\}$. The conditions in Theorem 1.4 are satisfied, since $s_1^* \approx 2.28,\ \ s_2^* \approx 0.63,\ \ s_3^{***} \approx 1.31$

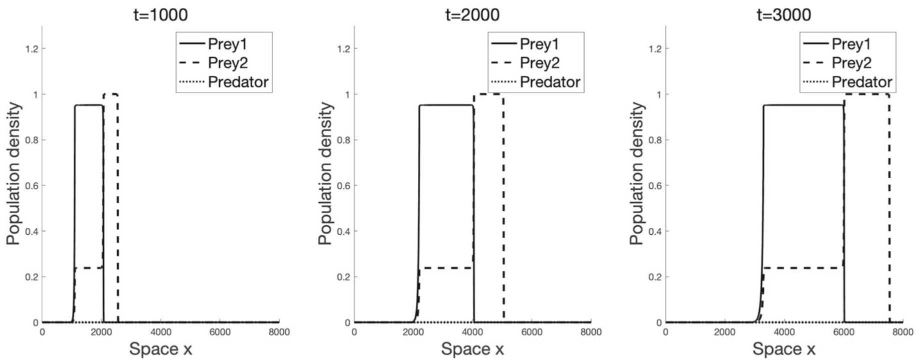

Fig. 3 Spreading of two preys for given parameters $\{d_1 = 1.3, d_2 = 1.8, d_3 = 0.1, s = 1.1\}$. The conditions in Theorem 1.5 are satisfied, since $s_1^{**} \approx 2.04,\ \ s_2^{**} = 1.2,\ \ s_3^* \approx 1.03$

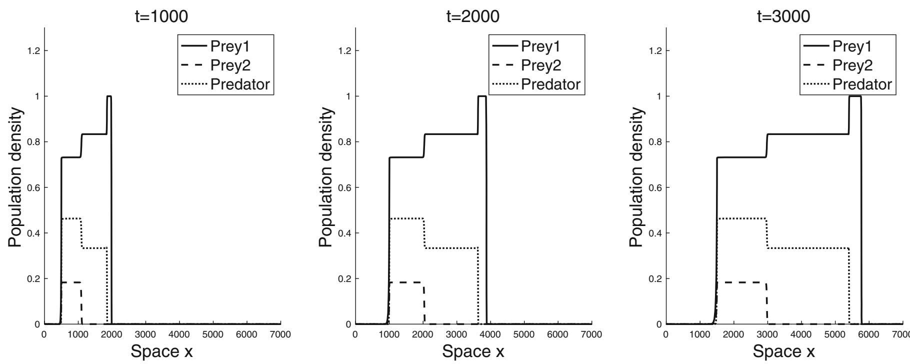

Fig. 4 Spreading of all species for given parameters $\{d_1 = 1.0, d_2 = 0.8, d_3 = 1.2, s = 0.5\}$. The conditions in Theorem 1.6 are satisfied, since $s_1^{***} \approx 1.63,\ \ s_2^{***} \approx 0.73,\ \ s_3^{***} \approx 1.86$

in Figs. 3 and 4 are larger than those in (1.12) and (1.13) stated in Theorems 1.5 and 1.6, respectively. This indicates that the upper bounds of s in Theorems 1.5 and 1.6 are not optimal. This is an open question to be explored in the future.

Finally, we add a remark on the spreading dynamics when condition $\alpha(\infty) = 0$ in (1.4) is replaced by $\alpha(\infty) > 0$.

Remark 4.1 In fact, when $\alpha_0 := \alpha(\infty) > 0$ in (1.4), since we assume that the predator cannot survive without any prey, we must have $\alpha_0 \in (0, 1)$. Then, by setting

$$\tilde{u} = u/(1+\alpha_0),\ \tilde{v} = v/(1+\alpha_0),\ \tilde{w} = w/(1+\alpha_0),\ \tilde{\alpha} = (\alpha(z) - \alpha_0)/(1+\alpha_0),$$
$$\tilde{r}_i = (1+\alpha_0)r_i,\ i = 1, 2, 3,\ \gamma = \frac{1-\alpha_0}{1+\alpha_0},$$

we end up with, after dropping the tilde, the system

$$\begin{aligned}
u_t(x,t) &= d_1 u_{xx}(x,t) + r_1 u(x,t)[1 + \alpha(x - st) - (u + hv + aw)(x,t)],\ x \in \mathbb{R},\ t > 0,\\
v_t(x,t) &= d_2 v_{xx}(x,t) + r_2 v(x,t)[1 + \alpha(x - st) - (ku + v + aw)(x,t)],\ x \in \mathbb{R},\ t > 0,\\
w_t(x,t) &= d_3 w_{xx}(x,t) + r_3 w(x,t)[-\gamma + \alpha(x - st) + (bu + bv - w)(x,t)],\ x \in \mathbb{R},\ t > 0,
\end{aligned}$$

in which $\alpha(\infty) = 0$. Note that $\gamma > 0$ and so the predator cannot survive without any prey.

On the other hand, we now assume $b > \gamma$ to ensure that the predator can survive if there is enough supply from the preys, namely, $u + v > \gamma/b$ which is possible since $\gamma/b < 1$ (recalling that the maximal carrying capacity of u (or v) is 1). Then the spreading dynamics described in Theorems 1.1–1.6 hold with the following changes.

(1) The constant equilibria:

$$\begin{cases} v_p = u_p = \frac{1+a\gamma}{1+ab},\ w_p = \frac{b-\gamma}{1+ab},\\ u_* = \frac{(1+a\gamma)(1-h)}{1-hk+ab(2-h-k)},\ v_* = \frac{(1+a\gamma)(1-k)}{1-hk+ab(2-h-k)},\ w_* = \frac{b(2-h-k)-\gamma(1-hk)}{1-hk+ab(2-h-k)}, \end{cases}$$

(2) The linear speeds:

$$s_3^* = 2\sqrt{d_3 r_3 (2b - \gamma)},\ s_3^{**} = 2\sqrt{d_3 r_3 [b(u_c + v_c) - \gamma]},\ s_3^{***} = 2\sqrt{d_3 r_3 (b - \gamma)}.$$

(3) The function space $X_1 \times X_1 \times X_{2b-1}$ is replaced by $X_1 \times X_1 \times X_{2b-\gamma}$.

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Conflict of interests We declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

Ethics approval and consent to participate The results in this manuscript have not been published elsewhere, nor are they under consideration (from you or one of your Contributing Authors) by another publisher.

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