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# The Existence of Periodic Solutions for Second-Order Delay Differential Systems

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**Abstract**
In this paper, we consider a kind of second-order delay differential system. By taking some transforms, the property of delay is reflected in the boundary condition. The wonder is that the corrseponding first-order system is exactly the so-called $P$-boundary value problem of Hamiltonian system which has been studied deeply by many mathematicians, including the authors of this paper. Firstly, we define the relative Morse index $\mu_Q(A, B)$ for the delay system and give the relationship with the $P$-index $i_P(\gamma_R)$ of Hamiltonian system. Secondly, by this index, topology degree and saddle point reduction, the existence of periodic solutions is established for this kind of delay differential system.

**Keywords** Delay differential systems · Second-order · Periodic solution · Variational methods

## 1 Introduction and Main Results

Delay models usually appear in some biological modeling. They have been used to describe several aspects of infectious disease dynamics: primary infection [8], drug therapy [34] and immune response [9], to name a few. Delays have also appeared in the study of chemostat models [48], circadian rhythms [39], epidemiology [10], the respiratory system [42], tumor growth [43] and neural networks [4]. Delay effects even appeared in the population dynamics of many species [40, 41].

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In 1974, Kaplan and Yorke [21] considered the periodic solutions of the following kinds of delay differential equations

$$\dot{x}(t) = f(x(t-1)),$$

and

$$\dot{x}(t) = f(x(t-1)) + f(x(t-2)),$$

with odd function $f$. They turned their problems into the problems of periodic solution of autonomous Hamiltonian system, it was proved that there existed an energy surface of the Hamiltonian function containing at least one periodic solution. Since then many papers (see [15, 16, 22, 23, 25] and the references therein) used Kaplan and Yorke's original idea to search for periodic solutions of more general differential delay equations of the following form

$$\dot{x}(t) = f(x(t-1)) + f(x(t-2)) + \cdots + f(x(t-m+1)).$$

The existence of periodic solutions of above delay differential equation has been investigated by Nussbaum in [35] using different techniques. Recently, many results on delay differential systems were obtained, readers may refer to the references [7, 18–20, 24, 33, 36, 46, 47] and the references therein. Specially, delay differential equation also has been used to study the COVID-19, the time-delay process is introduced to describe the latent period and treatment cycle, see [45], where the delay differential system derived from the COVID-19 model is a first order equation coupling with some second order delay differential-integral equations. The authors of this paper also have some results on the existence of periodic solutions of above delay differential equation, see [27, 28, 30, 44].

In 1994, Bainov and Domoshnitsky [3] considered the stability of the following second-order delay differential systems

$$z''(t) + \sum_{i=1}^{n} p_i(t) z'(t-\theta_i(t)) + \sum_{j=1}^{m} z(t-\tau_j(t)) = f(t),$$

and Agarwal et al. [1] further improved their result. For other results on second-order delay differential equations, readers may refer to [11, 14, 26], and the references therein.

In this paper, we will consider the following delay differential system

$$\begin{cases} \ddot{x}(t) = -[\nabla v(x(t-\tau)) + \cdots + \nabla v(x(t-(m-1)\tau))], \\ x(t+m\tau) = x(t), \end{cases} \quad \forall t \in \mathbb{R}, \qquad (DDS)$$

with $x \in C^2(\mathbb{R}, \mathbb{R}^n)$, $v \in C^1(\mathbb{R}^n, \mathbb{R})$. By taking some transformations, we will transfer the delay differential system (DDS) into the following second order Hamiltonian system

$$\begin{cases} A\ddot{z}(t) = V'(z(t)), \\ z(1) = Qz(0), \\ \dot{z}(1) = Q\dot{z}(0), \end{cases} \quad \forall t \in [0, 1], \qquad (HS)$$

where $z : \mathbb{R} \to \mathbb{R}^N$, $A, Q \in \mathcal{L}(\mathbb{R}^N)$ with $\mathcal{L}(\mathbb{R}^N)$ the set of real square matrixes on $\mathbb{R}^N$. More generally, in the system (HS) we assume $A$, $Q$ satisfy the following conditions which including (DDS) as a special case (see Section 3.2 below for details)

$$A^{-1} \in \mathcal{L}(\mathbb{R}^N), \quad A^T = A, \quad Q^T = Q^{-1}, \quad AQ = QA, \qquad (1.1)$$

and there exists $k \in \mathbb{N}^+$ such that

$$Q^k = I_N, \tag{1.2}$$

with $I_N$ the identity map on $\mathbb{R}^N$. The function $V$ satisfies the following condition
$(V_0)$ $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and

$$V(Qz) = V(z), \forall z \in \mathbb{R}^N. \tag{1.3}$$

The system (HS) is different from the classical second order Hamiltonian system since the matrix $A$ is neither positive or negative definite and the corresponding variational problem is strongly indefinite.

If $V \in C^2(\mathbb{R}^N, \mathbb{R})$, for any solution $\hat{z}(t)$ of (HS), linearized system at $\hat{z}(t)$ is

$$\begin{cases} A\ddot{z}(t) = B(t)z(t), \\ z(1) = Qz(0), \qquad \forall t \in [0, 1], \\ \dot{z}(1) = Q\dot{z}(0), \end{cases} \tag{$LHS$}$$

with $B(t) = V''(\hat{z}(t))$, and we have $B(1)Q = QB(0)$. So, we define the space as

$$\mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N)) := \{B \in C([0, 1], \mathcal{L}(\mathbb{R}^N)) | B(t + 1)Q = QB(t),\ B^T(t) = B(t)\}.$$

In the next section, we will define the index pair $(\mu_Q(A, B), \upsilon_Q(A, B))$. With this index, we have the following results.

**Theorem 1.1** *Assume $V$ satisfies $(V_0)$ and the following condition.*
*$(V_1)$ $V' : \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous*

$$\|V'(z + y) - V'(z)\|_{\mathbb{R}^N} \le l_V \|y\|_{\mathbb{R}^N},\ \forall z, y \in \mathbb{R}^N, \tag{1.4}$$

*with its Lipschitz constant $l_V > 0$.*

*$(V_2^{\pm})$ There exists $M_1,\ M_2,\ K > 0$, $B(t) \in \mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N))$ with $B(t) \equiv B$, such that*

$$V'(z) = Bz + G(z), \quad \forall z \in \mathbb{R}^N,$$

*with*

$$|G(z)| \le M_1, \quad \forall z \in \mathbb{R}^N,$$

*and*

$$\pm (G(z), z)_{\mathbb{R}^N} \ge M_2|z|_{\mathbb{R}^N},\ \forall |z|_{\mathbb{R}^N} > K. \tag{1.5}$$

*Then (HS) has at least one solution.*

**Theorem 1.2** *Assume $V$ satisfying conditions $(V_0)$, $(V_1)$ and the following condition*
*$(V_3)$ There exists $B \in C(\mathbb{R}^N, \mathcal{L}_s(\mathbb{R}^N))$ such that*

$$V'(z) = B(z)z + G(z),\ \forall z \in \mathbb{R}^N,$$

*with*

$$G(z) = o(z),\ |z| \to \infty.$$

*$(V_4)$ There exist $B_1,\ B_2 \in \mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N))$ satisfying*

$$\mu_Q(A, B_1) = \mu_Q(A, B_2), \quad \upsilon_Q(A, B_2) = 0,$$

*and*

$$B_1(t) \le B(z) \le B_2(t),\ \forall (t,z) \in S^1 \times \mathbb{R}^N.$$

*Then (HS) has at least one solution.*

Since $Q^k = I_N$, the solutions obtained in Theorem 1.1-1.2 in fact are $k$-periodic with $Q$-symmetric. As application of the above two theorems, we will treat the delay differential system (DDS) and obtain two results which are stated in Theorems 3.2–3.3.

## 2 Variational Setting

Let $S^1 := \mathbb{R}/(k\mathbb{Z})$ and $\mathbf{E}$ the closed subspace of $W^{1,2}(S^1, \mathbb{R}^N)$ defined by

$$\mathbf{E} = \{z \in W^{1,2}(S^1, \mathbb{R}^N) | z(t+1) = Qz(t)\}, \tag{2.1}$$

with the norm

$$\|z\|^2_{\mathbf{E}} := \|\dot{z}\|^2_{L^2} + \|z\|^2_{L^2}$$

and the corresponding inner product $(\cdot, \cdot)_{\mathbf{E}}$. Define the functional $\varphi$ on $\mathbf{E}$ by

$$\varphi(z) := \frac{1}{2}\int_0^1 (A\dot{z}(t), \dot{z}(t))dt + \int_0^1 V(z(t))dt, \quad z \in \mathbf{E}.$$

The critical points of $\varphi$ are the solutions of (HS).

Define the Hilbert space

$$\mathbf{L} := \{z \in L^2(S^1, \mathbb{R}^N) | z(t+1) = Qz(t)\}.$$

Define the unbounded self-adjoint operator $\hat{A} : \mathbf{L} \to \mathbf{L}$ by

$$\hat{A}z := A\ddot{z}(t), \quad z \in D(A) \tag{2.2}$$

with $D(\hat{A}) \subset \mathbf{L}$ the domain of $\hat{A}$ and we have $\mathbf{E} = D(|\hat{A}|^{1/2})$. Without confusion, we still denote it by $A$ for simplicity. We have $A$ is unbounded from below and above, so the functional $\varphi$ is strongly indefinite in this sense. Now, we will use the method of saddle point reduction to overcome this difficulty and get the definition of relative Morse index. On the other hand, the system (HS) can be translated to the first order Hamiltonian systems which has the $P$-index defined in [12, 13, 29, 31, 32], we will give the relation between these two indices.

### 2.1 Relative Morse Index

For any $B \in \mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N))$, consider the linearized system (LHS). We know that $B$ defines a self-adjoint operator on $\mathbf{L}$ by

$$z(t) \mapsto B(t)z(t), \quad \forall z \in \mathbf{L}. \tag{2.3}$$

Without confusion, we still denote it by $B$, that is to say $\mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N)) \subset \mathcal{L}_s(\mathbf{L})$ the set of bounded self-adjoint operators on $\mathbf{L}$. So (LHS) can also be rewritten as the following linear operator equation

$$Az = Bz, \quad z \in D(A) \subset \mathbf{L}, \tag{LOE}$$

with $A = \hat{A}$ defined in (2.2) and $B \in \mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N)) \subset \mathcal{L}_s(\mathbf{L})$.

Now, we will give the definition of the relative Morse index and display the relationship with spectral flow. Generally, for any bounded self-adjoint Fredholm operator $F$ on $\mathbf{E}$, there is a unique $F$-invariant orthogonal splitting

$$\mathbf{E} = \mathbf{E}^+(F) \oplus \mathbf{E}^-(F) \oplus \mathbf{E}^0(F), \tag{2.4}$$

where $\mathbf{E}^0(F)$ is the null space of $F$, $F$ is positive definite on $\mathbf{E}^+(F)$ and negative definite on $\mathbf{E}^-(F)$. We denote by $P_F$ the orthogonal projection from $\mathbf{E}$ to $\mathbf{E}^-(F)$. For any compact self-adjoint operator $T$ on $\mathbf{E}$, $P_F - P_{F-T}$ is compact (see Lemma 2.7 of [49]). Then by Fredholm operator theory, $P_F|_{\mathbf{E}^-(F-T)} : \mathbf{E}^-(F-T) \to \mathbf{E}^-(F)$ is a Fredholm operator. Here and in the sequel, we denote by $\mathrm{ind}(\cdot)$ the Fredholm index of a Fredholm operator.

**Definition 2.1** For any bounded self-adjoint Fredholm operator $F$ and a compact self-adjoint operator $T$ on $\mathbf{E}$, the relative Fredholm index pair $(\mu_F(T), \upsilon_F(T))$ is defined by

$$\mu_F(T) = \mathrm{ind}(\mathrm{P_F}|_{\mathbf{E}^-(\mathrm{F-T})}) \tag{2.5}$$

and

$$\upsilon_F(T) = \dim \mathbf{E}^0(F - T). \tag{2.6}$$

On the other hand, let $\{F_\theta | \theta \in [0, 1]\}$ be a continuous path of self-adjoint Fredholm operators on the Hilbert space $\mathbf{E}$. The following proposition displays the relationship between spectral flow and the relative Fredholm index defined above. It is well known that the concept of spectral flow $Sf(F_\theta)$ was first introduced by Atiyah, Patodi and Singer in [2], and then extensively studied in [5, 17, 37, 38, 49].

**Proposition 2.2** (See [6, Proposition 3].) *Suppose that, for each $\theta \in [0, 1]$, $F_\theta - F_0$ is a compact operator on* $\mathbf{E}$*, then*

$$\mathrm{ind}(P_{F_0}|_{\mathbf{E}^-(F_1)}) = -Sf(F_\theta).$$

Thus, from Definition 2.1,

$$\mu_F(T) = -Sf(F_\theta,\ 0 \le \theta \le 1),$$

where $F_\theta = F - \theta T$. Moreover, if $\sigma(T) \subset [0, \infty)$ and $0 \notin \sigma_P(T)$ the set of point spectrum of $T$, from the definition of spectral flow, we have

$$\begin{aligned}\mu_F(T) &= -Sf(F_\theta,\ 0 \le \theta \le 1)\\ &= \sum_{\theta \in [0,1)} \upsilon_F(\theta T)\\ &= \sum_{\theta \in [0,1)} \dim \mathbf{E}^0(F - \theta T).\end{aligned} \tag{2.7}$$

Up to now, we have defined the relative Fredholm index pair $(\mu_F(T), \upsilon_F(T))$ in general abstract setting and displayed the relationship with the spectral flow. Now, we can define our relative Morse index pair $(\mu_Q(A, B), \upsilon_Q(A, B))$ for our problem. The operator $A$ defined a bounded self-adjoint Fredholm operator $\tilde{A}$ on $\mathbf{E}$ by

$$(\tilde{A}z, w)_{\mathbf{E}} := (Az, w)_{\mathbf{L}},\ \forall z, w \in \mathbf{E}. \tag{2.8}$$

On the other hand, since the embedding map $i : \mathbf{E} \hookrightarrow \mathbf{L}$ is compact, the dual operator $i^* : \mathbf{L} \to \mathbf{E}$ is compact and for any $B \in \mathcal{L}_s(\mathbf{L})$ , $i^*B$ is a compact self-adjoint operator on

**E**. By Definition 2.1, we have the relative Fredholm index pair $(\mu_{\tilde{A}}(i^*B), \upsilon_{\tilde{A}}(i^*B))$, so we have the following definition.

**Definition 2.3** Let the operator $A$ defined in (2.2), for any $B \in \mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N))$, the index pair $(\mu_Q(A, B), \upsilon_Q(A, B))$ is defined by

$$\begin{cases} \mu_Q(A, B) = \mu_{\tilde{A}}(i^*B), \\ \upsilon_Q(A, B) = \upsilon_{\tilde{A}}(i^*B). \end{cases} \tag{2.9}$$

### 2.2 Relationship with *P*-Index

In this part, we will transfer (HS) into a first-order Hamiltonian system with $P$-boundary condition which is called $P$-boundary problem, and we will get that the index pair $(\mu_Q(A, B), \upsilon_Q(A, B))$ coincides with the $P$-index pair $(i_P(\gamma), \nu_P(\gamma))$ defined in [12, 13, 29, 31, 32].

Firstly, let us recall the $P$-boundary problem. Recall that the symplectic group is defined as

$$Sp(2N) \equiv Sp(2N, \mathbb{R}) = \{M \in \mathcal{L}(\mathbb{R}^{2N}) | M^T JM = J\},$$

where $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$, $I_N$ is the identity matrix on $\mathbb{R}^N$, and $\mathcal{L}(\mathbb{R}^{2N})$ is the space of $2N \times 2N$ real matrices. The $P$-boundary problem is the following Hamiltonian system

$$\begin{cases} \dot{x}(t) = JH'(t, x), \\ x(1) = Px(0), \end{cases} \tag{2.10}$$

where $P \in Sp(2N)$ and $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ satisfying

$$H(t + 1, Px) = H(t, x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^{2N}.$$

Clearly, if $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$, we have $P^T H''(t+1, Px)P = H''(t, x)$. Let $x : [0, 1] \to \mathbb{R}^{2N}$ be a solution of (2.10), linearzing the Hamiltonian system $\dot{x}(t) = JH'(t, x(t))$ at $x$ we get a Hamiltonian system

$$\dot{y}(t) = JR(t)y(t), \quad y(t) \in \mathbb{R}^{2N}, \tag{2.11}$$

with $R(t) = H''(t, x(t))$ satisfying

$$R(t + 1) = (P^{-1})^T R(t)P^{-1}. \tag{2.12}$$

The fundamental solution of (2.11) is a symplectic path $\gamma_R \in C^1([0, +\infty), Sp(2N))$ with $\gamma_R(0) = I$. For any such symplectic path $\gamma$, there is a so called Maslov $P$-index pair $(i_P(\gamma), \nu_P(\gamma)) \in \mathbb{Z} \times \{0, 1, \cdots, 2N\}$. The Maslov $P$-index theory for a symplectic path was first studied in [12, 29] independently for any symplectic matrix $P$ with different treatment. The Maslov $P$-index theory was generalized in [31] to the Maslov $(P, \omega)$-index theory for any $P \in Sp(2N)$ and all $\omega \in U = \{z \in \mathbb{C} | |z| = 1\}$. When the symplectic matrix $P$ is orthogonal, the $(P, \omega)$-index theory and its iteration theory were studied in [13] and it has been generalized in [32] . When $\omega = 1$, the Maslov $(P, \omega)$-index theory coincides with the Maslov P-index theory.

Secondly, let us consider system (HS). Denote

$$y := A\dot{z}, \quad x := \begin{pmatrix} z \\ y \end{pmatrix}, \text{ and } J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}, \tag{2.13}$$

then system (HS) can be transformed to the following first-order Hamiltonian system

$$\begin{cases} \dot{x} = JH'(x), \\ x(1) = Px(0), \end{cases} \tag{2.14}$$

with

$$H(x) := V(z) - \frac{1}{2}(A^{-1}y, y), \;\; P := \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}.$$

From (1.1), we have $H(Px) = H(x)$ and $P^T JP = J$, so $P \in SP(2N)$. That is to say by (2.13), we can transform system (HS) into the so called $P$-boundary problem (2.10). Similarly, by (2.13), for any $B \in \mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N))$, we can transform system (LHS) into the linear system

$$\dot{x}(t) = JR(t)x(t)$$

with $R := \begin{pmatrix} B(t) & 0 \\ 0 & -A^{-1} \end{pmatrix}$ and satisfying $R(t+1) = (P^{-1})^T R(t)P^{-1}$. Denote the corresponding fundamental solution by $\gamma_R$, so we have the $P$-pair $(i_P(\gamma_R), \nu_P(\gamma_R))$.

Lastly, from the property of the $P$-index pair, Proposition 2.2 and Definition 2.3, we have

$$\upsilon_Q(A, B) = \nu_P(\gamma_R), \;\; \mu_Q(A, B) = i_P(\gamma_R) + k_0, \;\; \forall B \in \mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N)),$$

where the constant $k_0 \in \mathbb{Z}$.

### 2.3 Saddle Point Reduction

Consider system (HS) with $V$ satisfying conditions $(V_0)$ and $(V_1)$. We will consider the method of saddle point reduction without assuming the nonlinear term $V \in C^2(\mathbb{R}^N, \mathbb{R})$, then we will give some abstract critical point theorems. Let $E_A(z)$ the spectrum measure of $A$, since $A$ has compact resolvent, we can choose $l > l_V$, such that

$$-l, l \notin \sigma(A) \;\text{ and }\; (-l, l) \cap \sigma(A) \neq \emptyset. \tag{2.15}$$

Consider the following projection maps on $\mathbf{L}$

$$P_A^0 := \int_{-l}^{l} dE_A(z), \;\; P_A^{\perp} = I - P_A^0, \tag{2.16}$$

with $I$ the identity map on $\mathbf{L}$. Then we have the following decomposition

$$\mathbf{L} = \mathbf{L}_A^{\perp} \oplus \mathbf{L}_A^0, \tag{2.17}$$

where $\mathbf{L}_A^* := P_A^*\mathbf{L}(* = \perp, 0)$ and $\mathbf{L}_A^0$ is finite dimensional subspace of $\mathbf{L}$. Denote $A^*$ the restriction of $A$ on $\mathbf{L}^*(* = \perp, 0)$, thus we have $(A^{\perp})^{-1}$ are bounded self-adjoint linear operators on $\mathbf{L}^{\perp}$ respectively and satisfying

$$\|(A^{\perp})^{-1}\| \leq \frac{1}{l}. \tag{2.18}$$

Define the functional $\Phi$ on $\mathbf{L}$ by

$$\Phi(z) = \int_0^1 V(z(t))dt, \; z \in \mathbf{L}. \tag{2.19}$$

Then from conditions $(V_0)$ and $(V_1)$, we have $\Phi \in C^1(\mathbf{L}, \mathbb{R})$ and

$$\|\Phi'(z+y) - \Phi'(z)\|_{\mathbf{L}} \leq l_V \|y\|_{\mathbf{L}}, \forall z, y \in \mathbf{L}. \tag{2.20}$$

System (HS) can be rewritten as the following abstract self-adjoint operator equation on $\mathbf{L}$

$$Az = \Phi'(z), \quad z \in D(A) \subset \mathbf{L}, \tag{$OE$}$$

which is equivalent to the following operator equations

$$z^\perp = (A^\perp)^{-1} P_A^\perp \Phi'(z^\perp + z^0), \tag{2.21}$$

and

$$A^0 z^0 = P_A^0 \Phi'(z^\perp + z^0), \tag{2.22}$$

where $z^* = P_A^* z(* = \perp, 0)$, for simplicity, we rewrite $x := z^0$. From (2.18) and(2.20), we have $(A^\perp)^{-1} P_A^\perp \Phi'$ is contraction map on $\mathbf{L}^+ \oplus \mathbf{L}^-$ for any $x \in \mathbf{L}^0$. So there is a map $z^\perp(x) : \mathbf{L}^0 \to \mathbf{L}^\perp$ satisfying

$$z^\perp(x) = (A^\perp)^{-1} P^\perp \Phi'(z^\perp(x) + x), \ \forall x \in \mathbf{L}^0, \tag{2.23}$$

and we have the following properties.

**Proposition 2.4** *(1) The map $z^\perp(x) : \mathbf{L}^0 \to \mathbf{L}^\perp$ is continuous, in fact we have*

$$\|z^\perp(x+h) - z^\perp(x)\|_{\mathbf{L}} \leq \frac{l_V}{l - l_V} \|h\|_{\mathbf{L}}, \quad \forall x, h \in \mathbf{L}^0. \tag{2.24}$$

*(2)* $\|z^\perp(x)\|_{\mathbf{L}} \leq \dfrac{l_V}{l - l_V} \|x\|_{\mathbf{L}} + \dfrac{1}{l - l_V} \|\Phi'(0)\|_{\mathbf{L}}$.

***Proof*** (1) For any $x, \ h \in \mathbf{L}^0$, we have

$$\begin{aligned}
&\|z^\perp(x+h) - z^\perp(x)\|_{\mathbf{L}} \\
&\quad = \|(A^\perp)^{-1} P_A^\perp \Phi'(z^\perp(x+h) + x + h) - (A^\perp)^{-1} P_A^\perp \Phi'(z^\perp(x) + x)\|_{\mathbf{L}} \\
&\quad \leq \frac{1}{l} \|\Phi'(z^\perp(x+h) + x + h) - \Phi'(z^\perp(x) + x)\|_{\mathbf{L}} \\
&\quad \leq \frac{l_V}{l} \|z^\perp(x+h) - z^\perp(x) + h\|_{\mathbf{L}} \\
&\quad \leq \frac{l_V}{l} \|z^\perp(x+h) - z^\perp(x)\|_{\mathbf{L}} + \frac{l_V}{l} \|h\|_{\mathbf{L}}.
\end{aligned}$$

So we have $\|z^\perp(x+h) - z^\perp(x)\|_{\mathbf{L}} \leq \frac{l_V}{l - l_V} \|h\|_{\mathbf{L}}$ and the map $z^\perp(x) : \mathbf{L}^0 \to \mathbf{L}^\perp$ is continuous.

(2) Similarly,

$$\begin{aligned}
\|z^\perp(x)\|_{\mathbf{L}} &= \|(A^\perp)^{-1} P_A^\perp \Phi'(z^\perp(x) + x)\|_{\mathbf{L}} \\
&\leq \frac{1}{l} \|\Phi'(z^\perp(x) + x)\|_{\mathbf{L}} \\
&\leq \frac{1}{l} \|\Phi'(z^\perp(x) + x) - \Phi'(0)\|_{\mathbf{L}} + \frac{1}{l} \|\Phi'(0)\|_{\mathbf{L}} \\
&\leq \frac{l_V}{l} (\|z^\perp(x)\|_{\mathbf{L}} + \|x\|_{\mathbf{L}}) + \frac{1}{l} \|\Phi'(0)\|_{\mathbf{L}}.
\end{aligned}$$

So we have $\|z^\perp(x)\|_{\mathbf{L}} \leq \frac{l_V}{l - l_V} \|x\|_{\mathbf{L}} + \frac{1}{l - l_V} \|\Phi'(0)\|_{\mathbf{L}}$.

□

***Remark 2.5*** It's easy to see $\mathbf{E} = D(|A|^{\frac{1}{2}})$, with the equivalent norm

$$\|z\|_{\mathbf{E}}^2 := \||A|^{\frac{1}{2}}(z^{\perp})\|_{\mathbf{L}}^2 + \|x\|_{\mathbf{L}}^2, \ z \in \mathbf{E}.$$

From (2.23), we have $z^{\perp}(x) \in D(A) \subset \mathbf{E}$, and

(1) The map $z^{\perp}(x) : \mathbf{L}^0 \to \mathbf{E}$ is continuous, and

$$\|(z^{\perp})(x+h) - (z^{\perp})(x)\|_{\mathbf{E}} \leq \frac{l_V \cdot l^{\frac{1}{2}}}{l - l_V}\|h\|_{\mathbf{L}}, \quad \forall x, h \in \mathbf{L}^0. \tag{2.25}$$

(2) $\|(z^{\perp})(x)\|_{\mathbf{E}} \leq \dfrac{l^{\frac{1}{2}}}{l - l_V}(l_V \cdot \|x\|_{\mathbf{L}} + \|\Phi'(0)\|_{\mathbf{L}})$.

***Proof*** The proof is similar to Proposition 2.4, we only prove (1).

$$\begin{aligned}
\|z^{\perp}(x+h) - z^{\perp}(x)\|_{\mathbf{E}} &= \|(|A|^{\frac{1}{2}})[z^{\perp}(x+h) - z^{\perp}(x)]\|_{\mathbf{L}} \\
&= \|(A^{\perp})^{-\frac{1}{2}}[P_A^{\perp}\Phi'(z^{\perp}(x+h) + x + h) - P_A^{\perp}\Phi'(z^{\perp}(x) + x)]\|_{\mathbf{L}} \\
&\leq \frac{1}{l^{\frac{1}{2}}}\|\Phi'(z^{\perp}(x+h) + x + h) - \Phi'(z^{\perp}(x) + x)\|_{\mathbf{L}} \\
&\leq \frac{l_V}{l^{\frac{1}{2}}}\|z^{\perp}(x+h) - z^{\perp}(x) + h\|_{\mathbf{L}} \\
&\leq \frac{l_V}{l^{\frac{1}{2}}}\|z^{\perp}(x+h) - z^{\perp}(x)\|_{\mathbf{L}} + \frac{l_V}{l^{\frac{1}{2}}}\|h\|_{\mathbf{L}} \\
&\leq \frac{l_V}{l}\|z^{\perp}(x+h) - z^{\perp}(x)\|_{\mathbf{E}} + \frac{l_V}{l^{\frac{1}{2}}}\|h\|_{\mathbf{L}},
\end{aligned}$$

where the last inequality depends on the fact that $\|z^{\perp}\|_{\mathbf{E}} \geq l^{\frac{1}{2}}\|z^{\perp}\|_{\mathbf{L}}$, so we have (2.25).

Now, define the map $z : \mathbf{L}^0 \to \mathbf{L}$ by

$$z(x) = x + z^{\perp}(x).$$

Define the functional $a : \mathbf{L}^0 \to \mathbb{R}$ by

$$a(x) = \frac{1}{2}(Az(x), z(x))_{\mathbf{L}} - \Phi(z(x)), \ x \in \mathbf{L}^0. \tag{2.26}$$

With standard discussion, the critical points of $a$ correspond to the solutions of (OE), and we have

**Lemma 2.6** *Assume $V$ satisfies $(V_0)$, $(V_1)$, then we have $a \in C^1(\mathbf{L}^0, \mathbb{R})$,*

$$a'(x) = Az(x) - \Phi'(z(x)), \quad \forall x \in \mathbf{L}^0. \tag{2.27}$$

*Further more, if $V \in C^2(\mathbb{R}^N, \mathbb{R})$, we have $a \in C^2(\mathbf{L}^0, \mathbb{R})$ and*

$$a''(x) = A|_{\mathbf{L}_0} - P^0\Phi''(z(x))z'(x), \tag{2.28}$$

***Proof*** For any $x, h \in \mathbf{L}^0$, write

$$\eta(x, h) := z^{\perp}(x+h) - z^{\perp}(x) + h$$

for simplicity, that is to say

$$z(x+h) = z(x) + \eta(x, h), \quad \forall x, h \in \mathbf{L}^0,$$

and from (2.24), we have

$$\|\eta(x,h)\|_{\mathbf{L}} \le C\|h\|_{\mathbf{L}}, \quad \forall x, h \in \mathbf{L}^0, \tag{2.29}$$

where $C = \dfrac{l}{l - l_V}$. Let $h \to 0$ in $\mathbf{L}^0$, and for any $x \in \mathbf{L}^0$, we have

$$\begin{aligned} &a(x+h) - a(x) \\ &= \frac{1}{2}[(Az(x+h), z(x+h))_{\mathbf{L}} - (Az(x), z(x))_{\mathbf{L}}] - [\Phi(z(x+h)) - \Phi(z(x))] \\ &= (Az(x), \eta(x,h))_{\mathbf{L}} + \frac{1}{2}(A\eta(x,h), \eta(x,h))_{\mathbf{L}} \\ &\quad - (\Phi'(z(x)), \eta(x,h))_{\mathbf{L}} + o(\|\eta(x,h)\|_{\mathbf{L}}). \end{aligned}$$

From (2.29) we have

$$a(x+h) - a(x) = (Az(x) - \Phi'(z(x)), \eta(x,h))_{\mathbf{L}} + o(\|h\|_{\mathbf{L}}), \quad \forall x \in \mathbf{L}^0, \text{ and } \|h\|_{\mathbf{L}} \to 0.$$

Since $z^{\pm}(x)$ is the solution of (2.23) and from the definition of $\eta(x,h)$, we have

$$(Az(x) - \Phi'(z(x)), \eta(x,h))_{\mathbf{L}} = (Az(x) - \Phi'(z(x)), h)_{\mathbf{L}}, \quad \forall x, h \in \mathbf{L}^0,$$

so we have

$$a(x+h) - a(x) = (Az(x) - \Phi'(z(x)), h)_{\mathbf{L}} + o(\|h\|_{\mathbf{L}}), \quad \forall x \in \mathbf{L}^0, \text{ and } \|h\|_{\mathbf{L}} \to 0,$$

and we have proved (2.27). If $\Phi \in C^2(\mathbf{L}, \mathbb{R})$, from (2.23) and by implicit function theorem, we have $z^{\pm} \in C^1(\mathbf{L}^0, \mathbf{L}^{\pm})$. From (2.23) and (2.27), we have

$$a'(x) = Ax - P^0\Phi'(z(x))$$

and

$$a''(x) = A|_{\mathbf{L}_0} - P^0\Phi''(z(x))z'(x),$$

that is to say $a \in C^2(\mathbf{L}^0, \mathbb{R})$. □

# 3 The Proofs of Main Results with Applications

## 3.1 The Proofs of Main Results

***Proof of Theorem 1.1*** Now, we consider the case of $(V_2^-)$. Since $A$ has compact resolvent, 0 is at most an isolate point spectrum of $A - B$ with finite dimensional eigenspace, that is to say there exists $\varepsilon_0 > 0$ small enough, such that $(-\varepsilon_0, 0) \cap \sigma(A - B) = \emptyset$. For any $\varepsilon \in (0, \varepsilon_0)$ and $\lambda \in [0, 1]$, consider the following two-parameters equation

$$(\varepsilon \cdot I + A - B)z = \lambda G(z), \tag{HS$_{\varepsilon,\lambda}$}$$

with $I$ the identity map on $\mathbf{L}$. If $\varepsilon = 0$ and $\lambda = 1$, it is (HS). We divide the following proof into four steps.

*Step 1* There exists a constant $C$ independent of $\varepsilon$ and $\lambda$, such that if $z_{\varepsilon,\lambda}$ is a solution of $(HS_{\varepsilon,\lambda})$,

$$\varepsilon\|z_{\varepsilon,\lambda}\|_{\mathbf{L}} \le C, \ \forall(\varepsilon, \lambda) \in (0, \frac{\varepsilon_0}{2}) \times [0, 1].$$

Since $(-\varepsilon_0, 0) \cap \sigma(A - B) = \emptyset$, we have $(\varepsilon - \varepsilon_0, \varepsilon) \cap \sigma(\varepsilon \cdot I + A - B) = \emptyset$. Consider the orthogonal splitting

$$\mathbf{L} = \mathbf{L}^-_{\varepsilon \cdot I + A - B} \oplus \mathbf{L}^+_{\varepsilon \cdot I + A - B},$$

where $\varepsilon \cdot I + A - B$ is negative definite on $\mathbf{L}^-_{\varepsilon \cdot I + A - B}$, and positive define on $\mathbf{L}^+_{\varepsilon \cdot I + A - B}$. Thus, if $z \in \mathbf{L}$, we have the splitting

$$z = z^- + z^+,$$

with $z^- \in \mathbf{L}^-_{\varepsilon \cdot I + A - B}$ and $z^+ \in \mathbf{L}^+_{\varepsilon \cdot I + A - B}$. If $z_{\varepsilon,\lambda}$ is a solution of $(HS_{\varepsilon,\lambda})$ with its splitting $z_{\varepsilon,\lambda} = z^-_{\varepsilon,\lambda} + z^+_{\varepsilon,\lambda}$ defined above, then we have

$$((\varepsilon \cdot I + A - B)z_{\varepsilon,\lambda}, z^+_{\varepsilon,\lambda} - z^-_{\varepsilon,\lambda})_{\mathbf{L}} = \lambda(G(z_{\varepsilon,\lambda}), z^+_{\varepsilon,\lambda} - z^-_{\varepsilon,\lambda})_{\mathbf{L}}.$$

Since $(\varepsilon - \varepsilon_0, \varepsilon) \cap \sigma(\varepsilon \cdot I + A - B) = \emptyset$, we have

$$((\varepsilon \cdot I + A - B)z_{\varepsilon,\lambda}, z^+_{\varepsilon,\lambda} - z^-_{\varepsilon,\lambda})_{\mathbf{L}} \geq \min\{\varepsilon_0 - \varepsilon, \varepsilon\}\|z_{\varepsilon,\lambda}\|^2_{\mathbf{L}}.$$

Since $r$ is bounded, for $(\varepsilon, \lambda) \in (0, \frac{\varepsilon_0}{2}) \times [0, 1]$, we have

$$C\|z_{\varepsilon,\lambda}\|_{\mathbf{L}} \geq \lambda(G(z_{\varepsilon,\lambda}), z^+_{\varepsilon,\lambda} - z^-_{\varepsilon,\lambda})_{\mathbf{L}} \geq \varepsilon\|z_{\varepsilon,\lambda}\|^2_{\mathbf{L}}.$$

Therefor, we have

$$\varepsilon\|z_{\varepsilon,\lambda}\|_{\mathbf{L}} \leq C, \ \forall(\varepsilon, \lambda) \in (0, \frac{\varepsilon_0}{2}) \times [0, 1].$$

*Step 2* For any $(\varepsilon, \lambda) \in (0, \frac{\varepsilon_0}{2}) \times [0, 1]$, $(HS_{\varepsilon,\lambda})$ has at least one solution. Here, we use the topology degree theory. Since $0 \notin \sigma(\varepsilon \cdot I + A - B)$, $(HS_{\varepsilon,\lambda})$ can be rewritten as

$$z = \lambda(\varepsilon \cdot I + A - B)^{-1}G(z).$$

Denote by $f(\varepsilon, \lambda, z) := \lambda(\varepsilon \cdot I + A - B)^{-1}G(z)$ for simplicity. From the compactness of $(\varepsilon \cdot I + A - B)^{-1}$ and condition $(V_2^-)$, Leray Schauder degree theory can be used to the map

$$z \mapsto z - f(\varepsilon, \lambda, z), \ z \in \mathbf{L}.$$

From the result received in Step 1, we have

$$\begin{aligned} deg(I - f(\varepsilon, \lambda, \cdot), B(0, R_\varepsilon), 0) &\equiv deg(I - f(\varepsilon, 0, \cdot), B(0, R_\varepsilon), 0) \\ &= deg(I, B(0, R(\varepsilon)), 0) \\ &= 1, \end{aligned}$$

where $R_\varepsilon > \frac{C}{\varepsilon}$ is a constant only depends on $\varepsilon$, and $B(0, R_\varepsilon) := \{z \in \mathbf{L} | \|z\|_{\mathbf{L}} < R_\varepsilon\}$.
*Step 3* For $\lambda = 1$, $\varepsilon \in (0, \varepsilon_0/2)$, denote by $z_\varepsilon$ one of the solutions of $(HS_{\varepsilon,1})$. We have $\|z_\varepsilon\|_{\mathbf{H}} \leq C$. In this step, $C$ denotes various constants independent of $\varepsilon$.
From the boundedness received in Step 1, we have

$$\|(A - B)z_\varepsilon\|_{\mathbf{L}} = \|\varepsilon z_\varepsilon - r(t, x, z_\varepsilon)\|_{\mathbf{L}} \leq C. \tag{3.1}$$

Now, consider the orthogonal splitting

$$\mathbf{L} = \mathbf{L}^0_{A-B} \oplus \mathbf{L}^\perp_{A-B},$$

where $A - B$ is zero on $\mathbf{L}^0_{A-B}$, and $\mathbf{L}^{\perp}_{A-B}$ is the orthonormal complement space of $\mathbf{L}^0_{A-B}$. Let $z_\varepsilon = z^0_\varepsilon + z^{\perp}_\varepsilon$ with $z^0_\varepsilon \in \mathbf{L}^0_{A-B}$ and $z^{\perp}_\varepsilon \in \mathbf{L}^{\perp}_{A-B}$. Since 0 is an isolated point in $\sigma(A - B)$, from (3.1), we have

$$\|z^{\perp}_\varepsilon\|_{\mathbf{L}} \le C \tag{3.2}$$

Additionally, since $G(z_\varepsilon)$, $z^{\perp}_\varepsilon$ and $\varepsilon z_\varepsilon$ are bounded in $\mathbf{L}$, we have

$$\begin{aligned}(G(z_\varepsilon), z_\varepsilon)_{\mathbf{L}} &= (G(z_\varepsilon), z^{\perp}_\varepsilon)_{\mathbf{L}} + (G(z_\varepsilon), z^0_\varepsilon)_{\mathbf{L}} \\ &= (G(z_\varepsilon), z^{\perp}_\varepsilon)_{\mathbf{L}} + (\varepsilon z_\varepsilon + (A - B)z_\varepsilon, z^0_\varepsilon)_{\mathbf{L}} \\ &= (G(z_\varepsilon), z^{\perp}_\varepsilon)_{\mathbf{L}} + \varepsilon(z^0_\varepsilon, z^0_\varepsilon)_{\mathbf{L}} \\ &\ge C. \end{aligned} \tag{3.3}$$

On the other hand, from (3.14) in $(H_2^-)$, we have

$$\begin{aligned}(G(z_\varepsilon), z_\varepsilon)_{\mathbf{L}} &= \int_{\Omega(K)} (G(z_\varepsilon), z_\varepsilon)dt + \int_{S^1/\Omega(K)} (G(z_\varepsilon), z_\varepsilon)dt \\ &\le -M_2 \int_{\Omega(K)} |z_\varepsilon| dt + C \\ &\le -M_2 \|z_\varepsilon\|_{L^1(S^1)} + C, \end{aligned} \tag{3.4}$$

where $\Omega(K) := \{t \in S^1 | |z_\varepsilon(t)| > K\}$. From (3.3) and (3.4), we have

$$\|z_\varepsilon\|_{L^1(S^1)} \le C. \tag{3.5}$$

Moreover, since $\mathbf{L}^0_{A-B}$ is a finite dimensional space, all norms are equivalent, from (3.2) and (3.5), we have prove the boundedness of $\|z_\varepsilon\|_{\mathbf{L}}$.

*Step 4* Passing to a sequence of $\varepsilon_n \to 0$, there exists $z \in \mathbf{L}$ such that

$$\lim_{\varepsilon_n \to 0} \|z_{\varepsilon_n} - z\|_{\mathbf{L}} = 0.$$

Here, we will use the method of saddle point reduction. Recall the constant $l > l_V$ satisfying condition (2.15), the projections $P^0_A$, $P^{\perp}_A$ defined in (2.16), the decomposition $\mathbf{L} = \mathbf{L}^0_A \oplus \mathbf{L}^{\perp}_A$ defined in (2.17), and we have

$$\|(A^{\perp})^{-1}\| \le \frac{1}{l_V + \delta},$$

with $\delta = l - l_V$. Let $\varepsilon' := \min\{\varepsilon_0, \delta\}$, for $\varepsilon \in (0, \frac{\varepsilon'}{2})$, denote by $A_\varepsilon := \varepsilon \cdot I + A$. Then $A_\varepsilon$ has the same invariant subspace with $A$, so we can also denote by $A^*_\varepsilon := A_\varepsilon|_{\mathbf{L}^*}$ $(* = 0, \perp)$, and we have

$$\|(A^{\perp}_\varepsilon)^{-1}\| \le \frac{1}{l_V + \delta/2}. \tag{3.6}$$

Since $z_\varepsilon$ satisfies $(HS_{\varepsilon,1})$, so we have

$$A^{\perp}_\varepsilon z^{\perp}_\varepsilon = P^{\perp}_A \Phi'(z^{\perp}_\varepsilon + z^0_\varepsilon),$$

with $\Phi$ defined in (2.19), and

$$z^{\perp}_\varepsilon = (A^{\perp}_\varepsilon)^{-1} P^{\perp}_A \Phi'(z^{\perp}_\varepsilon + z^0_\varepsilon). \tag{3.7}$$

Since $\mathbf{L}^0$ is a finite dimensional space and $\|z_\varepsilon\|_{\mathbf{L}} \le C$, there exists a sequence $\varepsilon_n \to 0$ and $z^0 \in \mathbf{L}^0$, such that

$$\lim_{n\to\infty} z^0_{\varepsilon_n} = z^0.$$

For simplicity, we rewrite $z_n^* := z_{\varepsilon_n}^* (* = \perp, 0)$, $A_n := \varepsilon_n + A$ and $A_n^\perp := A_{\varepsilon_n}^\perp$. So, we have

$$\begin{aligned}\|z_n^\perp - z_m^\perp\|_{\mathbf{L}} =& \|(A_n^\perp)^{-1} P_A^\perp \Phi'(z_n) - (A_m^\perp)^{-1} P_A^\perp \Phi'(z_m)\|_{\mathbf{L}} \\ \le& \|(A_n^\perp)^{-1} P_A^\perp (\Phi'(z_n) - \Phi'(z_m))\|_{\mathbf{L}} + \|((A_n^\perp)^{-1} - (A_m^\perp)^{-1}) P_A^\perp \Phi'(z_m)\|_{\mathbf{L}} \\ \le& \frac{l_V}{l_V + \delta/2} \|z_n - z_m\|_{\mathbf{L}} + \|((A_n^\perp)^{-1} - (A_m^\perp)^{-1}) P_A^\perp \Phi'(z_m)\|_{\mathbf{L}} \\ \le& \frac{l_V}{l_V + \delta/2} (\|z_n^\perp - z_m^\perp\|_{\mathbf{L}} + \|z_n^0 - z_m^0\|_{\mathbf{L}}) + \|((A_n^\perp)^{-1} - (A_m^\perp)^{-1}) P_A^\perp \Phi'(z_m)\|_{\mathbf{L}}.\end{aligned}$$

Since $(A_n^\perp)^{-1} - (A_m^\perp)^{-1} = (\varepsilon_m - \varepsilon_n)(A_n^\perp)^{-1}(A_m^\perp)^{-1}$ and $\{z_n\}$ is bounded in $\mathbf{L}$, we have

$$\|((A_n^\perp)^{-1} - (A_m^\perp)^{-1}) P_A^\perp \Phi'(z_m)\|_{\mathbf{L}} = o(1), \quad n, m \to \infty.$$

So we have

$$\|z_n^\perp - z_m^\perp\|_{\mathbf{L}} \le \frac{2l_V}{\delta} \|z_n^0 - z_m^0\|_{\mathbf{L}} + o(1), \quad n, m \to \infty,$$

therefore, there exists $z^\perp \in \mathbf{L}^\perp$, such that $\lim_{n\to\infty} \|z_n^\perp - z^\perp\|_{\mathbf{L}} = 0$. Thus, we have

$$\lim_{n\to\infty} \|z_{\varepsilon_n} - z\|_{\mathbf{L}} = 0,$$

with $z = z^\perp + z^0$. Last, let $n \to \infty$ in $(HS_{\varepsilon_n,1})$, we have $z$ is a solution of $(HS)$. □

In the proof of Theorem 1.2, we need the following Lemma.

**Lemma 3.1** *Let $B_1, B_2 \in \mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N))$.*

*(1) If $B_1 < B_2$, then we have*

$$\mu_Q(A, B_2) - \mu_Q(A, B_1) = \sum_{\lambda\in[0,1)} \upsilon_Q(A, B_1 + \lambda(B_2 - B_1)). \tag{3.8}$$

*(2) If $B_1 \le B_2$, $\mu_Q(A, B_1) = \mu_Q(A, B_2)$, and $\upsilon_Q(A, B_2) = 0$, then there exists $\varepsilon > 0$, such that for all $B \in \mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N))$ with*

$$B_1 \le B \le B_2,$$

*we have*

$$\sigma(A - B) \cap (-\varepsilon, \varepsilon) = \emptyset.$$

***Proof*** (1) From Proposition 2.2 and (2.7), we have (3.8).
(2) Since $\upsilon_Q(A, B_2) = 0$, there is $\varepsilon > 0$, such that

$$\upsilon_Q(A, B_2 + \lambda\varepsilon) = 0, \quad \forall \lambda \in [0, 1].$$

From (3.8) we have $\mu_Q(A, B_2 + \varepsilon \cdot I) = \mu_Q(A, B_2)$, and from $\mu_Q(A, B_1) = \mu_Q(A, B_2)$, we have $\upsilon_Q(A, B_1) = 0$. So we can choose $\varepsilon > 0$ small enough, such that

$$\upsilon_Q(A, B_1 - \varepsilon \cdot I) = \upsilon_Q(A, B_2 + \varepsilon \cdot I) = 0.$$

Since

$$B_1 - \varepsilon \cdot I \leq B - \varepsilon I < B + \varepsilon I \leq B_2 + \varepsilon \cdot I,$$

it follows that $\mu_Q(A, B - \varepsilon I) = \mu_Q(A, B + \varepsilon I)$. Note that by (2.7)

$$\sum_{-\varepsilon < t \leq \varepsilon} \upsilon_Q(A, B - t \cdot I) = \mu_Q(A, B + \varepsilon) - \mu_Q(A, B - \varepsilon) = 0.$$

We have $0 \notin \sigma(A - B - \eta),\ \forall \eta \in (-\varepsilon, \varepsilon)$, thus the proof is complete. □

***Proof of Theorem 1.2.*** Consider the following one-parameter equation

$$Az = (1 - \lambda) B_1 z + \lambda \Phi'(z), \qquad (\mathrm{HS}_\lambda)$$

with $\lambda \in [0, 1]$. Denote by

$$\Phi_\lambda(z) = \frac{1 - \lambda}{2} (B_1 z, z)_{\mathbf{L}} + \lambda \Phi(z),\ \ \forall z \in \mathbf{L}.$$

Since $V$ satisfies condition $(V_1)$ and $B_1 \in \mathcal{L}_Q(S_1, \mathcal{L}(\mathbb{R}^N))$, we have $\Phi'_\lambda : \mathbf{L} \to \mathbf{L}$ is Lipschitz continuous, and there exists $l' > 0$ independed of $\lambda$ such that $l' \notin \sigma(A)$ and

$$\|\Phi'_\lambda(z + h) - \Phi'_\lambda(z)\|_{\mathbf{L}} \leq l' \|h\|_{\mathbf{L}},\ \ \forall z, h \in \mathbf{L}, \lambda \in [0, 1].$$

Now, replace $l_V$ by $l'$ in (2.16), we have the projections $P^*_{A,l'}(* = \perp, 0)$ and the splitting

$$\mathbf{L} = \mathbf{L}^\perp_{A,l'} \oplus \mathbf{L}^0_{A,l'},$$

with $\mathbf{L}^*_{A,l'} = P^*_{A,l'}\mathbf{L}(* = \perp, 0)$. Thus $A^\perp$ has bounded inverse on $\mathbf{L}^\perp_{A,l'}$ with

$$\|(A^\perp)^{-1}\| < \frac{1}{l' + c},$$

for some $c > 0$. Without confusion, we still use $z^\perp$ and $z^0$ to represent the splitting

$$z = z^\perp + z^0,$$

with $z^* \in \mathbf{L}^*_{A,l'}(* = \perp, 0)$. Now, we divide the remainder of the proof into three steps. The number $C > 0$ denotes various constants independent of $\lambda$.

*Step 1* If $z$ is a solution of $(HS_\lambda)$, then we have $\|z^\perp(z^0)\|_{\mathbf{L}} \leq C\|z^0\|_{\mathbf{L}} + C$ Since $Az = \Phi'_\lambda(z)$, we have

$$z^\perp = (A^\perp)^{-1} P^\perp_{A,l'} \Phi'_\lambda(z)$$

$$\begin{aligned}
\|z^\perp(x)\|_{\mathbf{L}} &= \|(A^\perp)^{-1} P^\perp_{A,l'} \Phi'_\lambda(z^\perp(z^0) + z^0)\|_{\mathbf{L}} \\
&\leq \frac{1}{l' + c} \|\Phi'_\lambda(z^\perp(z^0) + z^0)\|_{\mathbf{L}} \\
&\leq \frac{1}{l' + c} \|\Phi'_\lambda(z^\perp(z^0) + z^0) - \Phi'_\lambda(0)\|_{\mathbf{L}} + \frac{1}{l' + c} \|\Phi'_\lambda(0)\|_{\mathbf{L}} \\
&\leq \frac{l'}{l' + c} (\|z^\perp(z^0)\|_{\mathbf{L}} + \|z^0\|_{\mathbf{L}}) + \frac{1}{l' + c} \|\Phi'_\lambda(0)\|_{\mathbf{L}}.
\end{aligned}$$

So we have $\|z^\perp(z^0)\|_{\mathbf{L}} \leq \frac{l'}{c} \|z^0\|_{\mathbf{L}} + \frac{1}{c} \|\Phi'_\lambda(0)\|_{\mathbf{L}}$. Thus, we have prove this step.
*Step 2* We claim that the set of all the solutions $(z, \lambda)$ of $(HS_\lambda)$ are a priori bounded.

If not, there exists a sequence $\{(z_n, \lambda_n)\}$ with $\lambda_n \in [0, 1]$ solving the problem $(HS_\lambda)$ with $\|z_n\|_{\mathbf{L}} \to \infty$. Without lose of generality, assume $\lambda_n \to \lambda_0 \in [0, 1]$. From step 1, we have $\|z_n^0\|_{\mathbf{L}} \to \infty$. Denote by

$$y_n = \frac{z_n}{\|z_n\|_{\mathbf{L}}},$$

and $\bar{B}_n := (1-\lambda_n)B_1 + \lambda_n B(z_n)$, from condition ($V_3$) we have $Ay_n = \bar{B}_n y_n + \frac{o(\|z_n\|_{\mathbf{L}})}{\|z_n\|_{\mathbf{L}}}$, that is

$$(A - \bar{B}_n)y_n = \frac{o(\|z_n\|_{\mathbf{L}})}{\|z_n\|_{\mathbf{L}}}. \tag{3.9}$$

Decompose $y_n = y_n^\perp + y_n^0$ with $y_n^* = z_n^*/\|z_n\|_{\mathbf{L}},\ * = \perp, 0$, we have

$$\begin{aligned}\|y_n^0\|_{\mathbf{L}} &= \frac{\|z_n^0\|_{\mathbf{L}}}{\|z_n\|_{\mathbf{L}}}\\ &\geq \frac{\|z_n^0\|_{\mathbf{L}}}{\|z_n^0\|_{\mathbf{L}} + \|z^\perp\|_{\mathbf{L}}}\\ &\geq \frac{\|z^0\|_{\mathbf{L}}}{C\|z^0\|_{\mathbf{L}} + C}.\end{aligned}$$

That is to say

$$\|y_n^0\|_{\mathbf{L}} \geq C > 0, \tag{3.10}$$

for $n$ large enough. Since $B_1(t) \leq B(z) \leq B_2(t)$, we have $B_1 \leq \bar{B}_n \leq B_2$. Duo to condition ($V_4$) and Lemma 3.1, we write $\mathbf{L} = \mathbf{L}^+_{A-\bar{B}_n} \bigoplus \mathbf{L}^-_{A-\bar{B}_n}$ with $A - \bar{B}_n$ is positive and negative define on $\mathbf{L}^+_{A-\bar{B}_n}$ and $\mathbf{L}^-_{A-\bar{B}_n}$ respectively. Re-decompose $y_n = \bar{y}_n^+ + \bar{y}_n^-$ respect to $\mathbf{L}^+_{A-\bar{B}_n}$ and $\mathbf{L}^-_{A-\bar{B}_n}$. From $(V_4)$ and (3.9), we have

$$\begin{aligned}\|y_n^0\|^2_{\mathbf{L}} &\leq \|y_n\|^2_{\mathbf{L}}\\ &\leq C((A-\bar{B}_n)y_n, \bar{y}_n^+ + \bar{y}_n^-)_{\mathbf{L}}\\ &\leq C\|(A-\bar{B}_n)y_n\|_{\mathbf{L}} \cdot \|y_n\|_{\mathbf{L}}\\ &\leq \frac{|o(\|z_n\|_{\mathbf{L}})|}{\|z_n\|_{\mathbf{L}}}\|y_n\|_{\mathbf{L}}.\end{aligned} \tag{3.11}$$

Since $\|z_n\|_{\mathbf{L}} \to \infty$ and $\|y_n\|_{\mathbf{L}} = 1$, we have $\|y_n^0\|_{\mathbf{L}} \to 0$ which contradicts to (3.10), so we have $\{z_n\}$ is bounded.

*Step 3* By Leray–Schauder degree, there is a solution of (HS).

Since the solutions of $(HS_\lambda)$ are bounded, there is a number $R > 0$ large eoungh, such that all of the solutions $z_\lambda$ of $(HS_\lambda)$ are in the ball $B(0, R) := \{z \in \mathbf{L} | \|z\|_{\mathbf{L}} < R\}$. So we have the Larey-Schauder degree

$$deg(I - (A - B_1)^{-1}(\lambda\Phi'(z) - \lambda B_1 z), B(0, R), 0)$$

is well defined and independent of $\lambda \in [0, 1]$, so

$$deg(I - (A - B_1)^{-1}(\Phi'(z) - B_1 z), B(0, R), 0) = deg(I, B(0, R), 0) = 1.$$

That is to say (HS) has at least one solution.

### 3.2 Applications

Consider the following delay differential system

$$\begin{cases} \ddot{x}(t) = -[\nabla v(x(t-\tau)) + \cdots + \nabla v(x(t-(m-1)\tau))], \\ x(t+m\tau) = x(t), \end{cases} \quad \forall t \in \mathbb{R}, \qquad (DDS)$$

with $x \in C^2(\mathbb{R}, \mathbb{R}^n)$, $v \in C^1(\mathbb{R}^n, \mathbb{R})$.

Now, we will transform the system (DDS) into the system (HS). Set $\tau = 1$ for simplicity. Let

$$z(t) = (x_1(t), \cdots, x_m(t))^T, \tag{3.12}$$

with $x_k(t) = x(t-(k-1))$ and $k = 1, 2, \cdots, m$. If $x(t)$ is a solution of (DDS), $z(t)$ is a solution of the following system

$$\begin{cases} -A_{nm}^{-1}\ddot{z}(t) = V'(z(t)), \\ \qquad z(1) = Q_{nm}z(0), \quad \forall t \in \mathbb{R}, \\ \qquad \dot{z}(1) = Q_{nm}\dot{z}(0), \end{cases} \qquad (\mathrm{HS}_2)$$

where $A_{nm} = \begin{pmatrix} 0 & I_n & \cdots & I_n \\ I_n & 0 & \cdots & I_n \\ \cdots & \cdots & \cdots & \cdots \\ I_n & I_n & \cdots & 0 \end{pmatrix}_{nm}$, and $Q_{nm} = \begin{pmatrix} 0 & 0 & \cdots & 0 & I_n \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_n & 0 \end{pmatrix}_{nm}$, with $I_n$ the identity map on $\mathbb{R}^n$, $z : \mathbb{R} \to \mathbb{R}^{nm}$. The function $V : \mathbf{R}^{nm} \to \mathbf{R}$,

$$V(z) := v(x_1) + v(x_2) + \cdots v(x_m).$$

On the other hand, if $z(t)$ is a solution of $(\mathrm{HS}_2)$, $x_1(t)$ is a solution of (DDS).

Let $A := -A_{nm}^{-1}$ and $Q := Q_{nm}$, it is easy to see that $A$, $Q$ and $V$ defined here satisfy the conditions in (1.1), (1.2) and (1.3), so $(\mathrm{HS}_2)$ is a specific case of (HS). Corresponding to Theorem 1.1 and 1.2, we have the following results.

**Theorem 3.2** *Assume $v \in C^1(\mathbf{R}^n, \mathbf{R})$ satisfies the following conditions.*
*$(v_1)$ $v' : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous*

$$\|v'(x+y) - v'(x)\|_{\mathbb{R}^n} \le l_v \|y\|_{\mathbb{R}^n}, \ \forall x, y \in \mathbb{R}^n, \tag{3.13}$$

*with its Lipschitz constant $l_v > 0$.*

*$(v_2^{\pm})$ There exists $M_1,\ M_2,\ K > 0$, $b(t) \in \mathcal{L}(S_1, \mathcal{L}(\mathbb{R}^n))$ with $b(t) \equiv b$, such that*

$$v'(x) = bx + r(x), \quad \forall x \in \mathbb{R}^n,$$

*with*

$$|r(x)| \le M_1, \quad \forall x \in \mathbb{R}^n,$$

*and*

$$\pm (r(x), x)_{\mathbb{R}^n} \ge M_2 |x|_{\mathbb{R}^N}, \ \forall |x|_{\mathbb{R}^n} > K. \tag{3.14}$$

*Then (DDS) has at least one solution.*

**Theorem 3.3** *Assume $v \in C^1(\mathbf{R}^n, \mathbf{R})$ satisfies conditions ($v_1$) and the following condition ($v_3$) There exists $b \in C(\mathbb{R}^n, \mathcal{L}_s(\mathbb{R}^n))$ such that*

$$v'(x) = b(x)x + r(x),\ \forall x \in \mathbb{R}^n,$$

*with*

$$r(x) = o(x),\ \text{uniformly for} |x| \to \infty.$$

*($v_4$) There exist $b_1,\ b_2 \in \mathcal{L}([0,1], \mathcal{L}(\mathbb{R}^n))$ satisfying*

$$b_1(t) \le b(x) \le b_2(t),\ \forall (t,x) \in [0,1] \times \mathbb{R}^n,$$

*and $B_1,\ B_2 \in \mathcal{L}_Q(S^1, \mathcal{L}(\mathbf{R}^{nm}))$ satisfying*

$$\mu_Q(A, B_1) = \mu_Q(A, B_2), \quad \upsilon_Q(A, B_2) = 0,$$

*with*

$$B_i := \begin{pmatrix} b_i & 0 & \cdots & 0 \\ 0 & b_i & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_i \end{pmatrix}_{nm},\ i = 1, 2.$$

*Then (DDS) has at least one solution.*

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## Declarations

**Conflict of interest** The authors declare that there is no conflict of interest regarding the publication of this paper.

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