Journal of Dynamics and Differential Equations (2024) 36:2555–2592
https://doi.org/10.1007/s10884-022-10215-5

Global Stability in a Two-species Attraction–Repulsion System with Competitive and Nonlocal Kinetics

Runlin Hu[1] · **Pan Zheng**[1,2,3]

Received: 4 August 2022 / Revised: 28 September 2022 / Accepted: 1 October 2022 /
Published online: 17 October 2022

Abstract

This paper deals with a two-species attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u - \xi_1 \nabla \cdot (u \nabla v) + \chi_1 \nabla \cdot (u \nabla z) + f_1(u, w), & (x, t) \in \Omega \times (0, \infty), \\ \tau v_t = \Delta v + w - v, & (x, t) \in \Omega \times (0, \infty), \\ w_t = \Delta w - \xi_2 \nabla \cdot (w \nabla z) + \chi_2 \nabla \cdot (w \nabla v) + f_2(u, w), & (x, t) \in \Omega \times (0, \infty), \\ \tau z_t = \Delta z + u - z, & (x, t) \in \Omega \times (0, \infty), \end{cases}$$

under homogeneous Neumann boundary conditions in a smoothly bounded domain $\Omega \subseteq \mathbb{R}^n$, where $\tau \in \{0, 1\}$, $\xi_i, \chi_i > 0$ and $f_i(u, w) (i = 1, 2)$ satisfy

$$\begin{cases} f_1(u, w) = u\left(a_0 - a_1 u - a_2 w + a_3 \int_\Omega u dx + a_4 \int_\Omega w dx\right), \\ f_2(u, w) = w\left(b_0 - b_1 u - b_2 w + b_3 \int_\Omega u dx + b_4 \int_\Omega w dx\right) \end{cases}$$

with $a_i, b_i > 0 (i = 0, 1, 2)$, $a_j, b_j \in \mathbb{R} (j = 3, 4)$. It is proved that in any space dimension $n \geq 1$, the above system possesses a unique global and uniformly bounded classical solution regardless of $\tau = 0$ or $\tau = 1$ under some suitable assumptions. Moreover, by constructing Lyapunov functionals, we establish the globally asymptotic stabilization of coexistence and semi-coexistence steady states.

Keywords Two-species · Attraction–repulsion · Competition · Nonlocal kinetics

✉ Pan Zheng
zhengpan52@sina.com

Runlin Hu
hurl134@163.com

[1] College of Science, Chongqing University of Posts and Telecommunications, Chongqing 400065, People's Republic of China

[2] Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, People's Republic of China

[3] College of Mathematics and Statistics, Yunnan University, Kunming 650091, People's Republic of China

Mathematics Subject Classification 35B35 · 35B40 · 35K15 · 35K55 · 92C17

1 Introduction

To describe some complex biological processes (such as cell sorting process [30]) in two species, we consider the following two-species attraction–repulsion chemotaxis system with competitive and nonlocal kinetic terms

$$\begin{cases} u_t = \Delta u - \xi_1 \nabla \cdot (u \nabla v) + \chi_1 \nabla \cdot (u \nabla z) + f_1(u, w), & (x, t) \in \Omega \times (0, \infty), \\ \tau v_t = \Delta v + w - v, & (x, t) \in \Omega \times (0, \infty), \\ w_t = \Delta w - \xi_2 \nabla \cdot (w \nabla z) + \chi_2 \nabla \cdot (w \nabla v) + f_2(u, w), & (x, t) \in \Omega \times (0, \infty), \\ \tau z_t = \Delta z + u - z, & (x, t) \in \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ (u, \tau v, w, \tau z)(x, 0) = (u_0(x), \tau v_0(x), w_0(x), \tau z_0(x)), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\tau \in \{0, 1\}, \xi_i, \chi_i > 0 (i = 1, 2), \Omega \subseteq \mathbb{R}^n (n \geq 1)$ is a smoothly bounded domain and f_1, f_2 satisfy

$$\begin{cases} f_1(u, w) = u\left(a_0 - a_1 u - a_2 w + a_3 \int_\Omega u dx + a_4 \int_\Omega w dx\right), \\ f_2(u, w) = w\left(b_0 - b_1 u - b_2 w + b_3 \int_\Omega u dx + b_4 \int_\Omega w dx\right) \end{cases} \tag{1.2}$$

with $a_i, b_i > 0 (i = 0, 1, 2), a_j, b_j \in \mathbb{R} (j = 3, 4)$. Here the movements of the two populations are described by random diffusion (i.e. $\Delta u, \Delta w$), chemoattractant (i.e. $-\xi_1 \nabla \cdot (u \nabla v), -\xi_2 \nabla \cdot (w \nabla z)$) and chemorepellent (i.e. $+\chi_1 \nabla \cdot (u \nabla z), +\chi_2 \nabla \cdot (w \nabla v)$). Moreover, in view of the classical Lotka–Volterra [26] and nonlocal dynamics, we assume that both populations reproduce and compete (or cooperate) themselves, and mutually compete (or cooperate) with the other. $u(x, t)$ and $w(x, t)$ represent the density of two populations, and $v(x, t), z(x, t)$ stand for the concentrations of the chemical substances. ξ_1, ξ_2 refer to the chemoattraction sensitivity coefficients and χ_1, χ_2 stand for the chemorepulsion sensitivity coefficients. The biological significance of these parameters $a_i, b_i (i = 0, 1, \ldots \cdot \cdot, 4)$ is explained in [28]. The initial data $(u_0, \tau v_0, w_0, \tau z_0)$ is nonnegative and satisfies

$$(u_0, \tau v_0, w_0, \tau z_0) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\bar{\Omega}) \times C^0(\bar{\Omega}) \times W^{1,\infty}(\bar{\Omega}). \tag{1.3}$$

Chemotaxis, a directed movement of biological cells or organisms in response to the concentration gradient of a chemical signal, is well known to play a significant role in a wide range of biological applications, such as pattern formation [5], embryonic development [18], wound healing [32] and blood vessel formation [7], etc. Notably, Keller and Segel [16] introduced the pioneering works of the chemotaxis which describes the aggregation of cellular slime molds Dictyostelium discoideum

$$\begin{cases} u_t = \Delta u - \xi \nabla \cdot (u \nabla v), & (x, t) \in \Omega \times (0, \infty), \\ \tau v_t = \Delta v + u - v, & (x, t) \in \Omega \times (0, \infty), \end{cases} \tag{1.4}$$

where $\xi \in \mathbb{R}$ and $\tau \in \{0, 1\}$. Here $u(x, t)$ and $v(x, t)$ represent the density of Dictyostelium discotylum and the concentration of chemical signals, respectively. Many scholars have studied deeply the system (1.4) in the past decades. For instance, when $\xi = \tau = 1$, all solutions of system (1.4) are globally bounded in one-dimensional domains [29]. If the initial mass $\int_\Omega u_0 dx$ of cells is small (i.e. $\int_\Omega u_0 dx < 4\pi$), then the solution of system (1.4) also is global and bounded in two-dimensional domains [9], whereas the initial mass of cells is large enough (i.e. $\int_\Omega u_0 dx > 4\pi$ and $\int_\Omega u_0 dx \in \{4K\pi, K \in \mathbb{N}\}$) then the aggregation occurs either in finite or infinite time [33]. Considering the volume-filling effect [31], Winkler [41] proved that the aggregation of cells happens in finite time. Moreover, Tao and Winkler [35] obtained that solutions of the above system are global and uniformly bounded. Of course, there are a large amount of results that include the global existence, boundedness and blow-up behavior of solutions to variants of the above system, we can refer to [6, 10–13, 17, 27, 36, 43–45].

To further understand system (1.1), let us mention some previous advances. When the system (1.1) has no repulsive mechanisms (i.e. $\chi_1 = \chi_2 = 0$) and nonlocal dynamics (i.e. $a_3 = a_4 = b_3 = b_4 = 0$), system (1.1) becomes the following two-species and two-stimuli chemotaxis system with Lotka–Volterra competitive terms

$$\begin{cases} u_t = \Delta u - \xi_1 \nabla \cdot (u \nabla v) + \mu_1 u(1 - u - a_1 w), & (x, t) \in \Omega \times (0, \infty), \\ \tau v_t = \Delta v + \beta w - \alpha v, & (x, t) \in \Omega \times (0, \infty), \\ w_t = \Delta w - \xi_2 \nabla \cdot (w \nabla z) + \mu_2 w(1 - w - a_2 u), & (x, t) \in \Omega \times (0, \infty), \\ \tau z_t = \Delta z + \delta u - \gamma z, & (x, t) \in \Omega \times (0, \infty), \end{cases} \tag{1.5}$$

where $\tau \in \{0, 1\}$ and $\alpha, \beta, \gamma, \delta, \xi_i, \mu_i, a_i > 0 (i = 1, 2)$. For the parabolic-elliptic-parabolic-elliptic case (i.e. $\tau = 0$), if the production efficiency of the signals v, z is the same as the consumption (i.e. $\alpha = \beta = \delta = \gamma = 1$), Zheng and Mu [51] obtained that all solutions of system (1.5) are globally bounded in two-dimensional domains. When $a_1, a_2 \in (0, 1)$, Zheng et al. [52] showed that the two species can maintain constant coexistence stable state. Furthermore, in two-dimensional domains, when $\mu_1 = \mu_2 = 0$ in (1.5), Yu et al. [46] proved that aggregations of species occurs if the initial masses $\int_\Omega u_0 dx$ and $\int_\Omega w_0 dx$ are large sufficiently (i.e. $\int_\Omega u_0 dx \cdot \int_\Omega u_0 dx - 2\pi(\frac{\int_\Omega u_0 dx}{\xi_1 \beta} + \frac{\int_\Omega w_0 dx}{\xi_2 \delta}) > 0$) and that globally bounded solutions exist if the initial masses $\int_\Omega u_0 dx$ and $\int_\Omega w_0 dx$ are small enough (i.e. $\max\{\int_\Omega u_0 dx, \int_\Omega w_0 dx\} < 4\pi$) and $\xi_1 = \xi_2 = \alpha = \beta = \delta = \gamma = 1$. On the other hand, due to existence of Lotka–Volterra competition, Tu et al. [38] derived the coexistence stable state of two species under the weak competition (i.e. $a_1, a_2 \in (0, 1)$) and semi-trivial equilibrium under strong competition ($a_1 \geq 1 > a_2 > 0$). Recently, the previous results of [38] was improved by Wang and Mu in [39]. For the fully parabolic case (i.e. $\tau = 1$), relying on the maximal Sobolev regularity, Zheng and Mu [51] derived that when the chemotactic sensitivities are small enough as related to the Lotka–Volterra competitive terms in the sense that $\frac{\xi_2}{\mu_1} < \theta_0$ and $\frac{\xi_1}{\mu_2} < \theta_0$ for some $\theta_0 > 0$, then system (1.5) possesses a globally bounded classical solution in any space dimension $n \geq 1$. For more related contents, we can refer to [19, 20, 22, 47–49, 53]. However, to the best of our knowledge the literature does not provide any qualitative analysis on the solution behavior when attraction–repulsion chemotaxis as well as nonlocal competitive Lokta-Volterra competitive terms involving both species are present.

So far several special cases for system (1.1) are studied by some scholars. For example, when $\tau = 0, n = 2$ and system (1.1) has no kinetic terms (i.e. $f_1 = f_2 \equiv 0$), the nonradial solutions blow up in finite time [21]. Considering the small initial masses and repulsive mechanisms (i.e. $\chi_1, \chi_2 > 0$), the global boundedness of solutions is established under the conditions that $n = 2$ and $\max\{\int_\Omega u_0 dx, \int_\Omega w_0 dx\} < \frac{4}{(\chi_1+\chi_2+\xi_1+\xi_2)C_{GN}}$ or $\chi_1 = \chi_2 = \xi_1 =$

$\xi_2 > 0$ in [21]. Similarly, due to the effect of repulsive mechanisms (i.e. $\chi_1, \chi_2 > 0$), Liu and Dai [23] proved the global boundedness of solutions to system (1.1) if $\min\{\chi_1, \chi_2\} > \xi_1 + \xi_2$ holds. Without nonlocal terms in (1.1) (i.e. $a_3 = a_4 = b_3 = b_4 = 0$), Zheng and Hu [50] obtained the constant equilibrium and semi-trivial equilibrium of two species. In addition, when f_1 and f_2 satisfy (1.2), Zheng et al. [54] studied the fully parabolic two-species chemotaxis system with indirect signal production, and derived that the global boundedness and stability of the constant steady state under some suitable assumptions. Recently, when f_1 and f_2 satisfy (1.2) with $a_0, a_1, b_0, b_2 > 0, a_2, a_3, a_4, b_1, b_3, b_4 \in \mathbb{R}$, the boundedness and stabilization of global solutions for system (1.1) were derived by Hu et al. in [14]. However, the authors only considered the constant coexistence stable state under the locally intraspecific competition and globally interspecific cooperation (i.e. $a_1, a_4, b_2, b_3 > 0, a_2, b_1 < 0$) in [14].

As analyzed above, the boundedness of global solutions to system (1.1) still remains open under locally competitive and nonlocal kinetic terms. For the fully competitive case (i.e. $a_i, b_i > 0 (i = 1, 2), a_j, b_j < 0 (j = 3, 4)$), the large time behavior of global solutions to system (1.1) is also an unsolved question. Hence, this paper gives an affirmative respond. For simplicity, we introduce a notation that is $(a)_+ := \max\{0, a\}$ for all $a \in \mathbb{R}$. Our main results are stated as follows.

Firstly, we consider the boundedness of solutions for (1.1) under the case $\tau = 0$.

Theorem 1.1 *Let $\tau = 0, \xi_i, \chi_i > 0 (i = 1, 2)$ and $\Omega \subseteq \mathbb{R}^n$ be a smoothly bounded domain. Assume that f_1, f_2 satisfy (1.2) with $a_i, b_i > 0 (i = 0, 1, 2), a_j, b_j \in \mathbb{R} (j = 3, 4)$, and the initial data (u_0, w_0) satisfies (1.3). If the following conditions hold:*

- *$n = 1$ and*
$$a_1 > \left((a_3)_+ + \frac{(a_4)_+ + (b_3)_+}{2}\right)|\Omega|, b_2 > \left((b_4)_+ + \frac{(a_4)_+ + (b_3)_+}{2}\right)|\Omega|; \quad (1.6)$$
- *$n \geq 2$, (1.6) and*
$$\min\left\{\chi_1 + a_1, \chi_2 + b_2\right\} > \xi_1 + \xi_2 \quad (1.7)$$
or
$$\xi_1 < \frac{na_2}{n-2}, \xi_2 < \frac{nb_1}{n-2}. \quad (1.8)$$

Then system (1.1) admits a unique global classical solution (u, v, w, z), which is uniformly bounded in $\Omega \times (0, \infty)$ in the sense that there exists $C > 0$ independent of t such that
$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)} \leq C \quad for\ all\ t > 0.$$

Remark 1.1 When $\tau = 0$ and $\chi_1 = \chi_2 = a_3 = a_4 = b_3 = b_4 = 0$, the results of Theorem 1.1 cover those of [38, 50]. When $n \geq 2$ and $a_2, b_1 > 0$, Theorem 1.1 not only includes the conditions $\min\left\{\chi_1 + a_1, \chi_2 + b_2\right\} > \xi_1 + \xi_2$ of [14], but also expands the conditions $\xi_1 < \frac{na_2}{n-2}$ and $\xi_2 < \frac{nb_1}{n-2}$. Moreover, compared with the results in [21], we can remove the small initial condition $\max\left\{\|u_0\|_{L^1(\Omega)}, \|w_0\|_{L^1(\Omega)}\right\} < \frac{4}{(\chi_1+\chi_2+\xi_1+\xi_2)C_{GN}}$ in this paper.

Next, we study the global stability of bounded solution for (1.1) when $\tau = 0$. When weak interspecific competition occurs, i.e.,
$$\frac{a_2 - a_4|\Omega|}{b_2 - b_4|\Omega|} < \frac{a_0}{b_0} < \frac{a_1 - a_3|\Omega|}{b_1 - b_3|\Omega|} \quad (1.9)$$

holds with $a_i, b_i > 0 (i = 0, 1, 2), a_j, b_j < 0 (j = 3, 4)$, then the unique positive constant steady state (u_*, v_*, w_*, z_*) can be recorded as

$$
\begin{aligned}
u_* &:= \frac{a_0 (b_2 - b_4|\Omega|) - b_0 (a_2 - a_4|\Omega|)}{(b_2 - b_4|\Omega|)(a_1 - a_3|\Omega|) - (a_2 - a_4|\Omega|)(b_1 - b_3|\Omega|)},\\
v_* &:= \frac{a_0 (b_1 - b_3|\Omega|) - b_0 (a_1 - a_3|\Omega|)}{(a_2 - a_4|\Omega|)(b_1 - b_3|\Omega|) - (b_2 - b_4|\Omega|)(a_1 - a_3|\Omega|)},\\
w_* &:= \frac{a_0 (b_1 - b_3|\Omega|) - b_0 (a_1 - a_3|\Omega|)}{(a_2 - a_4|\Omega|)(b_1 - b_3|\Omega|) - (b_2 - b_4|\Omega|)(a_1 - a_3|\Omega|)},\\
z_* &:= \frac{a_0 (b_2 - b_4|\Omega|) - b_0 (a_2 - a_4|\Omega|)}{(b_2 - b_4|\Omega|)(a_1 - a_3|\Omega|) - (a_2 - a_4|\Omega|)(b_1 - b_3|\Omega|)}.
\end{aligned}
\tag{1.10}
$$

Theorem 1.2 *Let the conditions in Theorem 1.1 and (1.9) with $a_i, b_i > 0 (i = 0, 1, 2), a_j, b_j < 0 (j = 3, 4)$ hold. Assume that system (1.1) admits a unique global classical solution (u, v, w, z) with the property*

$$
\begin{aligned}
&\|u\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[t,t+1])} + \|v\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[t,t+1])}\\
&\quad + \|w\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[t,t+1])} + \|z\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\overline{\Omega}\times[t,t+1])} \le K
\end{aligned}
\tag{1.11}
$$

for all $t \ge 1$, where $K > 0$ and $\vartheta \in (0, 1)$. Suppose that there exist $\theta_1, \theta_2 \in (0, 1)$ such that

$$
a_1 > -\frac{a_4 + b_3}{2}|\Omega| + \frac{u_* \chi_1^2 + w_* \xi_2^2}{8(1 - \theta_1)}
\tag{1.12}
$$

and

$$
\begin{aligned}
b_2 > \max\Bigg\{ &-\frac{a_4 + b_3}{2}|\Omega| + \frac{(a_2 + b_1)^2}{2\theta_1\theta_2\big[2a_1 - (a_4 + b_3)|\Omega|\big]},\\
&-\frac{a_4 + b_3}{2}|\Omega| + \frac{u_* \xi_1^2 + w_* \chi_2^2}{8(1 - \theta_2)}\Bigg\}.
\end{aligned}
\tag{1.13}
$$

Then for some fixed time $t_1 > 0$, there exist $C > 0$ and $\lambda_1 > 0$ such that

$$
\|u - u_*\|_{L^\infty(\Omega)} + \|v - v_*\|_{L^\infty(\Omega)} + \|w - w_*\|_{L^\infty(\Omega)} + \|z - z_*\|_{L^\infty(\Omega)} \le Ce^{-\lambda_1 t}
\tag{1.14}
$$

for all $t > t_1$, where (u_*, v_*, w_*, z_*) is given in (1.10).

Now, we consider the strong competition case, i.e.,

$$
\frac{a_2 - a_4|\Omega|}{b_2 - b_4|\Omega|} < \frac{a_1 - a_3|\Omega|}{b_1 - b_3|\Omega|} \le \frac{a_0}{b_0}
\tag{1.15}
$$

with $a_i, b_i > 0 (i = 0, 1, 2), a_j, b_j < 0 (j = 3, 4)$, then the unique semi-trivial equilibrium $(u_\star, 0, 0, z_\star)$ is

$$
u_\star = \frac{a_0}{a_1 - a_3|\Omega|}, z_\star = \frac{a_0}{a_1 - a_3|\Omega|}.
\tag{1.16}
$$

Theorem 1.3 *Let the conditions in Theorem 1.1 and (1.15) with $a_i, b_i > 0 (i = 0, 1, 2), a_j, b_j < 0 (j = 3, 4)$ hold. Assume that system (1.1) admits a unique global classical solution (u, v, w, z) with the property (1.11). Suppose that there exist $\theta_3, \theta_4 \in (0, 1)$ such that*

$$
a_1 > -\frac{a_4 + b_3}{2}|\Omega| + \frac{u_\star \chi_1^2}{8(1 - \theta_3)}
\tag{1.17}
$$

and

$$b_2 > \max\left\{ -\frac{a_4+b_3}{2}|\Omega| + \frac{(a_2+b_1)^2}{2\theta_1\theta_2\big[2a_1-(a_4+b_3)|\Omega|\big]}, \right. \\ \left. -\frac{a_4+b_3}{2}|\Omega| + \frac{u_\star \xi_1^2}{8(1-\theta_4)} \right\}. \tag{1.18}$$

Then the unique global solution (u, v, w, z) *has the following properties:*

(i) *If* $\frac{a_2-a_4|\Omega|}{b_2-b_4|\Omega|} < \frac{a_1-a_3|\Omega|}{b_1-b_3|\Omega|} < \frac{a_0}{b_0}$, *then for some fixed time* $t_2 > 0$ *there exist* $C > 0$ *and* $\lambda_2 > 0$ *such that*

$$\|u-u_\star\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z-z_\star\|_{L^\infty(\Omega)} \le Ce^{-\lambda_2 t} \tag{1.19}$$

for all $t > t_2$*, where* $(u_\star, z_\star)$ *is given by* (1.16).

(ii) *If* $\frac{a_2-a_4|\Omega|}{b_2-b_4|\Omega|} < \frac{a_1-a_3|\Omega|}{b_1-b_3|\Omega|} = \frac{a_0}{b_0}$, *then for some fixed time* $t_3 > 0$ *there exist* $C > 0$ *and* $\lambda_3 > 0$ *satisfying*

$$\|u-u_\star\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z-z_\star\|_{L^\infty(\Omega)} \le C(t-t_3)^{-\lambda_3} \tag{1.20}$$

for all $t > t_3$*, where* $(u_\star, z_\star)$ *is given by* (1.16).

On the other hand, we are concerned with the fully parabolic case, i.e. $\tau = 1$.

Theorem 1.4 *Let* $\tau = 1, \xi_i, \chi_i > 0 (i = 1, 2)$ *and* $\Omega \subseteq \mathbb{R}^n$ *be a smoothly bounded domain. Assume that* f_1, f_2 *satisfy* (1.2) *with* $a_i, b_i > 0(i = 0, 1, 2), a_j, b_j \in \mathbb{R}(j = 3, 4)$ *and the initial data* (u_0, v_0, w_0, z_0) *satisfies* (1.3). *Moreover, suppose that*

- $n = 1, 2$*, the condition* (1.6)*;*
- $n = 3$*, the condition*

$$\begin{cases} a_1 > \max\left\{\left((a_3)_+ + \frac{(a_4)_+ + (b_3)_+}{2}\right)|\Omega|, \frac{7\xi_1^2}{b_1} + 2b_1, \frac{7\chi_1^2}{b_1} + 2b_1 + 28\left(\frac{a_2^2}{b_1} + b_1\right)\right\}, \\ b_2 > \max\left\{\left((b_4)_+ + \frac{(a_4)_+ + (b_3)_+}{2}\right)|\Omega|, \frac{7\chi_2^2}{a_2} + 2a_2, \frac{7\xi_2^2}{a_2} + 2a_2 + 28\left(\frac{b_1^2}{a_2} + a_2\right)\right\}; \end{cases} \tag{1.21}$$

- $n \ge 4$*, the condition*

$$\begin{cases} a_1 > \max\left\{\left((a_3)_+ + \frac{(a_4)_+ + (b_3)_+}{2}\right)|\Omega|, \xi_1 + \chi_1 + C_S(\chi_1 + \xi_2)\right\}, \\ b_2 > \max\left\{\left((b_4)_+ + \frac{(a_4)_+ + (b_3)_+}{2}\right)|\Omega|, \xi_2 + \chi_2 + C_S(\chi_2 + \xi_1)\right\} \end{cases} \tag{1.22}$$

are satisfied, where C_S *is given in Lemma* 2.7. *Then system* (1.1) *admits a unique global classical solution* (u, v, w, z)*, which is uniformly bounded in* $\Omega \times (0, \infty)$ *in the sense that there exists* $C > 0$ *independent of* t *such that*

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)} \le C \quad for\ all\ t > 0.$$

Remark 1.2 When $\tau = 1$ and $a_i, b_i > 0(i = 0, 1, 2), a_j, b_j \in \mathbb{R}(j = 3, 4)$, Theorem 1.4 includes the result of [14] under the effect of locally interspecific competition. If $n = 2$ and $\chi_1 = \chi_2 = a_3 = a_4 = b_3 = b_4 = 0$, compared with the results in [22], Theorem 1.4

only requires the conditions $a_1, b_2 > 0$, which improves the conditions $a_1 > \xi_1 + \xi_2 C_S$ and $b_2 > \xi_2 + \xi_1 C_S$ in [22]. Moreover, the small initial condition in [23] can also be removed in this paper.

Finally, we study the global stability of bounded solutions for (1.1) when $\tau = 1$.

Theorem 1.5 *Let the conditions in Theorem 1.4 and (1.9) with $a_i, b_i > 0 (i = 0, 1, 2), a_j, b_j < 0 (j = 3, 4)$ hold. Assume that system (1.1) admits a unique global classical solution (u, v, w, z) with the property (1.11). Suppose that the conditions (1.12) and (1.13) are satisfied, then there exist $C > 0$ and $\kappa_1 > 0$ such that*

$$\|u - u_*\|_{L^\infty(\Omega)} + \|v - v_*\|_{L^\infty(\Omega)} + \|w - w_*\|_{L^\infty(\Omega)} + \|z - z_*\|_{L^\infty(\Omega)} \le Ce^{-\kappa_1 t} \tag{1.23}$$

for all $t > t_4$, where $t_4 > 0$ is some fixed time and (u_*, v_*, w_*, z_*) is given in (1.10).

Theorem 1.6 *Let the conditions in Theorem 1.4 and (1.15) with $a_i, b_i > 0 (i = 0, 1, 2), a_j, b_j < 0 (j = 3, 4)$ hold. Assume that system (1.1) admits a unique global classical solution (u, v, w, z) with the property (1.11). Suppose that the conditions (1.17) and (1.18) are satisfied. Then the unique global solution (u, v, w, z) has the following properties:*

(i) *If $\frac{a_2 - a_4|\Omega|}{b_2 - b_4|\Omega|} < \frac{a_1 - a_3|\Omega|}{b_1 - b_3|\Omega|} < \frac{a_0}{b_0}$, then for some fixed time $t_5 > 0$ there exist $C > 0$ and $\kappa_2 > 0$ satisfying*

$$\|u - u_\star\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z - z_\star\|_{L^\infty(\Omega)} \le Ce^{-\kappa_2 t} \tag{1.24}$$

for all $t > t_5$, where $(u_\star, z_\star)$ is given by (1.16).

(ii) *If $\frac{a_2 - a_4|\Omega|}{b_2 - b_4|\Omega|} < \frac{a_1 - a_3|\Omega|}{b_1 - b_3|\Omega|} = \frac{a_0}{b_0}$, then for some fixed time $t_6 > 0$ there exist $C > 0$ and $\kappa_3 > 0$ such that*

$$\|u - u_\star\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z - z_\star\|_{L^\infty(\Omega)} \le C(t - t_6)^{-\kappa_3} \tag{1.25}$$

for all $t > t_6$, where $(u_\star, z_\star)$ is given by (1.16).

Remark 1.3 In this paper, we only consider the large time behavior of the global solutions to system (1.1) under the $\frac{a_2 - a_4|\Omega|}{b_2 - b_4|\Omega|} < \frac{a_0}{b_0} < \frac{a_1 - a_3|\Omega|}{b_1 - b_3|\Omega|}$ and $\frac{a_2 - a_4|\Omega|}{b_2 - b_4|\Omega|} < \frac{a_1 - a_3|\Omega|}{b_1 - b_3|\Omega|} \le \frac{a_0}{b_0}$, respectively. For the strong competition case, it is not difficult to obtain the large time behavior by the same method used in the proof of Theorem 1.2 (or Theorem 1.6). However, for the fully strong competition case $\frac{a_1 - a_3|\Omega|}{b_1 - b_3|\Omega|} < \frac{a_0}{b_0} < \frac{a_2 - a_4|\Omega|}{b_2 - b_4|\Omega|}$, there is still an open problem about stabilization of global bounded solutions.

This paper is organized as follows. In Sect. 2, we give the local existence of solution to system (1.1) and some preliminary lemmas. In Sect. 3, we study the global existence and boundedness of solutions to system (1.1) with $\tau = 0$, and prove Theorem 1.1. In Sect. 4, we study the asymptotic behavior of global solutions to system (1.1) with $\tau = 0$, and prove Theorem 1.2 and Theorem 1.3. In Sect. 5, we discuss the global existence and boundedness of solutions to system (1.1) with $\tau = 1$, and prove Theorem 1.4. In Sect. 6, we consider the asymptotic behavior of global solutions to system (1.1) when $\tau = 1$, and prove Theorem 1.5 and Theorem 1.6. In addition, we let $u(\cdot, t) := u(x, t)$ and omit signs dx during integrating for brevity thoughout this paper.

2 Preliminaries

In this section, we shall give several preliminary lemmas. Firstly, we state the local existence of solutions for (1.1).

Lemma 2.1 *Let* $\tau \in \{0, 1\}$, $\xi_i, \chi_i > 0 (i = 1, 2)$ *and* $\Omega \subseteq \mathbb{R}^n (n \geq 1)$ *be a smoothly bounded domain. Assume that* f_1, f_2 *satisfy* (1.2)*, and the initial data* $(u_0, \tau v_0, w_0, \tau z_0)$ *satisfies* (1.3). *Then there exist* $T_{\max} \in (0, \infty]$ *and uniquely determined nonnegative functions*

$$\begin{cases} u, w \in C^0\big(\bar{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\bar{\Omega} \times (0, T_{\max})\big), \\ v, z \in C^0\big(\bar{\Omega} \times [0, T_{\max})\big) \cap C^{2,1}\big(\bar{\Omega} \times (0, T_{\max})\big) \cap L^{\infty}_{loc}\big([0, T_{\max}); W^{1,q}(\Omega)\big) \end{cases}$$

such that (u, v, w, z) *solves system* (1.1) *classically in* $\Omega \times (0, T_{\max})$*, where* $q > n$*. Moreover, if* $T_{max} < \infty$*, then*

$$\limsup_{t \nearrow T_{\max}} \Big(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{W^{1,q}(\Omega)} \Big) = \infty. \tag{2.1}$$

Proof This proof is similar to [42, Theorem 1.1] or [34, Lemma 2.1]. For more details, please refer to [14, Lemma 2.1]. The proof of Lemma 2.1 is complete. □

Secondly, we shall give a uniform L^1-bound of (u, v, w, z) for (1.1).

Lemma 2.2 *Let* (u, v, w, z) *be a solution to system* (1.1) *and the nonnegative initial data* $(u_0, \tau v_0, w_0, \tau z_0)$ *satisfies* (1.3) *with* $\tau \in \{0, 1\}$. *Suppose that the condition* (1.6) *holds. Then*

$$\|u(\cdot, t)\|_{L^1(\Omega)} + \|w(\cdot, t)\|_{L^1(\Omega)} \leq \max\left\{ \|u_0\|_{L^1(\Omega)} + \|w_0\|_{L^1(\Omega)}, 2|\Omega| \frac{\max\{a_0, b_0\}}{\min\{l_1, l_2\}} \right\} := M_0 \tag{2.2}$$

and

$$\|v(\cdot, t)\|_{L^1(\Omega)} + \|z(\cdot, t)\|_{L^1(\Omega)} \leq \max\left\{ \|v_0\|_{L^1(\Omega)} + \|z_0\|_{L^1(\Omega)}, M_0 \right\} := m_0 \tag{2.3}$$

for all $t \in (0, T_{\max})$*, where*

$$\begin{cases} l_1 = a_1 - |\Omega|(a_3)_+ - \dfrac{1}{2}|\Omega|\Big((a_4)_+ + (b_3)_+\Big), \\ l_2 = b_2 - |\Omega|(b_4)_+ - \dfrac{1}{2}|\Omega|\Big((a_4)_+ + (b_3)_+\Big). \end{cases} \tag{2.4}$$

Proof Integrating the first equation in (1.1), we deduce from Young's and Hölder's inequalities that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u &= \int_\Omega u\Big(a_0 - a_1 u - a_2 w + a_3 \int_\Omega u + a_4 \int_\Omega w\Big) \\ &\leq a_0 \int_\Omega u - a_1 \int_\Omega u^2 + \Big((a_3)_+ + \frac{(a_4)_+}{2}\Big)\Big(\int_\Omega u\Big)^2 + \frac{(a_4)_+}{2}\Big(\int_\Omega w\Big)^2 \\ &\leq a_0 \int_\Omega u - \Big(a_1 - \Big((a_3)_+ + \frac{(a_4)_+}{2}\Big)|\Omega|\Big)\int_\Omega u^2 + \frac{(a_4)_+}{2}|\Omega|\int_\Omega w^2 \end{aligned} \tag{2.5}$$

for all $t \in (0, T_{\max})$, where we have used that $a_2 > 0$. By the similar method to $w-$equation and $b_1 > 0$, we get

$$\frac{d}{dt}\int_\Omega w \le b_0 \int_\Omega w - \left(b_2 - \left((b_4)_+ + \frac{(b_3)_+}{2}\right)|\Omega|\right)\int_\Omega w^2 + \frac{(b_3)_+}{2}|\Omega|\int_\Omega u^2. \tag{2.6}$$

Then, combining (2.5) and (2.6), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+w) \le a_0 \int_\Omega u - \left(a_1 - \left((a_3)_+ + \frac{(a_4)_+ + (b_3)_+}{2}\right)|\Omega|\right)\int_\Omega u^2 \\ + b_0\int_\Omega w - \left(b_2 - \left((b_4)_+ + \frac{(a_4)_+ + (b_3)_+}{2}\right)|\Omega|\right)\int_\Omega w^2.\end{aligned} \tag{2.7}$$

Then the following proof of this lemma is similar to [14, Lemma 2.2]. Here we omit the details. The proof of Lemma 2.2 is complete. □

Lemma 2.3 (See [37, ChapterIII, Lemma 5.1]) *Let $\Phi(t) \ge 0$ satisfy*

$$\begin{cases}\Phi'(t) + k_1 y \Phi^\theta(t) \le k_2, \quad t > 0,\\ \Phi(0) = \Phi_0,\end{cases} \tag{2.8}$$

for $\Phi_0 \ge 0$ with some constants $k_1, k_2 > 0$ and $\theta > 0$. Then

$$\Phi(t) \le \max\left\{\Phi_0, \left(\frac{k_2}{k_1}\right)^{\frac{1}{\theta}}\right\} \text{ for all } t > 0.$$

Lemma 2.4 (See [8]) *Let $\Omega \subseteq \mathbb{R}^n (n \ge 1)$ be a bounded domain with smooth boundary. Assume that $p, k > 0, m \in [0, k)$ and $q, r \in [1, \infty]$ hold. Then for any $\Psi \in W^{k,q}(\Omega) \cap L^r(\Omega)$, there exists $C_{GN} = C(k, q, r, \Omega) > 0$ such that*

$$\|D^m\Psi\|_{L^p(\Omega)} \le C_{GN}\|D^k\Psi\|^{\alpha}_{L^q(\Omega)}\|\Psi\|^{1-\alpha}_{L^r(\Omega)} + C_{GN}\|\Psi\|_{L^r(\Omega)}, \tag{2.9}$$

where α satisfies

$$\frac{1}{p} - \frac{m}{n} = \alpha\left(\frac{1}{q} - \frac{k}{n}\right) + \frac{1}{r}\left(1-\alpha\right) \Leftrightarrow \alpha = \frac{\frac{1}{p} - \frac{m}{n} - \frac{1}{r}}{\frac{1}{q} - \frac{k}{n} - \frac{1}{r}} \in \left(\frac{m}{k}, 1\right)$$

and $D^k\psi$ is expressed as Fréchet derivative of order k.

Lemma 2.5 (See [24, Lemma 4.2]) *Let $\Omega \subseteq \mathbb{R}^n (n \ge 1)$ be a bounded domain with smooth boundary and $\Phi \in C^2(\Omega)$ satisfy $\frac{\partial \Phi}{\partial \nu} = 0$. Then*

$$\frac{\partial |\nabla \Phi|^2}{\partial \nu} \le C_\Omega |\nabla \Phi|^2, \tag{2.10}$$

where C_Ω is a positive constant depending only on the curvatures of $\partial\Omega$.

Lemma 2.6 (See [15]) *Let $\Omega \subseteq \mathbb{R}^n (n \ge 1)$ be a bounded domain with smooth boundary. Then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\int_{\partial\Omega} \Phi^2 \le \varepsilon \int_\Omega |\nabla\Phi|^2 + C_\varepsilon\left(\int_\Omega |\Phi|\right)^2 \text{ for all } \Phi \in W^{1,2}(\Omega). \tag{2.11}$$

Lemma 2.7 (See [40, Lemma 2.3]) *Let $\Omega \subseteq \mathbb{R}^n (n \geq 1)$ be a smoothly bounded domain and $0 \leq t_0 < T_{\max} \leq \infty$. Assume that $u_0 \in W^{2,p}(\Omega)(p > n)$ with $\partial_\nu u_0 = 0$ on $\partial\Omega$. Then for each $g \in L^p\big((0, T_{\max}); L^p(\Omega)\big)$, the problem*

$$\begin{cases} u_t = \Delta u - u + g, & (x,t) \in \Omega \times (0, T_{\max}), \\ \dfrac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0, T_{\max}), \\ u(\cdot, 0) = u_0, & x \in \Omega, \end{cases} \tag{2.12}$$

possesses a unique solution $u \in W^{1,p}\big((0, T_{\max}); L^p(\Omega)\big) \cap L^p\big((0, T_{\max}); W^{2,p}(\Omega)\big)$. Moreover, if $u(\cdot, t_0) \in W^{2,p}(\Omega)$ satisfies $\frac{\partial u(\cdot,t_0)}{\partial \nu} = 0$ on $\partial\Omega$, then there exists $C_S = C(p) > 0$ such that

$$\int_{t_0}^{t} e^{ps} \|\Delta u(\cdot, s)\|^p_{L^p(\Omega)} ds \leq C_S \int_{t_0}^{t} e^{ps} \|g(\cdot, s)\|^p_{L^p(\Omega)} ds + C_S e^{pt_0} \Big(\|u(\cdot, t_0)\|^p_{L^p(\Omega)} + \|\Delta u(\cdot, t_0)\|^p_{L^p(\Omega)} \Big)$$

for all $t \in (t_0, T_{\max})$.

3 Boundedness for $\tau = 0$

The aim of this section is to show the global boundedness of solution for (1.1) with $\tau = 0$ and prove Theorem 1.1.

Lemma 3.1 *Let $\tau = 0$ and $\Omega \subseteq \mathbb{R}$ be a smoothly bounded domain. Suppose that $a_2, b_1 > 0$ and (1.6) hold. Then there exists $C_1 > 0$ such that*

$$\|u\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} \leq C_1 \tag{3.1}$$

for all $t \in (0, T_{\max})$.

Proof By a straightforward computation and Lemma 2.2, we deduce from $a_2 > 0$ and Young's inequality that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u^2 + \int_\Omega u^2 \\ &= -2\int_\Omega |\nabla u|^2 - \xi_1 \int_\Omega u^2(v - w) + \chi_1 \int_\Omega u^2(z - u) \\ &\quad + \int_\Omega u^2 \Big((2a_0 + 1) - 2a_1 u - 2a_2 w + 2a_3 \int_\Omega u + 2a_4 \int_\Omega w \Big) \\ &\leq -2\int_\Omega |\nabla u|^2 + \xi_1 \int_\Omega u^2 w + \chi_1 \int_\Omega u^2 z + 2\Big(a_0 + 1 + 2a_5 M_0\Big) \int_\Omega u^2 - 2a_1 \int_\Omega u^3 \\ &\leq -2\int_\Omega |\nabla u|^2 + \Big(\xi_1 + \chi_1\Big) \int_\Omega u^3 + \xi_1 \int_\Omega w^3 + \chi_1 \int_\Omega z^3 + C_2 \end{aligned} \tag{3.2}$$

for all $t \in (0, T_{\max})$, where $a_5 := \max\left\{(a_3)_+, (a_4)_+\right\}$ and $C_2 = \frac{(2a_0+1+4a_5M_0)^3}{27a_1^2}|\Omega|$. Similarly, we get

$$\frac{d}{dt}\int_\Omega w^2 + \int_\Omega w^2 \le -2\int_\Omega |\nabla w|^2 + \left(\xi_2 + \chi_2\right)\int_\Omega w^3 + \xi_2\int_\Omega u^3 + \chi_2\int_\Omega v^3 + C_3 \tag{3.3}$$

with $C_3 = \frac{(2b_0+1+4b_5M_0)^3}{27b_2^2}|\Omega|$, where we have used that $b_1 > 0$.

Combining (3.2) with (3.3), we get

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega u^2 + \int_\Omega w^2\right) + \left(\int_\Omega u^2 + \int_\Omega w^2\right)\\
&\le -2\int_\Omega |\nabla u|^2 + \left(\xi_1 + \xi_2 + \chi_1\right)\int_\Omega u^3 + \chi_1\int_\Omega z^3\\
&\quad - 2\int_\Omega |\nabla w|^2 + \left(\xi_1 + \xi_2 + \chi_2\right)\int_\Omega w^3 + \chi_2\int_\Omega v^3 + C_2 + C_3
\end{aligned} \tag{3.4}$$

for all $t \in (0, T_{\max})$.

Then the following proof of this lemma is similar to [14, Lemma 3.1]. Here we omit the details. The proof of Lemma 3.1 is complete. □

Lemma 3.2 *Let $\tau = 0$ and $\Omega \subseteq \mathbb{R}^n (n \ge 2)$ be a smoothly bounded domain. Suppose that $a_2, b_1 > 0$, (1.6) and (1.7) or (1.8) hold. Then for some $p > p_0$, there exists $C_4 > 0$ such that*

$$\|u\|_{L^p(\Omega)} + \|w\|_{L^p(\Omega)} \le C_4 \tag{3.5}$$

for all $t \in (0, T_{\max})$, where

$$p_0 := \max\left\{\frac{n}{2}, \frac{\xi_1+\xi_2-\chi_1}{\xi_1+\xi_2-\chi_1-a_1}, \frac{\xi_1+\xi_2-\chi_2}{\xi_1+\xi_2-\chi_2-b_2}\right\}.$$

Proof Multiplying the first equation in (1.1) by $pu^{p-1} (p > p_0 \ge 1)$ and integrating by parts over Ω, it follows from Young's inequality and Lemma 2.2 that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p\\
&= -\frac{4(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 - \xi_1(p-1)\int_\Omega u^p(v-w) + \chi_1(p-1)\int_\Omega u^p(z-u)\\
&\quad + p\int_\Omega u^p\left(a_0 - a_1u - a_2w + a_3\int_\Omega u + a_4\int_\Omega w\right) + \int_\Omega u^p\\
&\le -\frac{4(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \left(\xi_1(p-1) - a_2p\right)\int_\Omega u^pw + \chi_1(p-1)\int_\Omega u^pz\\
&\quad + \left((a_0 + 2a_5M_0)p + 1\right)\int_\Omega u^p - \left(a_1p + \chi_1(p-1)\right)\int_\Omega u^{p+1}
\end{aligned} \tag{3.6}$$

for all $t \in (0, T_{\max})$, where $a_5 := \max\{(a_3)_+, (a_4)_+\}$. Using Lemma 2.2, Lemma 2.4 and Young's inequality, there exists a positive constant C_5 such that

$$\begin{aligned} \Big((a_0+2a_5M_0)p+1\Big)\int_\Omega u^p &= \Big((a_0+2a_5M_0)p+1\Big)\|u^{\frac{p}{2}}\|^2_{L^2(\Omega)} \\ &\le C_5\Big(\|\nabla u^{\frac{p}{2}}\|^{2\alpha_1}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{2-2\alpha_1}_{L^{\frac{2}{p}}(\Omega)} + \|u^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\Big) \\ &= C_5M_0^{p(1-\alpha_1)}\Big(\int_\Omega |\nabla u^{\frac{p}{2}}|^2\Big)^{\alpha_1} + C_5M_0^p \\ &\le \frac{4(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + C_6 \end{aligned} \tag{3.7}$$

with $C_6 = (1-\alpha_1)C_5^{\frac{1}{1-\alpha_1}}\left(\frac{p\alpha_1}{4(p-1)}\right)^{\frac{\alpha_1}{1-\alpha_1}} + C_5M_0^p > 0$, where $\alpha_1 = \dfrac{\frac{np}{2}-\frac{n}{2}}{1-\frac{n}{2}+\frac{np}{2}} \in (0,1)$ because of $p > 1$. Inserting (3.7) into (3.6), one has

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p &\le \Big(\xi_1(p-1)-a_2p\Big)\int_\Omega u^p w + \chi_1(p-1) \\ &\times \int_\Omega u^p z - \Big(a_1p+\chi_1(p-1)\Big)\int_\Omega u^{p+1} + C_6 \end{aligned} \tag{3.8}$$

for all $t \in (0, T_{\max})$.

Similarly, there is a positive constant C_7 such that

$$\begin{aligned} \frac{d}{dt}\int_\Omega w^p + \int_\Omega w^p &\le \Big(\xi_2(p-1)-b_1p\Big)\int_\Omega w^p u + \chi_2(p-1) \\ &\times \int_\Omega w^p v - \Big(b_2p+\chi_2(p-1)\Big)\int_\Omega w^{p+1} + C_7 \end{aligned} \tag{3.9}$$

for all $t \in (0, T_{\max})$.

By (3.8) and (3.9), we have

$$\begin{aligned} &\frac{d}{dt}\Big(\int_\Omega u^p + \int_\Omega w^p\Big) + \Big(\int_\Omega u^p + \int_\Omega w^p\Big) \\ &\le \Big(\xi_1(p-1)-a_2p\Big)\int_\Omega u^p w + \chi_1(p-1)\int_\Omega u^p z \\ &- \Big(a_1p+\chi_1(p-1)\Big)\int_\Omega u^{p+1} \\ &+ \Big(\xi_2(p-1)-b_1p\Big)\int_\Omega w^p u + \chi_2(p-1)\int_\Omega w^p v \\ &- \Big(b_2p+\chi_2(p-1)\Big)\int_\Omega w^{p+1} + C_8 \end{aligned} \tag{3.10}$$

for all $t \in (0, T_{\max})$.

Case 1: $\xi_1+\xi_2 < \min\{\chi_1+a_1, \chi_2+b_2\}$.

It follows from (1.7) that

$$p > \max\left\{\frac{\xi_1+\xi_2-\chi_1}{\xi_1+\xi_2-\chi_1-a_1}, \frac{\xi_1+\xi_2-\chi_2}{\xi_1+\xi_2-\chi_2-b_2}\right\},$$

which implies that the constants $K_1 := a_1 p + \chi_1(p-1) - \big(\xi_1 + \xi_2\big)(p-1)$ and $K_2 := b_2 p + \chi_2(p-1) - \big(\xi_1 + \xi_2\big)(p-1)$ are positive. Then we make use of the method in [14, Lemma 3.2] to obtain (3.5). Here we omit the details.

Case 2: $\xi_1 < \frac{na_2}{n-2}, \xi_2 < \frac{nb_1}{n-2}$.

It follows from the condition (1.8) with $n \geq 2$ that the definitions

$$I_1 = \left(\frac{n}{2}, \frac{\xi_1}{(\xi_1 - a_2)_+}\right), I_2 = \left(\frac{n}{2}, \frac{\xi_2}{(\xi_2 - b_1)_+}\right)$$

are well-defined. Then we fix $p \in I_1 \cap I_2$, which implies $\xi_1(p-1) - a_2 p < 0$ and $\xi_2(p-1) - b_1 p < 0$. By (3.10) and Young's inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega u^p + \int_\Omega w^p\right) + \left(\int_\Omega u^p + \int_\Omega w^p\right)\\
&\leq \chi_1(p-1)\int_\Omega u^p z - \chi_1(p-1)\int_\Omega u^{p+1} - a_1 p\int_\Omega u^{p+1}\\
&\quad + \chi_2(p-1)\int_\Omega w^p v - \chi_2(p-1)\int_\Omega w^{p+1} - b_2 p\int_\Omega w^{p+1} + C_8\\
&\leq \chi_1(p-1)\int_\Omega u^{p+1} + \chi_1(p-1)\int_\Omega z^{p+1} - \chi_1(p-1)\int_\Omega u^{p+1} - a_1 p\int_\Omega u^{p+1}\\
&\quad + \chi_2(p-1)\int_\Omega w^{p+1} + \chi_2(p-1)\int_\Omega v^{p+1}\\
&\quad - \chi_2(p-1)\int_\Omega w^{p+1} - b_2 p\int_\Omega w^{p+1} + C_8\\
&= \chi_1(p-1)\int_\Omega z^{p+1} - a_1 p\int_\Omega u^{p+1}\\
&\quad + \chi_2(p-1)\int_\Omega v^{p+1} - b_2 p\int_\Omega w^{p+1} + C_8.
\end{aligned} \tag{3.11}$$

With aids of the Agmon-Douglis-Nirenberg L^p estimates (see [1, 2]) and (3.20) of [14], then for some $p > \frac{n}{2}$, there exist some positive constants C_9 and C_{10} such that

$$\chi_1(p-1)\int_\Omega z^{p+1} \leq a_1 p\int_\Omega u^{p+1} + C_9 \tag{3.12}$$

and

$$\chi_2(p-1)\int_\Omega v^{p+1} \leq b_2 p\int_\Omega w^{p+1} + C_{10} \tag{3.13}$$

for all $t \in (0, T_{\max})$.

Combining (3.11)–(3.13) yields

$$\frac{d}{dt}\left(\int_\Omega u^p + \int_\Omega w^p\right) + \left(\int_\Omega u^p + \int_\Omega w^p\right) \leq C_{11} \tag{3.14}$$

with $C_{11} = C_8 + C_9 + C_{10}$. Then according to Lemma 2.3, one obtains

$$\int_\Omega u^p + \int_\Omega w^p \leq \max\left\{\int_\Omega u_0^p + \int_\Omega w_0^p, C_{11}\right\}$$

for all $t \in (0, T_{\max})$ and some $p > \frac{n}{2}$, which implies (3.5). The proof of Lemma 3.2 is complete. □

Lemma 3.3 *Let $\tau = 0$ and $\Omega \subseteq \mathbb{R}^n (n \geq 1)$ be a smoothly bounded domain. Assume that $a_2, b_1 > 0$, (1.6) and (1.7) or (1.8) hold. Then for all nonnegative initial data (u_0, w_0) satisfying (1.3), there exists $C_{12} > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} \leq C_{12} \tag{3.15}$$

for all $t \in (0, T_{\max})$.

Proof When $n = 1$, relying on Lemma 3.1 and the method in [51, Lemma 3.6], we can obtain that (3.15) holds. When $n \geq 2$, it follows from Lemma 3.2 and the Moser-Alikakos iteration in [3] (or [35, Lemma A.1]) that (3.15) holds. The proof of Lemma 3.3 is complete. □

Proof of Theorem 1.1 By Lemma 3.3, we obtain the boundedness of $\|u\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)}$ for all $t \in (0, T_{\max})$. By the well-known elliptic maximum principle and global boundedness of (u, w), we derive that (v, z) is bounded in $W^{1,q}(\Omega)(q > n)$. Hence it follows from Lemma 2.1 that $T_{\max} = \infty$. The proof of Theorem 1.1 is complete. □

4 Asymptotic Behavior for $\tau = 0$

In this section, we prove Theorems 1.2 and 1.3 by constructing some suitable energy functionals. To do this, we need the following key lemmas.

Lemma 4.1 (See [4, Lemma 3.1]) *Let $f(t) : (1, \infty) \to \mathbb{R}$ be a nonnegative and uniformly continuous function that satisfies $\int_1^\infty f(t)dt < \infty$. Then $f(t) \to 0$ as $t \to \infty$.*

4.1 Proof of Theorem 1.2

Lemma 4.2 *Suppose that the conditions of Theorem 1.2 hold. Then there exists $\delta_1 > 0$ such that*

$$\frac{d}{dt}\mathcal{F}_1(t) \leq -\delta_1 \int_\Omega \left((u-u_*)^2 + (v-v_*)^2 + (w-w_*)^2 + (z-z_*)^2\right) \tag{4.1}$$

for all $t > 0$, where

$$\mathcal{F}_1(t) := \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) + \int_\Omega \left(w - w_* - w_* \ln \frac{w}{w_*}\right)$$

and (u_*, v_*, w_*, z_*) is given by (1.10). Moreover, we have

$$\int_0^\infty \int_\Omega (u-u_*)^2 + \int_0^\infty \int_\Omega (v-v_*)^2 + \int_0^\infty \int_\Omega (w-w_*)^2 + \int_0^\infty \int_\Omega (z-z_*)^2 < \infty.$$

Proof Letting

$$\mathcal{A}_1(t) = \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) \text{ and } \mathcal{B}_1(t) = \int_\Omega \left(w - w_* - w_* \ln \frac{w}{w_*}\right), \tag{4.2}$$

then $\mathcal{F}_1(t)$ can be rewritten as

$$\mathcal{F}_1(t) = \mathcal{A}_1(t) + \mathcal{B}_1(t) \tag{4.3}$$

for all $t > 0$.

Firstly, by (5.4) of [14], we get the nonnegativity of $\mathcal{F}_1(t)$ for all $t \geq 0$.

Now, we will prove (4.1). By a simple calculation with (1.1) that

$$\begin{aligned} \frac{d}{dt}\mathcal{A}_1(t) &= \frac{d}{dt}\int_\Omega \left(u - u_* - u_* \ln\frac{u}{u_*}\right) \\ &= \int_\Omega \left(1 - \frac{u_*}{u}\right)\left(\Delta u - \xi_1\nabla\cdot(u\nabla v) + \chi_1\nabla\cdot(u\nabla z)\right) \\ &\quad + \int_\Omega (u-u_*)\left(a_0 - a_1 u - a_2 w + a_3\int_\Omega u + a_4\int_\Omega w\right) \\ &= -u_*\int_\Omega \left|\frac{\nabla u}{u}\right|^2 + \xi_1 u_*\int_\Omega \frac{\nabla u}{u}\cdot\nabla v - \chi_1 u_*\int_\Omega \frac{\nabla u}{u}\cdot\nabla z \\ &\quad + \int_\Omega (u-u_*)\left(a_0 - a_1 u - a_2 w + a_3\int_\Omega u + a_4\int_\Omega w\right). \end{aligned} \tag{4.4}$$

It follows from Hölder's and Young's inequalities that

$$\begin{aligned} &\int_\Omega (u-u_*)\left(a_0 - a_1 u - a_2 w + a_3\int_\Omega u + a_4\int_\Omega w\right) \\ &= \int_\Omega (u-u_*)\left(a_1(u_*-u) + a_2(w_*-w) - a_3\int_\Omega (u_*-u) - a_4\int_\Omega (w_*-w)\right) \\ &= -a_1\int_\Omega (u-u_*)^2 - a_2\int_\Omega (u-u_*)(w-w_*) \\ &\quad + a_3\left(\int_\Omega (u-u_*)\right)^2 + a_4\int_\Omega (u-u_*)\cdot\int_\Omega (w-w_*) \\ &\le -\left(a_1 + \frac{a_4}{2}|\Omega|\right)\int_\Omega (u-u_*)^2 - a_2\int_\Omega (u-u_*)(w-w_*) - \frac{a_4}{2}|\Omega|\int_\Omega (w-w_*)^2 \end{aligned}$$

due to $a_3, a_4 < 0$ and $a_0 = (a_1 - a_3|\Omega|)u_* + (a_2 - a_4|\Omega|)w_*$. Therefore, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{A}_1(t) \le& -u_*\int_\Omega \left|\frac{\nabla u}{u}\right|^2 + \xi_1 u_*\int_\Omega \frac{\nabla u}{u}\cdot\nabla v - \chi_1 u_*\int_\Omega \frac{\nabla u}{u}\cdot\nabla z \\ &- \left(a_1 + \frac{a_4}{2}|\Omega|\right)\int_\Omega (u-u_*)^2 - a_2\int_\Omega (u-u_*)(w-w_*) \\ &- \frac{a_4}{2}|\Omega|\int_\Omega (w-w_*)^2. \end{aligned} \tag{4.5}$$

Making use of the similar method for $\mathcal{B}_1(t)$, we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{B}_1(t) \le& -w_*\int_\Omega \left|\frac{\nabla w}{w}\right|^2 + \xi_2 w_*\int_\Omega \frac{\nabla w}{w}\cdot\nabla z - \chi_2 w_*\int_\Omega \frac{\nabla w}{w}\cdot\nabla v \\ &- \left(b_2 + \frac{b_3}{2}|\Omega|\right)\int_\Omega (w-w_*)^2 - b_1\int_\Omega (u-u_*)(w-w_*) \\ &- \frac{b_3}{2}|\Omega|\int_\Omega (u-u_*)^2 \end{aligned} \tag{4.6}$$

because of $b_0 = (b_1 - b_3|\Omega|)u_* + (b_2 - b_4|\Omega|)w_*$ and $b_3, b_4 < 0$.

Collecting (4.3), (4.5) and (4.6) yields

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) \leq & -u_*\int_\Omega\left|\frac{\nabla u}{u}\right|^2 + \xi_1 u_*\int_\Omega \frac{\nabla u}{u}\cdot\nabla v - \chi_1 u_*\int_\Omega\frac{\nabla u}{u}\cdot\nabla z \\ & - w_*\int_\Omega\left|\frac{\nabla w}{w}\right|^2 + \xi_2 w_*\int_\Omega\frac{\nabla w}{w}\cdot\nabla z - \chi_2 w_*\int_\Omega\frac{\nabla w}{w}\cdot\nabla v \\ & - \left(a_1 + \frac{a_4+b_3}{2}|\Omega|\right)\int_\Omega (u-u_*)^2 - (a_2+b_1)\int_\Omega (u-u_*)(w-w_*) \\ & - \left(b_2 + \frac{a_4+b_3}{2}|\Omega|\right)\int_\Omega (w-w_*)^2.\end{aligned} \tag{4.7}$$

By Young's inequality, we derive

$$\begin{aligned}-\chi_1\int_\Omega\frac{\nabla u}{u}\cdot\nabla z &\leq \frac{1}{2}\int_\Omega\left|\frac{\nabla u}{u}\right|^2 + \frac{\chi_1^2}{2}\int_\Omega|\nabla z|^2, \\ \xi_1\int_\Omega\frac{\nabla u}{u}\cdot\nabla v &\leq \frac{1}{2}\int_\Omega\left|\frac{\nabla u}{u}\right|^2 + \frac{\xi_1^2}{2}\int_\Omega|\nabla v|^2\end{aligned}$$

and

$$-\chi_2\int_\Omega\frac{\nabla w}{w}\cdot\nabla v \leq \frac{1}{2}\int_\Omega\left|\frac{\nabla w}{w}\right|^2 + \frac{\chi_2^2}{2}\int_\Omega|\nabla v|^2$$

as well as

$$\xi_2\int_\Omega\frac{\nabla w}{w}\cdot\nabla z \leq \frac{1}{2}\int_\Omega\left|\frac{\nabla w}{w}\right|^2 + \frac{\xi_2^2}{2}\int_\Omega|\nabla z|^2.$$

Then,

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) \leq & -\left(a_1 + \frac{a_4+b_3}{2}|\Omega|\right)\int_\Omega (u-u_*)^2 - (a_2+b_1)\int_\Omega (u-u_*)(w-w_*) \\ & - \left(b_2 + \frac{a_4+b_3}{2}|\Omega|\right)\int_\Omega (w-w_*)^2 + \frac{1}{2}(\xi_1^2 u_* + \chi_2^2 w_*)\int_\Omega|\nabla v|^2 \\ & + \frac{1}{2}(\chi_1^2 u_* + \xi_2^2 w_*)\int_\Omega|\nabla z|^2.\end{aligned} \tag{4.8}$$

In view of $0 = \Delta v - v + w$ and $v_* = w_*$, we obtain

$$\int_\Omega|\nabla v|^2 = -\int_\Omega (v-v_*)^2 + \int_\Omega (v-v_*)(w-w_*). \tag{4.9}$$

Similarly, we have

$$\int_\Omega|\nabla z|^2 = -\int_\Omega (z-z_*)^2 + \int_\Omega (z-z_*)(u-u_*). \tag{4.10}$$

Together with (4.8)–(4.10), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) \le &-\left(a_1 + \frac{a_4+b_3}{2}|\Omega|\right)\int_\Omega (u-u_*)^2 - (a_2+b_1)\int_\Omega (u-u_*)(w-w_*)\\ &-\left(b_2 + \frac{a_4+b_3}{2}|\Omega|\right)\int_\Omega (w-w_*)^2 - \frac{1}{2}(\xi_1^2 u_* + \chi_2^2 w_*)\int_\Omega (v-v_*)^2\\ &+\frac{1}{2}(\xi_1^2 u_* + \chi_2^2 w_*)\int_\Omega (v-v_*)(w-w_*) - \frac{1}{2}(\chi_1^2 u_* + \xi_2^2 w_*)\int_\Omega (z-z_*)^2\\ &+\frac{1}{2}(\chi_1^2 u_* + \xi_2^2 w_*)\int_\Omega (z-z_*)(u-u_*).\end{aligned} \tag{4.11}$$

Since the conditions (1.12) and (1.13) hold, then the constants

$$\mathcal{K}_1 := a_1 + \frac{a_4+b_3}{2}|\Omega| \quad\text{and}\quad \mathcal{K}_2 := b_2 + \frac{a_4+b_3}{2}|\Omega|$$

are positive, and there exist $\theta_1, \theta_2 \in (0,1)$ such that

$$4\theta_1\theta_2\mathcal{K}_1\mathcal{K}_2 > (a_2+b_1)^2, \tag{4.12}$$

$$\mathcal{K}_1 > \frac{u_*\chi_1^2 + w_*\xi_2^2}{8(1-\theta_1)} \tag{4.13}$$

and

$$\mathcal{K}_2 > \frac{u_*\xi_1^2 + w_*\chi_2^2}{8(1-\theta_2)}. \tag{4.14}$$

By (4.12)–(4.14) and $u_*, w_* > 0$, one can find some $\delta_1 > 0$ to satisfy

$$\begin{aligned}\delta_1 \le \min\Bigg\{&\frac{4\theta_1\theta_2\mathcal{K}_1\mathcal{K}_2 - (a_2+b_1)^2}{\theta_1\theta_2(\mathcal{K}_1+\mathcal{K}_2)}, \frac{8(1-\theta_1)\mathcal{K}_1(u_*\chi_1^2 + w_*\xi_2^2) - (u_*\chi_1^2 + w_*\xi_2^2)^2}{8(1-\theta_1)(2\mathcal{K}_1 + u_*\chi_1^2 + w_*\xi_2^2)},\\ &\frac{8(1-\theta_2)\mathcal{K}_2(u_*\xi_1^2 + w_*\chi_2^2) - (u_*\xi_1^2 + w_*\chi_2^2)^2}{8(1-\theta_2)(2\mathcal{K}_2 + u_*\xi_1^2 + w_*\chi_2^2)}, \mathcal{K}_1, \mathcal{K}_2\Bigg\}.\end{aligned} \tag{4.15}$$

Therefore, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) \le &-\delta_1\int_\Omega \left((u-u_*)^2 + (v-v_*)^2 + (w-w_*)^2 + (z-z_*)^2\right)\\ &+\int_\Omega \left(y_1 + y_2 + y_3\right),\end{aligned} \tag{4.16}$$

where

$$y_1 = -\theta_1(\mathcal{K}_1 - \delta_1)(u-u_*)^2 - (a_2+b_1)(u-u_*)(w-w_*) - \theta_2(\mathcal{K}_2 - \delta_1)(w-w_*)^2,$$

$$\begin{aligned}y_2 = &-(1-\theta_1)(\mathcal{K}_1 - \delta_1)(u-u_*)^2 + \frac{u_*\chi_1^2 + w_*\xi_2^2}{2}(u-u_*)(z-z_*)\\ &-\left(\frac{u_*\chi_1^2 + w_*\xi_2^2}{2} - \delta_1\right)(z-z_*)^2,\end{aligned}$$

$$\begin{aligned}y_3 = &-(1-\theta_2)(\mathcal{K}_2 - \delta_1)(w-w_*)^2 + \frac{u_*\xi_1^2 + w_*\chi_2^2}{2}(w-w_*)(v-v_*)\\ &-\left(\frac{u_*\xi_1^2 + w_*\chi_2^2}{2} - \delta_1\right)(v-v_*)^2.\end{aligned}$$

For each the discriminant of $y_i (i = 1, 2, 3)$, we dedure from (4.15) that

$$\begin{aligned}\triangle_1 &= (a_2 + b_1)^2 (w - w_*)^2 - 4\theta_1\theta_2(\mathcal{K}_1 - \delta_1)(\mathcal{K}_2 - \delta_1)(w - w_*)^2 \le 0,\\ \triangle_2 &= \frac{(u_*\chi_1^2 + w_*\xi_2^2)^2}{4}(z - z_*)^2\\ &\quad - 4(1-\theta_1)(\mathcal{K}_1 - \delta_1)\left(\frac{u_*\chi_1^2 + w_*\xi_2^2}{2} - \delta_1\right)(z - z_*)^2 \le 0,\\ \triangle_3 &= \frac{(u_*\xi_1^2 + w_*\chi_2^2)^2}{4}(v - v_*)^2\\ &\quad - 4(1-\theta_2)(\mathcal{K}_2 - \delta_1)\left(\frac{u_*\xi_1^2 + w_*\chi_2^2}{2} - \delta_1\right)(v - v_*)^2 \le 0,\end{aligned} \tag{4.17}$$

which concludes

$$y_i \le 0, i = 1, 2, 3. \tag{4.18}$$

By (4.16) and (4.18), we directly obtain (4.1). Finally, integrating (4.1) over $(0, \infty)$, we get

$$\int_0^\infty \int_\Omega (u - u_*)^2 + \int_0^\infty \int_\Omega (v - v_*)^2 + \int_0^\infty \int_\Omega (w - w_*)^2 + \int_0^\infty \int_\Omega (z - z_*)^2 < \infty.$$

The proof of Lemma 4.2 is complete. □

Lemma 4.3 *Suppose that the conditions of Theorem 1.2 hold. Then the solution (u, v, w, z) to (1.1) satisfies*

$$\|u - u_*\|_{L^\infty(\Omega)} + \|v - v_*\|_{L^\infty(\Omega)} + \|w - w_*\|_{L^\infty(\Omega)} + \|z - z_*\|_{L^\infty(\Omega)} \to 0 \tag{4.19}$$

as $t \to \infty$.

Proof A combination of Lemmas 4.1 and 4.2 implies this lemma. The proof of Lemma 4.3 is complete. □

Lemma 4.4 *Suppose that the conditions of Theorem 1.2 hold. Then there exist $C_1 > 0$ and $\lambda_1 > 0$ such that the solution (u, v, w, z) to (1.1) satisfies*

$$\|u - u_*\|_{L^\infty(\Omega)} + \|v - v_*\|_{L^\infty(\Omega)} + \|w - w_*\|_{L^\infty(\Omega)} + \|z - z_*\|_{L^\infty(\Omega)} \le C_1 e^{-\lambda_1 t} \tag{4.20}$$

for all $t > t_1$, where $t_1 > 0$ is some fixed time.

Proof This idea of proof is similar to [4, Lemma 3.7].

Set $g_1(s) = s - u_* \ln s$ and $g_2(s) = s - w_* \ln s$ for $s > 0$. By L'Hôpital's rule, we have

$$\lim_{s\to u_*} \frac{g_1(s) - g_1(u_*)}{(s - u_*)^2} = \frac{1}{2u_*} \tag{4.21}$$

and

$$\lim_{s\to w_*} \frac{g_2(s) - g_2(w_*)}{(s - w_*)^2} = \frac{1}{2w_*}. \tag{4.22}$$

By Lemma 4.3, one can see that there exist some $t_1 > 0$ and $C_2, C_3 > 0$ such that

$$C_2 \int_\Omega (u - u_*)^2 \le \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*} \right) \le C_3 \int_\Omega (u - u_*)^2 \tag{4.23}$$

and

$$C_2 \int_\Omega (w - w_*)^2 \le \int_\Omega \left(w - w_* - w_* \ln \frac{w}{w_*} \right) \le C_3 \int_\Omega (w - w_*)^2 \tag{4.24}$$

for all $t > t_1$.

By means of the definitions of $\mathcal{F}_1(t)$, it follows from the second inequalities in (4.23) and (4.24) that there exists $C_4 > 0$ such that

$$C_4 \mathcal{F}_1(t) \le \int_\Omega \left((u - u_*)^2 + (v - v_*)^2 + (w - w_*)^2 + (z - z_*)^2 \right) \tag{4.25}$$

for all $t > t_1$. With the aid of Lemma 4.2, we get

$$\begin{aligned} \frac{d}{dt} \mathcal{F}_1(t) \le -\delta_1 \int_\Omega \Big((u - u_*)^2 + (v - v_*)^2 \\ + (w - w_*)^2 + (z - z_*)^2 \Big) \le -C_4 \delta_1 \mathcal{F}_1(t), \end{aligned} \tag{4.26}$$

which implies

$$\mathcal{F}_1(t) \le \mathcal{F}_1(t_1) e^{-C_4 \delta_1 (t - t_1)} \tag{4.27}$$

for all $t > t_1$. Then combining (4.27) with the first inequalities in (4.23) and (4.24), there exists $C_5 > 0$ such that

$$\int_\Omega \left((u - u_*)^2 + (w - w_*)^2 \right) \le C_5 \mathcal{F}_1(t) \le C_5 \mathcal{F}_1(t_1) e^{-C_4 \delta_1 (t - t_1)} \tag{4.28}$$

for all $t > t_1$.

By applying the Gagliardo-Nirenberg inequality and (1.15), there exist some positive constants C_6, C_7 and C_8 such that

$$\begin{aligned} &\|u - u_*\|_{L^\infty(\Omega)} + \|w - w_*\|_{L^\infty(\Omega)} \\ &\le C_6 \|u - u_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|u - u_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} + C_6 \|w - w_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|w - w_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} \\ &\le C_7 \|u - u_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} + C_7 \|w - w_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} \\ &\le C_8 \left(\|u - u_*\|_{L^2(\Omega)}^2 + \|w - w_*\|_{L^2(\Omega)}^2 \right)^{\frac{1}{n+2}} \\ &\le C_8 \big(C_5 \mathcal{F}_1(t_1) \big)^{\frac{1}{n+2}} e^{-\frac{C_4 \delta_1 (t - t_1)}{n+2}} \end{aligned} \tag{4.29}$$

for all $t > t_1$.

By the application of the elliptic maximum principle and (4.29), we get

$$\begin{aligned} &\|v - v_*\|_{L^\infty(\Omega)} + \|z - z_*\|_{L^\infty(\Omega)} \\ &\le \|w - w_*\|_{L^\infty(\Omega)} + \|u - u_*\|_{L^\infty(\Omega)} \\ &\le C_8 \big(C_5 \mathcal{F}_1(t_1) \big)^{\frac{1}{n+2}} e^{-\frac{C_4 \delta_1 (t - t_1)}{n+2}} \end{aligned} \tag{4.30}$$

where we have used that $\tau = 0$ and $v_* = w_*$, $z_* = u_*$. The proof of Lemma 4.4 is complete. □

Proof of Theorem 1.2 Lemma 4.4 directly shows the results of Theorem 1.2. □

4.2 Proof of Theorem 1.3

Now, we introduce the following functionals

$$\mathcal{F}_2(t) := \int_\Omega \left(u - u_\star - u_\star \ln \frac{u}{u_\star}\right) + \int_\Omega w,$$

where $(u_\star, z_\star)$ is given by (1.16).

Lemma 4.5 *Let (u, v, w, z) be a global bounded classical solution to (1.1). Suppose that the conditions of Theorem 1.3 hold. Then there exists $\delta_2 > 0$ such that*

$$\frac{d}{dt}\mathcal{F}_2(t) \le -\delta_2 \int_\Omega \left((u-u_\star)^2 + v^2 + w^2 + (z - z_\star)^2\right) \\ +(b_0 - b_1 u_\star + b_3 u_\star |\Omega|) \int_\Omega w \tag{4.31}$$

for all $t > 0$. Moreover, we have

$$\int_0^\infty \int_\Omega (u-u_\star)^2 + \int_0^\infty \int_\Omega v^2 + \int_0^\infty \int_\Omega w^2 + \int_0^\infty \int_\Omega (z - z_\star)^2 < \infty.$$

Proof This proof is similar to Lemma 4.2. For reader's convenience, we give the sketch of the proof. Firstly, $\mathcal{F}_2(t)$ can be rewritten as

$$\mathcal{F}_2(t) = \mathcal{A}_2(t) + \mathcal{B}_2(t), \tag{4.32}$$

where $\mathcal{A}_2(t) := \int_\Omega \left(u - u_\star - u_\star \ln \frac{u}{u_\star}\right)$ and $\mathcal{B}_2(t) := \int_\Omega w$. Similarly, we obtain the nonnegativity of $\mathcal{F}_2(t)$ by (5.4) of [14].

Next, we will prove (4.31). It follows from Young's and Hölder's inequalities that

$$\begin{aligned} \frac{d}{dt}\mathcal{A}_2(t) &= -u_\star \int_\Omega \left|\frac{\nabla u}{u}\right|^2 + \xi_1 u_\star \int_\Omega \frac{\nabla u}{u} \cdot \nabla v - \chi_1 u_\star \int_\Omega \frac{\nabla u}{u} \cdot \nabla z \\ &\quad + \int_\Omega (u - u_\star)\left(a_0 - a_1 u - a_2 w + a_3 \int_\Omega u + a_4 \int_\Omega w\right) \\ &\le \frac{u_\star \xi_1^2}{2} \int_\Omega |\nabla v|^2 + \frac{u_\star \chi_1^2}{2} \int_\Omega |\nabla z|^2 \\ &\quad + \int_\Omega (u - u_\star)\left(a_1 u_\star - a_3 u_\star |\Omega| - a_1 u - a_2 w + a_3 \int_\Omega u + a_4 \int_\Omega w\right) \\ &\le \frac{u_\star \xi_1^2}{2} \int_\Omega |\nabla v|^2 + \frac{u_\star \chi_1^2}{2} \int_\Omega |\nabla z|^2 - \left(a_1 + \frac{a_4}{2}|\Omega|\right) \int_\Omega (u - u_\star)^2 \\ &\quad - a_2 \int_\Omega (u - u_\star) w - \frac{a_4}{2}|\Omega| \int_\Omega w^2, \end{aligned} \tag{4.33}$$

where we have used that $a_0 = a_1 u_\star - a_3 u_\star |\Omega|$ and $a_3, a_4 < 0$.

Based on the third equation of (1.1), we deduce from $b_3, b_4 < 0$, Hölder's and Young's inequalities that

$$\begin{aligned}\frac{d}{dt}\mathcal{B}_2(t) &= \int_\Omega w\left(b_0 - b_1 u - b_2 w + b_3\int_\Omega u + b_4\int_\Omega w\right)\\ &= \int_\Omega w\left(b_0 - b_1(u-u_\star) - b_1u_\star - b_2 w + b_3\int_\Omega (u-u_\star) + b_3u_\star|\Omega| + b_4\int_\Omega w\right)\\ &\le (b_0 - b_1u_\star + b_3u_\star|\Omega|)\int_\Omega w - \left(b_2 + \frac{b_3}{2}|\Omega|\right)\int_\Omega w^2 - b_1\int_\Omega (u-u_\star)w\\ &\quad - \frac{b_3}{2}|\Omega|\int_\Omega (u-u_\star)^2.\end{aligned} \tag{4.34}$$

Combining (4.33) with (4.34) obtains

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t) \le{}& \frac{u_\star\xi_1^2}{2}\int_\Omega |\nabla v|^2 + \frac{u_\star\chi_1^2}{2}\int_\Omega |\nabla z|^2 - \left(a_1 + \frac{a_4}{2}|\Omega| + \frac{b_3}{2}|\Omega|\right)\int_\Omega (u-u_\star)^2\\ &- (a_2+b_1)\int_\Omega (u-u_\star)w - \left(b_2 + \frac{a_4}{2}|\Omega| + \frac{b_3}{2}|\Omega|\right)\int_\Omega w^2\\ &+ (b_0 - b_1u_\star + b_3u_\star|\Omega|)\int_\Omega w.\end{aligned}$$

Then making once again use of (4.9) and (4.10), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t) \le{}& -\frac{u_\star\xi_1^2}{2}\int_\Omega v^2 + \frac{u_\star\xi_1^2}{2}\int_\Omega vw - \frac{u_\star\chi_1^2}{2}\int_\Omega (z-z_\star)^2\\ &+ \frac{u_\star\chi_1^2}{2}\int_\Omega (z-z_\star)(u-u_\star) - \left(a_1 + \frac{a_4}{2}|\Omega| + \frac{b_3}{2}|\Omega|\right)\int_\Omega (u-u_\star)^2\\ &- (a_2+b_1)\int_\Omega (u-u_\star)w - \left(b_2 + \frac{a_4}{2}|\Omega| + \frac{b_3}{2}|\Omega|\right)\int_\Omega w^2\\ &+ (b_0 - b_1u_\star + b_3u_\star|\Omega|)\int_\Omega w.\end{aligned} \tag{4.35}$$

It follows from (1.17) and (1.18) that $\mathcal{K}_3 := a_1 + \frac{a_4}{2}|\Omega| + \frac{b_3}{2}|\Omega|$, $\mathcal{K}_4 := b_2 + \frac{a_4}{2}|\Omega| + \frac{b_3}{2}|\Omega|$ are positive, and there exist $\theta_3, \theta_4 \in (0, 1)$ such that

$$4\theta_3\theta_4\mathcal{K}_3\mathcal{K}_4 > (a_2+b_1)^2 \tag{4.36}$$

and

$$\mathcal{K}_3 > \frac{u_\star\chi_1^2}{8(1-\theta_3)} \tag{4.37}$$

as well as

$$\mathcal{K}_4 > \frac{u_\star\xi_1^2}{8(1-\theta_4)}. \tag{4.38}$$

According to (4.36)–(4.38) and $u_\star, w_\star > 0$, one can find some $\delta_2 > 0$ to satisfy

$$\delta_2 \leq \min\left\{\frac{4\theta_3\theta_4\mathcal{K}_3\mathcal{K}_4-(a_2+b_1)^2}{\theta_3\theta_4(\mathcal{K}_3+\mathcal{K}_4)}, \frac{8(1-\theta_3)\mathcal{K}_3u_\star\chi_1^2-u_\star^2\chi_1^4}{8(1-\theta_3)(2\mathcal{K}_3+u_\star\chi_1^2)},\right. \\ \left.\frac{8(1-\theta_4)\mathcal{K}_4u_\star\xi_1^2-u_\star^2\xi_1^4}{8(1-\theta_4)(2\mathcal{K}_4+u_\star\xi_1^2)}, \mathcal{K}_3, \mathcal{K}_4\right\}. \tag{4.39}$$

Therefore, we get

$$\frac{d}{dt}\mathcal{F}_2(t) \leq -\delta_2\int_\Omega\left((u-u_\star)^2+v^2+w^2+(z-z_\star)^2\right)+(b_0-b_1u_\star+b_3u_\star|\Omega|)\int_\Omega w \\ +\int_\Omega\left(h_1+h_2+h_3\right), \tag{4.40}$$

where

$$h_1 = -\theta_3(\mathcal{K}_3-\delta_2)(u-u_\star)^2-(a_2+b_1)(u-u_\star)w-\theta_4(\mathcal{K}_4-\delta_2)w^2,$$
$$h_2 = -(1-\theta_3)(\mathcal{K}_3-\delta_2)(u-u_\star)^2+\frac{u_\star\chi_1^2}{2}(u-u_\star)(z-z_\star)-\left(\frac{u_\star\chi_1^2}{2}-\delta_2\right)(z-z_\star)^2,$$
$$h_3 = -(1-\theta_4)(\mathcal{K}_4-\delta_2)w^2+\frac{u_\star\xi_1^2}{2}vw-\left(\frac{u_\star\xi_1^2}{2}-\delta_2\right)v^2.$$

For each the discriminant of $h_i\,(i=1,2,3)$ and by (4.39), we have

$$\triangle_1 = (a_2+b_1)^2w^2-4\theta_3\theta_4(\mathcal{K}_1-\delta_2)(\mathcal{K}_2-\delta_2)w^2 \leq 0, \\ \triangle_2 = \frac{u_\star^2\chi_1^4}{4}(z-z_\star)^2-4(1-\theta_3)(\mathcal{K}_1-\delta_2)\left(\frac{u_\star\chi_1^2}{2}-\delta_2\right)(z-z_\star)^2 \leq 0, \\ \triangle_3 = \frac{u_\star^2\xi_1^4}{4}v^2-4(1-\theta_4)(\mathcal{K}_2-\delta_2)\left(\frac{u_\star\xi_1^2}{2}-\delta_2\right)v^2 \leq 0, \tag{4.41}$$

which concludes

$$h_i \leq 0, i = 1, 2, 3. \tag{4.42}$$

By (4.40) and (4.42), we directly obtain (4.31). Moreover, we get

$$\int_0^\infty\int_\Omega(u-u_\star)^2+\int_0^\infty\int_\Omega v^2+\int_0^\infty\int_\Omega w^2+\int_0^\infty\int_\Omega(z-z_\star)^2 < \infty$$

by integrating (4.31) over $(0, \infty)$. The proof of Lemma 4.5 is complete. □

Lemma 4.6 *Suppose that the conditions of Theorem 1.3 hold. Then the solution (u, v, w, z) to (1.1) satisfies*

$$\|u-u_\star\|_{L^\infty(\Omega)}+\|v\|_{L^\infty(\Omega)}+\|w\|_{L^\infty(\Omega)}+\|z-z_\star\|_{L^\infty(\Omega)} \to 0 \tag{4.43}$$

as $t \to \infty$.

Proof A combination of Lemma 4.1 and Lemma 4.5 implies this lemma. The proof of Lemma 4.6 is complete. □

Lemma 4.7 *Let the conditions of Theorem 1.3 hold.*

(i) *If* $\frac{a_2-a_4|\Omega|}{b_2-b_4|\Omega|} < \frac{a_1-a_3|\Omega|}{b_1-b_3|\Omega|} < \frac{a_0}{b_0}$, *then there exist* $C_9 > 0$ *and* $\lambda_2 > 0$ *such that the solution* (u, v, w, z) *to (1.1) satisfies*

$$\|u-u_\star\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z-z_\star\|_{L^\infty(\Omega)} \le C_9 e^{-\lambda_2 t} \tag{4.44}$$

for all $t > t_2$, *where* $t_2 > 0$ *is some fixed time.*

(ii) *If* $\frac{a_2-a_4|\Omega|}{b_2-b_4|\Omega|} < \frac{a_1-a_3|\Omega|}{b_1-b_3|\Omega|} = \frac{a_0}{b_0}$, *then there exist* $C_{10} > 0$ *and* $\lambda_3 > 0$ *such that the solution* (u, v, w, z) *to (1.1) satisfies*

$$\|u-u_\star\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z-z_\star\|_{L^\infty(\Omega)} \le C_{10}(t-t_3)^{-\lambda_3} \tag{4.45}$$

for all $t > t_3$, *where* $t_3 > 0$ *is some fixed time.*

Proof (i) When $\frac{a_2-a_4|\Omega|}{b_2-b_4|\Omega|} < \frac{a_1-a_3|\Omega|}{b_1-b_3|\Omega|} < \frac{a_0}{b_0}$, by using Lemma 4.6 and a similar argument as in the proof of [25, Lemma 4.3], there exist some $t_2 > 0$ and $C_{11}, C_{12} > 0$ such that

$$\begin{aligned} C_{11}\int_\Omega \left((u-u_\star)^2 + v^2 + w^2 + (z-z_\star)^2 + w\right) &\le \mathcal{F}_2(t) \\ &\le C_{12}\int_\Omega \left((u-u_\star)^2 + v^2 + w^2 + (z-z_\star)^2 + w\right) \end{aligned} \tag{4.46}$$

for all $t > t_2$. Then we infer from Lemma 4.4 that there exists $C_{13} > 0$ such that

$$\frac{d}{dt}\mathcal{F}_2(t) \le -C_{13}\delta_1\mathcal{F}_2(t), \tag{4.47}$$

which implies

$$\mathcal{F}_2(t) \le \mathcal{F}_2(t_2)e^{-C_{13}\delta_1(t-t_2)} \tag{4.48}$$

for all $t > t_2$. Finally, by the same argument as in the proof of Lemma 4.4, we obtain (4.44).

(ii) When $\frac{a_2-a_4|\Omega|}{b_2-b_4|\Omega|} < \frac{a_1-a_3|\Omega|}{b_1-b_3|\Omega|} = \frac{a_0}{b_0}$, it follows from the definition of $\mathcal{F}_2(t)$, (4.23) and Hölder's inequality that there exist $t_3 > 0$ and $C_{14} > 0$ such that

$$\begin{aligned} \mathcal{F}_2(t) &\le \int_\Omega \left(C_3(u-u_\star)^2 + v^2 + w + (z-z_\star)^2\right) \\ &\le C_3\left(\int_\Omega (u-u_\star)^2\right)^{\frac12}\left(\int_\Omega (u-u_\star)^2\right)^{\frac12} + \left(\int_\Omega v^2\right)^{\frac12}\left(\int_\Omega v^2\right)^{\frac12} \\ &\quad + |\Omega|^{\frac12}\left(\int_\Omega w^2\right)^{\frac12} + \left(\int_\Omega (z-z_\star)^2\right)^{\frac12}\left(\int_\Omega (z-z_\star)^2\right)^{\frac12} \\ &\le C_{14}\left(\int_\Omega (u-u_\star)^2 + \int_\Omega v^2 + \int_\Omega w^2 + \int_\Omega (z-z_\star)^2\right) \end{aligned} \tag{4.49}$$

for all $t > t_3$, where we have used the boundedness of solution (u, v, w, z). Thus we duduce from Lemma 4.5 that there exists $C_{15} > 0$ such that

$$\frac{d}{dt}\mathcal{F}_2(t) \le -C_{15}\delta_1\mathcal{F}_2^2(t), \tag{4.50}$$

which implies

$$\mathcal{F}_2(t) \le \frac{C_{16}}{t - t_3} \tag{4.51}$$

for all $t > t_3$ with some $C_{16} > 0$.

With the aids of (4.23) and (4.51), we find $C_{17} > 0$ such that

$$\int_\Omega \left((u - u_\star)^2 + w^2 \right) \le C_{17}\mathcal{F}_2(t) \le \frac{C_{16}C_{17}}{t - t_3} \tag{4.52}$$

for all $t > t_3$.

By the similar method as (4.29) and (4.30), we obtain (4.45). □

Proof of Theorem 1.3 Lemma 4.7 directly gives the statement of Theorem 1.3. □

5 Boundedness for $\tau = 1$

The aim of this section is to show the global boundedness of solution to (1.1) with $\tau = 1$ and prove Theorem 1.4. Firstly, we give the extensibility and regularity of solution to system (1.1), which will be used to confirm the global existence and boundedness of solutions.

Lemma 5.1 *Let $\tau = 1$ and $\Omega \subseteq \mathbb{R}^n (n \ge 1)$ be a smoothly bounded domain. Assume that $a_2, b_1 > 0$ and (1.6) hold. Suppose that there exists $p_0 \ge 1$ such that $p_0 > \frac{n}{2}$ and*

$$\sup_{t \in (0, T_{\max})} \left(\|u\|_{L^{p_0}(\Omega)} + \|w\|_{L^{p_0}(\Omega)} \right) < \infty. \tag{5.1}$$

Then $T_{\max} = \infty$ and

$$\sup_{t>0} \left(\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z\|_{L^\infty(\Omega)} \right) < \infty. \tag{5.2}$$

Moreover, there exist $\vartheta \in (0, 1)$ and $C_1 > 0$ independent of t such that

$$\begin{aligned} &\|u\|_{C^{\vartheta+2, \frac{\vartheta}{2}+1}(\bar{\Omega}\times[t,t+1])} + \|v\|_{C^{\vartheta+2, \frac{\vartheta}{2}+1}(\bar{\Omega}\times[t,t+1])} \\ &+ \|w\|_{C^{\vartheta+2, \frac{\vartheta}{2}+1}(\bar{\Omega}\times[t,t+1])} + \|z\|_{C^{\vartheta+2, \frac{\vartheta}{2}+1}(\bar{\Omega}\times[t,t+1])} \le C_1 \end{aligned} \tag{5.3}$$

for all $t > 1$.

Proof For the last term $f_1(u, w)$ in the $u-$equation of (1.1), we deduce

$$\begin{aligned} &u\left(a_0 - a_1 u - a_2 w + a_3 \int_\Omega u dx + a_4 \int_\Omega w dx\right) \\ &\le (a_0 + 2a_5 M_0)u - a_1 u^2 \\ &\le \frac{(a_0 + 2a_5 M_0)^2}{4a_1} \end{aligned} \tag{5.4}$$

with $a_5 := \max\left\{(a_3)_+, (a_4)_+\right\}$, where we have used $a_2 > 0$ and Lemma 2.2. We use the same method for $w-$equation. Then the following proof of this lemma is similar to [4, Lemma 2.6]. The proof of Lemma 5.1 is complete. □

Lemma 5.2 *Let $\tau = 1$ and $\Omega \subseteq \mathbb{R}^n (n \geq 1)$ be a smoothly bounded domain. Assume that $a_2, b_1 > 0$ and (1.6) hold. Then there exists $C_2 > 0$ such that*

$$\int_t^{t+\tau_0} \int_\Omega u^2(\cdot, s)ds + \int_t^{t+\tau_0} \int_\Omega w^2(\cdot, s)ds \leq C_2 \tag{5.5}$$

for all $t \in (0, T_{\max} - \tau_0)$, where $\tau_0 := \min\{1, \frac{T_{\max}}{2}\}$.

Proof This proof is similar to [14, Lemma 4.2]. Here we omit the details for brevity. □

Lemma 5.3 *Let $\tau = 1$ and $\Omega \subseteq \mathbb{R}^n (n \geq 1)$ be a smoothly bounded domain. Assume that $a_2, b_1 > 0$ and (1.6) hold. Then there exists $C_3 > 0$ such that*

$$\int_\Omega |\nabla v|^2 \leq C_3 \tag{5.6}$$

and

$$\int_\Omega |\nabla z|^2 \leq C_3 \tag{5.7}$$

for all $t \in (0, T_{\max})$, as well as

$$\int_t^{t+\tau_0} \int_\Omega |\Delta v(\cdot, s)|^2 ds \leq C_3 \tag{5.8}$$

and

$$\int_t^{t+\tau_0} \int_\Omega |\Delta z(\cdot, s)|^2 ds \leq C_3 \tag{5.9}$$

for all $t \in (0, T_{\max} - \tau_0)$, where $\tau_0 := \min\{1, \frac{T_{\max}}{2}\}$.

Proof This proof is similar to [14, Lemma 4.3]. The proof of Lemma 5.3 is complete. □

When $n = 2$, we shall establish L^2-bound of u and w, which is essential to obtain $L^\infty-$bound of u and w.

Lemma 5.4 *Let $\tau = 1$ and $\Omega \subseteq \mathbb{R}^2$ be a smoothly bounded domain. Assume that $a_2, b_1 > 0$ and (1.6) hold. Then there exists $C_4 > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq C_4 \tag{5.10}$$

and

$$\|w\|_{L^2(\Omega)} \leq C_4 \tag{5.11}$$

for all $t \in (0, T_{\max})$.

Proof Multiplying the first equation in (1.1) by $2u$ and integrating by parts over Ω, we deduce from Lemma 2.2 and Young's inequality that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^2 &= -2\int_\Omega |\nabla u|^2 - \xi_1 \int_\Omega u^2 \cdot \Delta v + \chi_1 \int_\Omega u^2 \cdot \Delta z \\ &\quad + 2\int_\Omega u^2\left(a_0 - a_1 u - a_2 w + a_3 \int_\Omega u + a_4 \int_\Omega w\right) \\ &\leq -2\int_\Omega |\nabla u|^2 - \xi_1 \int_\Omega u^2 \cdot \Delta v + \chi_1 \int_\Omega u^2 \cdot \Delta z \\ &\quad + 2(a_0 + 2a_5 M_0)\int_\Omega u^2 - 2a_1 \int_\Omega u^3 \end{aligned} \tag{5.12}$$

for all $t \in (0, T_{\max})$, where we have used that $a_2 > 0$ and $a_5 := \max\left\{(a_3)_+, (a_4)_+\right\}$.

Similarly, we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega w^2 \leq -2\int_\Omega |\nabla w|^2 - \xi_2\int_\Omega w^2\cdot\Delta z + \chi_2\int_\Omega w^2\cdot\Delta v \\ + 2(b_0 + 2b_5 M_0)\int_\Omega w^2 - 2b_2\int_\Omega w^3\end{aligned} \tag{5.13}$$

by $b_1 > 0$ for all $t \in (0, T_{\max})$, where $b_5 := \max\left\{(b_3)_+, (b_4)_+\right\}$. Then combining (5.12) with (5.13) and using Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u^2 + w^2) &\leq -2\int_\Omega |\nabla u|^2 - \xi_1\int_\Omega u^2\cdot\Delta v + \chi_1\int_\Omega u^2\cdot\Delta z \\ &\quad - 2\int_\Omega |\nabla w|^2 - \xi_2\int_\Omega w^2\cdot\Delta z + \chi_2\int_\Omega w^2\cdot\Delta v \\ &\quad + 2\int_\Omega u^2\Big((a_0 + a_5 M_0) - a_1 u\Big) \\ &\quad + 2\int_\Omega w^2\Big((b_0 + b_5 M_0) - b_2 w\Big) \\ &\leq -2\int_\Omega |\nabla u|^2 - \xi_1\int_\Omega u^2\cdot\Delta v + \chi_1\int_\Omega u^2\cdot\Delta z \\ &\quad - 2\int_\Omega |\nabla w|^2 - \xi_2\int_\Omega w^2\cdot\Delta z + \chi_2\int_\Omega w^2\cdot\Delta v + C_5\end{aligned} \tag{5.14}$$

for all $t \in (0, T_{\max})$, where $C_5 > 0$.

Then the following proof of this lemma is similar to [14, Lemma 4.4]. Here we omit the details. The proof of Lemma 5.4 is complete. □

When $n = 3$, in order to obtain the L^∞−bound of u and w, we still need to establish L^2−bound of u and w.

Lemma 5.5 *Let $\tau = 1$ and $\Omega \subseteq \mathbb{R}^3$ be a smoothly bounded domain. Assume that $a_2, b_1 > 0$ and (1.21) holds. Then there exists $C_6 > 0$ such that*

$$\|u\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} \leq C_6 \tag{5.15}$$

for all $t \in (0, T_{\max})$.

Proof We shall prove this lemma from the following steps:

Step 1. Based one (5.12), we apply Young's inequality and Lemma 2.2 to obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^2 &= -2\int_\Omega |\nabla u|^2 + 2\xi_1\int_\Omega u\nabla u\cdot\nabla v - 2\chi_1\int_\Omega u\nabla u\cdot\nabla z \\ &\quad + 2\int_\Omega u^2\Big(a_0 - a_1 u - a_2 w + a_3\int_\Omega u + a_4\int_\Omega w\Big) \\ &\leq -\int_\Omega |\nabla u|^2 + 2\xi_1^2\int_\Omega u^2|\nabla v|^2 + 2\chi_1^2\int_\Omega u^2|\nabla z|^2 \\ &\quad + 2(a_0 + 2a_5 M_0)\int_\Omega u^2 - 2a_1\int_\Omega u^3\end{aligned} \tag{5.16}$$

for all $t \in (0, T_{\max})$, where $a_5 := \max\left\{(a_3)_+, (a_4)_+\right\}$ and we have used the condition that $a_2 > 0$. By the similar way on $w-$equation, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2\\&\quad \le 2\xi_2^2\int_\Omega w^2|\nabla z|^2 + 2\chi_2^2\int_\Omega w^2|\nabla v|^2 + 2(b_0 + 2b_5M_0)\int_\Omega w^2 - 2b_2\int_\Omega w^3\end{aligned} \tag{5.17}$$

for all $t \in (0, T_{\max})$, where $b_5 := \max\left\{(b_3)_+, (b_4)_+\right\}$ and we have used the condition that $b_1 > 0$.

Step 2. Notice that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (b_1u + a_2w)(|\nabla v|^2 + |\nabla z|^2)\\&= \int_\Omega \Big(b_1u_t + a_2w_t\Big)\Big(|\nabla v|^2 + |\nabla z|^2\Big) + \int_\Omega \Big(b_1u + a_2w\Big)\Big((|\nabla v|^2)_t + (|\nabla z|^2)_t\Big)\end{aligned} \tag{5.18}$$

for all $t \in (0, T_{\max})$. Relying on the first and third equations in (1.1), it follows from Lemma 2.2 that

$$\begin{aligned}&\int_\Omega \Big(b_1u_t + a_2w_t\Big)\Big(|\nabla v|^2 + |\nabla z|^2\Big)\\&\le -\int_\Omega \nabla(b_1u + a_2w)\cdot\nabla(|\nabla v|^2 + |\nabla z|^2)\\&\quad + \int_\Omega (\xi_1b_1u - \chi_2a_2w)\nabla v\cdot\nabla(|\nabla v|^2 + |\nabla z|^2)\\&\quad + \int_\Omega (\xi_2a_2w - \chi_1b_1u)\nabla z\cdot\nabla(|\nabla v|^2 + |\nabla z|^2)\\&\quad + b_1\int_\Omega u(a_0 + 2a_5M_0 - a_1u)(|\nabla v|^2 + |\nabla z|^2)\\&\quad + a_2\int_\Omega w(b_0 + 2b_5M_0 - b_2w)(|\nabla v|^2 + |\nabla z|^2)\end{aligned} \tag{5.19}$$

for all $t \in (0, T_{\max})$. Applying Young's inequality once again, we obtain

$$\begin{aligned}&-\int_\Omega \nabla(b_1u + a_2w)\cdot\nabla(|\nabla v|^2 + |\nabla z|^2)\\&\le \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |\nabla w|^2 + (a_2^2 + b_1^2)\int_\Omega |\nabla|\nabla v|^2|^2 + (a_2^2 + b_1^2)\int_\Omega |\nabla|\nabla z|^2|^2\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}&\int_\Omega (\xi_1b_1u - \chi_2a_2w)\nabla v\cdot\nabla(|\nabla v|^2 + |\nabla z|^2)\\&\le \xi_1^2\int_\Omega u^2|\nabla v|^2 + \chi_2^2\int_\Omega w^2|\nabla v|^2 + \frac{a_2^2 + b_1^2}{2}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{a_2^2 + b_1^2}{2}\int_\Omega |\nabla|\nabla z|^2|^2\end{aligned} \tag{5.21}$$

as well as

$$\int_\Omega (\xi_2 a_2 w - \chi_1 b_1 u)\nabla z \cdot \nabla(|\nabla v|^2 + |\nabla z|^2) \\ \le \xi_2^2 \int_\Omega w^2 |\nabla z|^2 + \chi_1^2 \int_\Omega u^2 |\nabla z|^2 + \frac{a_2^2 + b_1^2}{2} \int_\Omega |\nabla |\nabla v|^2|^2 + \frac{a_2^2 + b_1^2}{2} \int_\Omega |\nabla |\nabla z|^2|^2 \tag{5.22}$$

for all $t \in (0, T_{\max})$.

It follows from the second and fourth equation in (1.1) that

$$\begin{aligned} &\int_\Omega \Big(b_1 u + a_2 w\Big)\Big((|\nabla v|^2)_t + (|\nabla z|^2)_t\Big) \\ &= \int_\Omega (b_1 u + a_2 w)\Big(\Delta |\nabla v|^2 - 2|D^2 v|^2 - 2|\nabla v|^2 + 2\nabla v \cdot \nabla w\Big) \\ &\quad + \int_\Omega (b_1 u + a_2 w)\Big(\Delta |\nabla z|^2 - 2|D^2 z|^2 - 2|\nabla z|^2 + 2\nabla z \cdot \nabla u\Big) \\ &= -\int_\Omega \nabla(b_1 u + a_2 w) \cdot \nabla(|\nabla v|^2 + |\nabla z|^2) + \int_{\partial\Omega} (b_1 u + a_2 w) \frac{\partial(|\nabla v|^2 + |\nabla z|^2)}{\partial \nu} \\ &\quad - 2\int_\Omega (b_1 u + a_2 w)\Big(|D^2 v|^2 + |D^2 z|^2\Big) - 2\int_\Omega (b_1 u + a_2 w)\Big(|\nabla v|^2 + |\nabla z|^2\Big) \\ &\quad + 2\int_\Omega (b_1 u + a_2 w)\Big(\nabla v \cdot \nabla w + \nabla z \cdot \nabla u\Big) \end{aligned} \tag{5.23}$$

for all $t \in (0, T_{\max})$, where we have used that $2\nabla v \cdot \nabla \Delta v = \Delta |\nabla v|^2 - 2\left|D^2 v\right|^2$ and $2\nabla z \cdot \nabla \Delta z = \Delta |\nabla z|^2 - 2\left|D^2 z\right|^2$. Likewise, applying Young's inequality for the last term in (5.23), there is

$$\begin{aligned} &2\int_\Omega (b_1 u + a_2 w)\big(\nabla v \cdot \nabla w + \nabla z \cdot \nabla u\big) \\ &\le \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla w|^2 \\ &\quad + 2b_1^2 \int_\Omega u^2 |\nabla v|^2 + 2b_1^2 \int_\Omega u^2 |\nabla z|^2 \\ &\quad + 2a_2^2 \int_\Omega w^2 |\nabla v|^2 + 2a_2^2 \int_\Omega u^2 |\nabla z|^2 \end{aligned} \tag{5.24}$$

for all $t \in (0, T_{\max})$.

Thus, in view of (5.18)–(5.24), we derive

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (b_1u+a_2w)(|\nabla v|^2+|\nabla z|^2)+2\int_\Omega (b_1u+a_2w)\big(|\nabla v|^2+|\nabla z|^2\big)\\
&\quad\le 2\int_\Omega |\nabla u|^2+2\int_\Omega |\nabla w|^2+(\xi_1^2+2b_1^2-a_1b_1)\int_\Omega u^2|\nabla v|^2\\
&\qquad+(\chi_1^2+2b_1^2-a_1b_1)\int_\Omega u^2|\nabla z|^2\\
&\qquad+(\chi_2^2+2a_2^2-a_2b_2)\int_\Omega w^2|\nabla v|^2+(\xi_2^2+2a_2^2-a_2b_2)\int_\Omega w^2|\nabla z|^2 \qquad (5.25)\\
&\qquad+3(a_2^2+b_1^2)\int_\Omega |\nabla|\nabla v|^2|^2+3(a_2^2+b_1^2)\int_\Omega |\nabla|\nabla z|^2|^2\\
&\qquad+b_1(a_0+2a_5M_0)\int_\Omega u(|\nabla v|^2+|\nabla z|^2)+a_2(b_0+2b_5M_0)\int_\Omega w(|\nabla v|^2+|\nabla z|^2)\\
&\qquad+\int_{\partial\Omega}(b_1u+a_2w)\frac{\partial(|\nabla v|^2+|\nabla z|^2)}{\partial\nu}
\end{aligned}$$

for all $t\in(0,T_{\max})$.

Step 3. It follows from the second equation in (1.1) and $2\nabla v\cdot\nabla\Delta v=\Delta|\nabla v|^2-2\left|D^2v\right|^2$ that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega|\nabla v|^4&=-2\int_\Omega|\nabla|\nabla v|^2|^2+2\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}-4\int_\Omega|\nabla v|^2|D^2v|^2\\
&\quad-4\int_\Omega|\nabla v|^4-4\int_\Omega w\Delta v|\nabla v|^2-4\int_\Omega w\nabla v\cdot\nabla|\nabla v|^2
\end{aligned}\qquad(5.26)$$

for all $t\in(0,T_{\max})$. By Young's inequality and the pointwise inequality $|\Delta v|^2\le n|D^2v|^2$ with $n=3$,

$$\begin{aligned}
-4\int_\Omega w\Delta v|\nabla v|^2&\le\frac{4}{3}\int_\Omega|\Delta v|^2|\nabla v|^2+3\int_\Omega w^2|\nabla v|^2\\
&\le 4\int_\Omega|\nabla v|^2|D^2v|^2+3\int_\Omega w^2|\nabla v|^2
\end{aligned}\qquad(5.27)$$

and

$$-4\int_\Omega w\nabla v\cdot\nabla|\nabla v|^2\le\int_\Omega|\nabla|\nabla v|^2|^2+4\int_\Omega w^2|\nabla v|^2.\qquad(5.28)$$

Thus, combining (5.26)–(5.28) implies

$$\begin{aligned}
\frac{d}{dt}\int_\Omega|\nabla v|^4&\le-\int_\Omega|\nabla|\nabla v|^2|^2+2\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}\\
&\quad+7\int_\Omega w^2|\nabla v|^2-4\int_\Omega|\nabla v|^4
\end{aligned}\qquad(5.29)$$

for all $t\in(0,T_{\max})$.

Similarly,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega|\nabla z|^4&\le-\int_\Omega|\nabla|\nabla z|^2|^2+2\int_{\partial\Omega}|\nabla z|^2\frac{\partial|\nabla z|^2}{\partial\nu}\\
&\quad+7\int_\Omega u^2|\nabla z|^2-4\int_\Omega|\nabla z|^4
\end{aligned}\qquad(5.30)$$

for all $t \in (0, T_{\max})$.

Step 4. As a consequence of (5.16), (5.17), (5.25), (5.29) and (5.30), we get

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(3(u^2+w^2)+(b_1u+a_2w)(|\nabla v|^2+|\nabla z|^2)+4(a_2^2+b_1^2)(|\nabla v|^4+|\nabla z|^4)\Big)\\
&\quad +2\int_\Omega (b_1u+a_2w)\big(|\nabla v|^2+|\nabla z|^2\big)+16(a_2^2+b_1^2)\int_\Omega\Big(|\nabla v|^4+|\nabla z|^4\Big)\\
&\quad +\int_\Omega\Big(|\nabla u|^2+|\nabla w|^2\Big)+(a_2^2+b_1^2)\int_\Omega\Big(|\nabla|\nabla v|^2|^2+|\nabla|\nabla z|^2|^2\Big)\\
&\le (7\xi_1^2+2b_1^2-a_1b_1)\int_\Omega u^2|\nabla v|^2+\Big(7\chi_1^2+2b_1^2+28(a_2^2+b_1^2)-a_1b_1\Big)\int_\Omega u^2|\nabla z|^2\\
&\quad +(7\chi_2^2+2a_2^2-a_2b_2)\int_\Omega w^2|\nabla v|^2+\Big(7\xi_2^2+2a_2^2+28(a_2^2+b_1^2)-a_2b_2\Big)\int_\Omega w^2|\nabla z|^2\\
&\quad +6(a_0+a_5M_0)\int_\Omega u^2-6a_1\int_\Omega u^3+6(b_0+b_5M_0)\int_\Omega w^2-6b_2\int_\Omega w^3\\
&\quad +b_1(a_0+2a_5M_0)\int_\Omega u\Big(|\nabla v|^2+|\nabla z|^2\Big)+a_2(b_0+2b_5M_0)\int_\Omega w\Big(|\nabla v|^2+|\nabla z|^2\Big)\\
&\quad +\int_{\partial\Omega}(b_1u+a_2w)\frac{\partial(|\nabla v|^2+|\nabla z|^2)}{\partial\nu}+8(a_2^2+b_1^2)\int_{\partial\Omega}\Big(|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}+|\nabla z|^2\frac{\partial|\nabla z|^2}{\partial\nu}\Big)
\end{aligned}
\tag{5.31}
$$

for all $t \in (0, T_{\max})$.

It follows from (1.21) that these constants $7\xi_1^2+2b_1^2-a_1b_1$, $7\chi_1^2+2b_1^2+28(a_2^2+b_1^2)-a_1b_1$, $7\chi_2^2+2a_2^2-a_2b_2$, $7\xi_2^2+2a_2^2+28(a_2^2+b_1^2)-a_2b_2$ are negative. Then, from (5.31) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(3(u^2+w^2)+(b_1u+a_2w)(|\nabla v|^2+|\nabla z|^2)+4(a_2^2+b_1^2)(|\nabla v|^4+|\nabla z|^4)\Big)\\
&\quad +2\int_\Omega (b_1u+a_2w)\big(|\nabla v|^2+|\nabla z|^2\big)+16(a_2^2+b_1^2)\int_\Omega\Big(|\nabla v|^4+|\nabla z|^4\Big)\\
&\quad +\int_\Omega\Big(|\nabla u|^2+|\nabla w|^2\Big)+(a_2^2+b_1^2)\int_\Omega\Big(|\nabla|\nabla v|^2|^2+|\nabla|\nabla z|^2|^2\Big)\\
&\le 6(a_0+2a_5M_0)\int_\Omega u^2-6a_1\int_\Omega u^3+6(b_0+2b_5M_0)\int_\Omega w^2-6b_2\int_\Omega w^3\\
&\quad +b_1(a_0+2a_5M_0)\int_\Omega u\Big(|\nabla v|^2+|\nabla z|^2\Big)+a_2(b_0+2b_5M_0)\int_\Omega w\Big(|\nabla v|^2+|\nabla z|^2\Big)\\
&\quad +\int_{\partial\Omega}(b_1u+a_2w)\frac{\partial(|\nabla v|^2+|\nabla z|^2)}{\partial\nu}+8(a_2^2+b_1^2)\int_{\partial\Omega}\Big(|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}+|\nabla z|^2\frac{\partial|\nabla z|^2}{\partial\nu}\Big)
\end{aligned}
\tag{5.32}
$$

for all $t \in (0, T_{\max})$.

Step 5. Let

$$
\begin{aligned}
\Phi(t) &:= 3\int_\Omega (u^2+w^2)+\int_\Omega (b_1u+a_2w)(|\nabla v|^2+|\nabla z|^2)+4(a_2^2+b_1^2)\int_\Omega\Big(|\nabla v|^4+|\nabla z|^4\Big),\\
\Psi(t) &:= \int_\Omega |\nabla u|^2+\int_\Omega|\nabla w|^2+(a_2^2+b_1^2)\int_\Omega\Big(|\nabla|\nabla v|^2|^2+|\nabla|\nabla z|^2|^2\Big)
\end{aligned}
$$

and

$$\Gamma(t) := 6\Big(a_0 + 2a_5 M_0 + \frac{1}{2}\Big)\int_\Omega u^2 - 6a_1\int_\Omega u^3 + 6\Big(b_0 + 2b_5 M_0 + \frac{1}{2}\Big)\int_\Omega w^2 - 6b_2\int_\Omega w^3$$
$$+ b_1(a_0 + 2a_5 M_0)\int_\Omega u\Big(|\nabla v|^2 + |\nabla z|^2\Big) + a_2(b_0 + 2b_5 M_0)\int_\Omega w\Big(|\nabla v|^2 + |\nabla z|^2\Big)$$
$$+ \int_{\partial\Omega}(b_1 u + a_2 w)\frac{\partial(|\nabla v|^2 + |\nabla z|^2)}{\partial\nu} + 8(a_2^2 + b_1^2)\int_{\partial\Omega}\Big(|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu} + |\nabla z|^2\frac{\partial|\nabla z|^2}{\partial\nu}\Big)$$

for all $t \in (0, T_{\max})$. Hence, we deduce from (5.32) that

$$\Phi'(t) + \Phi(t) + \Psi(t) \le \Gamma(t) \tag{5.33}$$

for all $t \in (0, T_{\max})$.

It follows from Young's inequality, Lemmas 2.4 and 4.3 that

$$\begin{aligned}
&b_1(a_0 + a_5 M_0)\int_\Omega u\Big(|\nabla v|^2 + |\nabla z|^2\Big) + a_2(b_0 + b_5 M_0)\int_\Omega w\Big(|\nabla v|^2 + |\nabla z|^2\Big)\\
&\le b_1^2(a_0 + a_5 M_0)^2\int_\Omega u^2 + a_2^2(b_0 + b_5 M_0)^2\int_\Omega w^2 + \int_\Omega |\nabla v|^4 + \int_\Omega |\nabla z|^4\\
&\le b_1^2(a_0 + a_5 M_0)^2\int_\Omega u^2 + a_2^2(b_0 + b_5 M_0)^2\int_\Omega w^2\\
&\quad + \frac{(a_2^2 + b_1^2)}{2}\int_\Omega\Big(|\nabla|\nabla v|^2|^2 + |\nabla|\nabla z|^2|^2\Big) + C_7
\end{aligned} \tag{5.34}$$

with some constants $C_7 > 0$. Applying Young's inequality, Lemma 2.2, Lemma 2.5 and Lemma 2.6 as well as Lemma 4.3, we obtain

$$\begin{aligned}
&\int_{\partial\Omega}(b_1 u + a_2 w)\frac{\partial(|\nabla v|^2 + |\nabla z|^2)}{\partial\nu} + 8(a_2^2 + b_1^2)\int_{\partial\Omega}\Big(|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu} + |\nabla z|^2\frac{\partial|\nabla z|^2}{\partial\nu}\Big)\\
&\le C_8\int_{\partial\Omega}(b_1 u + a_2 w)(|\nabla v|^2 + |\nabla z|^2) + 8C_8(a_2^2 + b_1^2)\int_{\partial\Omega}\Big(|\nabla v|^4 + |\nabla z|^4\Big)\\
&\le C_8\int_{\partial\Omega}(b_1^2 u^2 + a_2^2 w^2) + C_8\Big(8(a_2^2 + b_1^2) + 1\Big)\int_{\partial\Omega}\Big(|\nabla v|^4 + |\nabla z|^4\Big)\\
&\le \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla w|^2 + \frac{a_2^2 + b_1^2}{2}\int_\Omega\Big(|\nabla|\nabla v|^2|^2 + |\nabla|\nabla z|^2|^2\Big)\\
&\quad + C_9\Big(\int_\Omega u\Big)^2 + C_9\Big(\int_\Omega w\Big)^2 + C_9\Big(\int_\Omega |\nabla v|^2\Big)^2 + C_9\Big(\int_\Omega |\nabla z|^2\Big)^2\\
&\le \int_\Omega |\nabla u|^2 + \int_\Omega |\nabla w|^2 + \frac{a_2^2 + b_1^2}{2}\int_\Omega\Big(|\nabla|\nabla v|^2|^2 + |\nabla|\nabla z|^2|^2\Big) + C_{10}
\end{aligned} \tag{5.35}$$

for all $t \in (0, T_{\max})$, where C_8, C_9 and C_{10} are positive constants.

Inserting (5.34) and (5.35) into (5.33), we apply Young's inequality to obtain

$$\begin{aligned}\Phi'(t)+\Phi(t) &\le \left(6(a_0+a_5M_0+\frac{1}{2})+b_1^2(a_0+a_5M_0)^2\right)\int_\Omega u^2-6a_1\int_\Omega u^3\\ &\quad+\left(6(b_0+b_5M_0+\frac{1}{2})+a_2^2(b_0+b_5M_0)^2\right)\int_\Omega w^2-6b_2\int_\Omega w^3+C_7+C_{10}\\ &\le C_{11}\end{aligned} \tag{5.36}$$

with some $C_{11}>0$. One can get

$$\Phi(t)\le \max\{\Phi(0),C_{11}\} \tag{5.37}$$

in view of Lemma 2.3. The proof of Lemma 5.5 is complete. □

When $n\ge 4$, we shall show L^p−bound so as to obtain L^∞−bound of u and w.

Lemma 5.6 *Let $\tau=1$ and $\Omega\subseteq\mathbb{R}^n(n\ge 4)$ be a smoothly bounded domain. Assume that $a_2,b_1>0$ and (1.22) holds. Then for all $p>n$, there exists $C_{12}>0$ such that*

$$\|u\|_{L^p(\Omega)}+\|w\|_{L^p(\Omega)}\le C_{12} \tag{5.38}$$

for all $t\in(0,T_{\max})$.

Proof This proof is similar to [51, Lemma 4.1]. For reader's convenience, we give the sketch of the proof.

Multiplying the first equation in (1.1) by $pu^{p-1}(p>n)$ and integrating by parts over Ω, we infer from Young's inequality and Lemma 2.2 that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^p+(p+1)\int_\Omega u^p\\ &=-\frac{4(p-1)}{p}\int_\Omega|\nabla u^{\frac{p}{2}}|^2-\xi_1(p-1)\int_\Omega u^p\cdot\Delta v+\chi_1(p-1)\int_\Omega u^p\cdot\Delta z\\ &\quad+(pa_0+p+1)\int_\Omega u^p+p\int_\Omega u^p\left(-a_1u-a_2w+a_3\int_\Omega u+a_4\int_\Omega w\right)\\ &\le-\frac{4(p-1)}{p}\int_\Omega|\nabla u^{\frac{p}{2}}|^2+\left((a_0+2a_5M_0)p+p+1\right)\int_\Omega u^p\\ &\quad+\xi_1p\int_\Omega|\Delta v|^{p+1}+\chi_1p\int_\Omega|\Delta z|^{p+1}-\left(a_1-\xi_1-\chi_1\right)p\int_\Omega u^{p+1}\end{aligned} \tag{5.39}$$

for all $t\in(0,T_{\max})$, where we have used the condition that $a_2>0$. Applying Lemmas 2.2, 2.4 and Young's inequality, there exists a positive constant C_{13} such that

$$\begin{aligned}\left((a_0+2a_5M_0)p+p+1\right)\int_\Omega u^p &\le C_{13}\left(\|\nabla u^{\frac{p}{2}}\|^{2\alpha_1}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{2-2\alpha_1}_{L^{\frac{2}{p}}(\Omega)}+\|u^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\right)\\ &\le C_{13}M_0^{p(1-\alpha_1)}\left(\int_\Omega|\nabla u^{\frac{p}{2}}|^2\right)^{\alpha_1}+C_{13}M_0^p\\ &\le\frac{4(p-1)}{p}\int_\Omega|\nabla u^{\frac{p}{2}}|^2+C_{14}\end{aligned} \tag{5.40}$$

with $C_{14} = (1-\alpha_1)C_{13}^{\frac{1}{1-\alpha_1}}\left(\frac{p\alpha_1}{4(p-1)}\right)^{\frac{\alpha_1}{1-\alpha_1}} + C_{13}M_0^p > 0$, where $\alpha_1 = \frac{\frac{np}{2}-\frac{n}{2}}{1-\frac{n}{2}+\frac{np}{2}}$. Here it follows from $p > n$ that $\alpha_1 = \frac{\frac{np}{2}-\frac{n}{2}}{1-\frac{n}{2}+\frac{np}{2}} \in (0,1)$. By (5.39) and (5.40),

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^p + (p+1)\int_\Omega u^p \le \Big(\xi_1 + \chi_1 - a_1\Big)p\int_\Omega u^{p+1} \\ + \xi_1 p\int_\Omega |\Delta v|^{p+1} + \chi_1 p \int_\Omega |\Delta z|^{p+1} + C_8 \end{aligned} \tag{5.41}$$

for all $t \in (0, T_{\max})$.

Let $t_0 \in (0, T_{\max})$ such that $t_0 \le 1$. Applying the variation of constants formula to (5.41), there exists a positive constant C_{15} such that

$$\begin{aligned} \int_\Omega u^p(t) &\le \Big(\xi_1 + \chi_1 - a_1\Big)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega u^{p+1}(s)ds \\ &\quad + \xi_1 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta v(s)|^{p+1}ds + \chi_1 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta z(s)|^{p+1}ds \\ &\quad + e^{-(p+1)(t-t_0)}\int_\Omega u^p(t_0) + C_{11}\int_{t_0}^t e^{-(p+1)(t-s)}ds \\ &\le \xi_1 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta v(s)|^{p+1}ds + \chi_1 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta z(s)|^{p+1}ds \\ &\quad + \Big(\xi_1 + \chi_1 - a_1\Big)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega u^{p+1}(s)ds + C_{15} \end{aligned} \tag{5.42}$$

for all $t \in (t_0, T_{\max})$.

Similarly,

$$\begin{aligned} \int_\Omega w^p &\le \xi_2 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta z(s)|^{p+1}ds + \chi_2 p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta v(s)|^{p+1}ds \\ &\quad + \Big(\xi_2 + \chi_2 - b_2\Big)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega w^{p+1}(s)ds + C_{16} \end{aligned} \tag{5.43}$$

for all $t \in (t_0, T_{\max})$, where $C_{16} = \int_\Omega w^p(t_0) + \frac{C_{11}}{p+1}$.

Combining (5.42) with (5.43) yields

$$\begin{aligned} \int_\Omega u^p + \int_\Omega w^p &\le (\xi_1 + \chi_1 - a_1)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega u^{p+1}(s)ds \\ &\quad + (\chi_1 + \xi_2)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta z(s)|^{p+1}ds \\ &\quad + (\xi_2 + \chi_2 - b_2)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega w^{p+1}(s)ds \\ &\quad + (\xi_1 + \chi_2)p\int_{t_0}^t e^{-(p+1)(t-s)}\int_\Omega |\Delta v(s)|^{p+1}ds + C_{17} \end{aligned} \tag{5.44}$$

for all $t \in (t_0, T_{\max})$, where $C_{17} = C_{15} + C_{16}$.

Then the following proof of this lemma is similar to [14, Lemma 4.5]. Here we omit the details. The proof of Lemma 5.6 is complete. □

Proof of Theorem 1.4 For the case $n = 1$, it follows from Lemmas 2.2 and 5.1 that the solution of (1.1) is globally bounded. For the case $n = 2$, it follows from Lemma 5.1 and Lemma 5.4 that the solution of (1.1) is globally bounded. When $n = 3$, the global boundedness of solutions for (1.1) is obtained by Lemmas 5.1 and 5.5. When $n \geq 4$, it follows from Lemma 5.1 and Lemma 5.6 that the solution of (1.1) is globally bounded. The proof of Theorem 1.4 is complete. □

6 Asymptotic Behavior for $\tau = 1$

In this section, we discuss the asymptotic behavior for the fully parabolic system (1.1), and prove Theorem 1.5 and Theorem 1.6.

6.1 Proof of Theorem 1.5

Similar to the way used in Lemma 4.2, we introduce the following functionals

$$\mathcal{E}_1(t) := \int_\Omega \left(u - u_* - u_* \ln \frac{u}{u_*}\right) + \int_\Omega \left(w - w_* - w_* \ln \frac{w}{w_*}\right) + \frac{\rho_1}{2} \int_\Omega (v - v_*)^2 + \frac{\rho_2}{2} \int_\Omega (z - z_*)^2,$$

where (u_*, v_*, w_*, z_*) is given by (1.10) and $\rho_1, \rho_2 > 0$ shall be determined.

Lemma 6.1 *Let (u, v, w, z) be a global bounded classical solution to (1.1). Suppose that the conditions of Theorem 1.5 hold. Then there exists $\beta_1 > 0$ such that*

$$\frac{d}{dt}\mathcal{E}_1(t) \leq -\beta_1 \int_\Omega \left((u - u_*)^2 + (v - v_*)^2 + (w - w_*)^2 + (z - z_*)^2\right) \tag{6.1}$$

for all $t > 0$. Moreover,

$$\int_0^\infty \int_\Omega (u - u_*)^2 + \int_0^\infty \int_\Omega (v - v_*)^2 + \int_0^\infty \int_\Omega (w - w_*)^2 + \int_0^\infty \int_\Omega (z - z_*)^2 < \infty.$$

Proof This proof is similar to [14, Lemma 5.5] (or [4, Lemma 3.2]). The proof of Lemma 6.1 is complete. □

Lemma 6.2 *Suppose that the conditions of Theorem 1.5 hold. Then the solution (u, v, w, z) to (1.1) satisfies*

$$\|u - u_*\|_{L^\infty(\Omega)} + \|v - v_*\|_{L^\infty(\Omega)} + \|w - w_*\|_{L^\infty(\Omega)} + \|z - z_*\|_{L^\infty(\Omega)} \to 0 \tag{6.2}$$

as $t \to \infty$.

Proof A combination of Lemmas 4.1 and 6.1 implies this lemma. The proof of Lemma 6.2 is complete. □

Lemma 6.3 *Suppose that the conditions of Theorem 1.5 hold. Then there exist $C_1 > 0$ and $\kappa_1 > 0$ such that the solution (u, v, w, z) to (1.1) satisfies*

$$\|u - u_*\|_{L^\infty(\Omega)} + \|v - v_*\|_{L^\infty(\Omega)} + \|w - w_*\|_{L^\infty(\Omega)} + \|z - z_*\|_{L^\infty(\Omega)} \leq C_1 e^{-\kappa_1 t} \tag{6.3}$$

for all $t > t_4$, where $t_4 > 0$ is some fixed time.

Proof This proof is similar to [14, Lemma 5.7] (or [4, Lemma 3.6]). Here we omit the details. The proof of Lemma 6.3 is complete. □

Proof of Theorem 1.5 Lemma 6.3 directly shows the statement of Theorem 1.5. □

6.2 Proof of Theorem 1.6

Lemma 6.4 *Let (u, v, w, z) be a global bounded classical solution to (1.1). Suppose that the conditions of Theorem 1.6 hold. Then there exists $\beta_2 > 0$ such that*

$$\frac{d}{dt}\mathcal{E}_2(t) \le -\beta_2 \int_\Omega \Big((u-u_\star)^2 + v^2 + w^2 + (z-z_\star)^2\Big) \tag{6.4}$$

for all $t > 0$, where

$$\mathcal{E}_2(t) := \int_\Omega \big(u - u_\star - u_\star \ln\frac{u}{u_\star}\big) + \int_\Omega w + \frac{\rho_1}{2}\int_\Omega v^2 + \frac{\rho_2}{2}\int_\Omega \big(z - z_\star\big)^2.$$

Here $\rho_1, \rho_2 > 0$ shall be determined and $(u_\star, z_\star)$ is given by (1.16). Moreover,

$$\int_0^\infty \int_\Omega (u-u_\star)^2 + \int_0^\infty \int_\Omega v^2 + \int_0^\infty \int_\Omega w^2 + \int_0^\infty \int_\Omega (z-z_\star)^2 < \infty.$$

Proof This proof is similar to [14, Lemma 5.5] (or [4, Lemma 3.2]). Here we omit the details. The proof of Lemma 6.4 is complete. □

Lemma 6.5 *Suppose that the conditions of Theorem 1.6 hold. Then the solution (u, v, w, z) to (1.1) satisfies*

$$\|u-u_\star\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z-z_\star\|_{L^\infty(\Omega)} \to 0 \tag{6.5}$$

as $t \to \infty$.

Proof A combination of Lemmas 5.1 and 6.4 implies this lemma. The proof of Lemma 6.5 is complete. □

Lemma 6.6 *Suppose that the conditions of Theorem 1.6 hold.*

(i) *If $\frac{a_2-a_4|\Omega|}{b_2-b_4|\Omega|} < \frac{a_1-a_3|\Omega|}{b_1-b_3|\Omega|} < \frac{a_0}{b_0}$, then there exist $C_2 > 0$ and $\kappa_2 > 0$ such that the solution (u, v, w, z) to (1.1) satisfies*

$$\|u-u_\star\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z-z_\star\|_{L^\infty(\Omega)} \le C_2 e^{-\kappa_2 t} \tag{6.6}$$

for all $t > t_5$, where $t_5 > 0$ is some fixed time.

(ii) *If $\frac{a_2-a_4|\Omega|}{b_2-b_4|\Omega|} < \frac{a_1-a_3|\Omega|}{b_1-b_3|\Omega|} = \frac{a_0}{b_0}$, then there exist $C_3 > 0$ and $\kappa_3 > 0$ such that the solution (u, v, w, z) to (1.1) satisfies*

$$\|u-u_\star\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} + \|z-z_\star\|_{L^\infty(\Omega)} \le C_3 (t-t_6)^{-\kappa_3} \tag{6.7}$$

for all $t > t_6$, where $t_6 > 0$ is some fixed time.

Proof This proof is similar to Lemma 4.7 (or [4, Lemma 3.6]). Here we omit the details. The proof of Lemma 6.6 is complete. □

Proof of Theorem 1.6 Lemma 6.6 directly gives the results of Theorem 1.6. □

Acknowledgements The authors would like to deeply thank the editor and anonymous reviewers for their insightful and constructive comments. Pan Zheng is also deeply grateful to Professor Renjun Duan for his help and support at The Chinese University of Hong Kong.

Author Contributions Runlin Hu: Writing the original draft; Pan Zheng: Writing, Editing and Revising.

Funding The work is partially supported by National Natural Science Foundation of China (Grant Nos: 11601053, 11526042), Natural Science Foundation of Chongqing (Grant No. cstc2019jcyj-msxmX0082), China-South Africa Young Scientist Exchange Project in 2020, The Hong Kong Scholars Program (Grant Nos: XJ2021042, 2021-005), Young Hundred Talents Program of CQUPT in 2022–2024, Chongqing Municipal Education Commission Science and Technology Research Project (Grant No. KJZD-K202200602) and Chongqing Postgraduate Research and Innovation Project in 2022 (Grant Nos: CYS22451).

Availability of Data and Materials Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations

Ethical Approval Not applicable.

Competing interests The authors declare that this work does not have any conflicts of interests.

References

1. Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Commun. Pure Appl. Math. **12**, 623–727 (1959)
2. Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial diffferential equations satisfying general boundary conditions. II. Commun. Pure Appl. Math. **17**, 35–92 (1964)
3. Alikakos, N.: L^p-bounds of solutions of reaction-diffusion equations. Commun. Partial Differ. Equ. **4**, 827–868 (1979)
4. Bai, X., Winkler, M.: Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics. Indiana Univ. Math. J. **65**, 553–583 (2016)
5. Budrene, E., Berg, H.: Complex patterns formed by motile cells of Escherichia coli. Nature **349**, 630–633 (1991)
6. Burger, M., Francesco, M., Dolak, Y.: The Keller–Segel model for chemotaxis with prevention of overcrowding: linear vs. nonlinear diffusion. SIAM J. Math. Anal. **38**, 1288–1315 (2006)
7. Chaplain, M., Logas, G.: Mathematical modelling of cancer cell invasion of tissue: the role of the urokinase plasminogen activation system. Math. Models Methods Appl. Sci. **15**, 1685–1734 (2005)
8. Friedman, A.: Partial Differential Equations. Holt, Rinehart and Winston, New York (1969)
9. Gajewski, H., Zacharias, K.: Global behavior of a reaction-diffusion system modelling chemotaxis. Math. Nachr. **195**, 77–114 (1998)
10. Heihoff, F.: On the existence of global smooth solutions to the parabolic-elliptic Keller–Segel system with irregular initial data. J. Dyn. Differ. Equ. **9**, 1–25 (2021)
11. Horstmann, D., Winkler, M.: Boundedness vs. blow-up in a chemotaxis system. J. Differ. Equ. **215**, 52–107 (2005)
12. Hu, R., Zheng, P.: On a quasilinear fully parabolic attraction or repulsion chemotaxis system with nonlinear signal production. Discrete Contin. Dyn. Syst. Ser. B **12**, 7227–7244 (2022)
13. Hu, R., Zheng, P., Gao, Z.: Boundedness of solutions in a quasilinear chemo-repulsion system with nonlinear signal production. Evol. Equ. Control Theory **11**, 2209–2219 (2022)
14. Hu, R., Zheng, P., Shan, W.: On a two-species attraction-repulsion chemotaxis system with nonlocal terms, preprint
15. Ishida, S., Seki, K., Yokota, T.: Boundedness in quasilinear Keller–Segel systems of parabolic-parabolic type on non-convex bounded domains. J. Differ. Equ. **256**, 2993–3010 (2014)
16. Keller, E., Segel, L.: Initiation of slime mold aggregation viewed as an instability. J. Theoret. Biol. **26**, 399–415 (1970)

17. Kurt, H., Shen, W.: Finite-time blow-up prevention by logistic source in parabolic-elliptic chemotaxis models with singular sensitivity in any dimensional setting. SIAM J. Math. Anal. **53**, 973–1003 (2021)
18. Li, S., Muneoka, K.: Cell migration and chick limb development: chemotactic action of FGF-4 and the AER. Dev. Cell. **211**, 335–347 (1999)
19. Lin, K., Xiang, T.: On boundedness, blow-up and convergence in a two-species and two-stimuli chemotaxis system with/without loop. Calc. Var. **59**, 108 (2020)
20. Lin, K., Xiang, T.: On global solutions and blow-up for a short-ranged chemical signaling loop. J. Nonlinear Sci. **29**, 551–591 (2019)
21. Liu, A., Dai, B.: Blow-up vs boundedness in a two-species attraction-repulsion chemotaxis system with two chemicals. J. Math. Phys. **62**, 111508 (2021)
22. Liu, A., Dai, B.: Boundedness and stabilization in a two-species chemotaxis system with two chemicals. J. Math. Anal. Appl. **506**, 125609 (2022)
23. Liu, A., Dai, B., Chen, Y.: Boundedness in a two-species attraction-repulsion chemotaxis system with two chemicals. Discrete Contin. Dyn. Syst. Ser. B **27**, 6037–6062 (2022)
24. Mizoguchi, N., Souplet, P.: Nondegeneracy of blow-up points for the parabolic Keller–Segel system. Ann. Inst. H. Poincaré Anal. Non Linéaire **31**, 851–875 (2014)
25. Mizukami, M.: Boundedness and asymptotic stability in a two-species chemotaxis-competition model with signal-dependent sensitivity. Discrete Contin. Dyn. Syst. Ser. B **22**, 2301–2319 (2017)
26. Murray, J.: Mathematical Biology. Springer, Berlin (1993)
27. Nagai, T.: Blow-up of radially symmetric solutions of a chemotaxis system. Adv. Math. Sci. Appl. **5**, 581–601 (1995)
28. Negreanu, M., Tello, J.: On a competitive system under chemotactic effects with nonlocal terms. Nonlinearity **26**, 1083–1103 (2013)
29. Osaki, K., Yagi, A.: Finite dimensional attractors for one-dimensional Keller–Segel equations. Funkcial. Ekvac. **44**, 441–469 (2001)
30. Painter, K.: Continuous models for cell migration in tissues and applications to cell sorting via differential chemotaxis. Bull. Math. Biol. **71**, 1117–1147 (2009)
31. Painter, K., Hillen, T.: Volume-filling and quorum-sensing in models for chemosensitive movement. Can. Appl. Math. Q. **10**, 501–543 (2002)
32. Petter, G., Byrne, H., Mcelwain, D., Norbury, J.: A model of wound healing and angiogenesis in soft tissue. Math. Biosci. **136**, 35–63 (2003)
33. Senba, T., Suzuki, T.: Parabolic system of chemotaxis: blow up in a finite and the infinite time. Methods Appl. Anal. **8**, 349–367 (2001)
34. Stinner, C., Tello, J., Winkler, M.: Competitive exclusion in a two-species chemotaxis model. J. Math. Biol. **68**, 1607–1626 (2014)
35. Tao, Y., Winkler, M.: Boundedness in a quasilinear parabolic-parabolic Keller–Segel system with subcritical sensitivity. J. Differ. Equ. **252**, 692–715 (2012)
36. Tello, J., Winkler, M.: A chemotaxis system with logistic source. Commun. Partial Differ. Equ. **32**, 849–877 (2007)
37. Temam, R.: Infinite-Dimensional Dynamical Systemsin Mechanics and Physics, 2nd edition, Appl. Math. Sci. vol. 68, Springer, New York (1997)
38. Tu, X., Mu, C., Zheng, P., Lin, K.: Global dynamics in a two-species chemotaxis-competition system with two signals. Discrete Contin. Dyn. Syst. Ser. A **38**, 3617–3636 (2018)
39. Wang, L., Mu, C.: A new result for boundedness and stabilization in a two-species chemotaxis system with two chemicals. Discrete Contin. Dyn. Syst. Ser. B **25**, 4585–4601 (2020)
40. Wang, W., Zhuang, M., Zheng, S.: Positive effects of repulsion on boundedness in a fully parabolic attraction-repulsion chemotaxis system with logistic source. J. Differ. Equ. **264**, 2011–2027 (2018)
41. Winkler, M.: Does a ‘volume-filling effect’ always prevent chemotactic collapse? Math. Methods Appl. Sci. **33**, 12–24 (2010)
42. Winkler, M.: Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source. Commun. Partial Differ. Equ. **35**, 1516–1537 (2010)
43. Winkler, M.: Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model. J. Differ. Equ. **248**, 2889–2905 (2010)
44. Winkler, M.: Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening. J. Differ. Equ. **257**, 1056–1077 (2014)
45. Winkler, M.: Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities. SIAM J. Math. Anal. **47**, 3092–3115 (2015)
46. Yu, H., Wang, W., Zheng, S.: Criteria on global boundedness versus finite time blow-up to a two-species chemotaxis system with two chemicals. Nonlinearity **31**, 502–514 (2018)

47. Zhang, Q.: Competitive exclusion for a two-species chemotaxis system with two chemicals. Appl. Math. Lett. **83**, 27–32 (2018)
48. Zhang, Q., Liu, X., Yang, X.: Global existence and asymptotic behavior of solutions to a two-species chemotaxis system with two chemicals. J. Math. Phys. **58**, 111504 (2017)
49. Zheng, P.: Asymptotic stability in a chemotaxis-competition system with indirect signal production. Discrete Contin. Dyn. Syst. **41**, 1207–1223 (2021)
50. Zheng, P., Hu, R.: Boundedness and stabilization in a two-species attraction-repulsion chemotaxis-competition system, preprint
51. Zheng, P., Mu, C.: Global boundedness in a two-competing-species chemotaxis system with two chemicals. Acta Appl. Math. **148**, 157–177 (2017)
52. Zheng, P., Mu, C., Mi, Y.: Global stability in a two-competing-species chemotaxis system with two chemicals. Differ. Integral Equ. **31**, 547–558 (2018)
53. Zheng, P., Willie, R., Mu, C.: Global boundedness and stabilization in a two-competing-species chemotaxis-fluid system with two chemicals. J. Dyn. Differ Equ. **32**, 1371–1399 (2020)
54. Zheng, P., Xiang, Y., Xing, J.: On a two-species chemotaxis system with indirect signal production and general competition terms. Math. Models Methods Appl. Sci. **32**, 1385–1430 (2022)